Discrete Structures
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Lecture – 47
Algebraic Structures (Contd.)

So, in the last lecture, we have defined the algebraic systems, the Algebraic Structure and we have seen the general properties of the algebraic system. Now, in this lecture we will define some few algebraic systems and normally we use in real life problems.

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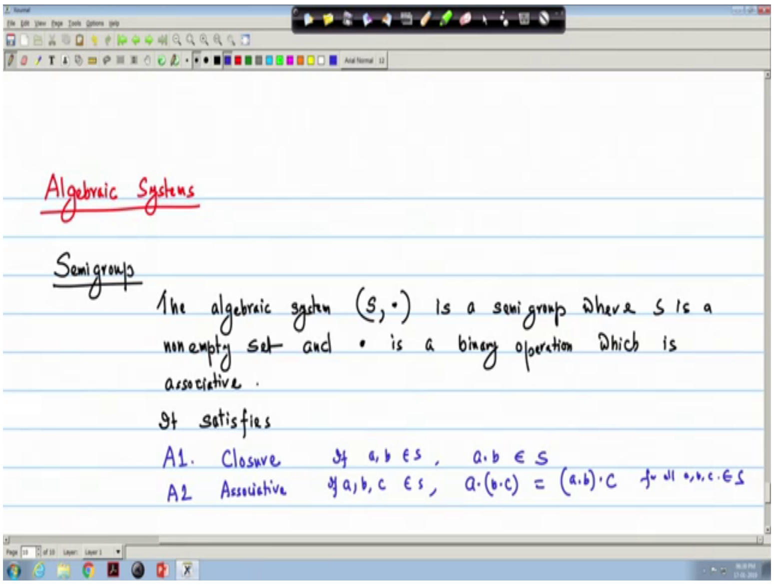

Now, so, again I can write that we will be reading the algebraic systems. So, the first algebraic system we will read or the simplest one is the semi group. So, this is the simplest algebraic structure. So, we define the as the algebraic system. Let this is a non-empty set is S and the operation only one operation we consider here is dot, the binary operation is a semi group where we write where S is a non-empty set and dot is a binary operation which is associative.

That means, if it is a semi group it satisfies the property of the following two properties it satisfies; that means, it satisfies the property of A 1, the closure; that means, if a, b belongs to S then a dot b belongs to S and property of A 2; that means, the associative;

that means, if a, b, c belongs to S we remember that associative is a dot b dot c is a dot b dot c and this is for all a, b, c belongs to for all a, b, c belongs to S.

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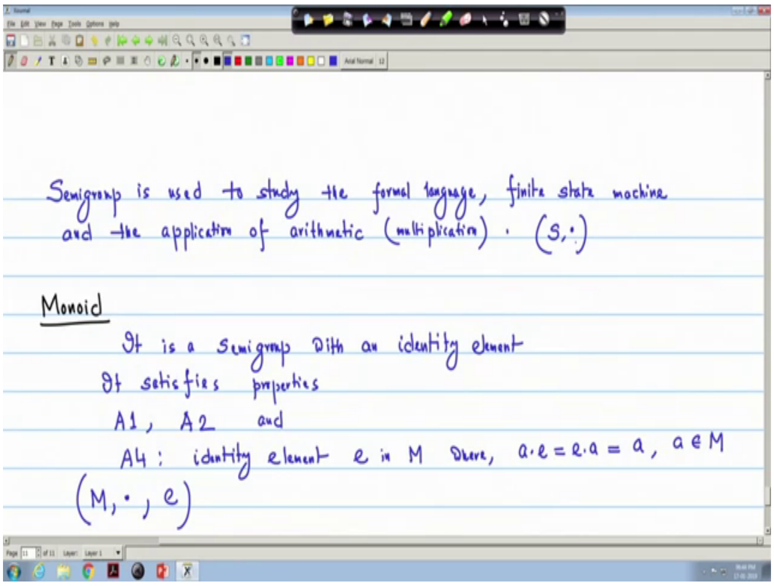

Now, normally this semi group is used to study the formal language, the finite state machine, Boolean algebra so, it has a number of applications. So, semi group is used to study its application area. The formal language, a finite state machine and the application of arithmetic say since we are taking the multiplication as separation so, any arithmetic operation associated with multiplication we can use this thing.

Now, I define another simple algebraic structure it is called the monoid again it is a non-empty set or I can tell that it is a semi group with the identity element. See we will in this study we will try to add more and more properties the algebraic system holds. So, in this way if I define that means it is closure it is associative in addition it has one identity element.

So, I can write that it is a semi group, it is a semi group with an identity element; that means, it satisfies A 1 properties A 1, A 2 since it is a semi group and A 3 the or A 4 we wrote that if I tell that it is identity element is A 4 as in the last lecture we have done. So, this is the identity element exists, identity element e say I am writing one identity element e in m where a dot e equal to e dot a equal to e, where for all a belongs to M.

So, normally this monoid we denote as M, e or give the operation if my operation is dot. So, M dot e whereas, earlier the semi group we write as we denote as simply S dot. So, we define two very simple type of algebraic structures. Now, we see some of the some of the examples that it holds ok.

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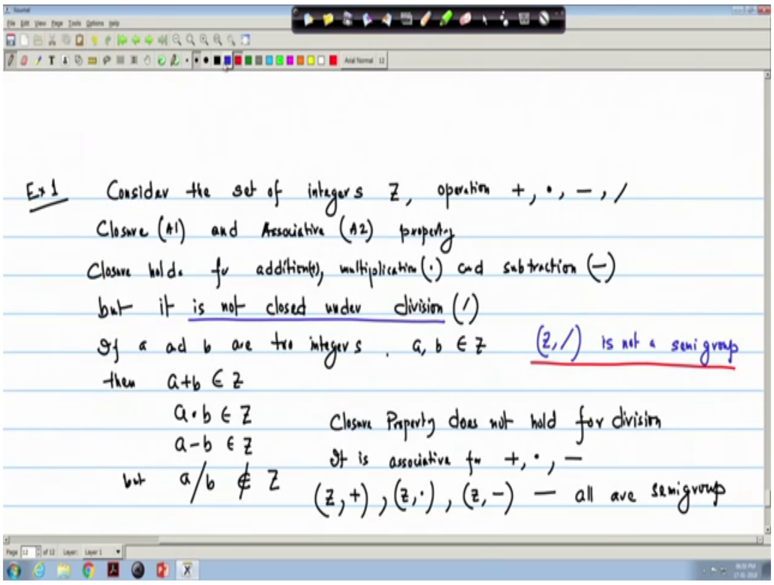

We take one example. So, we consider the set consider the set of integers say Z and operation plus addition multiplication subtraction and our division, ok. Now, see if I consider only the closure and the associative property; that means, closure consider only the closure; that means, my A 1 and associative property that is my A 2. Then it is closed the closure property holds; that means, closure holds for addition multiplication and subtraction, but it is not closed under division.

Since, if a and b are two integers; if a and b are two integers; that means, a b belongs to Z then a plus b belongs to Z a because a plus b is integer a dot b is integer a minus b is also integer, but a divided by b may not be a integer. So, closure property does not hold for division. So, closure property it is not closed under division. So, closure property does not hold for division; hold for division.

Now, if I consider my associative. Now, associate again it is associative for addition multiplication and subtraction. So, I can tell that it is a semi group; that means, Z plus then Z multiplication, then Z is subtraction all are semi group. But, say I can write here that Z division is not a, is not a semi group; since closure property does not hold. Now, we see another example of earlier we have seen that what about the odd integers.

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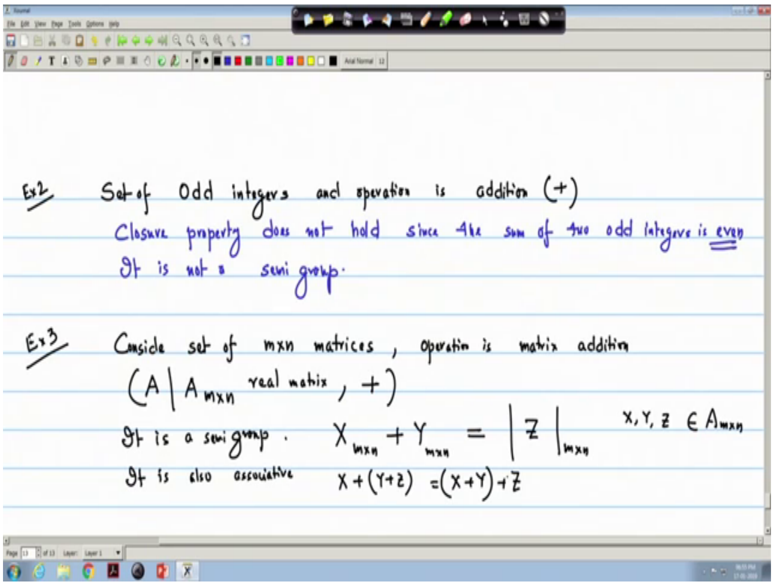

We take example 2, consider set of odd integers and operation is simple addition then it is not the or closure property does not hold. Since the sum of two odd integers are even, since the sum of two odd integers is even and which is not in the set because we have considered the set of odd integers. So, it is not a semi group; so, it is not a semi group.

Now, we consider all another example we consider that for a matrix. See if we consider set of m by n matrix; set of m by n matrices and the matrix addition operation is say matrix addition. So, I can write the system is A or A m by n matrix or m I can, it takes it can take the real values. So, I can write it is a real matrix and an operation is addition. So, this is the system we consider.

Now, it is a semi group; it is a semi group since it if we add it is a semi group since if we add 2 m by n matrices; that means, if I add a I take two different matrix a X m by n plus Y m by n then I will be getting some matrix a Z which is also m by n I will be getting a matrix Z m by n. Here X, Y Z all belongs to A m by n.

And, we know that it under addition that it is also associative because if I add X plus now, if I write that X plus Y plus Z will be get getting X plus Y plus Z. So, these are.

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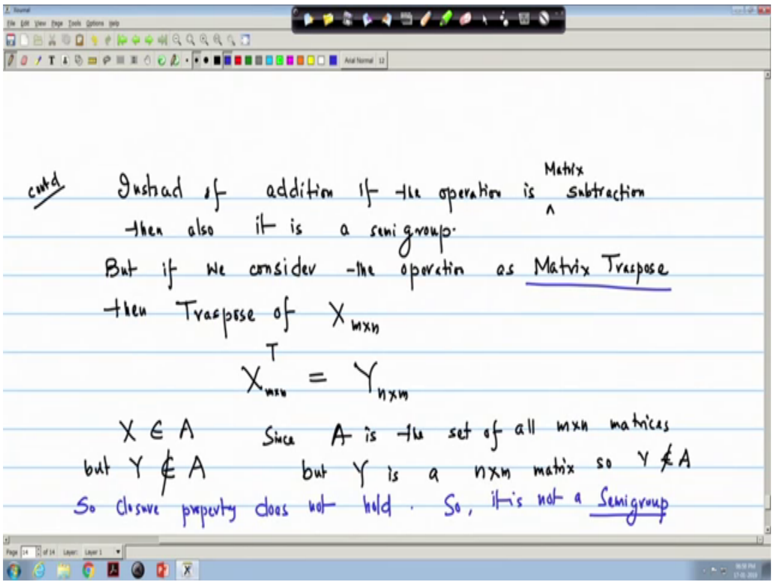

If instead of addition if we consider subtraction; if we consider subtraction; so, instead of addition if the operation is subtraction say matrix subtraction then also it is a semi group, but if we consider the operation as the transpose matrix transpose. It is a matrix subtraction ok; consider the operation as matrix transpose then transpose of X m by n; normally we write that thing as it is denoted as X transpose X T. So, X T if it is dimension m by n that will give you that some Y n by m.

Now, see the order is changed this becomes n by n. So, X belongs to A because A is the set of all m by n matrices, but Y does not belongs to A, since A is the set of all m by n matrices. Since, but Y is a n by m matrix; so, Y does not belongs to A. So, it is not closure. So, that means, closure property does not hold; so, closure property does not hold. So, it is not a semi group; it is not a semi group. This is the closure property does not hold for this thing.

Now, if it is not a semi group then; obviously, it is not a monoid. Since the monoid we have defined as the a semi group with an identity element.

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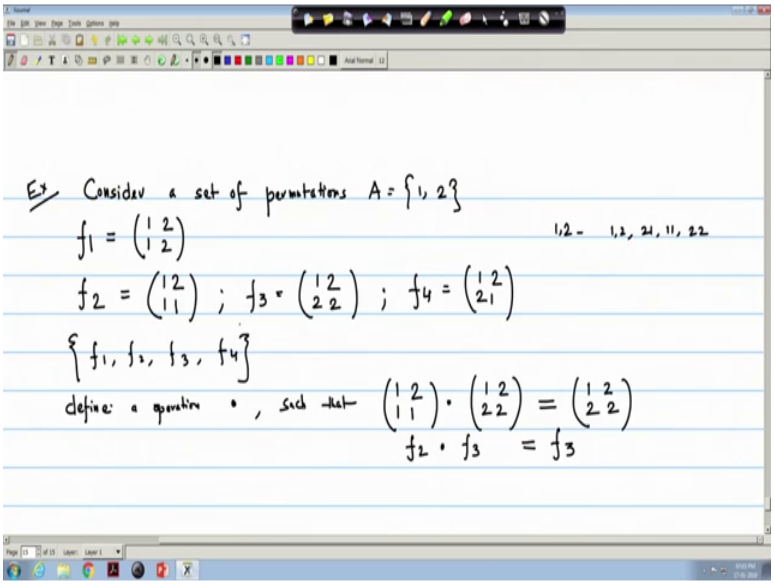

So, now we see another different type of example, say we consider the set of permutation of two of a two element set.

Consider set of permutations of a two element set A; say the elements are 1, 2. So, mainly the operations here because remember this we told that this can be a function, this can be transformation, this can be a mapping. So, we have considering here as if this is the set and that all the permutations we are taking. So, the permutations we define say like say f 1, I define as if 1 2 1 2. So, this is my permutation.

Now, other permutations possible are f 2 is 1 2 1 1; f 3 is 1 2 2 1 or 2 2 and f 4 I take 1 2 2 1. Since, I have only two element set 1 2, so, with these two elements I can get 1 2, 2 1, 1 1, 2 2 these are the combinations; that means, these are my permutations. So, this permutations is 1 2, it will permutated to 1 2 only, it is permutated to 1 1, it is permutated to 2 2, it is permutated. So, these are the only four possible permutations possible. So, these are my set, these are my set of permutations, ok. So, my set of permutations these are my sets are f 1, f 2, f 3, and f 4.

Now, we see that which class of algebraic system it forms. Now, we first define a operation, we define a operation dots like define a operation dot such that it will take let us say I give say 1 2 1 1 dot 1 2 2 2 I take this operation; that means what? 1 2 1 1 is my f 2 and 1 2 2 2 is my f 3 as if.

Now, permutation this will be permuted that see if it is 1 2 1 is permuted to 1 and here 1 is permuted to 2. So, I give this is 1 2; 2 it is 1 and then again 1 it becomes 2. So, this is 2 2. So, that 1 2 2 2 is my f 3. So, it becomes f 2 dot f 3 is f 3. So, the dot operation we define on these permutations like this, then see that if I do these operations we made on other elements since we have f 1, f 2, f 3, f 4 that four permutations possible.

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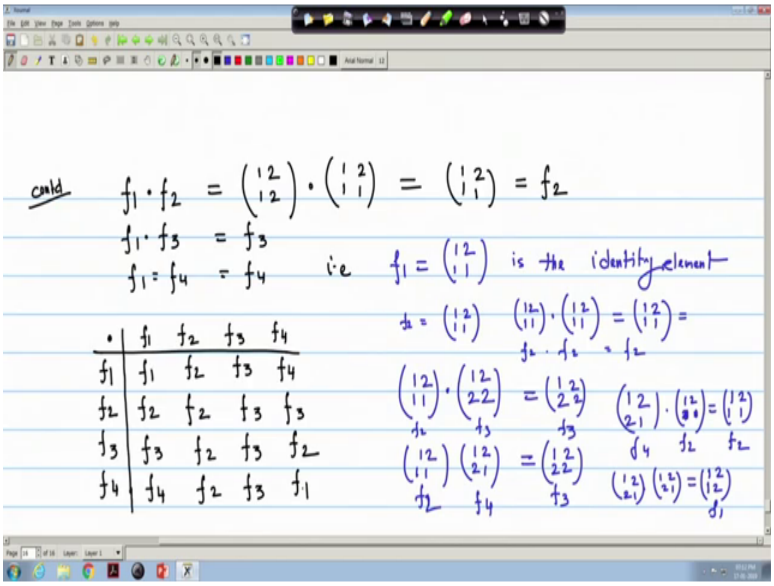

So, first we see that if we made f 1 dot f 2; f 1 we have done 1 2 1 2, f 2 we have given 1 2 1 1 and we will see the way we have defined it is 1 1. So, this becomes 1 1; 2 2 2 1 so, 2 2 2 1; that means, it is f 2 only. So, f 1 dot f 2 is f 2. Similarly, if I do the f 1 dot f 3 I will be getting f 3, f 1 dot f 4 I will be getting f 4 only; that means, that is the f 1 equal to 1 2 1 1 is the identity; is the identity element.

Now, if I complete all the operations under this dot we have defined; so, I have f 1, f 2, f 3, f 4 with these dot operation just we have defined on the permutations of two elements set then I get since, f 1 is the identity. So, f 1 dot f 1 it will f 1, it is f 2, it is f 3, it is f 4.

Now, f 2 dot f 1, if I because f 1 is the identity. So, I will be getting this is f 2 only f 2 dot f 2 if just we do f 2 is 1 2 1 1. So, 1 2 1 1 dot 1 2 1 1; this give 1 1 and then 2 1 1 1 so, this becomes 1 2 1 1 1 1 2 1 is the f 1 f 2 1 2 1 1 is the f 2. So, f 2 dot f 2 is f 2 only. So, in this way if I can fill up that I will be getting this is f 2 this is also f 2 dot f 2 is f 2 then f 2 dot f 3 is f 3 and f 2 dot f 4 is also f 3. I see f 2 dot f 4 if we do f 2 is 1 1, f 4 we have written f 4 is 1 2 2 1. So, this will be say 1 1 1 2; that means, this becomes 1 2 then 2 1 1 2 so, 2 2 2 1 1 2. So, this is 2 2.

So, it is f 4. Now, this is f 2, this is f 3. So, you have given it is just a minute, f 2 we have taken f 2 we have taken 1 2 1 1, f 3 is 1 1 2 2 2 f 3 is 1 3 2 2, f 3 is 1 2 2 2, this is 2 2, this is 2 2. Then this is 1 1 1 2, 1 2 2 1 2 2. So, this is f 3 only. Now, this is f 3 only. So, f 2 dot f 3 is f 3; then f 2 dot f 4 is also f 3, ok; f 2 dot 1 2 1 1 and f 4 was 1 2 2 1 and that becomes 1 2 2 1 just a minute f 4 what is f 4? f 4 is 2 1 ok. So, this becomes 2 2. So, that is also f 3.

So, this is f 1 f 2 dot f 4 that is it that is also f 3. So, we have given this is f 3. So, now, in similar way if I do then I will be getting a this is already we have done f 3 f 1 this becomes f 3 then this is f 2 we have done then f 3 f 3 is f 3 and f 3 f 4 will get f 2 and then it is f 4 f 1 is f 4; f 4 f 2 already we have done f 4 f 2 is f 3 or we will just do the f 4 f 2 f 4 is 1 2 1 2 2 1, this is my f 4 and I want f 2 so, 1 2 1 1; 1 2 1 1. So, this becomes 1 2 2 1; 1 2 2 1 and then 2 1 1 1; so, 2 1. So, this becomes also 1 2 1 1. So, 1 2 1 1 is f 2. So, this is also f 2. So, this becomes f 2, then this becomes f 3 and this becomes f 4, f 4 this becomes you will see that f 4 f 4 is 1 2 2 1; 1 2 2 1. So, this becomes 1 2 2 1; 1 2 2 1 1 1 2 1 1 2. So, 2 1 1 2 this is 2 2. So, this is my f 1 only, 1 2 1 2 is my f 1 only, ok.

So, this is my hope we get this operation in this. Now, what you see from this? That means, it is first thing is it is closure because I have the set only f 1, f 2, f 3, f 4 when we operate this thing on dot then we get again f 2, f 3, f 4.

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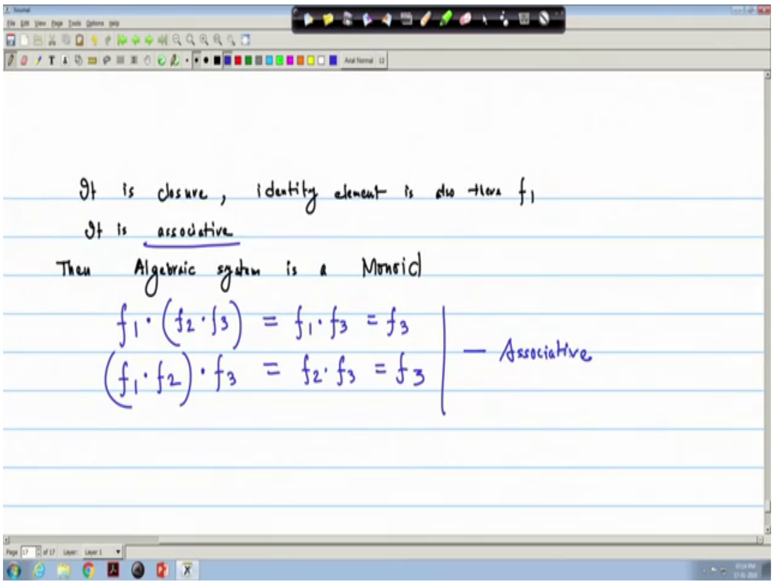

So, we get that it is closure, then identity element is there; identity element is also there is f 1 and it is associative then the system of the algebra with algebraic system is a monoid.

Since it has a and we can see that associative property also, we can see that it holds the associative property; that means, if I can do that f 1 dot f 2 dot f 3 we will see that just from the previous thing that f 2 dot f 3 is f 3 f 2 dot f 3 is f 3. So, f 1 dot f 3 which is f 3 only. Now, if we dot f 1 dot f 2 dot f 3 since f 1 dot f 2 is f 2 only and this is f 3. So, I get f 3. So, if it gives me the associative property, ok. So, we get a system that is set of permutation of two elements set which is a monoid and obviously, that is a semi group.