Discrete Structures
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Lecture – 14
Proof Techniques (Contd.)

So, we have learned the mathematical induction and now we will see that now there is another form of mathematical induction which is also very useful to solve many other mathematical problems which is the strong form of mathematical induction.

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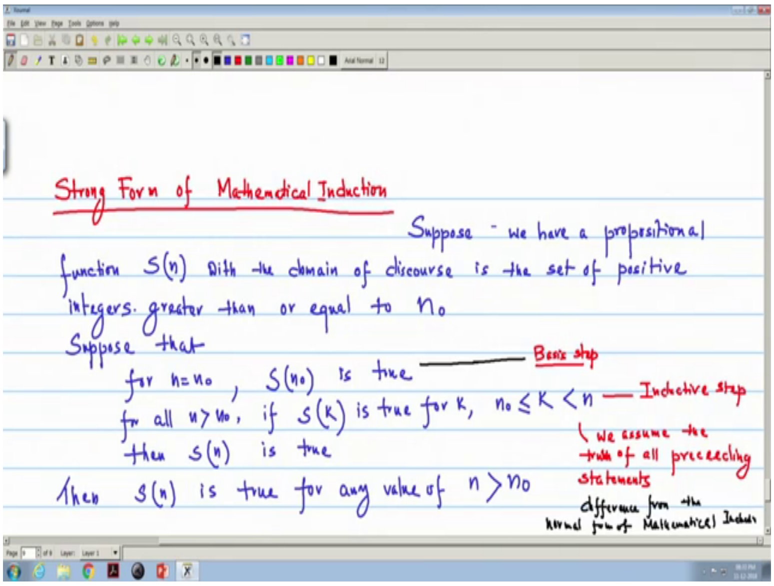

So, if we remember the principle of mathematical induction, then we have assumed that if we have to proof the formula for some value; then the immediate predecessor value or immediate predecessor statements is assumed to be true.

Now, the main difference from the normal mathematical induction and the strong form of mathematical induction is that here we assume that not only the immediate predecessor statements, but for all previous or preceding statements must be true. So, first we read, write the principle of strong form of mathematical induction; so, suppose we have a propositional function say S n with the domain of discourse is the set of positive integers. We put one restriction that set up positive integers greater than or equal to n 0.

Now suppose that for what you see for n equal to n 0; that means, S n 0 is true and for all n greater than n 0, if S k is true for k where n 0 less than equal to k less than n, then S n is true. Then we claim then S n is true for any value of n greater than n 0.

Now, we first see the difference from the simple mathematical induction. So, what is the basis step? This is our basis step. We have considered the general value that n equal to n 0; n 0 can be 1, but we take some other value we can take also that some other value of other than 1. And then for all n greater than n 0 if S k is true for k n 0 less than equal to k less than n; that means, this is my inductive step and here we do assume that all preceding statements are true. That means, I can write we the inductive state; we assume the truth values or truth values you can write truth of on preceding statements.

So, this is the main difference from the difference from the normal form of mathematical induction from the normal form of mathematical induction, where we have assumed only the immediate predecessor statements to be true. And many times these are strong form of mathematical induction are very useful to prove some complex problems. Now, we see some of the examples and how actually strong form of mathematical induction can be utilized.

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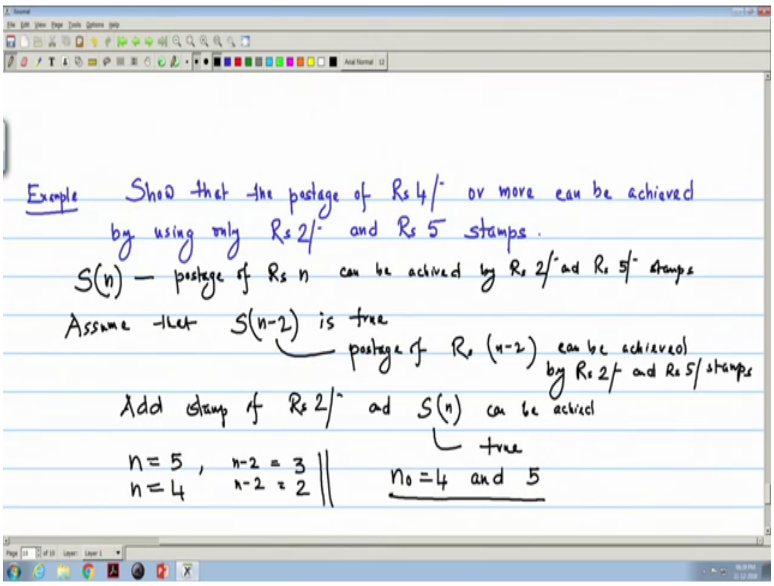

So, we see one example. Show that the postage of rupees 4 or more can be achieved by using only rupees 2 and rupees 5 stamps. So, first we have to show the basis step.

Now, here before writing the basis step and the inductive state, we try to explain that how what will be our approach that to use the strong form of mathematical induction. See since rupees 2 stamps are allowed rupees 2. So, if any n minus 2 postage it is true; that means, any n minus 2 postage can be achieved with rupees 2 and rupees 5, then what we can do? We can just add another stamp of rupees 2 and we will get the postage of rupees n; that means, if this is my proposition say proposition is S n that S n is the say proposition of that postage of rupees n; postage of rupees n can be achieved my rupees 2 and rupees 5 stamps.

Now, if we assume that if we assume that S n minus 2 is true; that means, that S n minus 2 is that postage of this is postage of rupees n minus 2 be achieved can be achieved by rupees 2 and rupees 5 stamps, then we will simply add rupees 2 more. So, we add stamp of rupees 2 and we will get S n and S n can be achieved or S n is true. So, this is our approach. So; that means, is very simple if we consider or if we can see that S n minus 2 is true then we will add each time only rupees 2 stamp and we will get the, we will get for all values of n it is true.

Now, say if it is 5 say n equal to 5 then or n minus 2 is n minus 2 is 3 so, but our statement is not given for n equal to 3 or say if n equal to 4, then n minus 2 is 2. So, it is not given it is particularly for n equal to 5. So, our statement for this is n equal to 3 or n equal 2 it is not given the statements are only rupees 4 or more rupees 4 or more. So, here our n 0 is; that means, our basis step that n 0 is equal to 4 and 5. We have to show separately that the statement is true for n equal to 4 and n equal to 5. So, this is this should be our basis step. So, now we see the proof.

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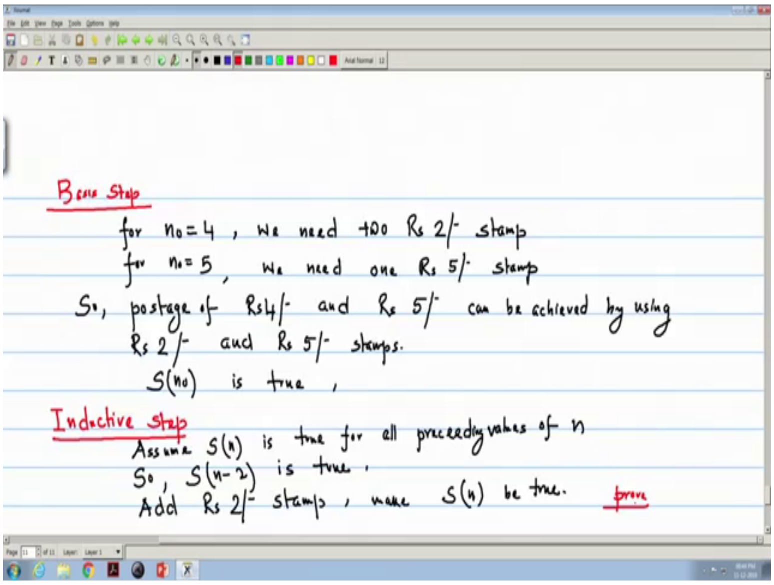

So, what basis step when for n 0 equal to for n 0 equal to 4, we need 2 rupees 2 stamp for n 0 equal to 5, we need 1 rupees 5 stamp. So, postage of rupees 4 and rupees 5 postage of rupees 4 and rupees 5 can be achieved by using rupees 2 and rupees 5 stamps. So, our basis step is true. So, our S n 0 is true or basis step is true. Now what about our inductive step? Already we have seen that strong form of induction that all preceding statements are true. So, assume S n is true for all preceding statements or all preceding values of n. So, it is true for n minus 2. So, S n minus 2 is true.

So, once S n minus 2 is true add rupees 2 stamp and make S n be true so, it is proved. So, one S n minus 2 is true we will just add rupees 2 and S n is true. So, it is proof that any postage of rupees 4 or more than 4 can be achieved by stamps of rupees 2 and rupees 5. So, this is proved. So, notice for this particular example it is very much required that that sum preceding values like for n we have that S n minus 2 must be true. Now, we say different type of examples that how mathematical induction can be used to compute the complexity or the number of steps because that is very important in computer science. So, we see a different type of example here.

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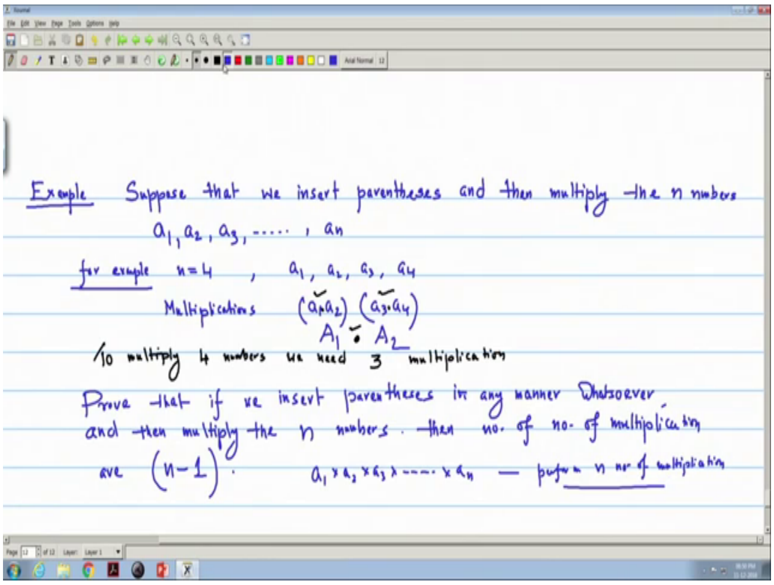

Suppose we have we insert parentheses and then multiply the n numbers. So, we have to multiply n numbers say a 1 a 2 a 3 up to a n, n and we insert parentheses and then multiply the n numbers. Say for example, n equal to 4 I have the numbers a1, a2, a3, a4 and the multiplications are and multiplications of a 1 a 2 say a 3 a 4. So, we insert parentheses and first we multiply a 1 a 1 dot a 2 we multiply a 1 dot a 2 and then a 3 dot a 4 say this is a 1 say this is capital A 1 product, this is capital A 2 and then I need another multiplication. So, to multiply 4 numbers; so to multiply 4 numbers we need 1 2 3 3 multiplication.

Now, proof that if we insert parentheses in any manner whatsoever and then multiply the n numbers, then the number of multiplications are n minus 1; that means, to multiply a 1 into a 2 into a 3 up to n n, we have to perform n number of multiplications. And if you insert parentheses and we multiply 2 numbers at a time; now, we see the proof.

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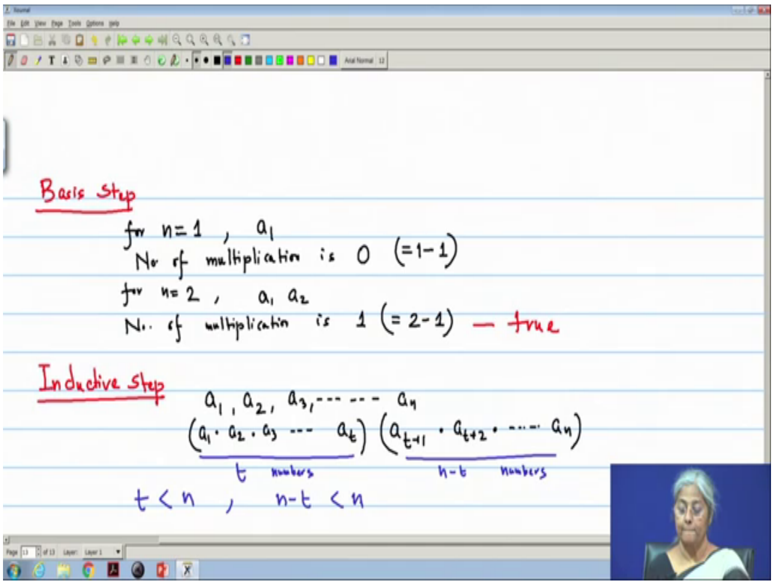

So, first is the our basis step. So, for n equal to 1, I have only one number a 1 only 1 number a 1. So, the number of multiplications is 1 which is equal to number multiplication is 0 0 which is equal to 1 minus 1 n minus 1. For n equal to 2 the numbers are a 1 and a 2. So, the number of multiplication is 1 equal to 2 minus 1. So, the basis step is true. So, we can write that it is basis step is true; that means, for n equal to 1 or n equal to 2; the multiplication is n minus 1 is true.

Now, for inductive step say I have n number of n numbers a 1, a 2, a 3 up to a n. So, now, we put parentheses. We first partition into 2 a 1 we multiply a 1 a 2 up to t a t and a t plus 1 a t plus 2 up to a n. Now if we apply the strong form of induction, then t is less than n first thing is that we partition or we insert parentheses so, that t number t numbers we multiply and rest n minus t numbers these are t numbers t numbers. And, these are n minus t numbers n minus t numbers; so, here t less than n as well as n minus t also less than n.

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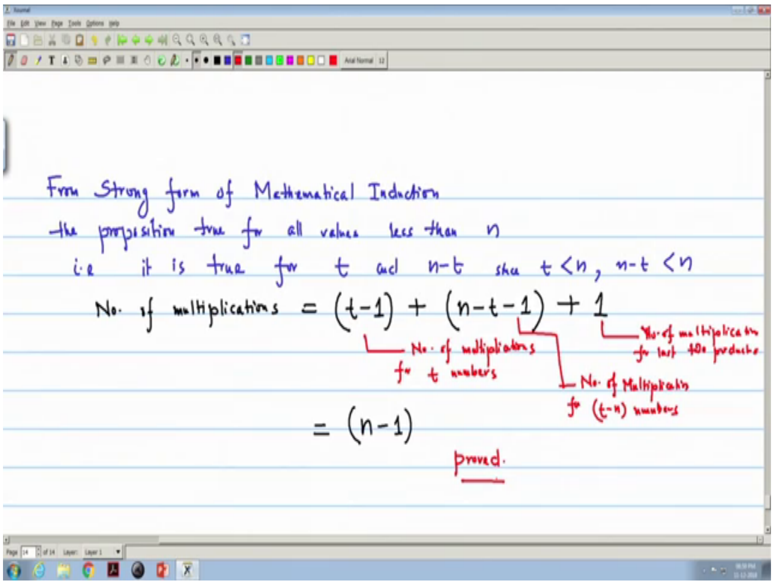

So, if we use the from strong form of induction mathematically induction is write, the proposition is true for all preceding statements true for all values less than n; that means, it is true for t and n minus t since t less than n and n minus t also less than n. Since, we have partition or we have inserted the parentheses so, that t numbers are multiplied in one part and rest n minus t are multiplied in another part.

So, if we continue the inductive step we can write that the number of multiplications; the number of is for the first part it is since t numbers are there. So, t minus 1 multiplications are required plus for n minus t for n minus t minus 1 multiplication of required and the last multiplication I need another one. So, this is for t number of multiplications or I should write multiplication of number of multiplications for t numbers and this is one number of multiplication for t minus n numbers and this is for number of multiplication for last 2 product.

So, this becomes n minus n minus 1. So, it is it is proved; see this is number of multiplications. So, similar way what will given a program or some computations like that this is a multiplication. So, what will be the number of computations required, number of multiplication, number of additions, a number of comparisons which actually directly gives a complexity of a program so, or complexity of some algorithm. So, using strong form of mathematical induction sometimes it is very easy to compute the complexity of some algorithm or the of some program.

We have now ones small thing that it is we have read the simple or normal form of mathematical induction and the strong form of mathematical induction. Now yeah if we know read the well ordering property which is actually the generalized concept of both of this form. So, I must I just mention or write this property.

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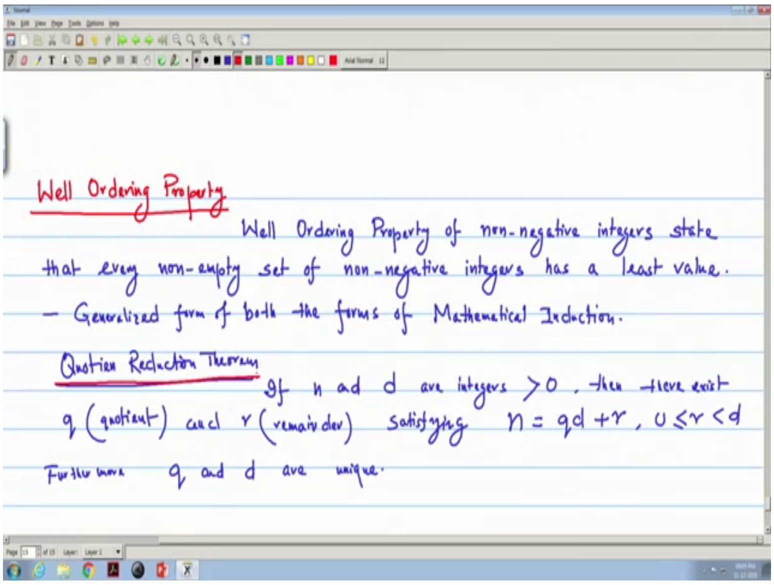

Now, it is the well ordering property. Now, well ordering property of non-negative integers that if state that every non-empty set of non-negative integers has a least value; and this is the generalize this simple statement is the generalized form of the normal mathematical induction and the form of normal mathematical induction as well as strong form of mathematical induction. We can write generalized form of both the forms of mathematical induction.

Only I mention that this well ordering property; this is mainly used to proof our well known that quotient reduction theorem. That quotient reduction theorem, it is used and the theorem that if a positive integer n is divided by d; that means, if n and d are integers greater than 0, then there exists q the quotient and r the remainder that satisfy n equal to q d plus r where, 0 less than equal to r less than d and furthermore that q and d are unique; furthermore q and d values are unique.

So, this is our very well known Quotient Reduction Theorem and this is when it is proved by very easily it can be proved by well ordering property. But, all of we know this simple form just I mentioned that we can prove this thing by this mathematical induction. So, with this we finish that concept or that how mathematical inductions are used to prove the many mathematical formulas.