Discrete Structures
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Lecture – 44
Combinatorics (Contd.)

So, we have read the r-permutation of a multiset; that means the ordered arrangements when the set contains the elements with repetition and that repetition we have considered finite repetition; that means, the each element appears finite number of times or it can be infinite number of times. Now, the way we have defined permutation and combination if we remember that it is the arrangement, but permutation when it is the ordered arrangement combination when it is unordered arrangement.

So, the similar way I can defined the combination of multi sets; that means, that we have we can count the arrangements or the counting in problems, but we can apply that where the unordered arrangement is considered from a set or set having elements with infinite repetition.

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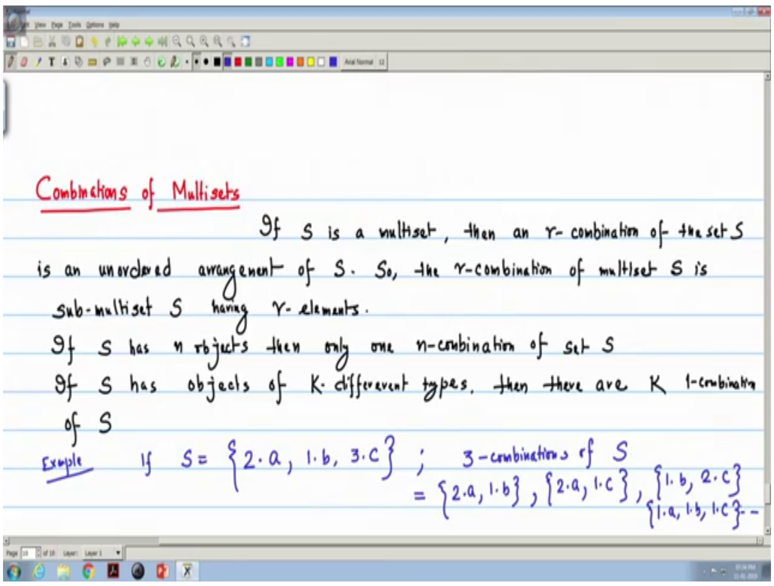

So, we will read the combinations of multi set. Similar way we can define the combinations only the difference is unordered arrangement. So, if S is a multiset then an r-combination of the set S is an unordered arrangement of S and similar way the way we have defined combinations of sets I can tell. So, the r-combinations of multiset of multi set

S is nothing, but the sub multisets of S is sub multi set since it is unordered. So, I can tell this is a sub set, so, some multi set S having r-elements because I am considering r-combination.

Now, if S is an n object then there is only one n-combination of this then only one n-combination and normally we call that is the combination of set n-combination of set S. If S has objects of K-different types then there are K number of 1-combination of S, these are similar to that normal set.

So, if we illustrate with an example say I consider a multi set S say if S I take that with finite repetition say I have a elements with 2 repetition, b with 2 repetition and c with 3 repetition. Then since it is an unordered, so, I can write some 3-combinations though sum of the 3-combinations of S is simply 3. So, I can write 2 a, 1 b I can write 2 a, 1 c I can write 1 b, 2 c or it can be 1 a, 1 b, 1 c. It can be you can write here 1 a, 1 b, 1 c and so on we can write. Now, we see how we can give the general statement of computing the r-combination.

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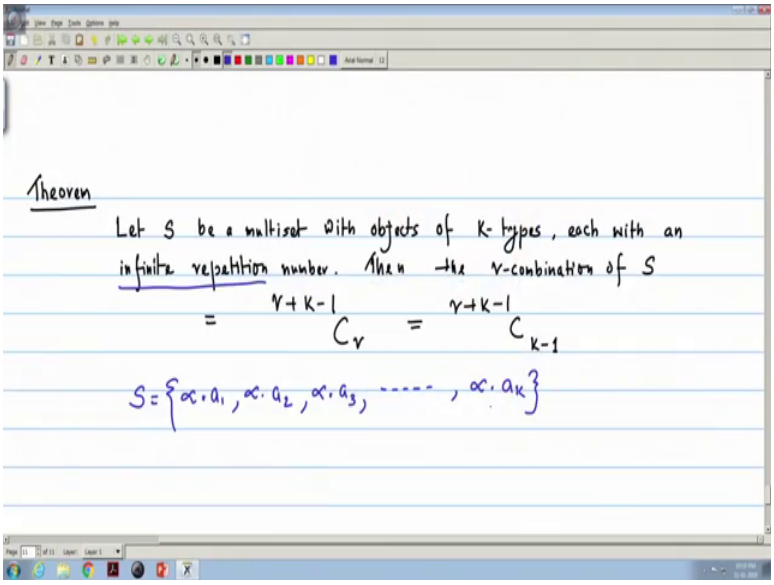

So, I give a theorem this statement I write that let S be a multiset with objects of K-types each with an infinite repetition then the r-combination of the set S we can write as r plus K minus 1 C r and which is same as that of r plus K minus 1 C K minus 1. And here it is infinite repetition, so, S is S is of type K type; so, infinite a 1, infinite repetition of a 2, infinite repetition of a 3 and so on infinite repetition of a K.

So, now, with this theorem statement we read some problems and how we can actually compute the combinations.

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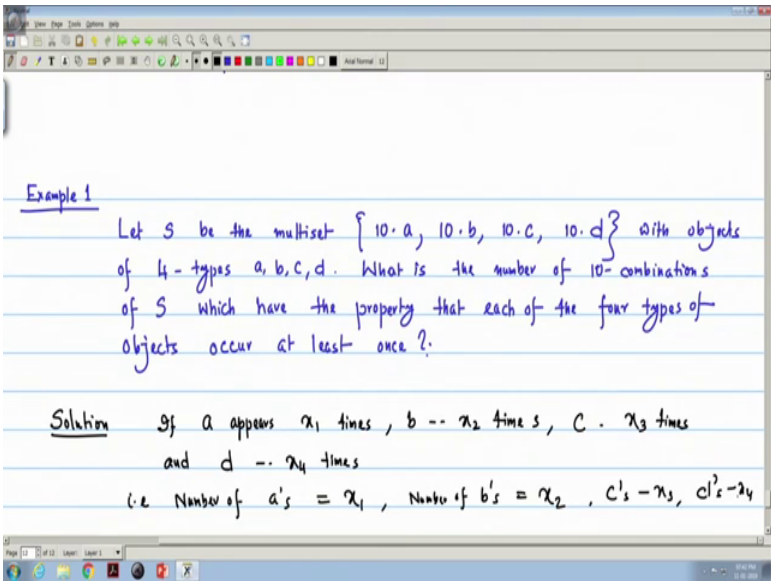

You first take a multiset one example we take as it is the like the theorem statement. Let S be the multiset say I have a with 10 repetitions, b with 10 repetitions, c with 10 repetitions and d. So, there are four different types a, b, c, d with objects of four types of four different types. Then the question is what is the number of 10 different or I can write 10-combinations of S which have the property that each of the four types of objects object occur at least once.

So, the just like the theorem statement here is system multiset having four different types of objects a, b, c, d and each type of objects appear with finite repetition and 10 repetition 10 is the 10 repetition number. Now, what is the number of 10-combinations of S which have the property that each of the four type of objects occur at least once.

Now, here if we consider ok, this solution if we think say if a appears x 1 times, b of x 2 times, c x 3 times and d x 4 times; that means, x 1, x 2, x 3, x 4 are my repetition number or other way I can tell that a number of a's number of that is number of a's equal to x 1, number of b's equal to x 2 and like c's number of c's x 3, number of d's x 4.

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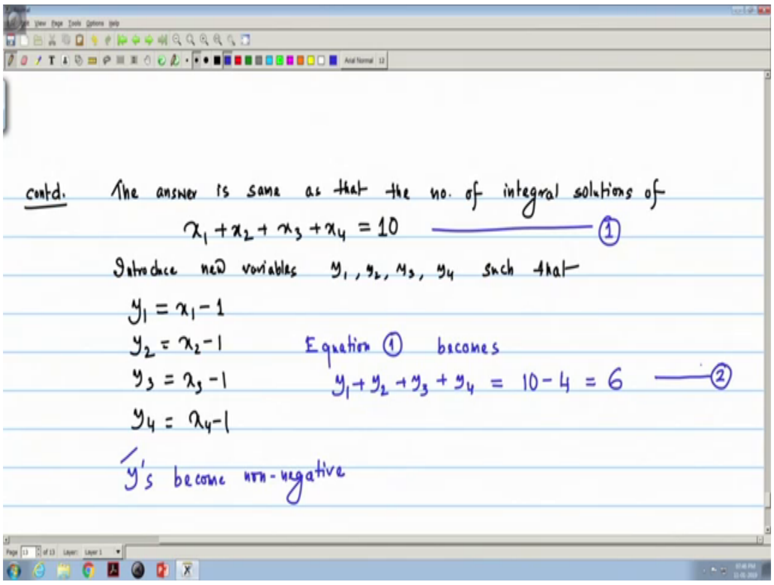

So, I can write that the answer is you can write that the answer is same as that of the is same as that of the number of integral solutions of x 1 plus x 2 plus x 3 plus x 4 equal to 10. Since each object appears at least once and it is a 10-combination so, the total objects or total object count should be 10. So, I can tell this is the 10-combinations of this thing.

So, I can write the I introduce an variable y 1, y 2, y 3, y 4. So, introduce new variables say y 1, y 2, y 3, y 4 such that this becomes y 1 equal to x 1 minus 1 y 2 equal to x 2 minus 1 y 3 equal to x 3 minus 1 and y 4 equal to x 4 minus 1.

So, my equation 1, becomes, so, the equation 1 becomes y 1 plus y 2 plus y 3 plus y 4 equal to 10 minus 4 equal to 6. So, that why we have done. So, that these y's become here the here the y's become non-negative; y's become non-negative. I have to find out the number of non-negative integral solution of this new equation I can write this new equation as the 2.

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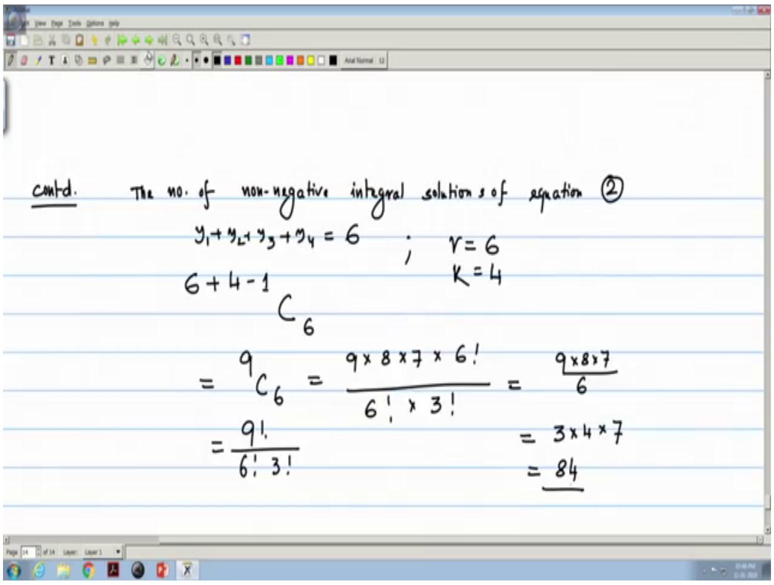

So, if I continue I have to find out the number of non-negative integral solution of equation 2, that is y 1 plus y 2 plus y 3 plus y 4 equal to 6.

So, now if we apply the theorem of the r-combinations of multiset S, then I will get that number of non negative solutions this becomes.

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I have any 6 ok, I write if we see the statement that r plus K minus 1 C r and this was the r-combination of S and K different types of objects. So, we identify here r; that means, here r equal to 6 and four different types. So, K equal to 4.

So, I can write r plus K minus 1, so, 6 plus 4 minus 1 C r 6; r is 6. So, this becomes 9 C 6 equal to 9 into 8, 7 into 6 factorial because this is 9 factorial divided by 6 factorial into 3 factorial. So, this is equal to 6 factorial into 3 factorial this is equal to 9 into 8 into 7 divided by 6 is equal to 3 into 4 7 is 84, so, I will get the total 84 solutions.

Now, one important application area of this particular problem that or these r-combinations of set with infinite repetition is the number of integral solution. How to find the number of integral solution? If we see the previous example see the examples we have actually converted into that form that initially it was only that combination it was given the statement, but we have converted the problem statement. So, that it becomes that how to find out the number of integral solutions of this equation.

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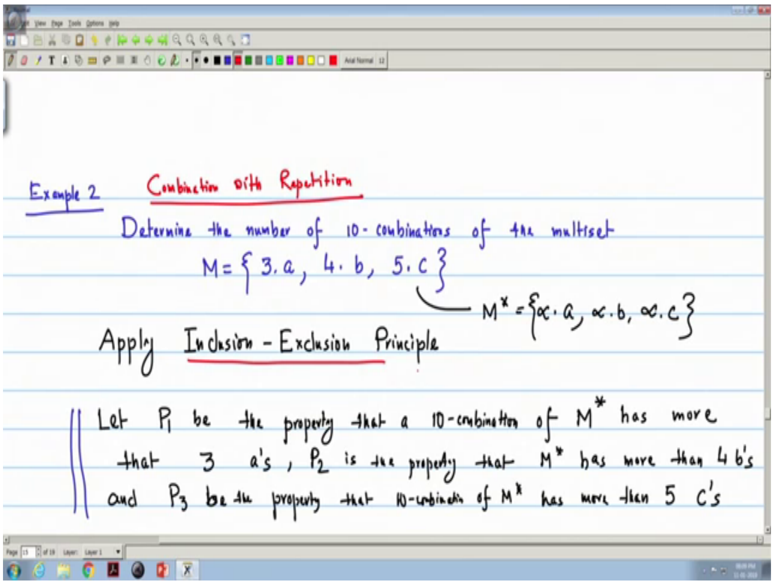

So, now if we see this type of examples that I give a different problem that say another example if you see and we give it different solution procedure ok. The statement is determine the number of 10-combinations of the multiset m equal to 3 a with repetition 3, b with repetition 4 and c with repetition 5.

Now, here the problem is this and we will apply the inclusion-exclusion principle. So, different solution technique will be using here we will apply inclusion exclusion principle. So, first we consider that we identify the property. So, let P 1 be the property that a 10-combination of say I am making this thing as a M star 1 multiset and M star I am giving that as if m star is that again three different types having infinite repetition. So, infinite b an infinite c and making this thing and property that 10-combination of M star has more than 3 a's see the original problem it was that there are 3 a; that means, a with finite repetition 3.

Now, the property I am taking that P 1 the property that it information of M star has more than 3 a's because that is why we have changed the size of the multiset and that we take as if this is M star which is infinite set. Otherwise, we cannot consider that v a element or the a have the more number of a will appear where the more than the repetition number. So, here the reputation number is 3. So, we consider that P 1 is that property.

Similarly that P 2 is the property that the a 10-combination of M star has more than M star has more than 4 b's; more than 4 b's and P 3 is the property that 10-combination of M star has more than 5 c's more; than 5 c's. So, these are my that property.

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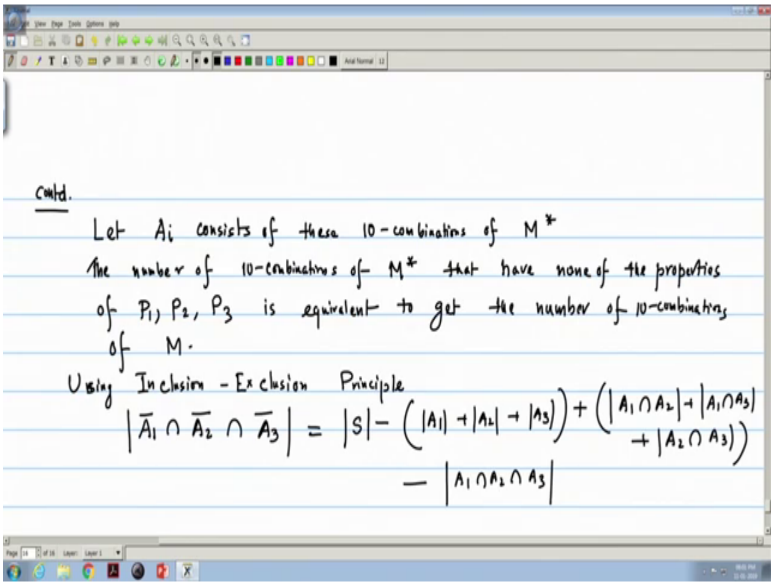

So, if we remember that inclusion-exclusion principle that actually the way we have defined that now let a 1 be the of this combination; that means, A i. If I continue let as usual we consider that A i consists of this 10-combination; that means, a 1 is the a 1 concerned with a's, a 2 is that of with b's and a 3 is that of with c's.

So, if we remember the inclusion-exclusion principle then I can write that the size of the set that it will give let the number of I can write the number of 10-combinations M star that have none of the properties of P 1, P 2, P 3 is equivalent to get the number of 10-combinations of M. Why?

If we see the way we have taken that M was that a with 3 repetitions, b with 4 repetitions C with 5 repetition then I have converted M star with infinite repetition and we have considered P 1 as if a appears more than that has more than 3 a's a appears more than 3, b appears more than 4, c appears more than 5. So, this is actually complement. So, that is why then I if I now apply so, using if I remember that using inclusion-exclusion principle because that tells that A complement the cardinality of A complement is the cardinality of set S minus cardinality of A.

So, inclusion-exclusion principle I can write the here it is 3. So, cardinality of A 1 complement, intersection A 2 complement, intersection A 3 complement these will be the set S minus sigma A i. So, here it is only three; that means, A 1 plus A 2 plus A 3 minus the 2-combinations plus 2-combinations then A 1 intersection A 2 plus A 1 intersection A 3 plus A 1, A 2 intersection A 3. Then I have another that all three together I have to take that A 1 intersection A 2 intersection A 3 this is a our inclusion exclusion principle.

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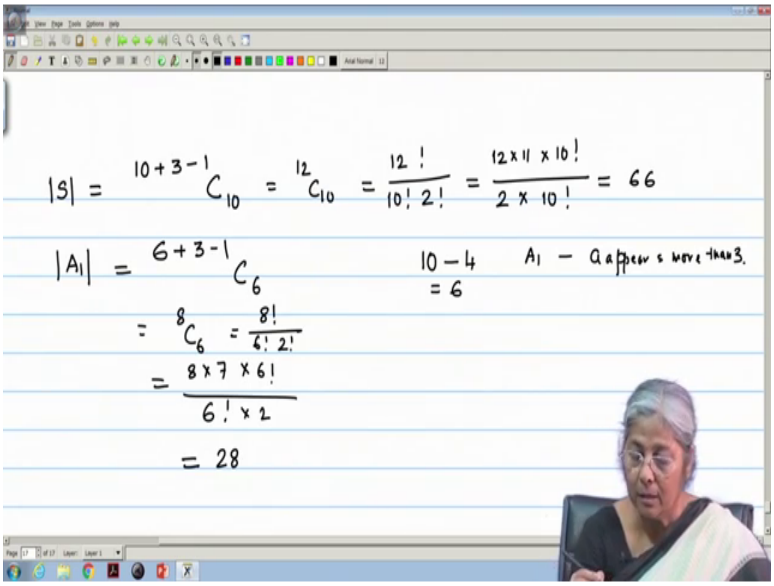

Now, we have to compute all the values. So, now, we use that r use the r-combination of how to compute that r-combination of set S. So, S is I can write that 10 because 10-combination 3 type. So, 10 plus 3 minus 1 C 10 would which is 12 C 10 equal to 12 factorial divided by 10 factorial into 2 factorial is 12 into 11 into 10 factorial and divided by 2 into 10 factorial equal to 66.

Now, I have to calculate the others. Now, how to compute that A 1? See I can write that because there are 10-combinations and A 1 we have taken the property that a appears more than 3. So, if a appears more than 3 since I have 10 numbers where that more than 3 means at least 4. So, I have 10 and for when I am considering the property of A 1 because that a appears a appears more than 3 ok. So, this will be 4; so, this becomes 6. So, I am taking that this is 6 plus 3 minus 1 C 6 and this becomes 8 C 6 and this is equal to into this is 8 factorial by 6 factorial into 2 factorial. So, 8 into 6 factorial divided by 6 factorial into 2, so, this becomes 28.

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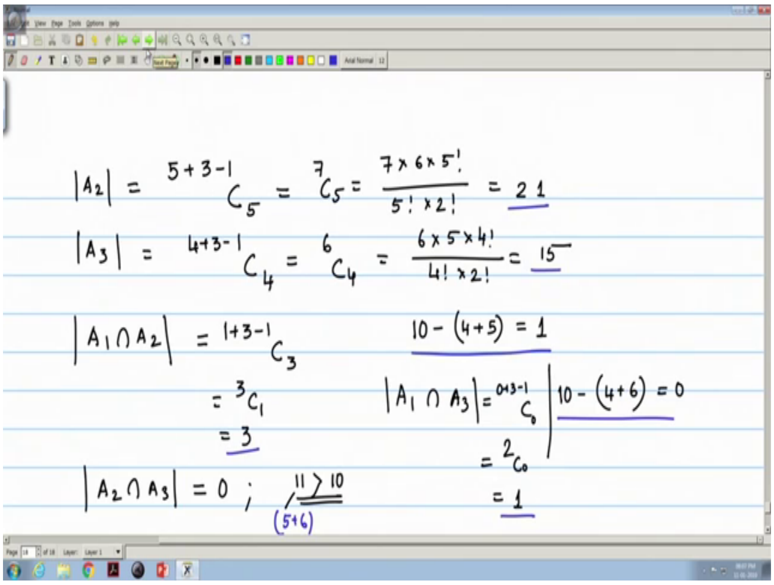

So, similar way I can compute A 2 which will be it will be 5 plus because more b appears more than 4; that means, 5. So, it is 5 plus 3 minus 1 C 5 and this value becomes 7 C 5 this becomes 7 into 6 into 5 factorial divided by 5 factorial into 2 factorial and this becomes 21. Then A 3 is 4 plus 3 minus 1 because more than 5 means 6 so, C appears. So, this becomes 6 C 4 equal to 6 into 5 into 4 factorial divided by 4 factorial into 2 factorial and this becomes 15.

Now, the two combinations I can get A 1, A 2, if I take together so, this becomes 10 minus 4 plus 5 because for a it is 4, for b it is 5, so, this becomes 1. So, I have 1 plus 3 minus 1 C 3 is 3 C 1 equal to 3. Now, for A 2 A 3; A 2 A 3 this becomes 0 for A 2 A 3 this becomes 0. Why? Because here this becomes 5 plus 6, 11 and 11 is greater than 10. So, it is not possible this type of combination.

Now, for A 1 A 3 so, A 1 A 3 this becomes 10 minus 4 plus 6. So, here these 11 is; these 11 is coming from 5 plus 6 b and c, so thing so, these becomes equal to 0. So, A 1, A 2, A 3 is 0 plus 3 minus 1 C 0 and this equal to 2 C 0; 2 C 0 is 1. So, I got this value 21, this is 15, this is 3, 1, this is 0 another value have to compute A 1, A 2, A 3.

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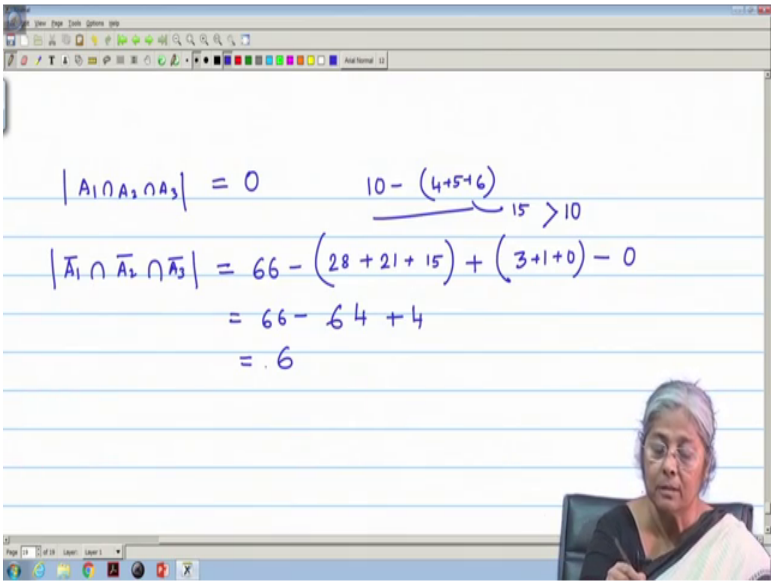

So, this is my A 1 A 2 A 3 and similarly this becomes 0 because here this is 10 minus 4 plus 5 plus 6. So, this is greater than I am sorry 4 plus 5 plus 6. So, 15; which is greater than 10, so, this type of combinations is not possible, so, this becomes 0.

So, now, if I put that values all those values with inclusion-exclusion principle A 1, A 2, A 3 then I got as that is 66 minus A 1 is 28 A 2 for A 2 this is 21, this is 15 then for 2-combination we got 3 plus 1 plus 0 then 3-combination it is 0 only. So, this becomes 66 minus 64 minus 4 plus 4 I am sorry, this term will be plus 4, so, this becomes 6.

So, see the when we are considering that combination with repetition, so, this is one example of combination with repetition I can apply the inclusion-exclusion principle. So, I can write this is an example of; this in an example of combination with repetition; is an example of combination with repetition where we use the inclusion exclusion principle and we can get this solution directly by applying this.

So, directly we can the earlier example directly we can we have applied the theorem of computing the r-combinations of a multiset S and that S has elements with finite or infinite repetition we have considered and one example we have seen that how to find the integral solution of an equation. And here another example with combination with repetition we have seen that we can use that inclusion-exclusion principle that we have read earlier. In this way we can handle the permutation and combinations.