Algorithmic Game Theory

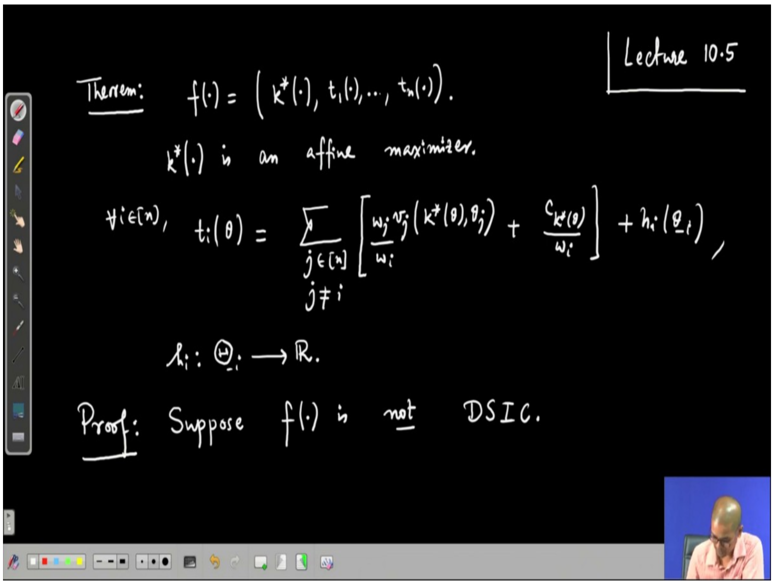
Prof. Palash Dey
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture - 50
Affine Maximizer

Welcome, in the last class we have seen an example of VCG mechanism in strategic network formation and then we were asking is there any other allocation rules other than allocatively efficient allocation rules which are implementable in DSIC in single parameter environment.

And, towards that we have stated a generalizations of Groves theorem which states that any affine maximizer is dominant strategy incentive compatible or they can be made domain strategy incentive compatible by a suitable payment formula and we have stated that theorem. So, today we will start with proving this result.

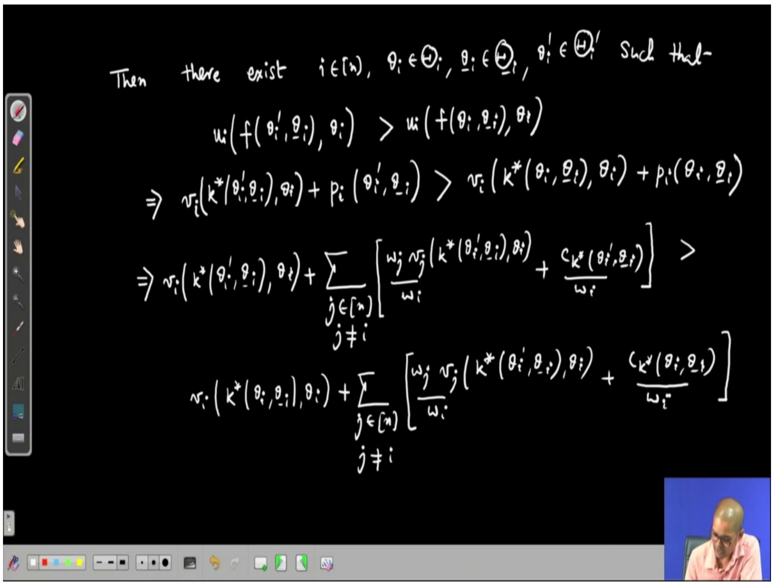
(Refer Slide Time: 01:12)



So, let us briefly state the result. So, what was the theorem, that we have a social choice function f which is $(k^*(.), t_1(.), \ldots, t_n(.))$, $k^*(.)$ is an affine maximizer and the payment is for all $i \in [n]$, $t_i(\theta) = \sum_{j \in [n], j \neq I} \left[v_j(k^*(\theta), \theta_j) \frac{w_j}{w_i} + \frac{c_{k^*(\theta)}}{w_i} \right] + h_i(\theta_{-i})$, where h_i is an arbitrary function of theta minus i to real numbers. Proof, these are proof by contradiction.

So, suppose if is not dominant strategy incentive compatible, suppose f is not dominant strategy incentive compatible.

(Refer Slide Time: 03:41)



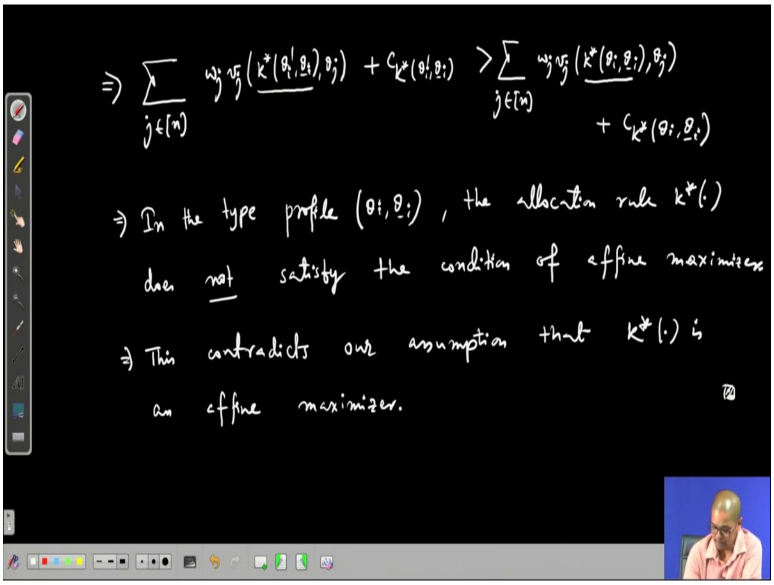
Then, there exists there exist a player $i \in [n]$, its type $\theta_i \in \Theta_i, \theta_{-i} \in \Theta_{-i}, \theta_i^{'} \in \Theta_i$ such that player i is better off reporting his type its type to be $\theta_i^{'}$, although its type is θ_i. So, what its utility? If it reports $\theta_i^{'}$ is $u_i(f(\theta_i^{'}, \theta_{-i}), \theta_i)$, because its two type is θ_i.

This is strictly greater than $u_i(f(\theta_i, \theta_{-i}), \theta_i)$. Now, what is utility? In the quasi-linear environment utility is valuation plus payment. So, $v_i(k^*(\theta_i^{'}, \theta_{-i}), \theta_i)$, this is the valuation plus payment $p_i(\theta_i, \theta_{-i})$. This is greater than $v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + p_i(\theta_i, \theta_{-i})$.

Now, let us plug in the payment formula. What is the payment formula? $v_i(k^*(\theta_i^{'}, \theta_{-i}), \theta_i)$. The payment formula is $\sum_{j \in [n], j \neq i} \left[w_j \frac{v_j(k^*(\theta_i^{'}, \theta_{-i}), \theta_i)}{w_i} + \frac{c_{k^*(\theta_i^{'}, \theta_{-i})}}{w_i} \right]$ plus $h_i(\theta_{-i})$ that will get cancelled from both side.

This is greater than $v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + \sum_{j \in [n], j \neq i} \left[w_j \frac{v_j(k^*(\theta_i^{'}, \theta_{-i}), \theta_i)}{w_i} + \frac{c_{k^*(\theta_i, \theta_{-i})}}{w_i} \right]$.

(Refer Slide Time: 08:49)



So, w i is at the denominator gets cancelled and what we get here is that $\sum_{j\in[n]} w_j v_j(k^*(\theta_i^{'},\theta_{-i}),\theta_j)+c_{k^*(\theta_i^{'},\theta_{-i})}>\sum_{j\in[n]} w_j v_j(k^*(\theta_i,\theta_{-i}),\theta_j)+c_{k^*(\theta_i,\theta_{-i})}$.

But, then in the type profile not in the type profile (θ_i,θ_{-i}), the allocation rule $k^*(.)$ does not satisfy the allocation rule $k^*(.)$ does not satisfy the condition of affine maximizer. Namely, it does not maximize sum of weighted valuations plus constant.

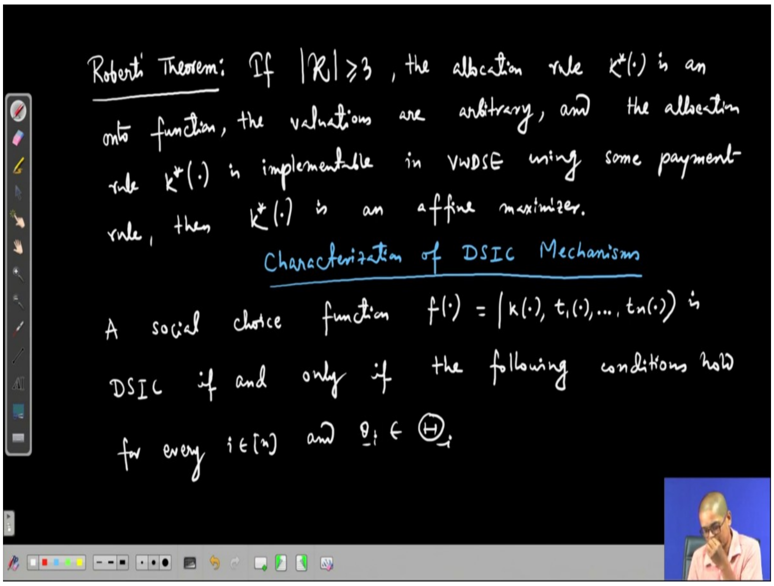
The outcome that at the allocation that it chooses is $k^*(.)(\theta_i,\theta_{-i})$ at the type profile (θ_i,θ_{-i}). But, at the same type profile (θ_i,θ_{-i}) another allocation, namely $k^*(.)(\theta_i^{'},\theta_{-i})$ gives strictly more weighted valuation plus the constant.

So, that contradicts. So, this contradicts our assumption that k star is an affine maximizer which is exactly what we need to prove. Because, if it is contradicts then what we assume to begin with is false; that means, it is not the case that it is not dominant strategy incentive compatible, it is the mechanism is dominant strategy incentive compatible. But, the, but again this also is a sufficient condition, it is not a necessary condition.

It only says that if the allocation rule is an affine maximizer, then there exist a payment rule which can make which results in the dominant strategy incentive compatible mechanism, but is it necessary or the same question remains. So, does there exist an allocation rule which is not a fine maximizer and still there exists a payment rule which makes the resulting mechanism dominant strategy incentive compatible.

And, the answer is no, affine maximizer is in some sense what is the maximum we can have and that is proved in Roberts' theorem.

(Refer Slide Time: 13:44)



Roberts' theorem, it says that if the number of allocations is at least 3 and the allocation rule is onto its an onto function, allocation rule is an onto function and the valuations can be arbitrary. Arbitrary valuations are allowed, valuations are arbitrary and the allocation rule k star is implementable in very weakly dominant strategy equilibrium using some payment rule, then k star must be an affine maximizer, k star is an affine maximizer.

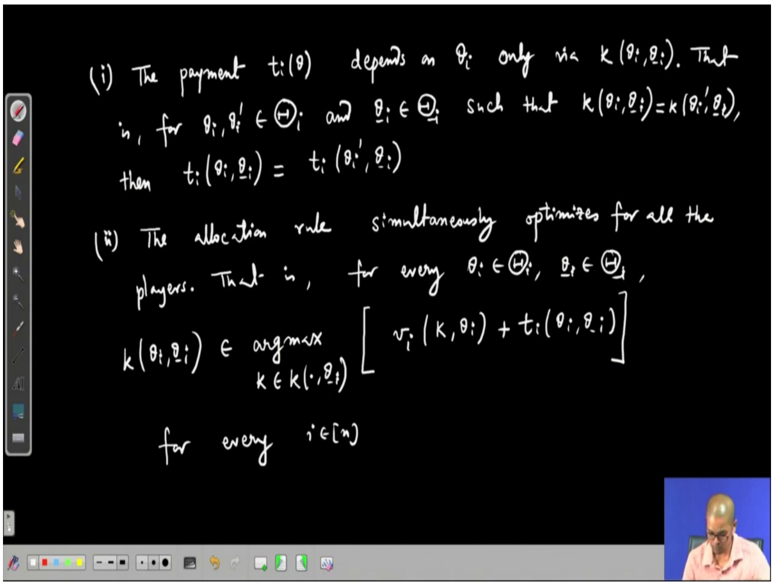
So, this Roberts' theorem and the because of the generalized Groves theorem, we have a satisfactory answer to characterize the set of all allocation rules which are dominant strategy incentive compatible in single in a quasi linear environment. What are they? They are simply the affine maximizer, if there is at least 3 allocations and the allocation rule is on to; that means, there is no some dummy allocation, it is just for staying it is there, but it is never selected.

If there is no such sort of dummy allocation; that means, to say that the allocation rule is onto then the then and if the allocation rule is implementable in dominant strategy equilibrium, very weakly dominant strategy equilibrium then $k^*(.)$ must be an affine maximizer. The proof of this result is out of scope, but this sort of resolves the question that what are the social choice functions which are implementable in dominant strategy equilibrium in the quasi-linear environment.

Now, what we will do is that we will see some more equivalent characterization of DSIC mechanisms in a quasi-linear environment which will be really useful further down the line. So, what we will see is some different kind of characterization of DSIC mechanisms in quasi linear environment. So, another kind of characterization of DSIC mechanisms ok.

So, a social choice function if this two parts, one is allocation, another is payment is dominant strategy incentive compatible if and only if, if and only if the following conditions hold conditions hold for every $i \in [n]$ and $\theta_{-i} \in \Theta_{-i}$. So, fix a player i with this put to this player for all type profiles of other players $\theta_{-i} \in \Theta_{-i}$, the following conditions will hold.

(Refer Slide Time: 20:08)



What is the first condition? The payment $t_i(\theta)$ depends on θ, depends on θ_i only via the allocation $k(\theta_i, \theta_{-i})$. So, in particular if i take two type profiles and if their allocation is same; that means, two type profiles theta and theta prime and if the allocations is same; that means, $k(\theta) = k(\theta')$, then the payment to player i will also be the same, if you only focus on θ_i.

So, if θ_{-i} remains same and theta i changes, but still allocation remains same then the payment also will remain same. So that means, let me let us write that is that is for $\theta_i, \theta_i' \in \Theta_i$ and $\theta_{-i} \in \Theta_{-i}$ such that. So, the type profile of other players remains same, remain the same and the type of the player i changes from θ_i to $\theta_i^{'}$, but still the allocation remains same.

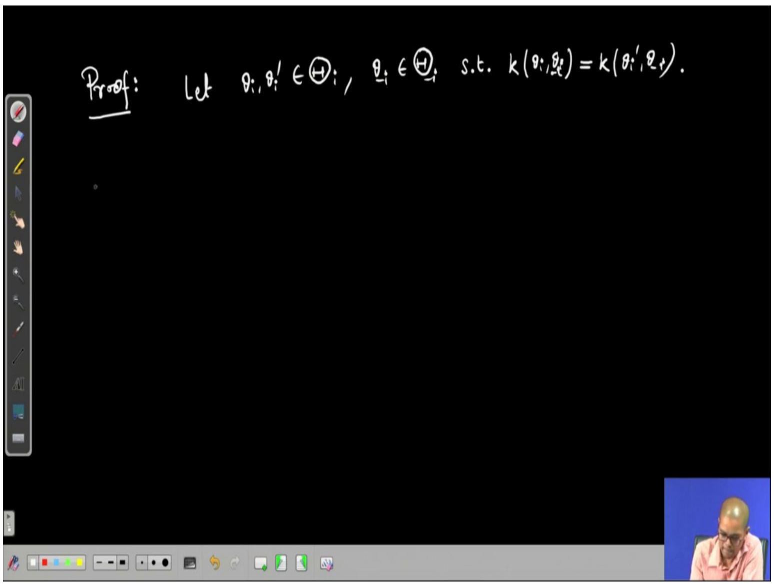
Such that $k(\theta_i, \theta_{-i})$ is same as $k(\theta_i^{'}, \theta_{-i})$. Then if this is the case; that means, for a type profile θ_{-i} of other players and two types θ_i and $\theta_i^{'}$ of player i, the allocation remains same; then the payment also remains same to player i.

This, then $t_i(\theta_i, \theta_{-i}) = t_i(\theta_i^{'}, \theta_{-i})$, this is what is what do you mean by payment depends on payment depends on θ_i only via k, via the allocation ok. So, that is the first condition. The second condition is that the allocation simultaneously maximizes the utility of all the players.

Let us write that the allocation rule simultaneously optimizes for all the players, what do you mean by that? That is for every small $\theta_i \in \Theta_i$ $\theta_{-i} \in \Theta_{-i}$, the allocation $k(\theta_i, \theta_{-i})$; if you look at from player is perspective, it is as if this is the best allocation chosen for player i which maximizes player is utility, $k_i \in arg\,max_{k \in k(.,\theta_{-i})} v_i(k, \theta_i) + t_i(\theta_i, \theta_{-i})$.

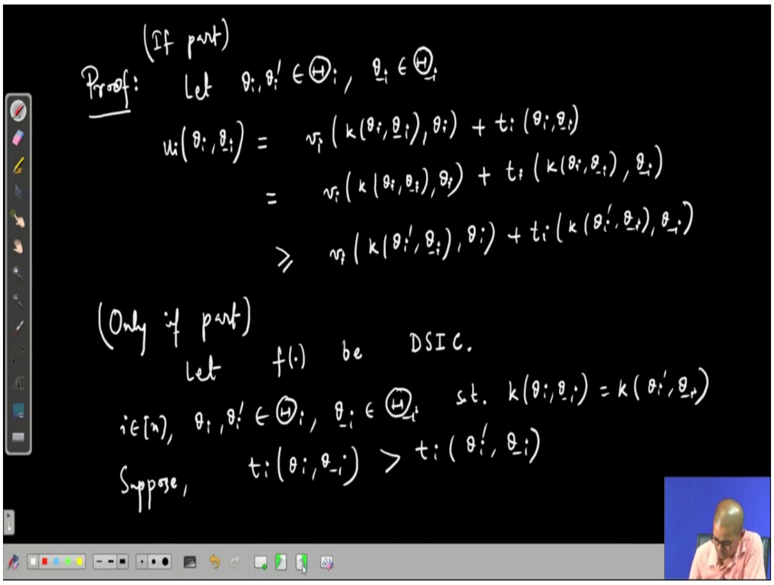
So, allocation function k takes the parameter θ_i and θ_{-i} as input, keep θ_{-i} fixed and vary θ_i and see where what are the allocations, what are the different allocations we have. And, among those allocations you pick the one which maximizes the utility $v_i(k, \theta_i) + t_i(\theta_i, \theta_{-i})$,. This should hold for every $i \in [n]$.

(Refer Slide Time: 26:18)



Proof, this will be this is a very useful characterization which we will see in next couple of classes, but let us prove it, the first part. So, let us take $\theta_i, \theta_i^{'} \in \Theta_i$ and $\theta_{-i} \in \Theta_{-i}$ such that the allocation remains same is $k(\theta_i^{'}, \theta_{-i})$, the allocation remains same. Then, you see so, first we are doing no cut, this first we are doing the if condition because this is the characterization.

(Refer Slide Time: 27:38)



So, first if part; that means, if there exist a social choice function which satisfies these two conditions then we will say that this is dominant strategy incentive compatible. So, then what is $u_i(\theta_i^{'}, \theta_{-i})$? We will show that in the type profile (θ_i, θ_{-i}) player i is not better off by reporting any other type say $\theta_i^{'}$.

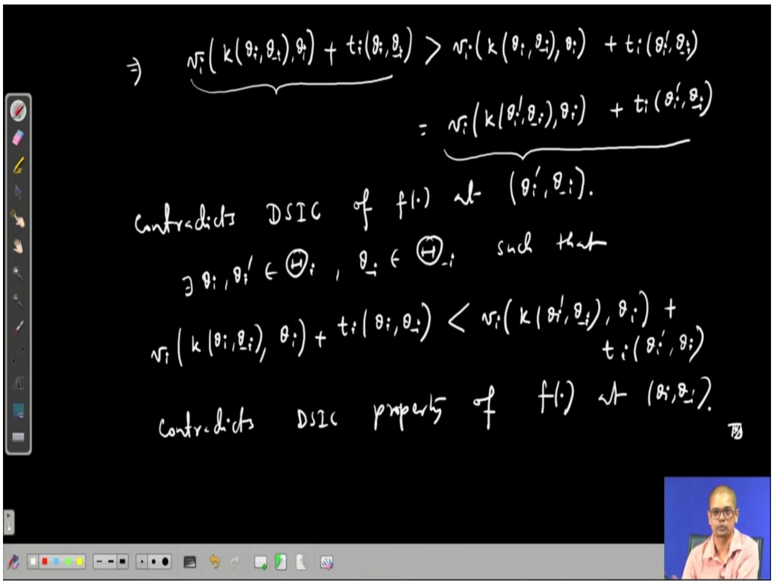
So, what is $u_i(\theta_i, \theta_{-i})$? This is valuation $v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i})$ ok, very good. So, now, this is now payment is a function of valuations only, this is $k(\theta_i, \theta_{-i})$. And, because this allocation only such that it maximizes this sum of the sum maximize this utility is greater than $v_i(k(\theta_i^{'}, \theta_{-i}), \theta_i) + t_i(k(\theta_i^{'}, \theta_{-i}), \theta_{-i})$.

So, allocation depends on θ_i only via k via this payment depends on θ_i only via the allocation. So, this proves the first part. There is no incentive for players to misreport. Now, only if part so, let f be dominant strategy incentive compatible. So, first I need to prove the first part; that means, it depends on the payment depends on allocation only.

So, let i be a player n and there are two types I take $\theta_i, \theta_i' \in \Theta_i, \theta_{-i} \in \Theta_{-i}$ such that $k(\theta_i, \theta_{-i}) = k(\theta_i', \theta_{-i})$, the payment must be the same. Suppose note so, suppose the payment is not same, although allocation remains same, although the type profile of other players remain the same.

Suppose, the payment is not same, suppose $t_i(\theta_i, \theta_{-i}) > t_i(\theta_i', \theta_{-i})$, although the allocation remains same.

(Refer Slide Time: 31:38)



So, what we do is that we add valuation $v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) < v_i(k(\theta_i', \theta_{-i}), \theta_i) + t_i(\theta_i', \theta_{-i})$.

Now, we have a contradiction to so, this means that this is the utility or so, this contradicts the dominant strategy incentive compatibility of player i. So, consider the type profile $\theta_i', \theta_i, \theta_{-i}$ and if it reports truthfully, then this is the utility that it gets.

On the other hand, if it misreports then and misreports is type to be θ_i, then the utility that it gets is the left hand side which is more. So, this contradicts dominant strategy incentive compatibility of f at type profile (θ_i', θ_{-i}) ok. So, similarly the second part, the only if part the second property.

So, suppose this does not maximize the utility. So that means, that there exist $\theta_i, \theta_i' \in \Theta_i, \theta_{-i} \in \Theta_{-i}$ such that $v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) < v_i(k(\theta_i', \theta_{-i}), \theta_i) + t_i(\theta_i', \theta_{-i})$

So, but the if this is the case, then again it is a contradiction to DSIC at $\theta_i^{'}, \theta_i, \theta_{-i}$, this also again contradicts DSIC property of if at (θ_i, θ_{-i}). So, which concludes the proof. So, we will conclude here today.