Algorithmic Game Theory
Prof. Palash Dey
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture - 47
VCG Mechanism

Welcome. So, from the last two lectures we have started studying quasi linear environment and in today's lecture we will study a class of dominant strategy incentive compatible mechanisms. This is not only one mechanism, but so many mechanisms.

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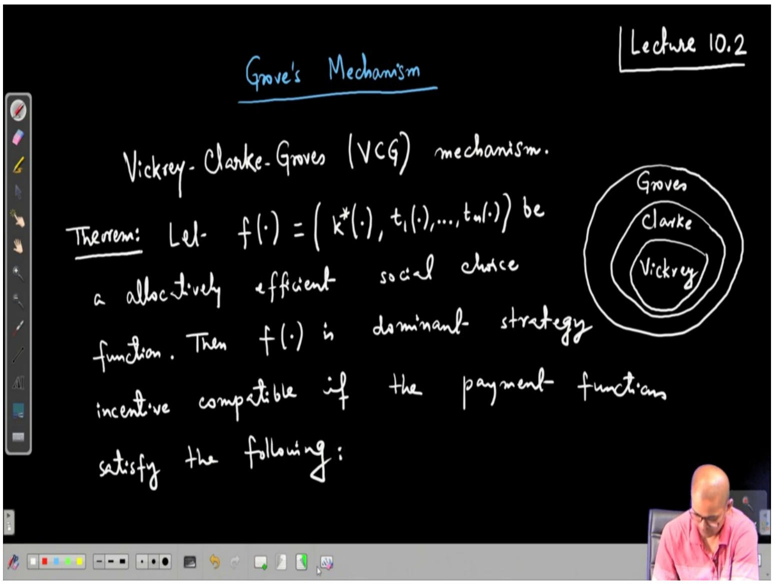

So, today's topic is Grove's mechanism. So, there is a whole class of mechanisms which is called a Vickrey – Clarke – Grove's mechanism, in short VCG mechanism and Grove's mechanism is the most general one it is like this is a class of mechanisms which are Grove's. A subclass of Grove's or Clarke mechanism and a further subclass is Vickrey's mechanism, ok.

So, today, we will study start with Grove's mechanism theorem what are Grove's mechanism? Let if is a social choice function which looks like of course; allocation rule let us call it k^* because k the allocation part is allocatively efficient. So, we often use k^* to stress or remember the fact that the allocation rule is allocatively efficient $t_1, \ldots, t_n$ be allocatively efficient social choice function.

A social choice function is called allocatively efficient if its allocation function is allocatively efficient. Then, f is dominant strategy incentive compatible if the payment function if the payment functions satisfy the following.

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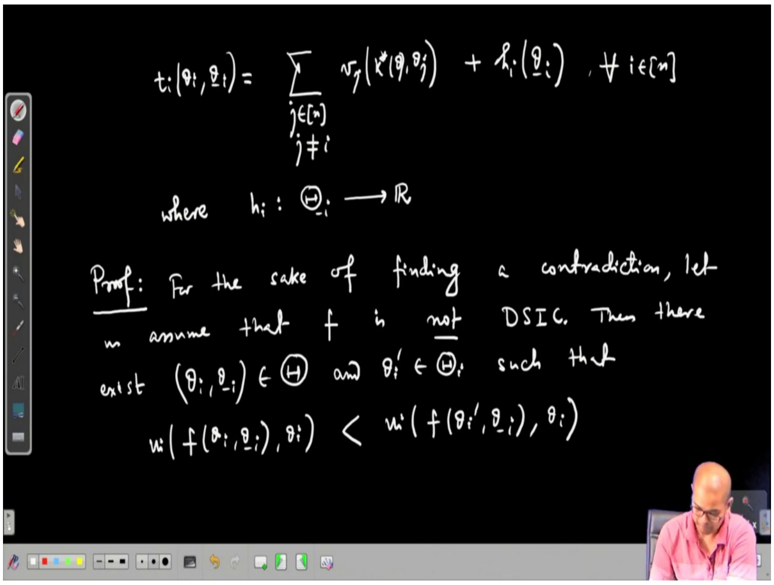

t i of it follows it follows certain structure certain condition is $\sum_{j\in[n], j\neq i} v_j(k^*(\theta), \theta_j)$. So, payment is sum of valuations of the outcome of every other players except the player i plus some function which is specific to player i, but that depends only on the type profile of other players. This this payment formula should be satisfied for all player $i \in [n]$.

So, how does the payment formula will look like payment should be a sum of valuations of all the players except player i plus some any arbitrary function or function which can vary from player to player. But, that function will depend only on the type profile of other players, where $h_i : \Theta_{-i} \to \mathbb{R}$ proof.

So, suppose this social choice function if satisfy this payment this structure; that means, k^* is allocatively efficient and the payment satisfy this structure, still it is not dominant strategy incentive compatible. So, for the sake of finding a contradiction let us assume that f is not dominant strategy incentive compatible, then there exists there exist a player i, there exist a type there exist like this way there exists a following thing (θ_i, θ_{-i}) type profile where player i benefits from misreporting from θ_i to $\theta_i^{'}$.

There exists a type profile (θ_i,θ_{-i}) and a type $\theta_i^{'}\in\Theta_i$ such that $u_i(f(\theta_i,\theta_{-i}),\theta_i)<u_i(f(\theta_i^{'},\theta_{-i}),\theta_i)$. So, when we calculate it is utility we will write θ_i. So, it receives strictly more utility by misreporting its type to be $\theta_i^{'}$, ok.

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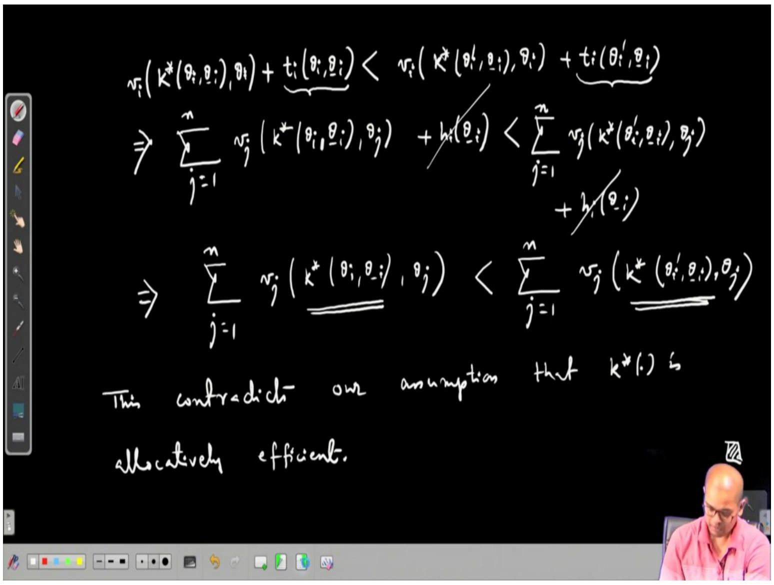

So, what is utility? Utility is nothing, but valuation plus payment. So, u_i is v_i plus payment $v_i(k^*(\theta_i,\theta_{-i}),\theta_i)+t_i(\theta_i,\theta_{-i})v_i(k^*(\theta_i^{'},\theta_{-i}),\theta_i)+t_i(\theta_i^{'},\theta_{-i})$.

Now, from here I can conclude that because of the Grove's payment structure, the payment t_i depends is a function of let us see t_i how does t_i look like t_i is this. So, let us put t_i there if I put t_i there, then t_i is sum of valuation of all players all other players. So, this is except i, but this i-th player valuation is here.

So, by putting the payment formula we get this is $\sum_{j=1}^{n} v_j(k^*(\theta_i,\theta_{-i}),\theta_j)+h_i(\theta_{-i})<\sum_{j=1}^{n} v_j(k^*(\theta_i^{'},\theta_{-i}),\theta_j)+h_i(\theta_{-i})$. So, this h_i gets cancelled and what we have is $\sum_{j=1}^{n} v_j(k^*(\theta_i,\theta_{-i}),\theta_j)<\sum_{j=1}^{n} v_j(k^*(\theta_i^{'},\theta_{-i}),\theta_j)$.

But, this contradicts that this allocation function is allocatively efficient k^* is allocatively efficient, it is not because then this contradicts our assumption that k^* is allocatively efficient. Why? Because look at the type θ in that type profile the outcome chosen by $k^*(\theta_i,\theta_{-i})$ and what is its sum of valuations? Sum of valuations is this the left hand side.

Now, you compare this outcome $k^*(\theta_i, \theta_{-i})$ with another outcome namely $k^*(\theta_i^{'}, \theta_{-i})$. This particular outcome in the type profile θ gives more sum of valuations which so, this contradicts that this allocation function k^* is allocatively efficient it is not because in this type profile theta I am able to find another outcome which have more sum of total valuations. So, this concludes the proof of Grove's theorem.

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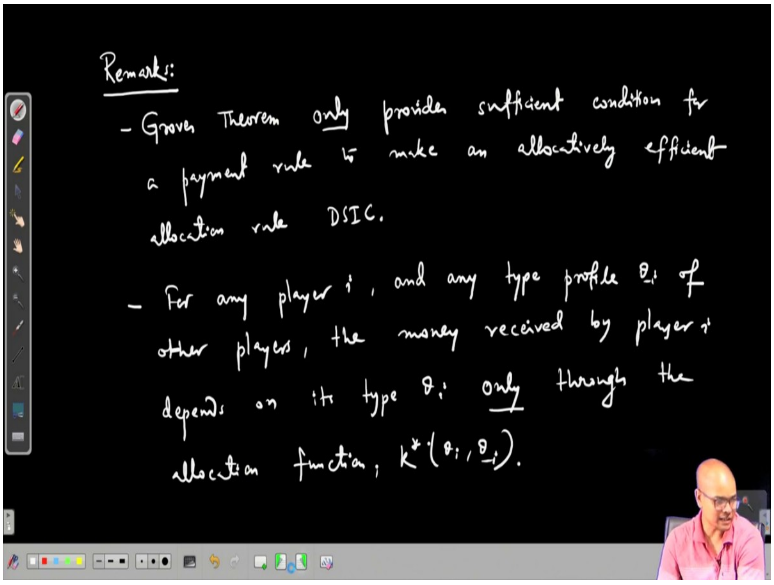

Couple of important remarks that Grove's theorem remarks, Grove's theorem only provides sufficient conditions sufficient condition for a payment rule to make an allocatively efficient allocation rule dominant strategy incentive compatible in particular this is not a necessary condition.

In particular, it is possible to have a payment rule to not satisfy this Grove's conditions and still make an allocatively efficient allocation rule to be dominant strategy incentive compatible. Then now, let us observe another important quantity. So, for any type profile of other players for any player i fix player i for any player i and any type profile θ_{-i} of other players.

You know the money received by player i money received by player i depends on its type θ_i only through the allocation function because the payment function, the payment rule does not depend on the valuation of the player of it is in the sum of valuations of other players. So, how does payment is connected with θ_i? It is only through the allocation rule k^* or allocation function it is $k^*(\theta_i, \theta_{-i})$.

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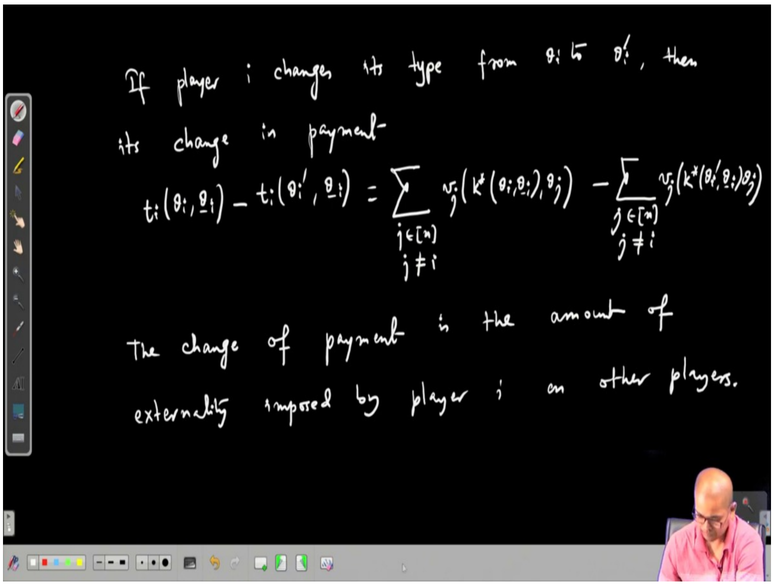

Not only that you know if player i changes its type from θ_i to say θ_i^{i}, then its change in payment what is change of change in payment? $t_i(\theta_i, \theta_{-i}) - t_i(\theta_i^{'}, \theta_{-i})$. What is t_i? t_i is $\sum_{j \in [n], j \neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) + h_i(\theta_{-i})$ that gets cancelled out from $t_i(\theta_i, \theta_{-i})$ and $t_i(\theta_i, \theta_{-i})$ both had $h_i(\theta_{-i})$.

So, that part gets cancelled. And, so, the only the first part remains $\sum_{j \in [n], j \neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j)$. So, you see in other words the change in payment is the externality imposed by player i to other players. How much player i is type getting being changed from θ_i to $\theta_i^{'}$ how much it changes how much it affects the other players what is the total amount of change of valuations in of other players that is the amount of difference of payments made to player i.

So, in other words the change of payment is the amount of externality imposed by player i on other players, ok.

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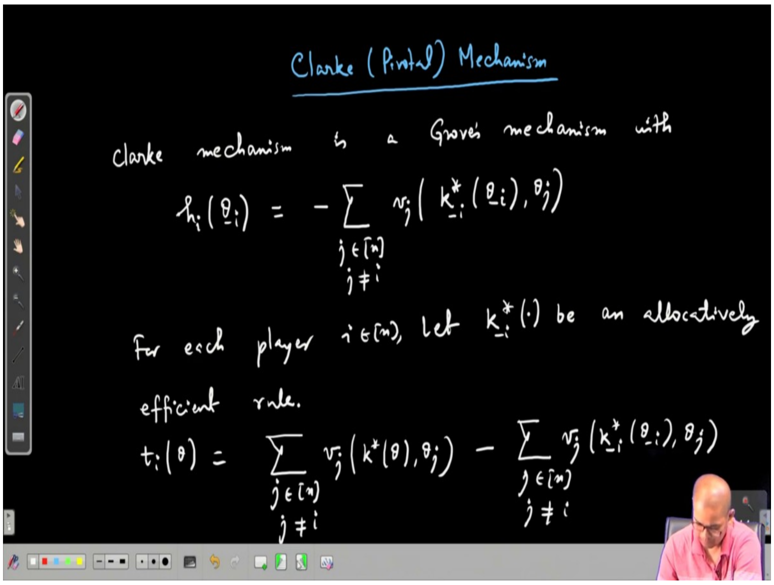

Next, we study let us study Clarke mechanism and this is also called Clarke pivotal mechanism. So, as I said it is a, it is a subclass of Grove's mechanism. Clarke mechanism is a subclass of. So, special case of Grove's mechanism Clarke mechanism is a Grove's mechanism with the payment t_i has two parts: one is sum of valuations of other plus h_i.

Now, h_i for Grove's mechanism it was arbitrary; for Clarke mechanism we are using some particular h_i. It is a function of type profile of other players it is $\sum_{j\in[n], j\neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j)$. So, suppose the scenario the setting is such that you know there are allocatively efficient allocation rules in a setting where i-th player is no longer there the i-th player is removed. So, we have many allocatively efficient rules.

So, K star is an allocatively efficient rule and for each player $i \in [n]$ let k^*_{-i} be an allocatively efficient rule ok. So, then it says that you know what should be $h_i(\theta_{-i})$? You remove player i from the system and see what is the best outcome or in that sense what is the outcome chosen by any allocatively efficient allocation rule and then what is the sum of the valuations of all the players in that outcome except i; i is not in the scenario and you take the minus of that.

So, with this $h_i(\theta_{-i})$ what is the payment? If I if we put the put h i of theta minus i in the Grove's payment rule $t_i(\theta) = \sum_{j\in[n], j\neq i} v_j(k^*(\theta), \theta_j) - \sum_{j\in[n], j\neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j)$.

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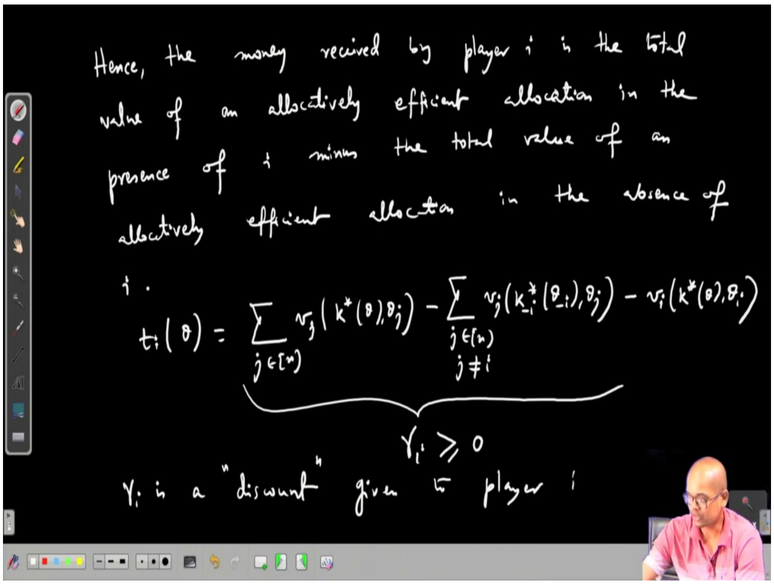

Hence, the money received by player i is the total value of an allocatively efficient allocation in the presence of i minus the total value of an allocatively efficient allocation in the absence of i total value means that the sum of the valuations of all the players, ok. Now, another interpretation of Clarke's payment is interesting.

So, Clarke's payment tool can also be thought of as or rewritten as $t_i(\theta)=\sum_{j\in[n]} v_j(k^*(\theta),\theta_j)-\left(\sum_{j\in[n], j\neq i} v_j(k^*_{-i}(\theta_{-i}),\theta_j)\right)-v_i(k^*(\theta),\theta_i)$.

And, let us call this particular expression γ_i and you know because k^* is a allocatively efficient rule this γ_i is always greater than equal to 0. So, you see that player i gets player's payment is its valuation and it is a given it is given a discount of γ_i.

So, gamma i is a discount given to player i because it the item it is receiving this allocation it values v_i and, but it is not paying exactly v_i it is paying slightly less and that is how much less that is γ_i and so, this is the discount. So, this particular discount is also called Clarke's discount, ok.

So, we will stop it stop here today. In the next class we will start looking at Vickrey mechanism.