Discrete Structures
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Lecture – 39
Counting Techniques and Pigeonhole Principle (Contd.)

So, we have learned the Pigeonhole Principle in Simple form. Today, we will read the Strong form of Pigeonhole principle or sometimes we call this is the generalized pigeonhole principle and how it is applied to solve the different type of problems. So, first we see the pigeonhole principle in strong form.

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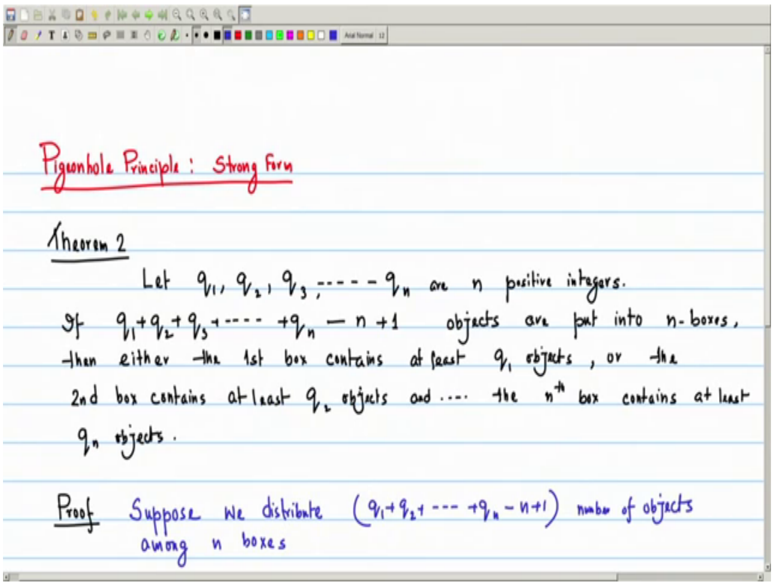

We write that as a theorem to give a theorem 2 because theorem 1 was the pigeonhole principle in simple form. So, I write let of q 1, q 2, q 3 and q n are integers on in positive integers. If the total number of objects say q 1 plus q 2 plus q 3 up to q n; minus n since I have n number of n positive integers I have taken plus 1. These many number of objects are distributed or are put into n boxes. Then, the theorem tells or the strong form of pigeonhole principle that then either the first box contains at least q 1 number of objects or the second box contains at least q 2 objects and so on. I can tell that the nth box contains at least q 1 objects.

So, I have these many positive integers, I have written q 1, q 2, q 3, q n; then, q 1 plus q 2 plus q 3 plus q n minus n plus 1 these many number of objects if I want to distribute among these n boxes. Then either the first box contains at least q 1 objects or the second box contains at least the q 2 objects and so on. In this way, I can tell that the nth box contains at least given objects. Now, how we can prove these thing; how to prove the ok?

Now, suppose we distribute the total objects that is q 1 plus q 2 plus q n minus n plus 1 this total objects; these number of objects into n boxes; among n boxes,

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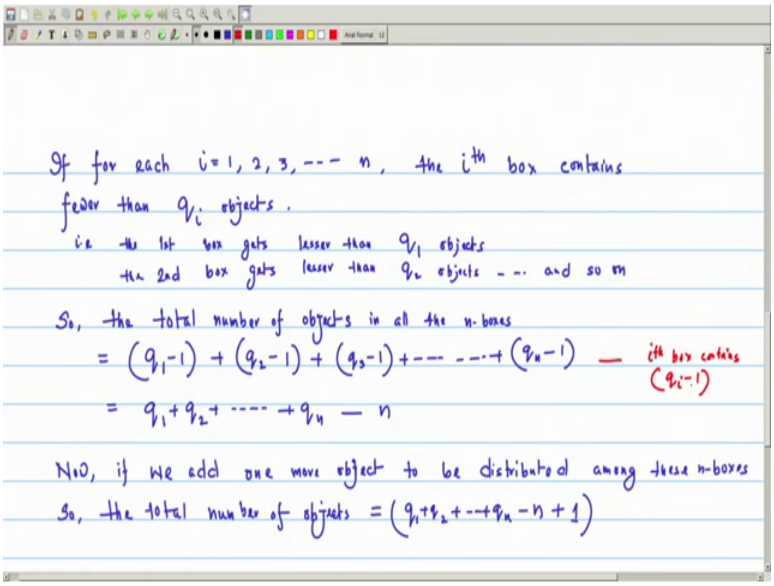

If the distribution is such that for each I equal to 1 to 3 to n, the ith box contains fewer than less than q i objects, fewer than q i objects. That means, the distribution is such that each box or say the ith box gets less than q i objects; that means, q 1 the first box that is the first box gets fewer than or lesser than q 1 objects; the second box gets lesser than q 2 objects and so on.

So, what will be the total number of objects, if each box get lesser than the objects that q ith objects? So, I can write the total number of objects in; so, the total number of objects in all the n boxes such that our distribution is like everyone that every box gets fewer objects is equal to say q 1 minus 1; at least 1 less than q 1. Similarly, q 2 minus 1, q 3 minus 1 plus q n minus 1 and this becomes q 1 plus q 2 plus q n minus n since there are n boxes. So, this type of I have n terms. Now if we add, so this is the total number of objects in the n boxes.

So, now if we add 1 more objects; one more object to be distributed; one more object to be distributed among these n boxes. So, my total number of objects become, objects equal to q 1 plus q 2 plus q n minus n plus 1 that the last object that we have added and the distribution we have done or we have assumed that each box, see these each box contains fewer than q i; that means, here ith box contains here we have considered that q i minus 1. So, when we have added 1 object.

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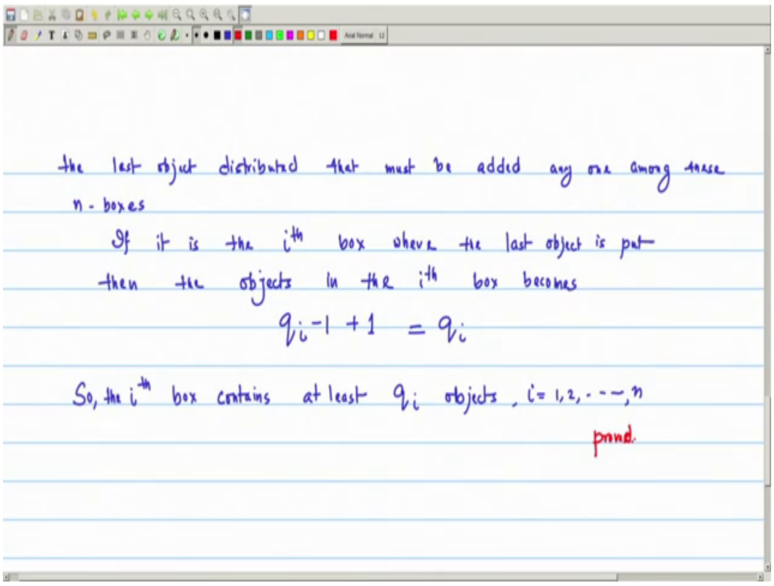

So, that object must have gone from the object. So, the last object distributed that must be added, anyone among these any one of the boxes among these n boxes. So, if it is ith box, if it is the ith box where the last item last object is put; then, the objects in the ith box becomes earlier it was q i minus 1. Now I have added 1. So, q i and minus 1 plus 1 and that is equal to q i. So, the ith box contains at least q i objects; where, i is 1 to n. So, it proves the statement in the strong form of pigeonhole principle.

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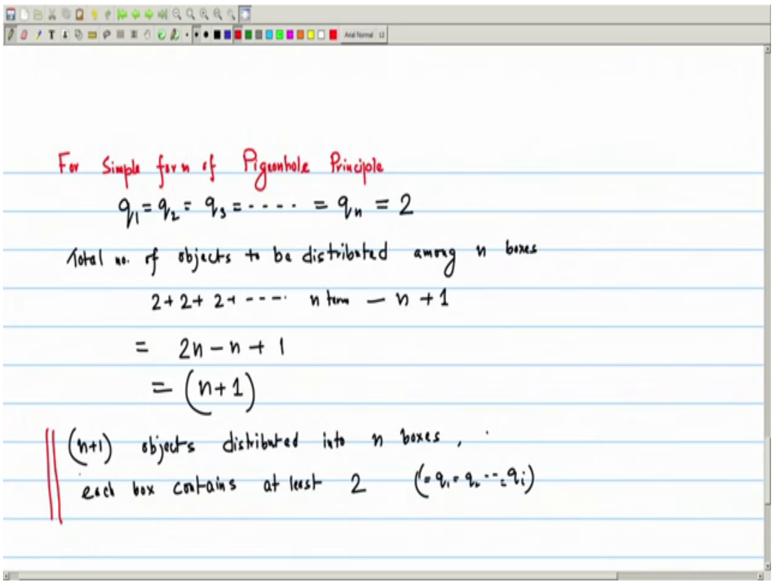

Now, we see what are the other different way we can state the strong form of the pigeonhole principle. Now before that one since we have told that many times the strong form of pigeonhole principle is called the generalized form. So, we see that we can get the simple form from the generalized form or the strong form; how we can get the simple form?

So, for simple form of pigeonhole principle, see if we put that q 1 equal to q 2 equal to q 3 equal to q n equal to 2; then, what are the total number of objects? So, total number of objects to be distributed among n boxes is 2 plus 2 plus 2 n terms, since I have n boxes then minus n plus 1. So, these becomes 2 n plus n minus n plus 1 equal to n plus 1.

So; that means, n plus 1 objects. Now these becomes n plus 1 objects distributed into n boxes and each one will get 2 or more than 2; that means, at least 2, since my q i q 1, q 2 is 2. So, in a strong form it tells that q i; here q i is 2. So, it gets at least each box contains at least 2 which is equal to that q 1, q 2 actually q i.

So, it is same as that of our simple form of pigeonhole principle and what last day we have mentioned last lecture that when we have discussed the simple form that we can take this thing as a also a coloring problem.

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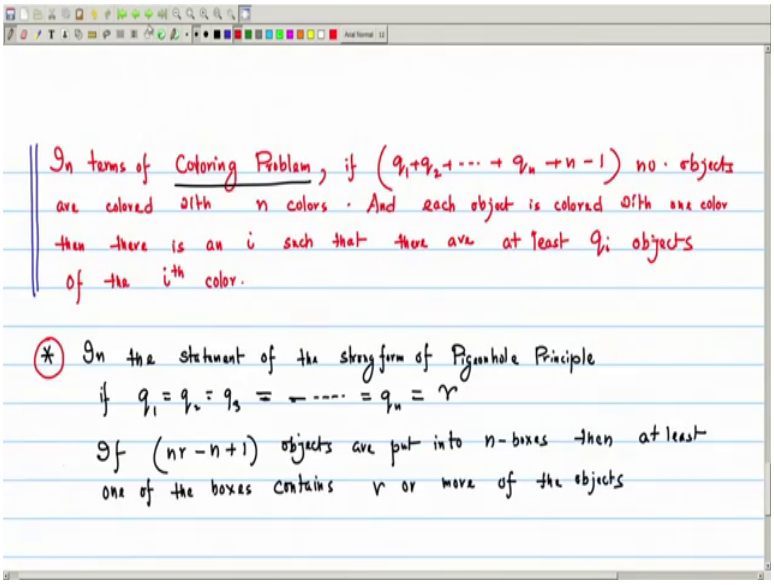

So, in terms of coloring problem, we can tell that if q 1 plus q 2 plus q n plus n minus 1 number of objects or colored with n colors and each one each object is colored with 1 color. Then, from the generalized principle, we can tell that there is an i, such that there are at least qi objects of the ith color.

So, in coloring problem, we can when we have to find out the number of colors or minimum number of colors to be is required that we can apply our pigeonhole principle. Now, we see that this is all coloring problem. Now, we see what are the different other form, we can write or we can state these generalized pigeonhole principle. You can write one let in a what in the strong form or in the statement of the strong form of pigeonhole, if q 1 equal to q 2, q 3 equal to q n equal to r.

That means, I can write that if r plus r plus up to n; that means, nr minus n plus 1, these objects are put into n boxes. Then at least since now all boxes have the same capacity and that is equal to r; then, at least 1 of the boxes contains r or more of the objects. So, in one form we can write.

Now, if I think from the elementary mathematics point of view, then also the strong form we can write a a similar type of statement as that of our generalized pigeonhole principle. So, I can write that from elementary mathematics.

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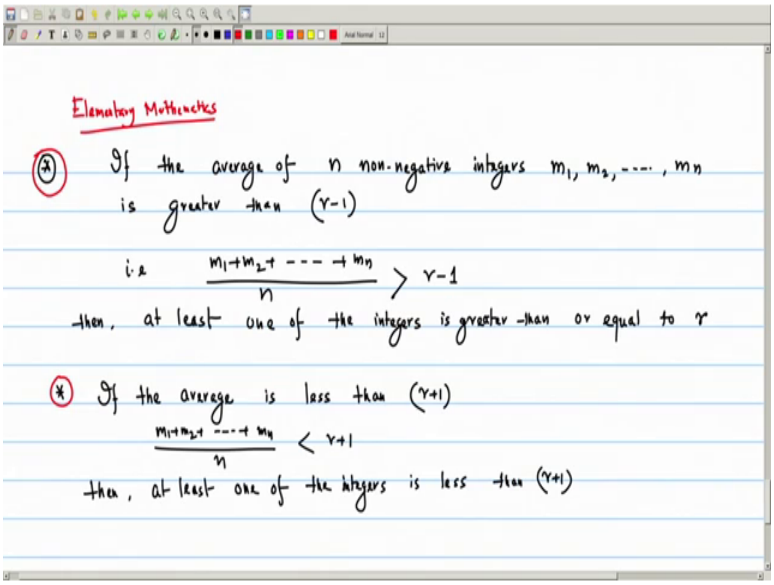

That if the now we take the average ok; average of all q 1 plus q 2 plus q n objects. So, if the average of in the number of objects I am taking some non negative integers, n non-negative integers and I am giving that integers are m 1, m 2 up to m n and average is if the average is greater than r minus 1; that means, that is m 1 plus m 2 plus m n divided by n. This is my average and this is greater than r minus 1.

Then, we can write that at least one of the integers is greater than or equal to r. Now, some other way another form I can tell, I can write that if the average is less than r plus 1; that means, m 1 plus m 2 plus m n divided by n is less than r plus 1. Then, at least 1 is less than then we can write then at least 1 of the integers, at least 1 of the integers is less than r plus 1.

So, the statement of strong form or generalized pigeonhole principle now we are writing for the elementary mathematics for we are applying that thing for or elementary mathematics and already for simple form we have seen the that how numerical problems are solved using the pigeonhole principle.

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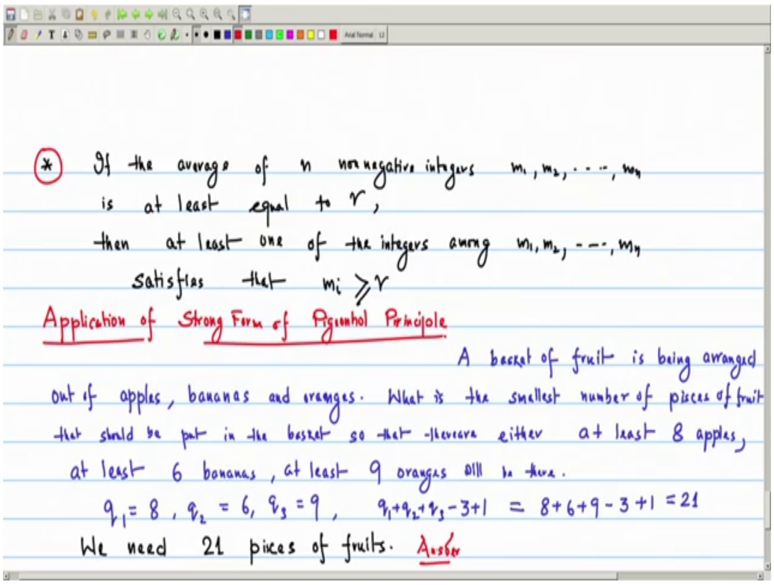

And that I think this is same as that of I can write, that if the average of n non-negative integers m 1, m 2 m n is equal to r is at least equal to r is at least equal to r; then, at least 1 of the integers among m 1, m 2 m n satisfies that m i n integer m i greater than equal to r.

So, this is what we see that simple elementary mathematics that if the objects we write as the integers and we can easily write that thing. Now, we see 1 application. We see 1 application of strong form. It is a very common problem, we write first the statement that a basket has 3 type of fruits or is being arranged out of 3 type of fruits say apples, banana and oranges.

Now, problem is that, what is the smallest number of pieces of fruit that should be kept on, that should be put in the basket so that there are either at least 8 apples or at least 6 bananas or at least 9 oranges will be there. So, as it is we can apply here q 1 the strong form of pigeonhole principle. Here q 1 is 8, q 2 is 6, q 3 is 9 and I have 3 different types of fruits.

So, the total number of objects or total number of fruits would be that to be distributed is q 1 plus q 2 plus q 3 minus 3 plus 1. So, these becomes 8 plus 6 plus 9 minus 3 plus 1 equal to 23 minus 3 plus 21; 21 fruits. So, we need 21 fruits. So, we need 21 pieces of pieces of fruits and this is my answer.

So, directly we can we can apply the strong form of pigeonhole principle to solve these type of problem. So, with this we finish the concepts of pigeonhole principles, the simple form; the strong form and some of the applications that how we have how we can solve using this principle that we have discussed. And next lecture, we will see again that how some different type of problems, we can apply the pigeonhole principle.