Discrete Structures
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Lecture - 12
Proof Techniques (Contd.)

So, we are discussing about the Proof Techniques and in the last lecture we have started the indirect proof techniques. And under indirect proof techniques we have read the proof by contradiction.

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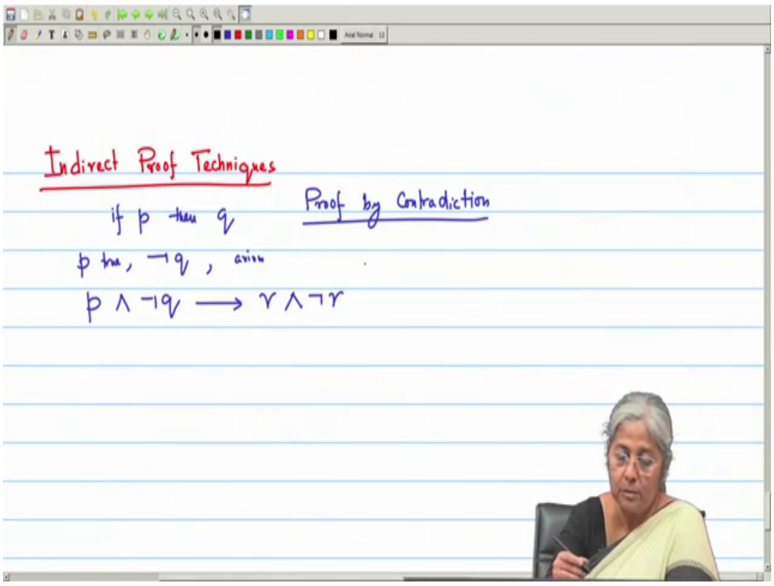

In the proof by contradiction we have assumed that when the theorem is represented as a if p then q, then using the hypothesis to be true p true the negation q; that means, conclusion which we have to prove true false and then other axioms definitions and previously derived theorems we have tried to prove that the conclusion is true or to give a contradiction. So, that was the indirect techniques or the proof by contradiction.

Now, how do we know that the proof by contradiction works correctly? Or more simple way if I tell that what we are assuming that negation q and if p is true and negation q together they imply a contradiction. That means, r and negation r whether that is working correctly or I can tell that whether they are equivalent, I can tell p and negation q whether they implied that r an negation r and whether this thing is equivalent to p and negation q, p implies q and r and negation r.

So, if we give it proved by the truth table method. So, we first give it to table that since here we have 3 propositions. So, I take a truth table give a truth table of next page we go.

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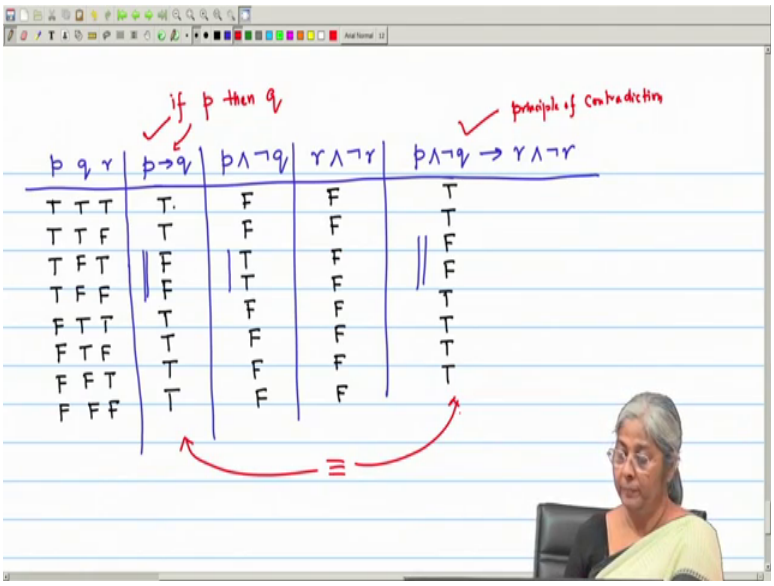

That p q r these are the 3 values then we take p implies q, if p then q then our assumption that p and negation q that we have assumed for contradiction and then the contradiction itself that r and negation r, then whether p and negation q this implies r and negation r.

Now we have to show that whether the p implies q; p implies q because this is the theorem that we want to prove; that means, if p then q which is which is p implies q and p and negation q implies r and negation r. What is the basic principle of contradiction? I write this is my principle of contradiction. So, first I show that whether this method is correct or it is coming correctly that the way we are applying the principle of contradiction, we take all the possible truth values of p q r. So, we take T T, T T F, T F T, and T F F then F T T, F T F, F F T and F F F.

So, what is p implies q? P T T implies T is true this is also true only T implies F this is false again T implies F this is false again when it is a all lay this is vacuously true so all are true. Then p and negation q, negation q is, for the first two cases it is false. So, this becomes false next to a true, then again negation q is false. So, this is false again these are false. So, accept this two, accept these two these are false. What is r and negation r? Always false r and negation r is always false. Now see p and negation q implies r and negation r. So, F F true, true only these two cases the T implies F we know these are false and remaining all are true, all are true. So, you see from p implies q the second column the T T, F F, F F and again remaining for T T T, F F and T.

So, these two are; these two are equivalent you see that this column and this column are equivalent. So, the technique that we are using in the contradiction method that it is proof that it should work properly; that means, p implies q that the theorem to be proved; that means, the proposition and which is equivalent to p and negation q implies r and negation r. So, this is the basic principle of contradiction.

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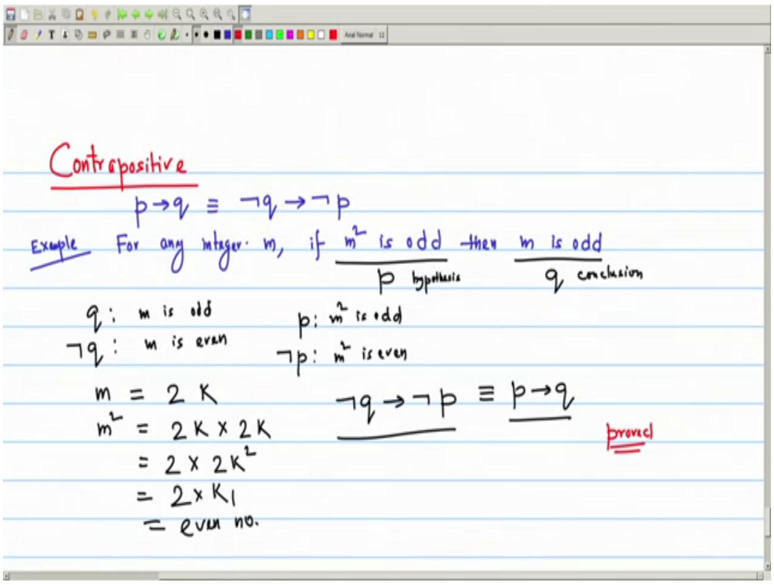

Now one special case of contradiction is the contrapositive and what is the principle of contrapositive? We know the when we have read the logic that if p implies q the conditional proposition that it is equivalent to that negation q implies negation p. So, sometimes to prove some statements or theorems that it is much easier to show that negation q implies negation p instead of p implies q.

We see one example, same for any integer m if m square is odd then m is odd this is one statement that we have to show that it is true or we have to prove. So, if m square is odd then m is odd. So, our m square is odd this is our hypothesis p, if p then q so m is odd this is my conclusion.

So, this is my hypothesis this is my conclusion and what contrapositive tells? That p implies q and negation is equivalent to negation q implies negation p. So, what is negation q? q is m is odd q is m is odd. So, negation q is m is even.

Now, we know the definition of even number. So, m is even is we can write 2 k. So, what is m square is 2 k into 2 k equal to 2 into 2 k square. So, again this is a even number since 2 into 2 k square we can write 2 into k 1 so that is a even number. So, what is even number? That hypothesis was m square is odd and we are getting m square is even; that means, it is negation p, this is my negation p because p was m square is odd.

So, negation p is m square is even. So, what we see we started with m is even; that means, the negation q and we got that it is negation p; that means, negation q implies negation p. So, and we know that this is equivalent to p implies q. So, since we get that this that this is true then p implies q so q is true. So, it is proved. So, this is by prove by contrapositive, this is a special case of contradiction.

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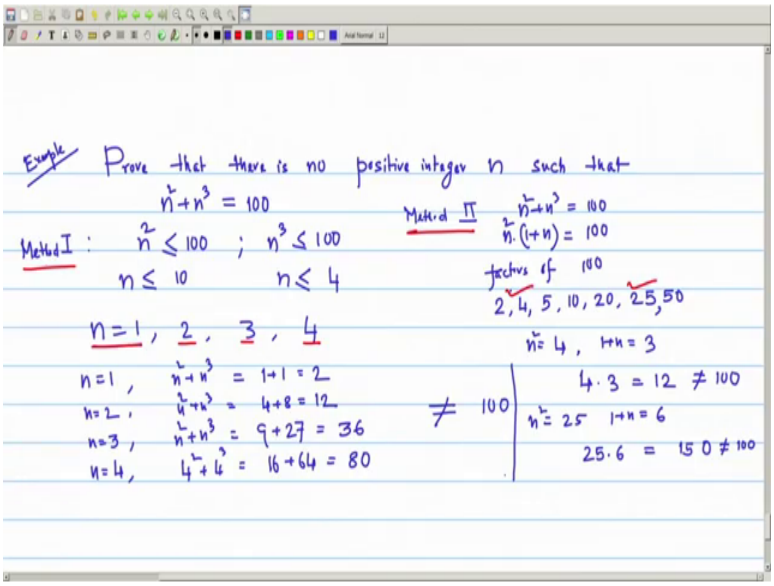

We see another example that, prove that there is no positive integer n such that n square plus n cube equal to 100. How to prove this thing? See two ways I can prove this method; very simple way.

So, one method see one way I can tell that n square plus n cube equal to 100. So, if I have two terms in the left hand side. So, if I think separately then n square must be less than equal to 100 and n cube also must be less than equal to 100, since the sum of n square plus n cube equal to 100. So, n square less than equal to 100; that means, n must be less than equal to 10 similarly, here I can tell n less than equal to it is 100s.

So, 4 because if it is 5 then it is 125. So, n less than equal to 4 because if I take both that thing together then only for n equal to 1, 2, 3, 4 these cases we have to check. Then what is the value of for n equal to 1? n square plus n cube is 1 plus 1 which is not equal to 100 for 2, for n square plus n cube equal to 4 plus 8 12, n equal to 3 36, n equal to 4 is it so, no one is these are not equal to 100.

Now, see here the proof is another way I can do, another way what better I tell that thing also that I can do in this way n square plus n cube equal to 100, I take n square 1 plus n is 100. So, since these are two factors we know the if when the factors of 100 these are 2, 4, 5, 10, 20, 25, 50. Now see n square this is a perfect square and in the factors of 100 there are only two perfect square, 4 and 25. So, if n square is 4 then n is 2. So, 1 plus n is 3, 1 plus n is 3. So, 4 into 3 this is equal to, 4 into 3 equal to 12 not equal to 100 and if n square equal to 25 then 1 plus n equal to 6, n equal to 5. So, then in that case 25 into 6 this is equal to 150 which is not equal to 100.

Now, these are the two ways I can prove that n n square plus n cube equal to 100 that it is there is no such positive integer n exists. Now see in this technique in both the cases the way we have proved, what we have done we have taken different cases of n or different values of n we have taken say the first method we have taken (Refer Time: 20:38) write method I, this is method II ok.

1st method we have taken n equal to 1, n equal to 2, 3, 4 and we have seen whether n square plus n cube equal to 100 or not. Here only the cases are we have taken n square equal to 4 and n square equal to 25, but see here also we have taken that 2 cases, here there are 4 cases here there are 2 cases. So, this is one technique which actually comes from the either direct method or indirect method, but this we categorize in a different way we will call that that proof by cases and many time this is very helpful or much easier to prove some theorems.

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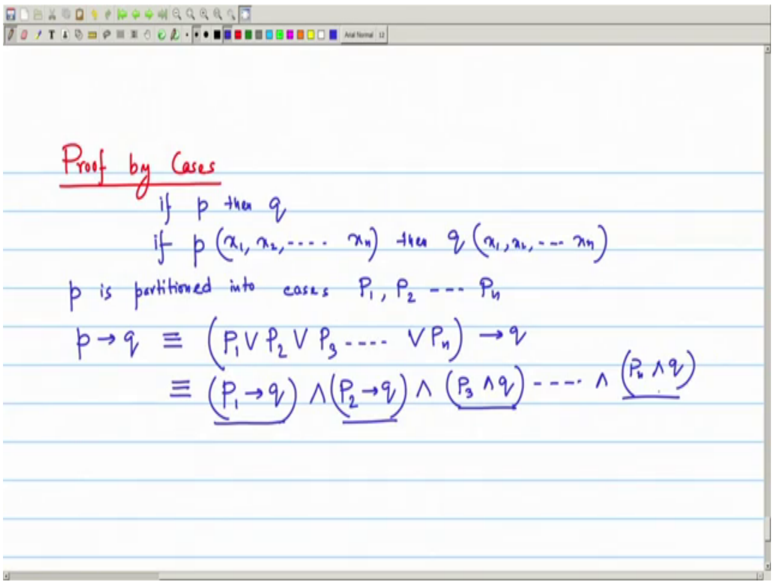

So, we can tell that this is the proof by proof by cases. How we define? Because our theorem is stated as or represented as if p then q.

And if we can elaborate if p it takes for very x 1, x 2, x n then q x 1, x 2, x n; now these if p is partitioned in to different cases. So, p is partitioned into cases see p 1, p 2 p n then I can write p I can write as or I can write the theorem statements p implies q. I can write p implies q is equivalent to p 1 implies q or p 2 implies q or first p 1 is p is partition. So, better I give p 1 or p 2 or p 3 p n. So, this is p, this whether this implies q and this is equivalent to p 1 implies q and p 2 implies q and p 3 implies q like that, p n implies q. So, we these are these are considered as the different cases and if I can prove then we can tell that this is actually proof by proof by cases for each cases. So, this is now we take one example.

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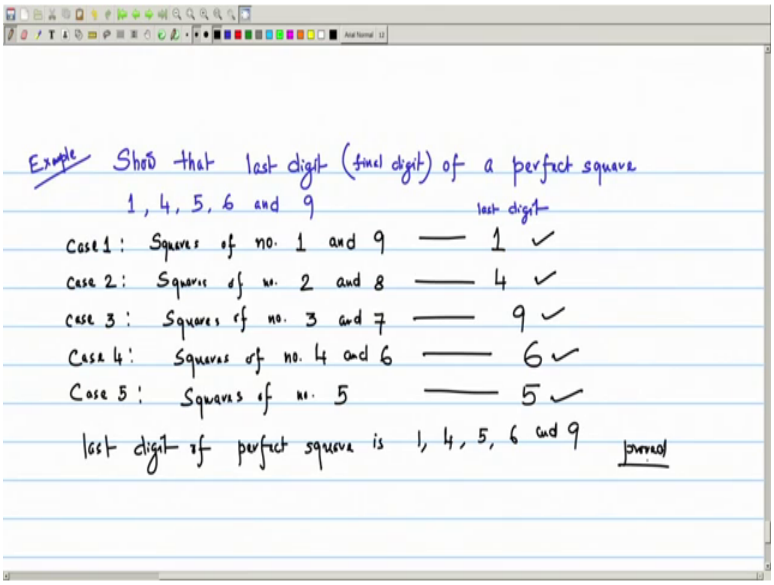

So, then the last digit or sometimes we call final digit of a perfect square are only are confined into some digits they are 4, 5, 6 and 9 ok. So, we have to show that the last digit of a perfect square are only confined to these num decimal numbers 1, 4, 5, 6 and 9. So, now, if we take by cases so I can think in this way that solution. So, this is case 1.

So, I take the squares of number 1 and 9 or we can think the number whose last digit is 1 and 9, then the last digit of that particular integer number is only 1 because 1 square 1 and 9 square is 81 so the last digit is 1. So, I can write this is my last digit. Now the case 2, the squares of number 2 and 8 and this is 4 and 6 plus digits will be 4 and 6 case 3 squares of number 3 and 7, both the cases it is only 9. Case 4, 4 and 6 both the cases it is only 6 also this is wrong, this is only 4 64. So, last digit is 64.

Then case 5, squares of number 5 is only 5. So, what we see that the whatever be the numbers that last digit is from 1 to 5 and then these are 1 for 2 and 8 is 4, 3 and 7 9, 4 and 6 6 and 5 so; that means, these are only the last digits are last digits last digit of perfect square is 1, 4, 5, 6 and 9 it is proved. So, this is a very simple example of the proof by cases, many time that one complex prove is becomes very simple if we can partition the case, it partition it into cases.

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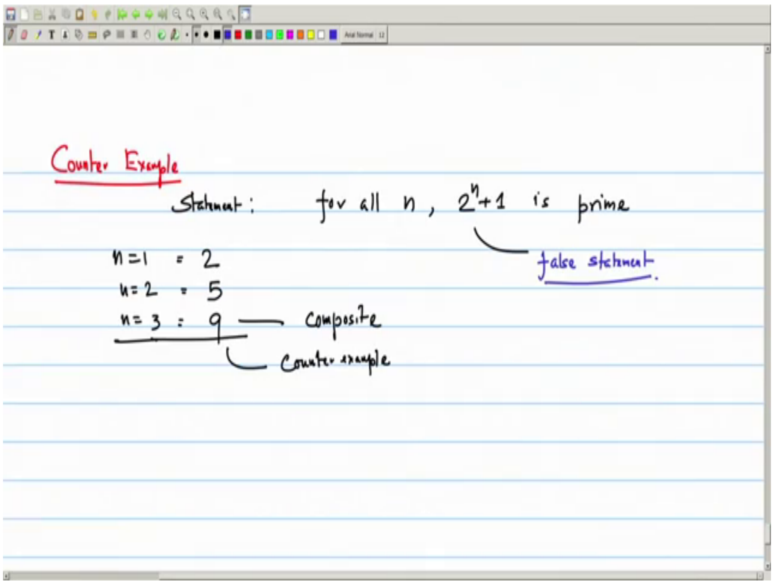

Now sometimes we call that there are many different, other different techniques which are actually derived from direct or indirect one such is that a counter example we created that one is that by counter example. So, just to give one example that show that for all n 2 to the power n plus 1 is prime, whether it is true or false or I make a statement rather that. So, for all, this is my statement on statement that for all n 2 to the power n plus 1 is prime.

Now, if we again do by cases say for n equal to 1 we see that this is 2 prime for n equal to 2 then it is 5 prime n equal to 3 it is 9. So, you see that for n equal to 3 that 2 to the power n plus 1 this is this is not a prime number, but this is a composite number 9. So, this is a counter example and we can tell that this is a counter example and we can tell that this is a false statement this is a false statement.

So, many time that we can use that this is proved by counter example. So, these are normally the different tech techniques direct or indirect and these are the prove by cases or the counter examples or the contradiction, the contrapositive which are slightly different variant of that direct indirect methods. And another way another proof techniques which is very important particularly in computer science that is that proof by induction. So, next step will read that thing.