Discrete Structures
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Lecture – 45
Combinatorics (Contd.)

We are discussing that how to handle the counting problems with permutation and combination. And mainly the very simple way we have defined the permutation combination problems as the arrangement of ordered and unordered sets or ordered and unordered selection or arrangement from a set.

Now, this lecture will be discussing one topic it is called the derangement. I can simply tell that this is a different type of arrangement or the reverse type of arrangement. So, normally arrangement some property is to be satisfied we have defined. Now, here also that property is that arrangement should not be there.

So, sometimes we will tell that this is actually that the inverse of the arrangement. And there also this is one counting techniques because we will see many real life problems that come comes under this type of topic. So, this is normally called that derangement. So, this is a important combinatorial problem ok.

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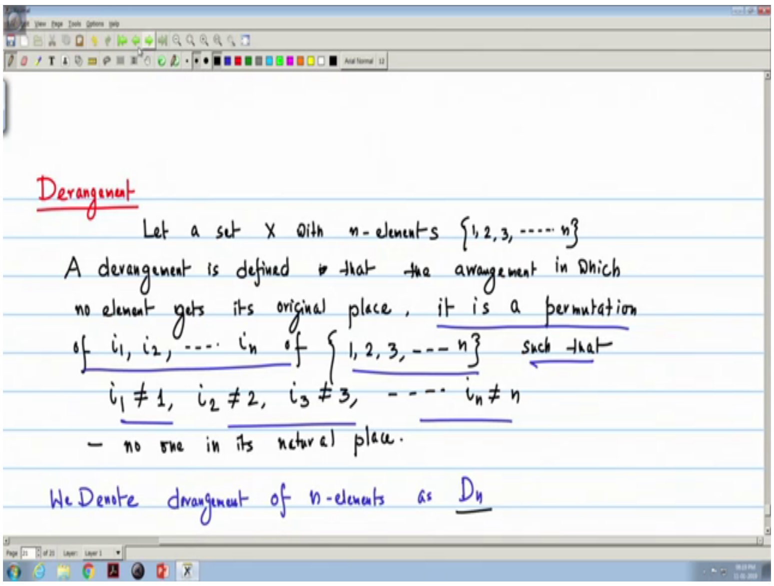

So, with one simple example first I try to explain that what is derangement? So, let in a party that 20 people joins and the 20 people reach their with a hat; that means, all people will have a hat. Now, when they leave then they will collect the hat.

And the if I give this type of condition that no one will get his or her own hat. And this is then how many ways it is possible that no one of that 20 people will get their own hat. So, normally an ordered arrangement will be we can tell that each people will get their hat. Now I am telling why I am telling it is reverse that means, this is some problem that will be handling that how many ways we have to tell that how many ways it is possible that no one will get their own hat. So, this is the problem of derangement.

Now, first we define the problem of derangement. So, we are the actual nature of the object is that we can take a set of n elements ok. Like we have consider like 20 person or 20 people join the party with hat. So, I define that thing that let a set X with n elements as 1, 2 n.

Now we define a derangement like that a derangement is defined that in is defined that the arrangement in which no element gets its original place. That means, I can tell that it is a permutation; it is a permutation of say write I give some variables permutation of i 1 i 2 i n i n of the elements 1 2 3 up to n.

Such that i 1 not equal to 1, i 2 not equal to 2, i 3 not equal to 3 and so on. That means, this is the permutation that no one gets it is place means the as if we are taking the place is 1 2 3 that is my right number or place. And then if it is a permutation of i 1, i 2, and I 3 and in then i 1 not equal to 1, i 2 not equal to 1, i 3 i, in not equal to n that should be. That means, that is no one in it is elements in its natural position no one in its natural place or natural position.

So, I define derangement as if the reverse of that thing. So, I can define it is a permutation of this if I can consider this is the natural position then no one will get the position this is my condition to be there. Now we normally denote we denote derangement of n elements. We denote derangement of since it is a permutation of n elements as D n, this is the convention. Now you see that how we can make the derangement.

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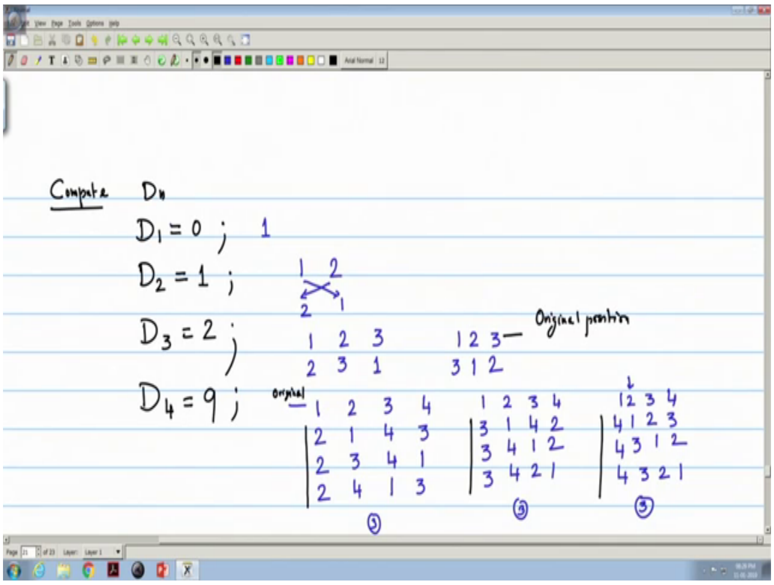

Now what will be D 1? So you can compute so compute some values of D n compute D n ok. So, what will be D 1? If D 1; that means, I have only 1 element and say that is position is 1, so, I give I have only 1. Then no derangement is possible because it is only 1 position and it is fixed.

So, my D 1equal to 0 because no derangement is possible; no derangement is possible. Now if I have so D 1 is 0. Now I have D 2 is 0 now see I get I have 1 and 2 then that output should be or the permutation output of the permutation will be. So, I get 2 should be here and 1 should be here so this is the permutation I want.

That means, only 1 permutation is possible with 2 elements that it will be they do not get their position. That means, here 1 then 2 comes here. So, my D 2 is D 2 is only 1 because I have only 1 permutation possible. Then what is D 3? So, this time I have 3 elements ok, I have 3 elements 1, 2, 3. So, I have 1, 2, 3 you see that how many are possible. So, if I have 1, 2, 3, then 1 permutation is there that in place of either 2 can come or if I take if it is 1, 2, 3 in place 3 can come.

Now if 2 then 1 can come here or here, but 1 I cannot put here then 3 will be in its position; that is not the definition. So, I must put 3 here so that everyone gets a different place, so this is a derangement and for 3 then again similarly 2 I cannot put here I can get 1 then 2 will be here.

So that means, these are the 2 permutation this the 1 and 1 this permutation where that no one gets its proper position 1 instead of 1 it is 2, 2 gets 3, 3 is 1, 1 3 1 2 so these are 2. So, I have D 3 equal to 2. I take D 4 then I have 4 numbers or 4 elements. Now if it is 4; this is my original position or the place.

Then what I can write that there can be 3 in place of 1 it can be 2. I can write in place of; in place of 1 it can be 3 or this is original. So, in place of 1 it can be 4. Now if it is 2 then what will be the other. If it is 2 then with 1 3 at 1 3 4 I have to keep, so, one can come here, then 4 must come here and 3.

Now again if it is 2 then see 3 can come here then 4 must come here 1. So, 2 1 4 3 2 3 4 1 and 4 can come here, but 3 must be there so 1. So, I get these three. Similarly I see that these 3 is here. So, then it can be 1 4 3 sorry 1 4 2 this can be 1 4 2 3.

This can be 4 4 1 2 this can be 4 2 1 so 3 1 4 2 3 4 1 2 3 4 2 1. So, I get another 3 starting with 3 similarly I will get 4. So, these can if it is 4 then these can be 1 this must be 2 it is 3. If it is 4 instead of 1 here it can come 2 sorry 3 it is the position up to 3 it must be 3 this can be 1 and 2 and this 4 then it can be 3 2 1. So, 4 1 2 3, 4 3 1 2, 4 3 2 1 so I get 9 here 3 here 3 here 3.

So, my D 4 is; D 4 is 9. So, I get that the this is actual how we can get the derangement. That means, what are the number of permutations we can get, so, that no element can get it is own position; so, this is the derangement.

But; obviously, that if and my number of elements will be more then it is very difficult to get in this way to count. So, how we can do that thing? So, now, if we put the a sum formula or that some generalised rule whether we can frame. I give a theorem of derangement.

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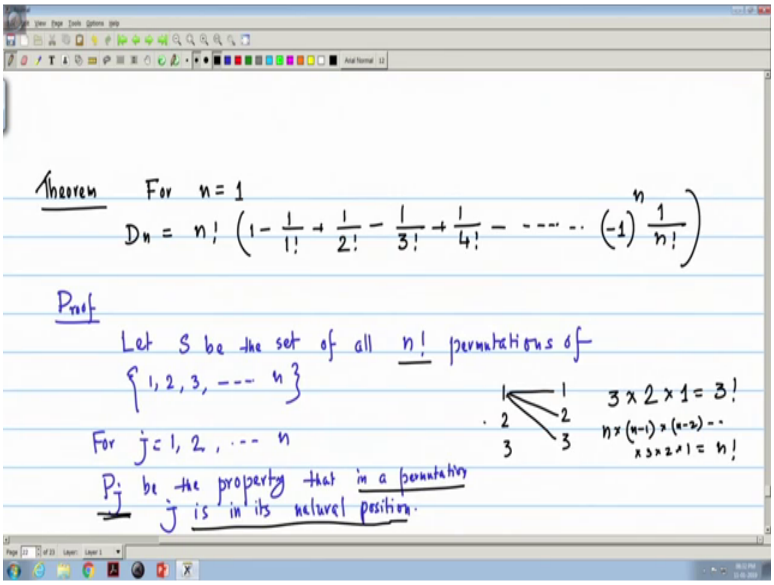

So, this is my theorem of derangement I can write. So, for I give that for n equal to 1 the derangement formula is D n equal to n factorial 1 minus 1 by 1 factorial plus 1 by 2 factorial minus 1 by 3 factorial plus 1 by 4 factorial minus in these way if I go. Since I do not know what is the value of n? So, whether it is odd or evens so minus 1 to the power n 1 by n factorial. So, this is my formula for derangement.

Now, how to proof these things how we can proof this derangement theorem. If we see that this is nothing, but the selection that as if we have the see the examples what this is when we have 4 elements as if we have some 4 original positions and what we are doing that we are placing the elements in such a way so that no one gets it is original position. As if we are considering this is the; this is the original position this is their original position.

Similarly, here it is the original position, this is my original position. So, in the similar way if I write that so let S be the set of all n factorial permutations. Since there will be an factorial permutations of n numbers, if we quickly see with this thing by how it is n factorial permutations; see I have 3 numbers 1 2 3. Then what are the different way I can permit. Because when I consider that 1; 1 can go here 3 different ways because 1 can get this position 1 can get this position or 1 can get this position.

So, when I am considering the first one it can be 3 positions, but once 1 is placed for 2 it is only 2 positions, similarly once 1 and 2 are placed 3 has only 1 place left. So, these will be 1, so, this is 3 into 2 into 1 since I have 3 elements. So, this is 3 factorial similarly if I have n, so, for n the first one will get the all n places for the second one. Since already 1 has got the place so, it will be n minus 1 then for n minus 2 and then up to 3 2 1.

So, these will be my n factorial. So, that we have written that n factorial permutations of 1 2 3 n. Now say for some j say for some j equal to 1 2 n I write from property some P j be the property like the one we have considered in our inclusion exclusion principle the property that in a permutation; in a permutation j is in its natural position; j is in it is natural position.

See or derangement is that no one will get its natural position, but we note that the property P j I have taken for all j equal to 1 to n that is P 1 P 2 P 3 that when I am considering j that it is a permutation where j is in it is natural position j will be in it is position.

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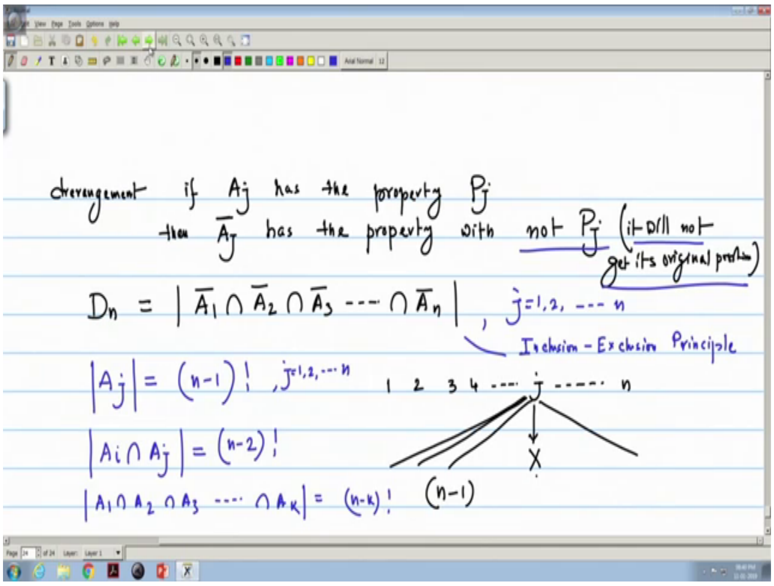

So, the derangement is if you remember the definition of derangement. So, derangement that when P j gets; that means, it should be my P j gets P j is the property that j gets it is position. And if A j is that number of the if I write if A j has the property of P j; has the property P j.

Then actually A j bar has the property has not the property P j; that means, it will not get the original position. So, A j bar has the property with not P j. That means, P j means it gets its original position; that means, it will not get; it will not get j will not get its original position. So, how actually it is the coming with normally the we have the inclusion and exclusion principle that we are taking.

So, now how; that means, D n I can write the D n. That this is nothing, but the arrangement of A 1 compliment, A 2 compliment, A 3 compliment. That means, if I apply the above statement for all n elements then no one will get it is original position. What I wrote, that A j bar is as the property with not P j; that means, it will not get its original position. Now if take for all j equal to 1 2 n then the derangement is that no element gets its original position.

That means, 1 will not get its position is A 1 bar, 2 will not get its position is to A 2 bar, 3 will not get its position is A 3 bar and similarly for n. So, D n equal to this A 1 bar intersection A 2 bar is A 1 bar which is nothing, but our inclusion, exclusion principle. This is my inclusion exclusion principle that we can apply; that we can apply. Then how will be doing that thing?

Now if I consider my first element; then first element that what are the way it will take since or if I consider that k kth element say. So, any a I first I consider any one element say A j. So, I can write A j can get n minus 1 factorial ways, why?

Because see I have elements like 1 2 3 4 some j say and then n. Now when it will be getting the place we can write in different way. See I am writing 1 2 3 4, so, I am considering some j n then what are the place j can take? J cannot take this place because this is its original position.

That means, other than this place; other than this place j can take any one of the place. So, j can come here 1 j can come here 2 come here and some j can go here so; that means, it has option n minus 1. So, what will be the permutation? That means, A j can go n minus 1 factorial which that it can get. So, I can write that it can be the number of permutation that for A j that it is n minus 1 factorial.

Now, for this is for j equal to say j equal to 1 2 any n; that means, which one we considering first this will be my n factorial. Now what will be my permutation then the for if I take the 2 permutation; that means, A i intersection A j, because if we remember my inclusion exclusion principle we have to take the two combinations also.

Then if it is 2 then simply it will be the similar logic I can give it is n minus 2 factorial. Because now already 2 elements are placed, so, I have to or 2 elements they will not get their original position. So, I have to discard those 2 places. So, these becomes a n minus 2 in this way I can write this thing that A 1 I can write that A 1 intersection A 2 intersection A 3 intersection A k.

That will be my n minus k factorial, these will be my n minus k factorial. So, now, if we try to utilize my inclusion exclusion principle, so, it will be a there are since another thing I require.

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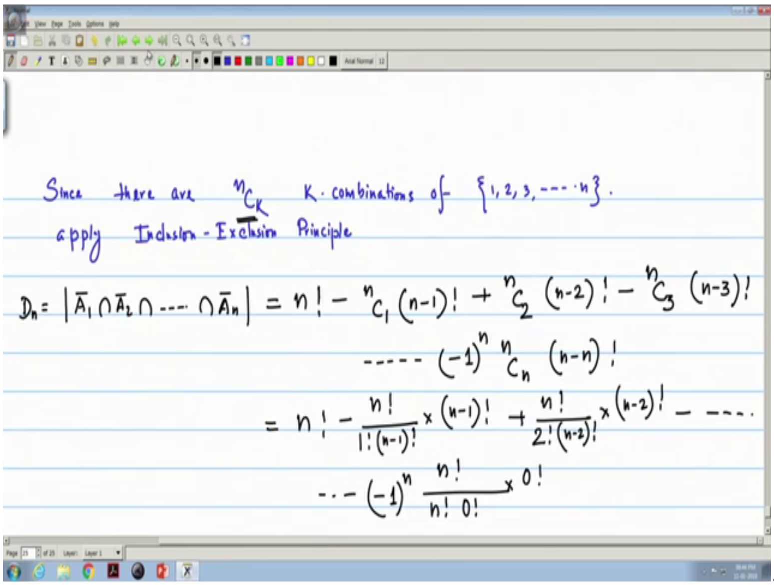

That since there are n C k ways I can choose n C k the k combinations of 1 to n elements that numbers that I want to put. So, now, if we apply inclusion, exclusion principle; inclusion exclusion principle.

Then how we get because already we have seen that D n I can define nothing, but my A 1 compliment intersection A 2 compliment like A n or this compliment of this thing. So, if I get that all permutation earlier what we have seen; that means, no restriction these will be in permutations minus this is my n C 1 I see that there are this can be there and for 1 there can be n minus 1 factorial permutation.

And that 1 I can choose from in this n C 1 which I can choose the first element and for each n minus 1 factorial. Now I can for 2; 2 elements I can choose for n C 2 and for 2 we have seen the number of permutation would be n minus 2 factorial these will be the next will be the similar n C 3 then n minus 3 factorial.

And if I go in that way then from n nth minus 1 to the power n, then n C n n minus n factorial n C n this is n minus n factorial. So, now, if I put this thing or if I compute n factorial minus n C 1 n minus 1, so, these n factorial divided by 1 factorial into n minus 1 factorial into n minus 1 factorial plus second term is that n factorial by 2 factorial into n minus 2 factorial into n minus 2 factorial minus.

If I go in this way these will be minus 1 to the power n C n, so, I can write n factorial divided by these n factorial into 0 factorial and n minus n is 0 factorial like that. So, see that here actually that each term from the denominator and the numerator that one term is always same. So, if these are equal also these will it will vanish.

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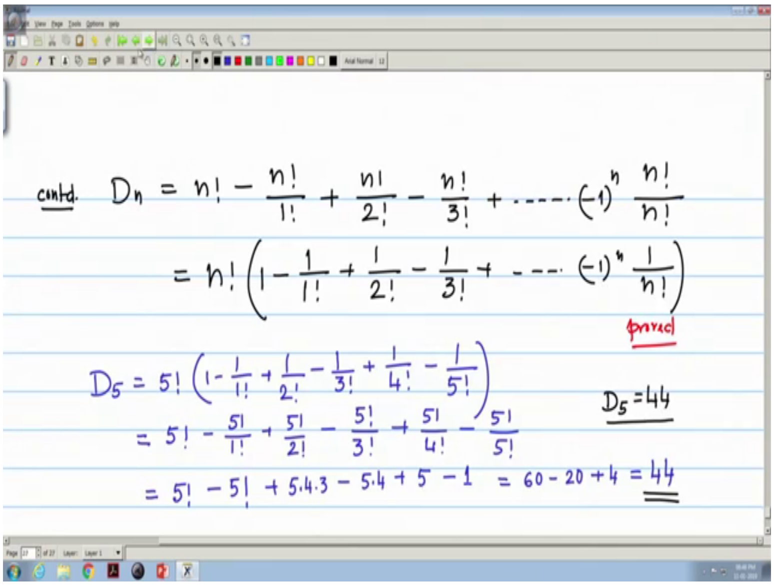

So, if I continue then I have the D n equal I can write D n equal to n factorial minus n factorial by 1 factorial plus n factorial by 2 factorial minus n factorial by 3 factorial. In these way if I go then minus 1 to the power n, then n factorial by n factorial.

This the last term if you see that 0 factorial it will cancel and then I get if I take n fact come out than I will be getting 1 minus 1 by 1 factorial 1 by 2 factorial minus 1 by 3 factorial plus minus 1 to the power n 1 by n factorial. So, this is the theorem statement that we get that we can prove the that this is my derangement expression.

Now, I can input some compute because earlier we have seen that only some simple values or some small values of n we can easily we can compute. So, now, we can put up to 4 we have done. So, if D 5 now I give the expression 5 factorial 1 minus 1 factorial 1 by 2 factorial minus 1 by 3 factorial plus 1 by 4 factorial minus 1 by 5 factorial.

So, it will be it comes that 5 factorial minus 5 factorial by 1 factorial 5 factorial by 2 factorial minus. So, each one term will be plus 1 will be minus 5 factorial by 3 factorial plus 5 factorial by 4 factorial minus 5 factorial by 5 factorial. So, if I compute will be getting that quickly I do that this is again 5 factorial both will cancel.

Then it is 5 factorial by 2 factorial means this is 5 into 4 into 3 minus this is 5 into 4 plus this 5 minus 1. So, these will cancel and it will be 60 minus 20 plus 4. So, this becomes 44. So, my D 5 is if I get another value D 5 is 44 I get 95 is 44. Now see we can compute since I got a expression, but fortunately there is another very nice expression we can get and that is some recursive statement that we can recursive expression that we can get recursive expression for D n.

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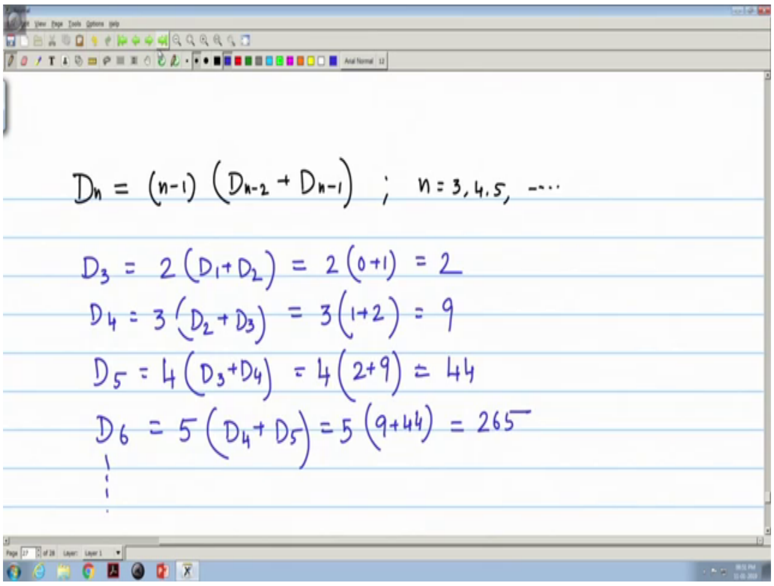

So, I can write it actually we can prove easily if you can if you use that expression for D n. Then I can write D n is n minus 1 into D n minus 2 plus D n minus 1 and this is for n equal to 3 4 5 I can write. So, now, easily we can if we can put we can get D 3 equal to 2 into D1plus D 2 and already we got D1D2 is 0 plus 1.

So, this becomes 2, then D 4 equal to 2 into D 1 plus D 2 plus D 3 this becomes 2 into 1 plus 2 this becomes 9 similarly I can 2 into D 3 plus D 4 is equal to 2 plus 9 it is ok, D 4 becomes 3. It is n minus 1 n minus 1 is 3 here n is 1 is 4. So, these becomes 44 similarly now I can compute more values that n minus 1 this is n minus 4 this is 5 D 4 plus D 5 is 5 into 9 plus 44, these becomes 265

And in this way I can compute and I can easily compute or all the derangements of these things. So, once we get the derangement now easily we can apply these expressions. And particularly these recursive expression will be very easy to get the derangements. And the problem that we start started the example that easily we can use that thing. So, one quickly we if we see one example.

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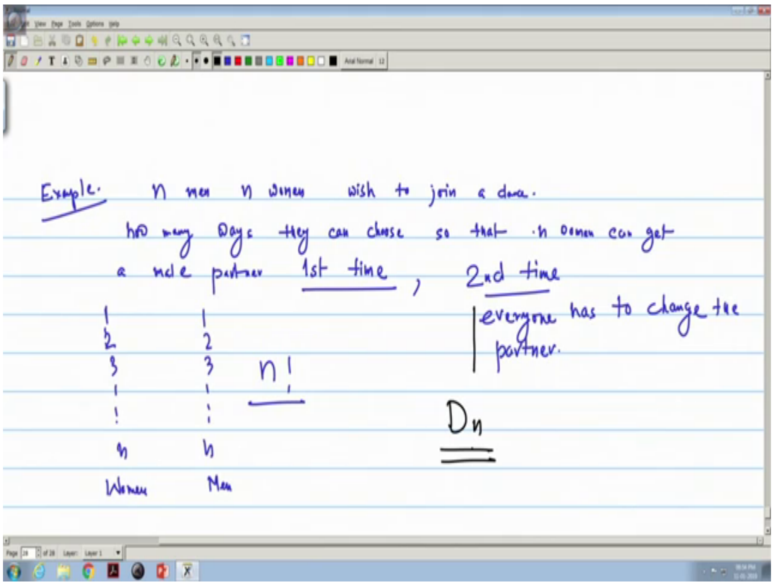

That what early one we are giving them that some in a party that if in n men and n women. Just they wish to join a dance and how many ways; how many ways that they can choose, so, that the n women can get a male partner; can get a male partner first time.

So, that is my n factorial. So, if it is first time we remember since I have 1 2 3 simply n, I have also 1 2 3 I have these women, n women, n men. So, simply there is no restriction n factorial always the first time they can get. Now, if second time we want the restriction is that they has to everyone has to change the partner this is my restriction. That means, that is my derangement everyone has to change the partner; everyone has to change the partner.

Then this is that actually that they the nothing, but the D n. So, this is actually the second time dance, so, if it is second time and this is my restriction so this is D n. So, simply that using D n we can solve this type of problem or combinatorics. So, with this lecture we finish our lectures with the combinatorics problem that how to handle the counting techniques with combinatorics problems.