Discrete Structures
Prof. Dipanwita Roychoudhury
Department of Computer Science & Engineering
Indian Institute of Technology, Kharagpur

Lecture - 27
Recursion (Contd.)

So, we are discussing the Recursion and last lecture we have learned how actually we can define recursively the sequences functions and the sets. Today we will see in details with some examples that how recursively defined sequences can be formed and what are their applications.

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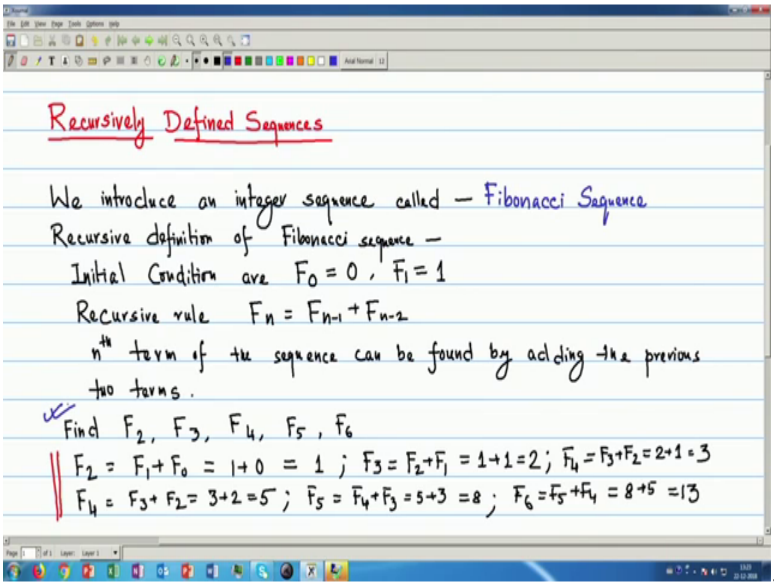

First we see the Recursively Defined Sequences. We introduce an integer sequence which is popularly used in combinatorics and in graph theory it is called the Fibonacci sequence. So, you introduce called Fibonacci sequence. So, first we see how these Fibonacci sequence are recursively defined, the recursive definition of.

So, as we discussed last day and we have defined recursively defined sequences functions and sets; that with some initial values and we can give some rule so that we get a particular term in the sequence from the previous term or previous set of terms. So, Fibonacci sequence it is defined that its initial condition, the initial condition or say F 0 equal to 0. F 1 equal to 1 where F represents the Fibonacci sequence and the recursive definitions are or you can tell recursive rule; that F n equal to F n minus 1 plus F n minus 2.

That means, the nth term of the sequence can be found by adding the previous 2 terms. So, how we can find the some of the terms using this recursive rule? So, since F 0 and F 1 are given so if we find that find F 2 F 3 F 4 F 5 F 6. Then I can apply the recursive rule as well as we take the initial values. So, F 2 is F 1 plus F 0 and F 1 is given 1 F 0 is 0 so this is 1. Now F 3 is F 2 plus F 1 just now we got F 2 equal to 1, and F 1 has the initial condition is given as 1 so it is 2. Similarly I get F 3 F 4 already you will got F 3 so F 4 is F 3 plus F 2 and F 3 is 2 F 2 is 1 so this is 3. We got F 4 is F 3 plus F 2 and that is 3 plus 2 is 5.

Then F 5 is F 4 plus F 3 is 5 plus 3 is 8, then F 6 is F 5 plus F 4 is 8 plus 5 equal to 13. And if I continue in this way we will be getting the values of the different terms of the sequences. So, in this way we can define the sequence recursively we can define a sequence. Now we can prove this sequence; that means, the recursive definition of Fibonacci sequence in conjunction with the mathematical induction we can prove, we can establish the some of the interesting properties using the Fibonacci sequences. And which is number of application in combinatorics as well as graph theory and other as branches of mathematics.

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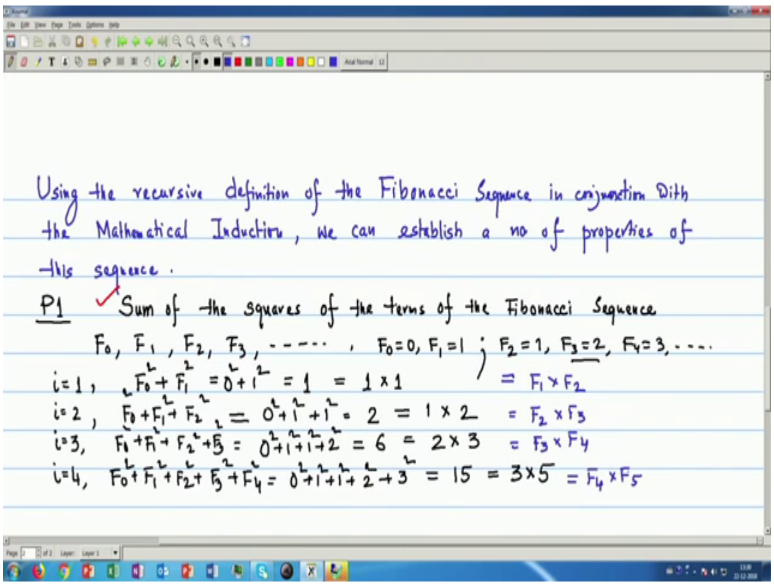

So, the recursive using the recursive definition of the Fibonacci sequence in conjunction with the mathematical induction. We can establish a number of properties of these sequences of these sequences. One very simple example we see so we see a property P 1 say it the property tells the sum of the squares of the terms of the Fibonacci sequence.

So, just now we have seen the terms are F 0 F 1 F 2 F 3 like that the Fibonacci sequence. And we have a initial condition we have seen that F 0 equal to 0, F 1 equal to 1. And using the recursive rules we have found out that F 2 equal to 1 F 3 equal to 2 F 4 equal to 3 like that already we have seen. Now we want to see that; what is the value of the or the sum of the squares of the terms of Fibonacci sequence. So, if I write the say for I equal to 1, then it is sum of squares is F 0 square plus F 1 square. And this is 0 square plus 1 square is 1 and say I am writing this is 1 into 1.

Now, if it is I equal to 2 if I equal to 2 then sum becomes F 0 square plus F 1 square plus F 2 square. And this becomes 0 square plus 1 square plus F 2 is also 1. So, again 1 square that is 2 and this becomes 1 into 2. Say I equal to 3 so similarly I can write F 0 square plus F 1 square plus F 2 square plus F 3 square is 0 square plus 1 square plus 1 square plus my F 3 is 2 so 2 square. So, this becomes 6 is 2 into 3.

Now, if I continue in this way F 2 square plus F 3 squared plus F 4 square again this becomes 0 square plus 1 square plus 1 square plus 2 square plus F 4 is 3. So, 3 square and this is 9 plus 6 this is 15 this becomes 3 into 5. Now if we observe the results or the sum of squares for different values of I. What we see that we can write that it is for I equal to 1 1 into 1 so this becomes F 1 into F 2 because my F 1 and F 2 both are 1.

Now, my F 2 F 3 is 2 so I can write. So, I can write this thing as F 1 into F 2 I can write this thing as the F 2 into F 3. I can write this thing as the F 3 into F 4. This as the F 4 into F 5 so we give we get a pattern.

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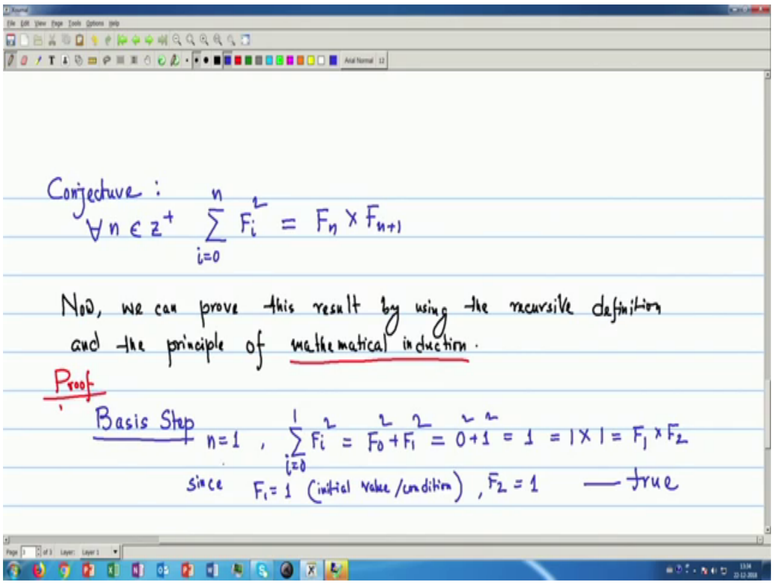

So, we can frame the formula that we can write that the conjecture or the result is that we can write the conjecture that for all n belongs to z plus the positive set of positive integers. The sum of squares of the Fibonacci terms that up to n terms if I write this is F i square and that becomes F n into F n plus 1. So, this is first we see the property that what will be the sum of squares of in Fibonacci terms and we get this formula.

Now we can prove now we can prove this result by using the recursive definition and the principle of mathematical induction. As in the last lecture we have described that recursive definition is closely related with the mathematical induction. Because the recursive rule that is framed from the concept of mathematical induction. Now how we can prove that thing.

So, we will be using the mathematical induction. So, if I get the proof so first thing is that as the we have want to apply mathematical induction. So, we have to write the basis step. So, for basis step we that for n equal to 1 we take assume n equal to n equal to 1 then the conjecture becomes the n 1 F 1 square or should write the similar way as it is written that I equal to 0 to 1.

F I square equal to F 0 square plus F 1 square and from the initial conditions this is 0 square plus 1 square is 1 and that I can write that 1 into 1 is F 1 into F 2. Since we know that F 2 is also 1, since F 1 equal to 1 from initial condition initial value or condition we can write. And F 2 equal to 1 so my conjecture is true for the basis step n equal to 1.

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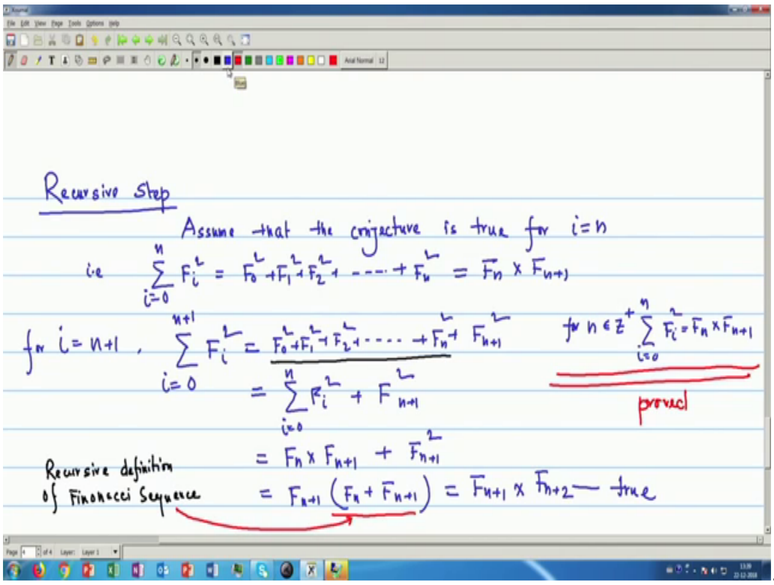

Now, we see the how we can prove the recursive step now to prove the recursive step we use mathematical induction. So, we assume that the conjecture is true for I equal to n; that means, for I equal to 0 to n F I square equal to F 0 square plus F 1 square plus F 2 square plus F n square this equal to F n into F n plus one.

Now, if we can show that this conjecture is true assuming that it is true for I equal to n the conjecture is true for I equal to n plus one. So, for I equal to n plus one we have to show that conjecture is true if we can show then 0 to n plus 1. And this is F i square so I can write this is F 0 square plus F 1 square plus F 2 square F n square plus F n plus 1 whole square F n plus 1. Now this is from F 0 square to F n square I can replace by this summation I equal to 0 to n F I square plus F n plus 1 square.

Now, already according to that principal of mathematical induction it is the conjecture is true for I equal to n so I can write this is F n into F n plus 1. And the next term is F n plus 1 whole square. So now, if I take F n plus 1 comma then this becomes F n plus F n plus 1 is F n plus 1 into F n plus F n plus 1 is according to the definition of Fibonacci sequence this becomes n plus 2. So, here I use the recursive definition of here we use the recursive definition of Fibonacci sequence and we use the principle of mathematical induction.

So, this is F n plus 1 F is for I equal to n plus 1 it is true it is true. So, I can write that the property for n belongs to for all n belongs to the set of positive integers that summation I equal to 0 2, n F I square is F n into F n plus 1 this we can the proof. And in this way in this way I can prove many such properties for this Fibonacci sequence. Now we observe or some the importance of the basis step or the initial condition.

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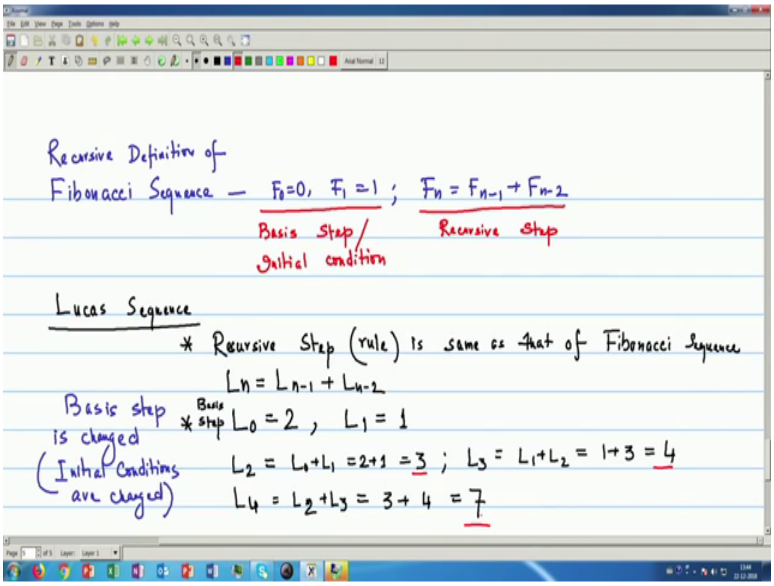

So, how we define that Fibonacci sequence? We define the Fibonacci recursive definition that we have given the Fibonacci sequence the recursive definition of Fibonacci sequence that with the initial condition F 0 equal to 0, F 1 equal to 1. The sequence is F n is F n minus 1 plus F n minus 2. Now if we change the only the basis step so this is my initial conditions or the basis step I can write this is my basis step or my initial condition of the sequence.

So, if I just change this the even if the rule is recursive rule is same then it will generate some new sequences so this is my recursive step. Now one such new sequence we call that are the Lucas sequence we see that Lucas sequence even if the recursive steps are same. So, here the recursive rule or step recursive rule or recursive step is same as that of Fibonacci sequence. So that means, if we can write that represent this thing as a L, Lucas sequence as L then L n is L n minus 1 plus L n minus 2.

But we change the basis step basis step is changed we write the basis step is changed or I should tell that initial conditions are changed, initial conditions are changed. So, initial conditions we write that L 0 equal to 2 and L 1 is same as that of Fibonacci F 1 is 1. Now if we see the sequences we see the sequences and we write this is my basis step say I can

I can find out that L 2 which is 0 plus L 1 this is 2 plus 1 equal to 3. L 3 is L 1 plus L 2 is L 1 is one L 2 is 3 so this is 4.

L 4 is L 3 plus L 2 plus L 3 so this is 3 plus 4 is 7. So, we see that terms are even the recursive rules are same as that of Fibonacci sequence, but the terms are different. Here L 2 is 3 L 3 is 4 L 4 is 7 and the terms are different. So, if we can if we prepare the list we see the, we can compare the sequences; the Fibonacci sequences sequence and Lucas sequence.

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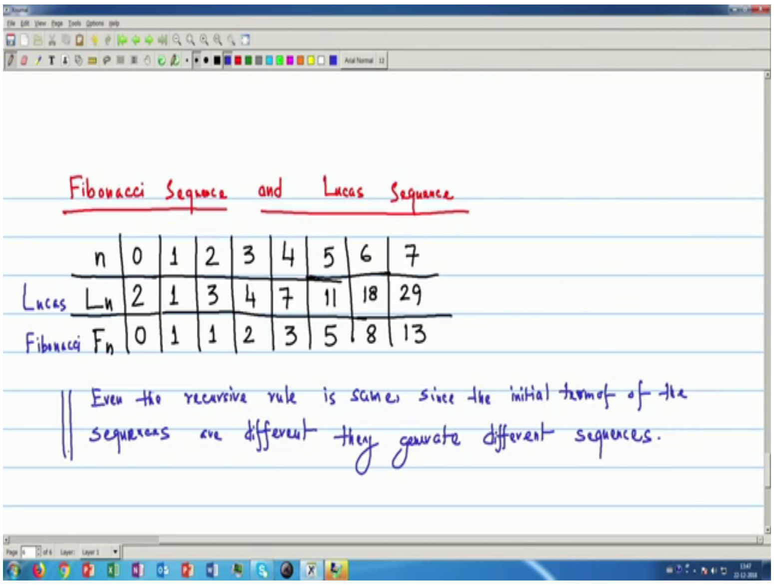

You can if we compare the values we see that if we can put n we give L n and we get give the F n and we see the different values of n. Say for n equal to 0 1 2 3 4 we take some more values 5 6 7. And we can see L n is 2 because it is given the initial conditions then 2 plus 1 3, 3 plus 1 4, 4 plus 3 7, 7 plus 4 11, 11 plus 7 18, 18 plus 29. And here the initial conditions are different for Fibonacci it is 0 and 1. So, it is 1 1 plus 1 2 it is 3 then it is 5 then it is 8, 13 like that. So, what we see that that this is my Lucas number and this is my Fibonacci number.

See even the so conclusion is even the recursive rule or the recursive definition is same; same since the initial condition or initial terms of the sequences are different they generate different sequences. Now since the recursive rule is same only the initial values are different. So, there are there is a close relation between the Lucas numbers and the Fibonacci numbers. So, we can see some; one property that again like.

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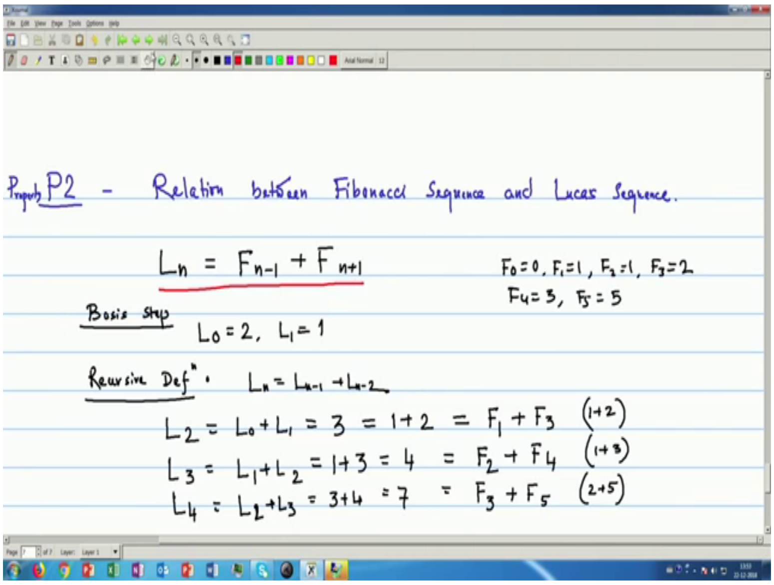

We have seen earlier only the Fibonacci property. So, one property we have seen that where we give either relation this gives some relation between Fibonacci sequence and Lucas sequence. I write one such property as the see if I take the sum of Fibonacci numbers I take Lucas number is i equal to I take a simple property that I can write the Lucas number say nth term of the Lucas sequence L n I can write it is F n minus one plus F n plus 1.

So, if I see the list of sequence quickly we can see that my basis step is L 0 equal to 2, L 1 equal to 1 so I can write and the recursive definition if I give recursive definition of Lucas number. We know that L n is L n minus 1 plus L n minus 2. So, I can write that it is I can find that L 2 since L 2 is L 0 plus L 1. And we have seen already the numbers are 3 etcetera ok. And I can write 3 as 1 plus 2 and if you remember that this is F 1 I can write F 1 plus F 3 is 2.

Because if we remember the Fibonacci F 0 equal to 0 F 1 equal to 1 F 2 equal to 1, F 3 equal to 2 then F 4 equal to 3 like that. So, what is my L 3 L 3 is L 1 plus L 2 and L 1 is 1 L 2 is 3. So, this is 4 and I can write that this is 4 is 3 plus 1 that means, F 2 plus F 4. If I continue in this way L 4 is L 2 plus L 3 and this is my L 2 is 3 plus L 3 is 4 this is 7 and I can write that this is F 3 2 plus 5 F 3 plus F 5.

So, I can write this is 2 plus 5 this is F 2 is 1 plus 3 this is also 1 plus F 1 is 1 F 3 is 2. So, I can see that the property holds. And the similar way we can write that. So, we can see that property holds for that L n is L n is F n minus 1 plus F n minus 2. And the similar way we can we can give the proof also.

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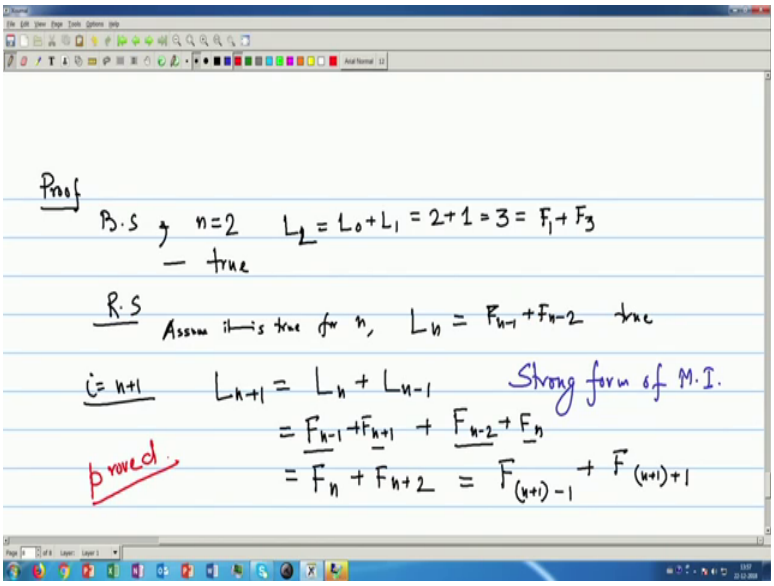

We can see that for basis step if it is for n equal to 1 let my L 1 or for n equal to 2 if I see that n equal to 2 L 2 is L 0 plus L 1 is 2 plus 1 equal to 3 equal to my F 1 plus F 3 1 plus 2. So, it is true and for recursive step assume for assume it is true for n assume it is true for n. So, L n equal to F n minus 1 plus F n minus 2 true; so, for i equal to n plus 1 what I can show L n plus 1 is L n plus L n minus 1.

Now, I can apply the strong form of mathematical induction here we can use strong form of mathematical induction. So, that for L n L n minus 1 for both we can apply the conjecture and then for L n it is F n minus 1 plus F n minus 2 and for L n minus 1 it is F n minus 2 plus F n. So, if I take together the for L n it is L n is L 0 plus L 1 L minus 1 plus F and n plus 1. This is L minus 1 plus F n plus 1 for n minus 1 it is n minus 2 and n.

So, this becomes if n minus 1 F n plus F n plus 2 is F n plus 2 and n plus n minus 2 and n minus 1 and n minus 2 this becomes F n and this becomes F n plus 2. So, which is the F n plus 1 minus 1 plus F n plus 1 plus 1 so it is proved; so, we see that how recursively defined sequences can be defined and for new sequences. And how we get the properties; how we can prove the properties using the mathematical induction as well as the recursive definitions of the sequence.