Discrete Structures

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Lecture - 13
Proof Techniques (Contd.)

We are discussing about the proof techniques and last lecture, we have read the direct proofs and the indirect technique of proofs. Today we will read the mathematical induction. And this is the most important proof technique that we use.

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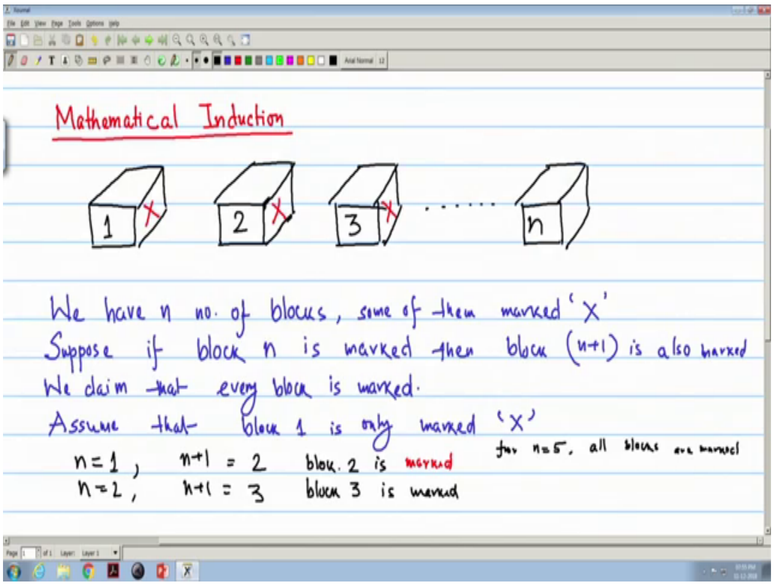

So, we will read mathematical induction. First I explained what do we mean by mathematical induction, and how it is used to prove the different formulas. Suppose, we have a sequence of blocks numbered 1, 2, 3 like that. We draw the block. So, first we will draw the blocks. So, these are number 1, 2, 3 and say up to n. So, we have n number of blocks. And some of the blocks here are marked as x, say some of the blocks marked x. And suppose that we have suppose so first we have n number of blocks, some of them marked cross. Suppose, if block n is marked, then block n plus 1 is also marked, this is given. Then we claim that every block is marked. Then how do we prove that our claim is true?

So, say first we see that mark n number of lick blocks, and some of them marked x. So, first we assume we assume that block 1 is block 1 is only marked. So, I have block 1 is only marked. Now, it is given that if block n is marked, then block in plus 1 is marked. So, if we n equal to 1, since block 1 is marked, so n plus 1 equal to 2. So, block 2 is also marked, block 2 is also marked. So, block 2 is marked. Now, if n equal to 2 is marked, so n plus 1 equal to 3, so block 3 is also marked block 3 is also marked.

Now, in this way, if I increase n and we add 1 to n, we can show that up to any number say n that all blocks are marked. Now, suppose some a few of the blocks are marked say for n equal to 4, the block is marked. Say for n equal to 5, block is marked all blocks are all blocks are marked. Then if I consider the n plus 1, that means, block 6 then it can be marked.

So, we can prove that for every n, and n greater than equal to 1 that all blocks are marked. So, this simple concept that if it is true for one basic value and then some statements are given, then based on that statements, we can prove that it is true for all values of n, all positive values of n and this is actually the mathematical induction.

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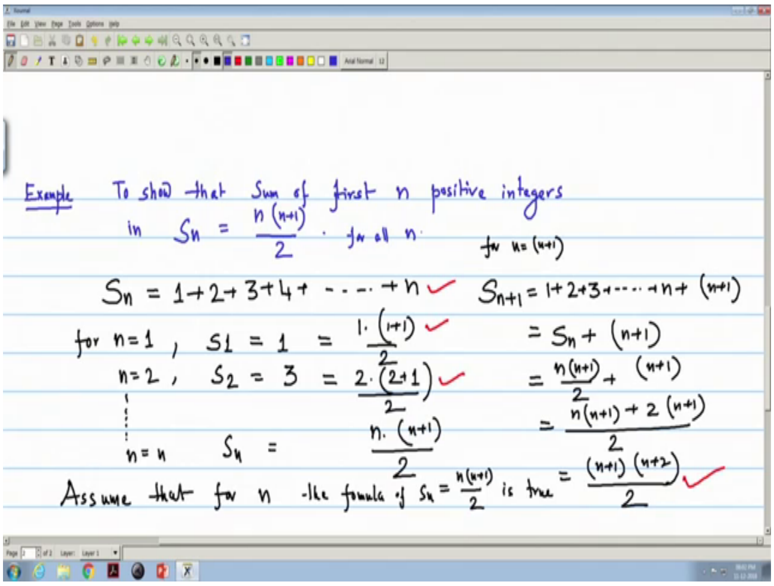

So, now, we use mathematical induction in more formal way. We first take an one simple example how we use the mathematical induction to prove some simple mathematical formula. So, we will take one example that to show that sum of first n positive integers is n into n plus 1 by 2, all of you know this formula, but we have to prove by mathematical induction.

Now, first we see it first sum of first n positive integers, so we can write that S n is we can write S n is 1 plus 2 plus 3 plus 4 plus n. So, for n equal to 1, S 1 is only 1. So, I can write this is 2, 1 plus 1 by 2. For n equal to 2 is 2 is 2 into 2 plus 3 by 2, so 2 plus 1. Now, if I write for this is for n, this value is n into n plus 1 by 2.

Now, from the givens series or the this formula, we can see that for S 1 equal to 1, this is true, we give a tick mark. For n equal to 2, it is 1 plus 2 is 3, so the formula is true. For n equal to n we assume we assume that for n the formula is true. So, we assume we assume that for n, the formula for S n formula of S n equal to n into n plus 1 by 2 is true.

Now, we have to show that or if we can show that for the next value of n, it is true, then we can take that for each value of n the formula is true. So, you see for n equal to n plus 1. So, for n equal to n plus 1, or from the definition we know that it is S n plus 1 is 1 plus 2 plus 3 plus n plus n plus 1. Now, these I can write is S n plus n plus 1, S n is the sum up to n terms, and we have assumed that for n it is true. So, write the formula for S n which is n into n plus 1 by 2 plus n plus 1.

If we do n into n plus 1 plus 2 into n plus 1 divided by 2 is n plus 1 n plus 2 divided by 2. So, we see that the formula for n plus 1, this is also true. So, now we can claim that S n is equal to n into n plus 1 by 2, this is true for any value of n. So, we can write for all n this is this is true for all n, true for all values of n. So, now we can give the principle of mathematical induction.

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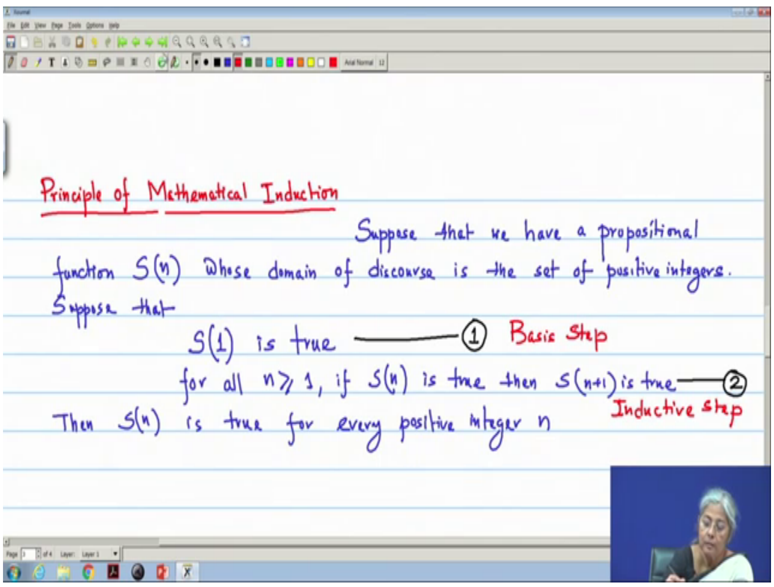

Suppose that we have a propositional function. Let the function be S n whose domain of discourse is the set of positive integers. Suppose, that S 1 is true, that means, the propositional function S n for n equal to 1 is true for all n greater than equal to 1 if S n is true, then S n plus 1 is true. Then S n is true for every positive integer n.

Now, we have taken one propositional function S n, and we assume that S 1 is true. We give this is number 1. And for all n greater than equal to 1 if S n is true then it is given that S n plus 1 is true. Then S n is true for every positive integer n. So, this is the principle of mathematical induction. And we will use this thing to prove the formula or to verify some equalities or inequalities in mathematics. Now, we see that some example we see with this thing.

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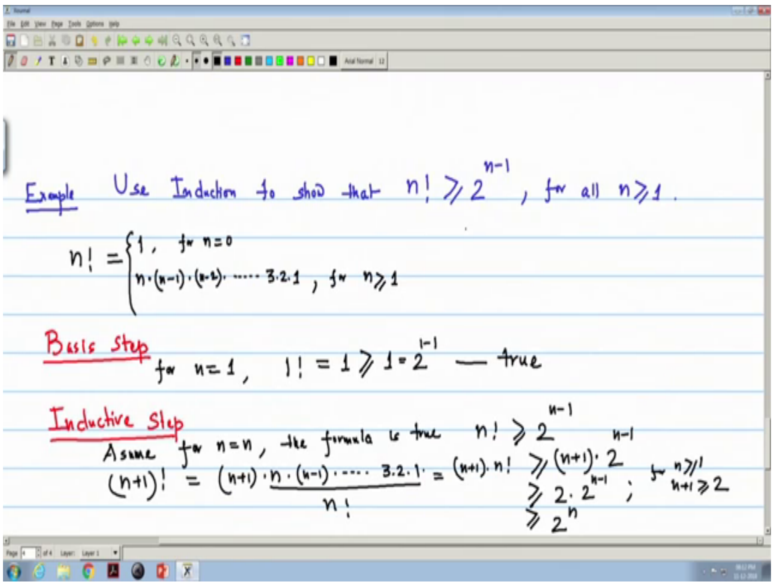

You take one example simple example that use induction to show that n factorial is greater than equal to 2 to the power n minus 1 for all n greater than equal to 1. So, we know the definition of factorial that n factorial equal to 1 for n equal to 0, and equal to n into n minus 1 into n minus 2 to 3 to 1 for n greater than equal to 1. So, the if we want to use the mathematical induction, then in the principle of mathematical induction, this one we S 1 is true these we call that the basis step, these we call the call this is our, and equation 2, this is all inductive step.

So, first we have to see that whether our basis step is true or not, because in the principle of mathematical induction we have assumed that our S 1 is true, so that S n for n equal to

1 is the base system. So, here for n equal to 1 for so I give that basis step for n equal to 1 one factorial equal to 1, which is greater than equal to 1 is 2 to the power 1 minus 1. So, it is so the basis it is true. So, the basis step is true.

Now, we see the inductive step. Now, we assume that for n for n equal to n it is true. So, assume for n the formula or the proposition is true, that means, n factorial greater than equal to 2 to the power n minus 1. Now, we have to show that it is true for n plus 1 factorial. So, we see that for n plus 1 factorial, we know from the definition that it is n plus 1 into n into n minus 1 up to 3 to 1. So, up to these this is nothing but n factorial. So, this is our n plus 1 into n factorial that all of we know. So, now, I can write this is greater than equal to n plus 1 into 2 to the power n minus 1.

Now, one can write this is greater than equal to 2 into 2 to the power n minus 1, because for n greater than equal to 1 n plus 1 greater than equal to 2. So, this is greater than equal to 2 to the power n. So, for n plus 1 n equal to n plus 1, it is proved. So, according to the mathematical induction the prince principle of mathematical induction that, we can claim that for all values of n that n factorial greater than equal to 2 to the power n minus 1 is true. So, it is proved.

Now from these two examples that what we see that one correct formula must be given. And we are actually proving that whether for some basic values that is for n equal to 1, the formula is true or not; and then for some n if we assume that for n it is true, then whether n plus 1 the formula is true or not. Then the question is that how we can get the formula or the correct formula. So, sometimes that from some the result for different values of n, we can or from the sequences of the results, we can frame some formula.

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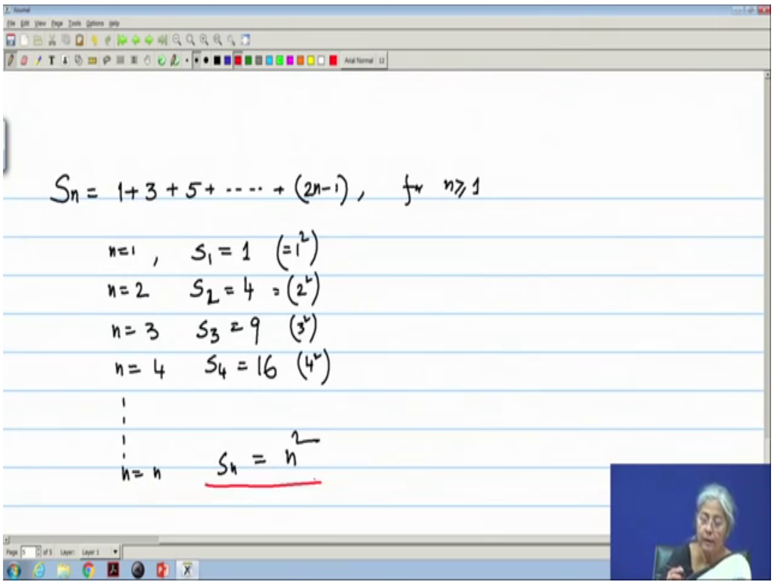

I give one small example say we want to add or take the sum of say S n equal to 1 all odd numbers 1 plus 3 plus 5 up to 2 n minus 1, for n greater than equal to 1. Now, I do not know the, what is the correct formula. So, what I will do. So, for S 1 that is for n equal to 1, I can write for n equal to 1, S 1 equal to 1; n equal to 2, S 2 equal to 4; n equal to 3 S 3 equal to 9; n equal to 4, S 4 equal to 16.

So, we get the pattern that which is this equal to 1 square, 2 square, 3 square, 4 square and so we can tell that for n equal to n that means, S n equal to n square. So, our formula is that is n equal to n square and then we can prove. So, this is one very simple technique to get the correct formula. Now, the scope of mathematical induction is again not only for proving the correct formula or verifying the equalities or inequalities, actually it can be used for many other mathematical proofs. So, one such example we see.

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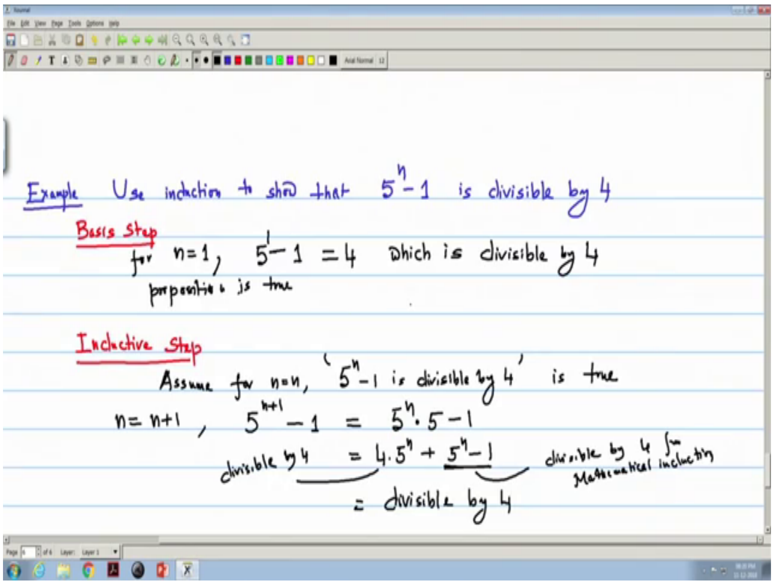

Use induction or mathematical induction to show that 5 to the power n minus 1 is divisible by 4. So, what will be the basis step, we take for n equal to 1, for n equal to 1, it is 5 to the power 1 minus 1 equal to 4, which is divisible by 4. So, the basis step, the proposition is true. So, the statement or the proposition is true.

Now, we see the inductive step. So, assume for n that means, 5 to the power n minus 1 is divisible by 4. The proposition is true is true. Now, I see whether it is true for n equal to for n plus 1. So, for n equal to n plus 1, it should be 5 to the power n plus 1 minus 1. So, I can write that is 5 to the power n 5 minus 1 is 4 into 5 to the power n plus 5 to the power n minus 1. So, this 5 to the power n minus 1 is already it is true.

And since it is 4 into 5 to the power n, so it is the first term is divisible by 4. So, the first term is divisible by 4 and this is for the mathematical induction, since it is true for n. So, it is divisible by 4 from mathematical induction. So, it is divisible by, so it is divisible by 4. So, the basis step, we sum value this is basis value of n that is for n equal to 1, we check whether the formula is true or not. And for the inductive step, we assume that the does the immediate predecessor value is true.

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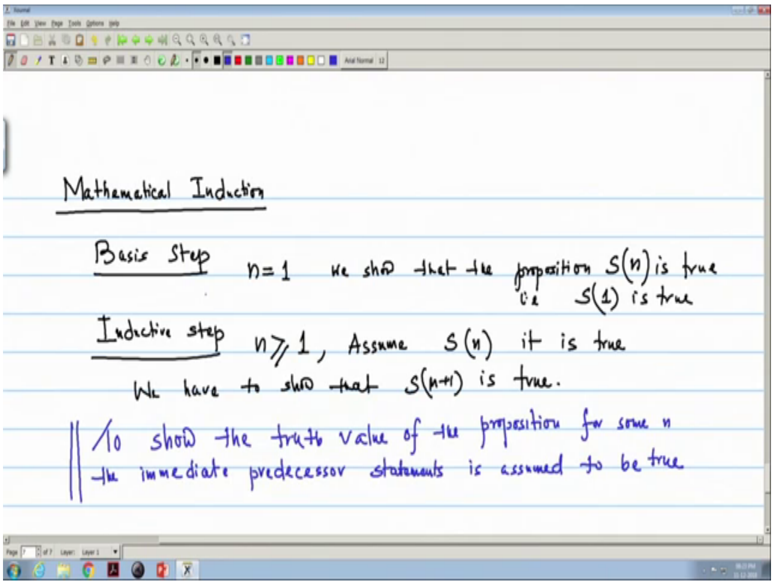

That means we can if we can write that that for mathematical induction that, how to use that for basis step some basic value of n normally for n equal to 1 n equal to 1. We show that the proposition show that the proposition, that means, the statement given statement or the formula say proposition S n is true; that means, S 1, S 1 is that means, S 1 is true.

And for inductive step inductive step that means, for all n greater than equal to 1, we first assume S n is for n it is true. Then we have to show we have to show that if n plus 1 is true, so that means, that to show the truth values of the proposition for some n, the immediate predecessor statements or the propositions is assumed to be true. And here the for the basis step for some basic value n equal to 1, it is true. Now, if for some formula or some proposition that n equal to 1 is not valid.

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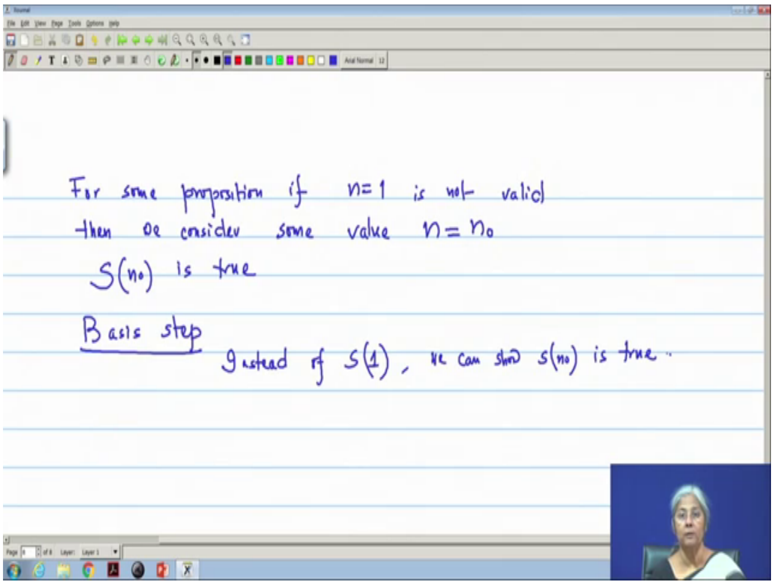

That means, for some proposition if n equal to 1 is not valid that means, there does not exist any result for this. Then we consider some value n 0, n equal to n 0. And we show that S n 0 is true. So, some basis step can be replaced that instead of S 1, we can show that S n 0 is true ok. So, this is the how that principle of mathematical induction is used to prove the formula or to verify equalities, inequalities and other proof techniques.