Parallel Algorithms
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Lecture - 33
Shuffle Exchange Graphs, de Bruijn Graphs

Welcome to the thirty third lecture of the MOOC on Parallel Algorithms. We will continue with our discussion of interconnection networks that are related to hypercubes in particular when the previous lecture stopped, we were talking about Shuffle Exchange Graphs. So, let us continue with that discussion.

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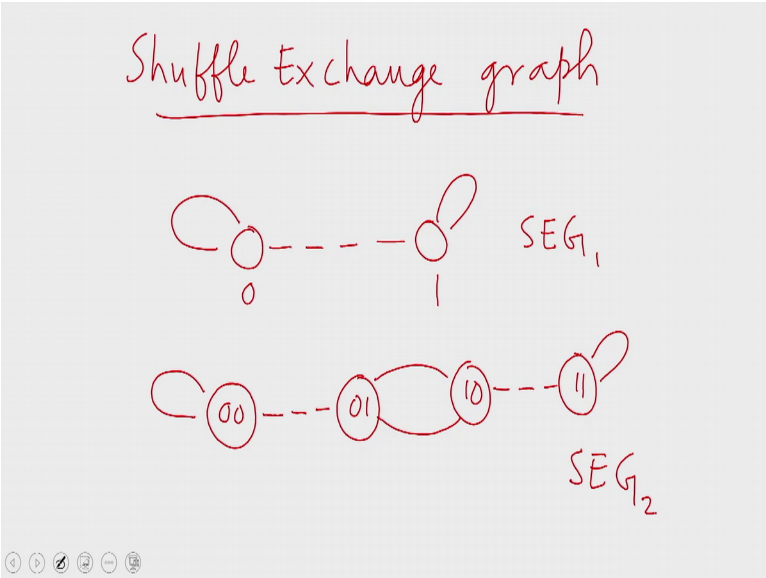

A shuffle exchange graph has two kinds of edges. There are shuffle edges and exchange edges. We saw a three-dimensions shuffle exchange graph in the previous class. Today, let us see first a shuffle exchange graph of dimension 1. A shuffle exchange graph of dimension 1 has two nodes named 0 and 1. There is an exchange edge between them because these two nodes differ in the least significant bit. And when 0 is cyclled to the left, we get 0 again. Therefore, we have a self loop involving the vertex 0 and when 1 is left shifted, we get 1 again. So, we have a self loop involving vertex 1 as well. So, this is a shuffle exchange graph of dimension 1.

The shuffle exchange graph of dimension 2 involves vertices 0 0 0 1 1 0 and 1 1. There is a self loop involving 0 0 and 1 1 because when these strings are left shifted, we keep getting the same string. There is an exchange edge between 1 0 and 1 1 and there is an exchange edge between 0 0 and 0 1. When 0 1 is left shifted, we get 1 0 and when 1 0 is left shifted, we get 0 1. So, there are two edges between 0 1 and 1 0. So, this is a shuffle exchange graph of dimension 2.

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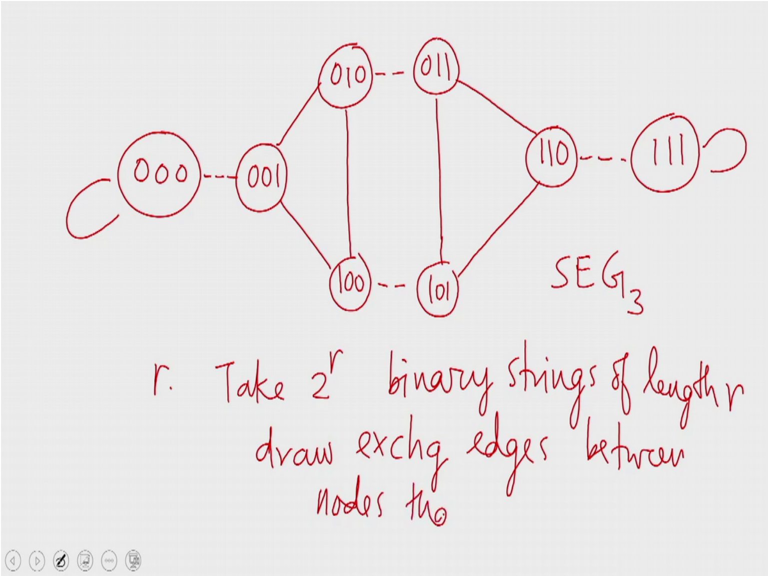

And then coming to dimension 3 which we saw in the last class, the nodes are numbered with 3 bit binary strings 0 0 left shiftedwill give us 0 0 0 triple 1 left shifted will give us triple 1. So, there are self loops to them, then there is an exchange edge from triple 0 to 0 0 1 because they differ in the least significant bit. And then a left cyclic shift of double 0 1 will give us 0 1 0 and a byte cyclic shift will give us 1 double 0 0 1 0 and 0 double 1 are adjacent through an exchange edge.1 double 0 and 1 0 1 are adjacent through an exchange edge again 1 1 0 and 1 1 1 are again adjacent through an exchange edge. When 0 double 1 is cyclically left shifted, we will get 1 1 0 and that is cyclically left shifted; we will get 1 0 1. So, this is a shuffle exchange graph of dimension 3.

So, you can construct shuffle exchange graphs of higher dimensions in this manner. In particular for dimension r take the 2 power r binary strings of length r and then draw the exchange edges between nodes that differ in the least significant bit.

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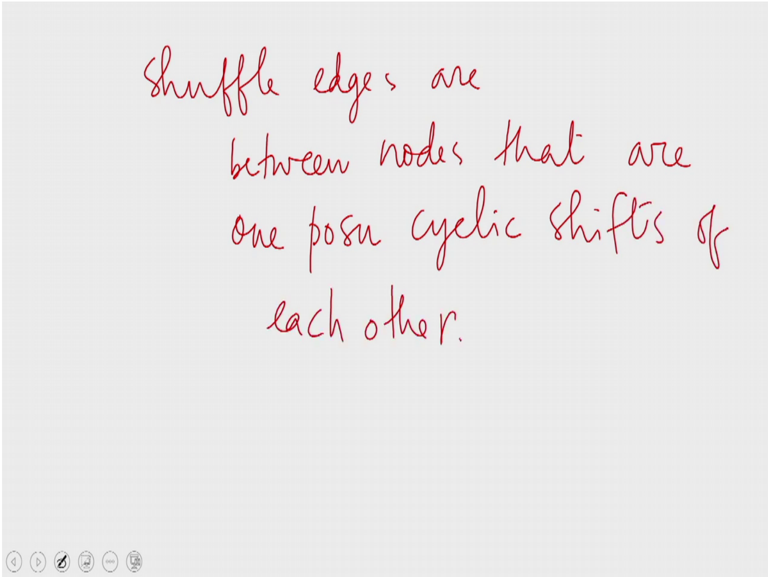

Shuffle edges are between vertices that are cyclic shifts of each other.

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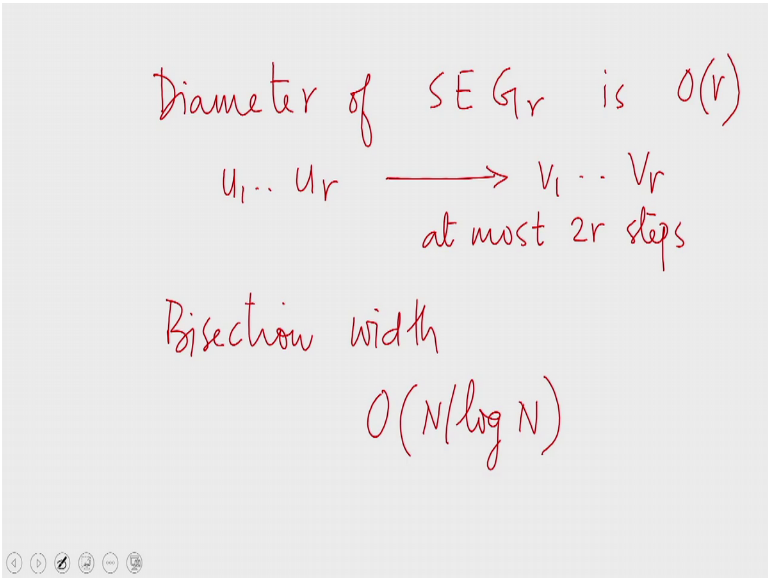

So, as we saw in the last class, the diameter of such a graph is order of r. That is in particular, you can go from vertex u 1 to u r to vertex v 1 through v r by fixing the bits 1 at a time and this will take at most 2r steps for any u and any v. Therefore, the diameter of a shuffle exchange graph of dimension r is order of r. What about bisection width? It can be shown that the bisection width is order of N by log N and the proof is quite interesting.

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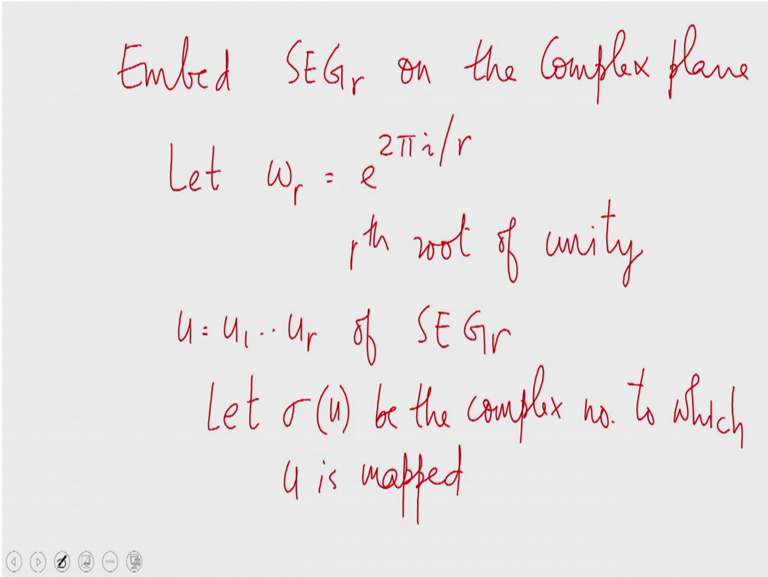

Let us embed the shuffle exchange graph on the complex plane. Let us define omega r as e power 2 pi i by r which is the r’th root of unity. Now consider node u equals u 1 through u r of SEG r. Let sigma of u be the complex number to which u is mapped. So, we map the network onto the complex plane using a function sigma of u.

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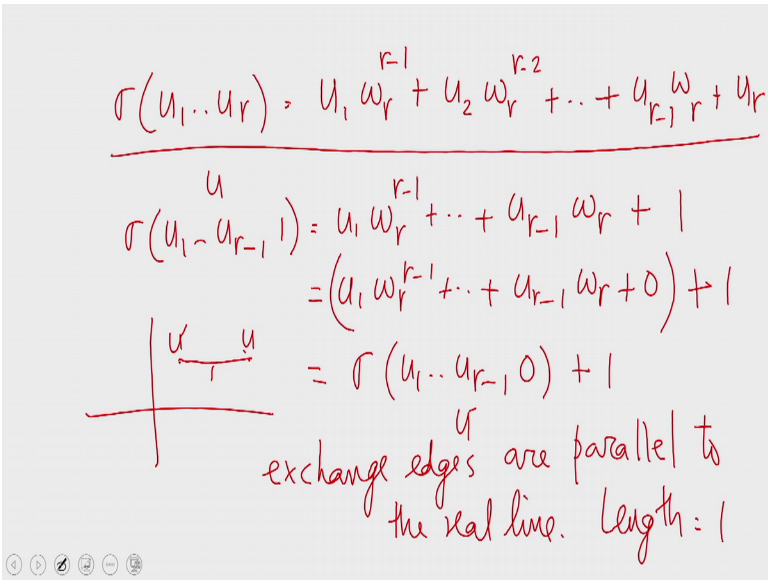

Now, sigma of u is defined in this manner. This is how we define sigma of u. Sigma of u is defined using the r’th root of unity. Now with this definition, we find that sigma of u 1 through u r minus 1 1 is that would be 1 at the last position that is because u r equal to 1 here. If the last bit happens to be 1, then this is what sigma of u was going to be. But this can be written as plus 1, but the quantity within the brackets happen to be sigma of u 1 through u r minus 1 0 plus 1.

So, what we find is that the complex number to which u 1 through u r minus 1 1 maps to is 1 more than the complex number to which u 1 through u r minus 1 0 maps. So, if in the complex plane, we consider these two vertices if I call this one u and this u prime, we find that u and u prime are mapped 1 unit apart. The difference between u prime and u is real number 1; sorry it is other way around u prime will be to the left and u will be to the right because u has 1 at the last bit.

Now, this is nothing, but an exchange edge. So, what this establishes is that exchange edges under this mapping are parallel to the real line and have a length of unit; the length of all exchange edges unity under this mapping. So, now, we are affecting a mapping of the shuffle exchange graph of dimensions r on to the complex plane and we find that every exchange edge are mapped 1 unit apart.

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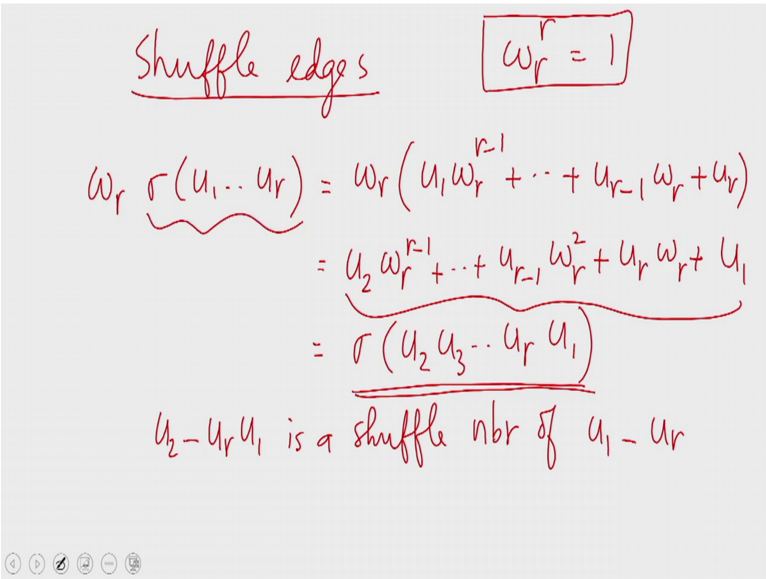

Now, let us see how the shuffle edges are mapped. To find out how shuffle edges are mapped, let us consider this quantity. Sigma of u is the complex number to which u is mapped we multiply that complex number with omega r substituting for sigma. This is what we get. Now taking omega r inside we find that we have plus u 1 that is because omega r power r equal to 1; omega r happens to be the r'th root of unity. Therefore, omega r power r equal to 1; therefore, this is what we get.

Now, this is nothing but sigma of u 2 through u r u 1, but u 2 through u r u 1 is a shuffle neighbor of u 1 through u r and on the complex plane, these two quantities are related by a multiplication with omega r.

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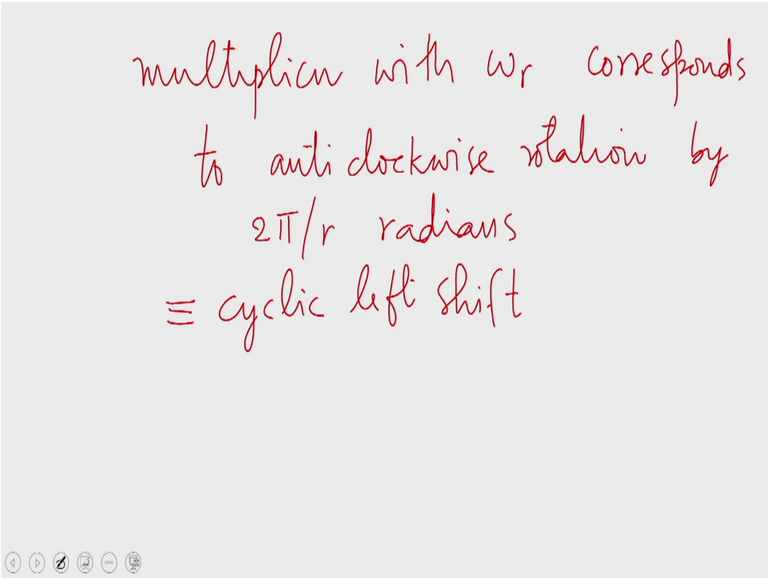

But multiplication with omega r corresponds to anti-clockwise rotation by 2 pi by r radians which means this is equivalent to cyclic left shift. Correspondingly cyclic right shift will correspond to clockwise rotation by 2 pi by r radians.

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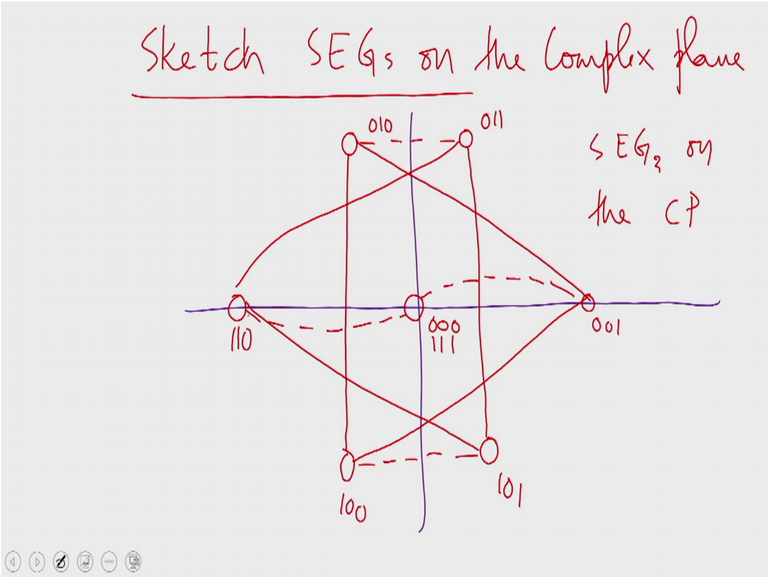

So, now this allows us to sketch shuffle exchange graphs on the complex plane. In particular, let us consider the 3 dimensional shuffle exchange graph. This is the imaginary axis and this is the real axis and the origin, we have 0 0 0 and 1 1 1 that is because the cube roots of unity when all added together will give us 0. The exchange neighbor of double 0 will be 0 0 1. When 0 0 1 is rotated anti-clockwise by 2 pi by 3 radians, we will get the node corresponding to 0 1 0. When that is rotated by another 2 pi by 3 radians, we will get 1 0 0 and when that is rotated again by 2 pi by 3 radians, we will come back to double 0 1. The exchange neighbor of triple 1 will be similarly 1 1 0.

So, there is an exchange edge from 1 1 0 to 1 1 1 just as there is an exchange edge from double 0 to triple 0 to double 0 1. The exchange neighbor of 0 1 0 will be 0 double 1. From 0 double 1, we come to 1 1 0 by a 120 degree rotation. From 1 1 0, we can go to 1 0 1 also by a 120 degree notation which also happens to be an exchange edge exchange neighbor of 1 double 0 and then there is a shuffle edge between 0 1 1 and 1 0 1. The 3 dimensional shuffle exchange graph has four exchange edges; all of them are marked now and shuffle edges are also marked. So, this is the plotting of a 3 dimensional shuffle exchange graph on the complex plane.

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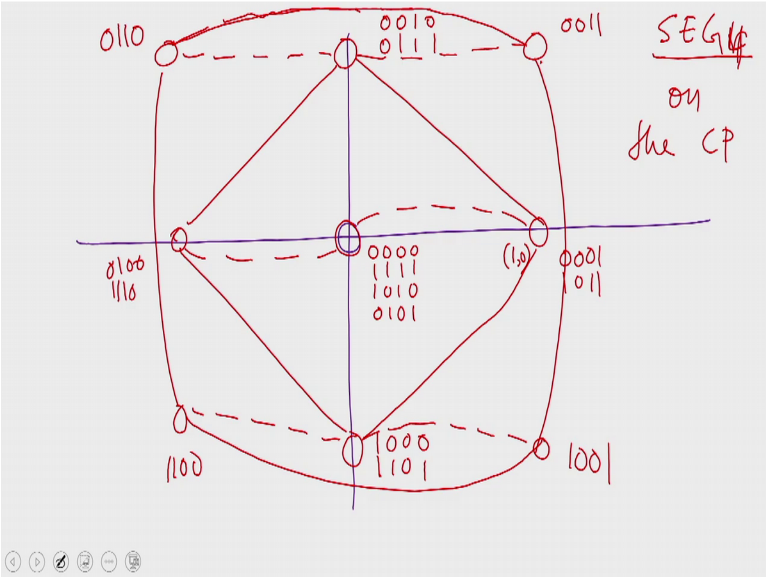

Similarly, if you plot the four dimensional shuffle exchange graph on the complex plane, we find that several nodes map to the origin as you can verify double 0 double 0 double 1 double 1 1 0 1 0 and 0 1 0 1 all mapped to the origin; both double 0 0 1 and 1 0 1 1 map to point 1, 0. Similarly 0 0 1 0 and 0 1 1 1 map to 0 1 and diametrically opposite, we have 1 0 double 0 and 1 1 0 1 mapping to 0 minus 1 and minus 1 0 has 0 1 0 0 and 1 1 0 1 1 1 0.

So, we have exchange edges of this sort. The shuffle edges between these nodes, a shuffle in this case corresponds to a pi by 2 radians rotation which is a 90 degree rotation on the complex plane. Now 0 0 1 0 has an exchange edge to 0 0 1 1 rotating from 0 0 1 1, the shuffle edge. So, these are the shuffle edges involving double 1 double 0 and its cyclic rotations. There are exchange edges like this, 0 1 1 0 to 0 1 1 1. Similarly here also we have an exchange edge; double 1 double 0 to double 1 0 1 and there is an exchange edge from there is an exchange edge from 1 0 double 0 to 1 0 0 1. So, those are the exchange edges and the shuffle edges of a four-dimensional shuffle exchange graph.

So, in the case of a four-dimensional shuffle exchange graph, we find that there are several vertices mapping to the same complex number that is because in this case we have what are called degenerate necklaces.

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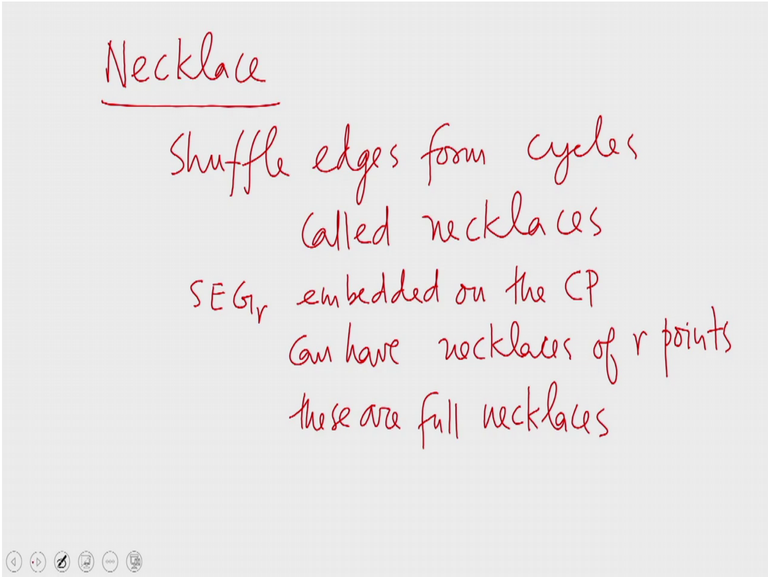

But first let me define what is called a necklace. On the complex plane, we find that necklace is formed. A necklace is a cycle formed by shuffle edges. If you look at the plotting of a 3 dimensional shuffle exchange graph, we find that we have two necklaces of length 3 each. So, in particular, in shuffle exchange graph r embedded on the complex plane can have necklaces of r points. Of course, no necklace can have more than r points because every shuffle edge corresponds to 2 pi by r radians rotation. But we can have necklaces of r points; these are full necklaces.

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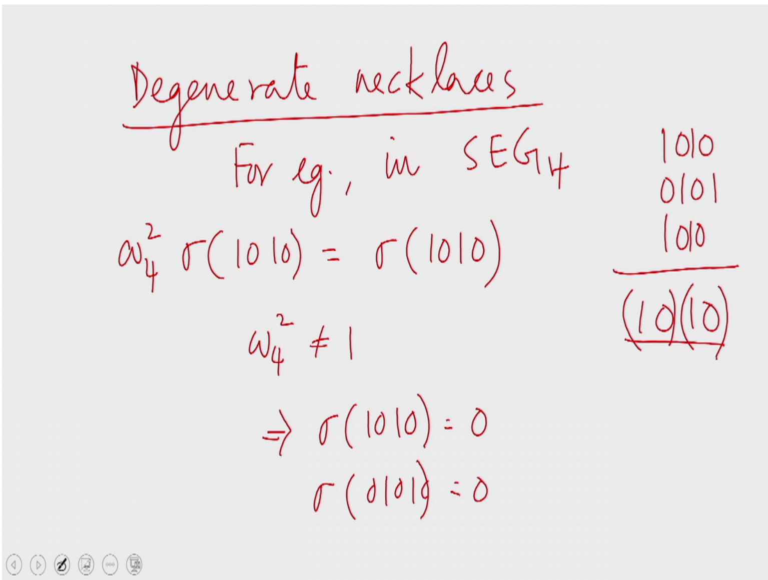

Some necklaces may not be full, these are called degenerate necklaces. For example, in SEG 4; consider sigma of 1 0 1 0. When 1 0 1 0 is cyclically left shifted, we get 0 1 0 1. When that is cyclically left shifted, we get 1 0 1 0 again what; that means, is that omega 4 squared multiplied by sigma of 1 0 1 0 is equal to sigma of 1 0 1 0 because every shuffle edge corresponds to a rotation by 2 pi by 4 radians. Since omega 4 squared is not equal to 1 this implies that sigma of 1 0 1 0 equal to 0.

Similarly, sigma of 0 1 0 1 also is 0 as we saw in the figure double 0 double 0 double 1 double 1 1 0 1 0 and 0 1 0 1 all mapped to the origin. These form the degenerate necklaces. So, degenerate necklaces happen when we have cycles within the binary representation. For example, the binary representation of 1 0 1 0 is cyclic. It has the same unit 1 0 1 0 repeating multiple times. So, when the binary representation is cyclic, then we have degenerate necklaces.

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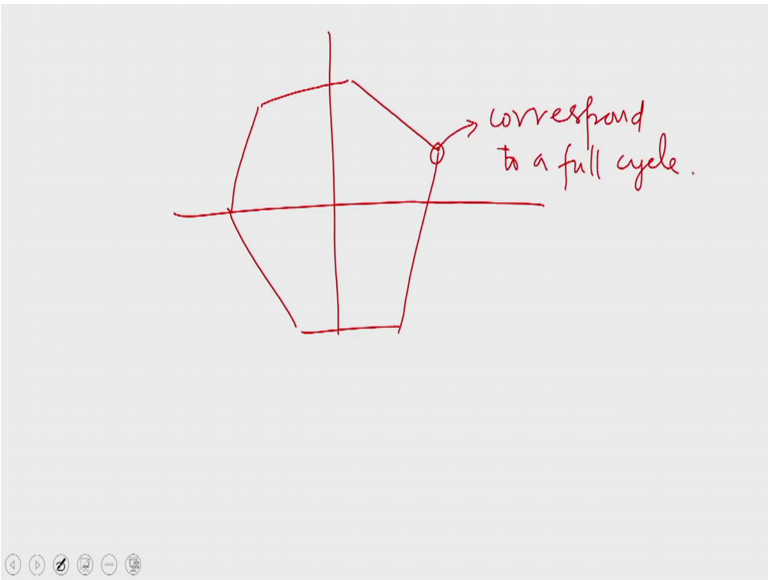

But then when you consider a full necklace; if any node in the shuffle exchange graph maps to a complex number which is not the origin, then by rotating by 2 pi by r radians from there, you will complete a full cycle and come back to the starting node. Therefore, any node which is mapped to a non-origin will correspond to a full cycle.

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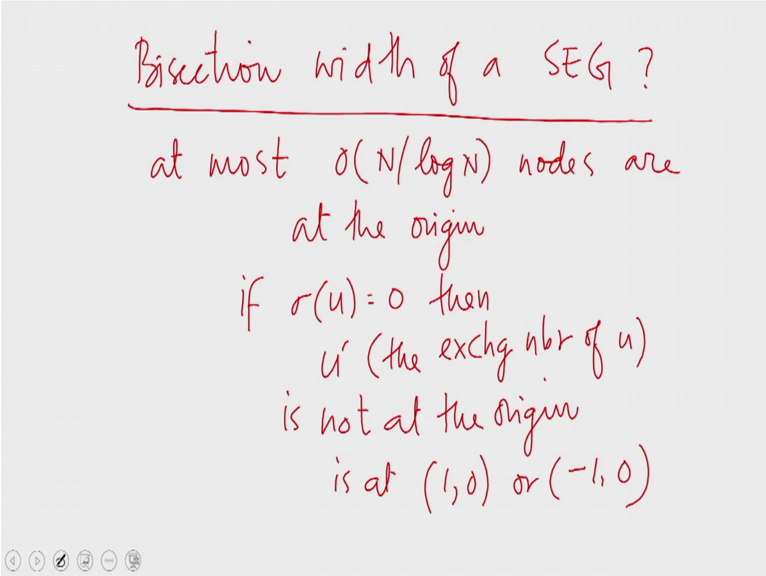

Now, why did we do all this? We want to find the bisection width of a shuffle exchange graph. First what I want to claim is that at most order of N by log N nodes are at the origin. This is because if sigma of u equal to 0 that is u is mapped to the origin, then u prime the exchange neighbor of u is not at the origin. It will have to be at either 1, 0 or minus 1, 0 because an exchange edge is of length 1 and is parallel to the real axis. Therefore, u prime will be mapping either to 1, 0 or minus 1, 0. In that case, there is a full necklace containing u prime.

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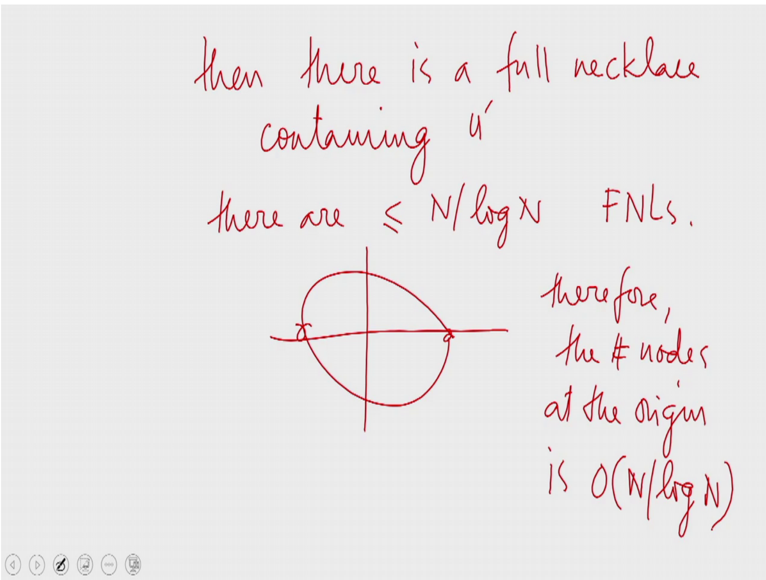

So, what we find is that for every node that maps to the origin its exchange neighbor is in a full necklace. There are at most N by log N full necklaces. There can be at most N by log N full necklaces because a full necklace by definition has log N nodes. So, there could be at most N by log N distinct full necklaces. When you consider a full necklace at most 2 nodes of the full necklace can be on the real line. Therefore, the number of nodes at the origin is order of N by log N. Once again if a node falls at the origin its exchanged neighbor falls in a full necklace, then that itself accounts for N by N log N vertices. There can be at most N by log N full necklaces. Therefore, at most order of N by log N vertices will fall at the origin.

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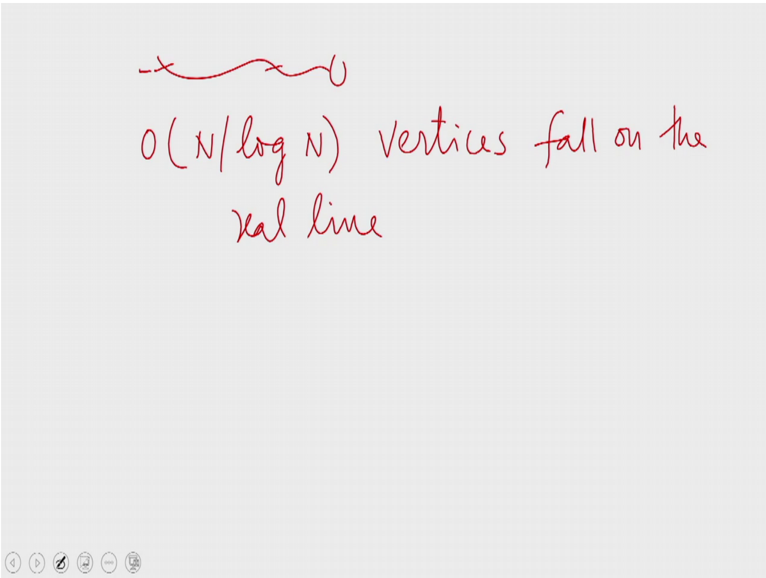

By the same argument, we can show that order of N by log N vertices fall on the real line. If a node falls on the real line, but it is not at the origin, then it will be a part of a full necklace and in any full necklace at most 2 nodes can fall on the real line and there are at most N by log N full necklaces. So, at most N by log N vertices can fall on the real line.

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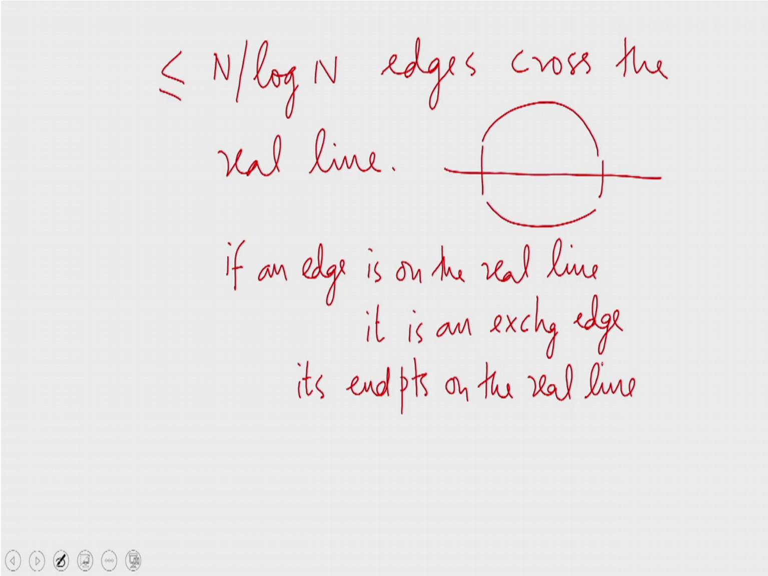

Similarly, at most N by log N edges cross the real line. Why is this so? When you consider a full necklace at most 2 edges in the full necklace can cross the real line. The remaining nodes are either above the real line or below the real line. Therefore, every other edge is strictly above the real line or strictly below the real line.

So, at most 2 edges of any full necklace can cross the real line and then if an edge is on the real line, it is an exchange edge and its endpoints are on the real line and we have already bound the number of nodes on the real line to be order of N by log N. Therefore, putting all this together, we can say that at most N by log N edges cross the real line or are on the real line.

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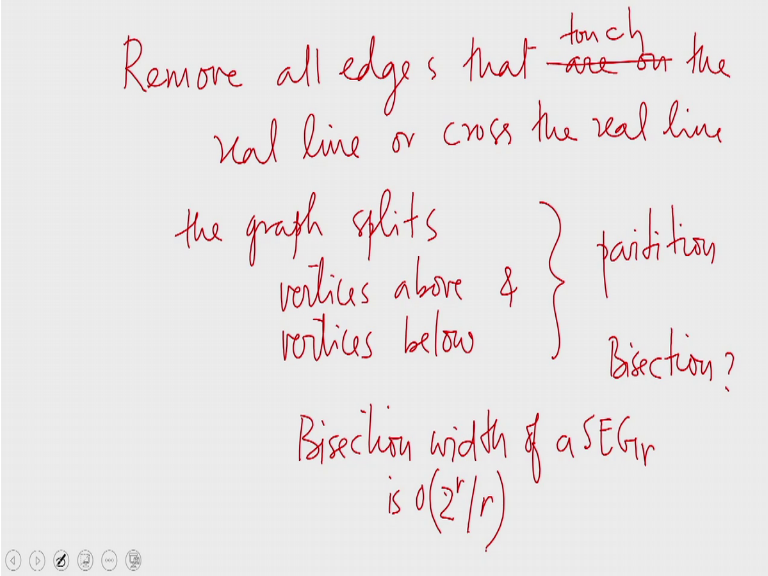

Now, let us remove all edges that are on the real line or cross the real line or let us say we remove all edges that touch the real line or cross the real line that is has one of the endpoints on the real line or cross the real line, the graph splits. Let us consider the vertices above the real line and the vertices below the real line. So, this is a partition of the graph, but is it a bisection? It will be a bisection of the number of vertices above the real line is equal to the number of vertices below the real line.

In fact, that indeed is the case therefore; this is indeed a bisection and the number of edges that we have removed is order of N by log N. Therefore, we are in fact, getting a bisection. But how do we show that the number of vertices above the real line is equal to the number of vertices below the real line?

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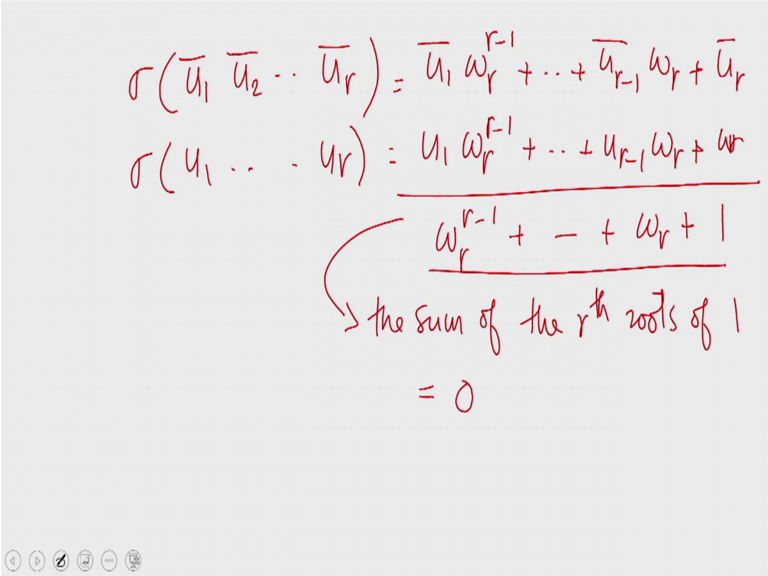

Let us consider sigma of the bitwise complement of u. This is going by the definition similarly sigma of u 1 through u r is u r. And when we add them together, we find that this is, but then what is this? This is the sum of the r'th roots of unity which we know is 0. What it means is that the bitwise complement of a vertex u gets mapped to the additive inverse of sigma of u. In other words, if sigma of u happens to be below the real line, then sigma of u bar happens to be above the real line and vice versa.

Therefore the number of nodes that are below the real line is equal to the number of nodes that are above the real line. So, putting this together with our earlier argument, we find that the bisection width of a shuffle exchange graph of dimension r is 2 power r by r order of 2 power r by r or in other words if r equal to log N we have a bisection width of order of N by log N. So, shuffle exchanged graphs like hypercubes have this nice property, they have a small diameter and a large bisection width.

Now, let us see a network which is closely related to shuffle exchange networks and also have some nice properties. After that we will see some simulation results in both of them.

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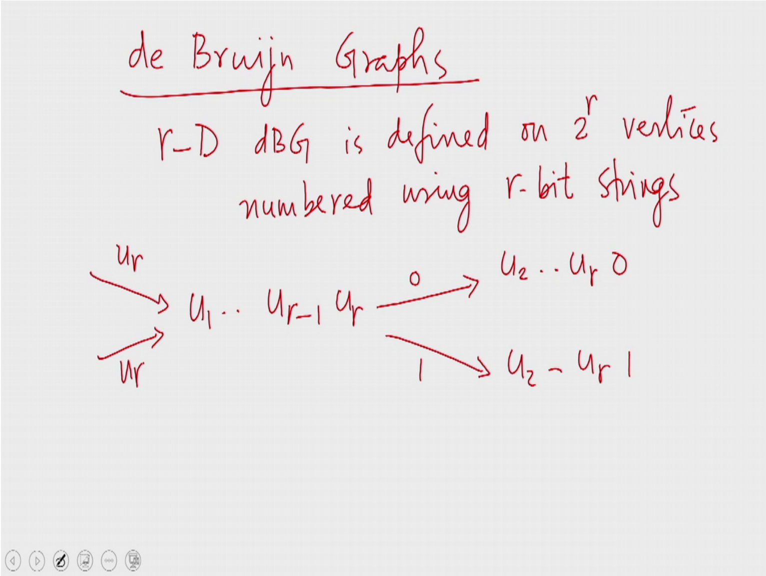

These are de Bruijn graphs. An r dimensional de Bruijn graph defined on 2 power r vertices. These vertices are numbered using r bit strings. The edges of this network are defined like this from u 1 through u r minus 1 u r, we have an edge to u 2 through u r 0 and we have an edge to u 2 through u r 1. That is to obtain an out neighbor of a vertex; we have to cyclically left shift the binary representation of the vertex.

In that case u 1 will come at the least significant position, we will have u 2 through u r at the most significant positions and then we get one neighbor and then by exchanging the last bit, we will get the other neighbor. So, as we can see the correspondence between shuffle exchange graphs is immediately evident. When you cyclically left shift, you get one neighbor which is the same as the shuffled neighbor and then by flipping the last bit, we will get another neighbor which happens to be the exchanged neighbor of the shuffled neighbor. In other words, de Bruijn graph is obtained by contracting the exchanged edges of a shuffle exchange graph of the same dimension. We will come to that later.

So, once the edges are defined; in this in this graph, we also defined some edge labels. The edge which is directed to u 2 through u r 0 will be labeled 0 and the edge which is labeled to u 2 through u r 1 will be labeled 1. So, if you label the edges in this fashion, we find that u 1 through u r minus 1 u r by the extension of by an extension of the definition. We find that there are two incoming edges to it. These are from these are this will be labeled u r each.

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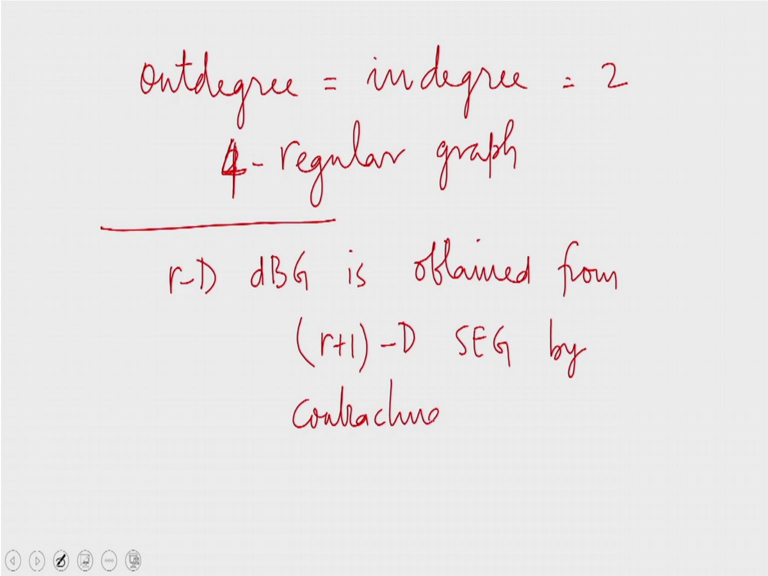

So, you can see that in this case the out degree of every vertex is equal to the in degree of every vertex which is equal to 2. So, this is a 2 regular graph or the underlying undirected graph is a 4 regular graph every vertex has a vertex degree of 4. So, you could say the underlying undirected graph is 4 regular; that is because every vertex has 4 neighbors 2 in neighbors and 2 out neighbors.

There are some interesting recursive properties to de Bruijn graph. You find that an r dimensional de Bruijn graph as I mentioned just now is obtained by from an r plus 1 dimensional shuffle exchange graph by contracting the exchange edges.

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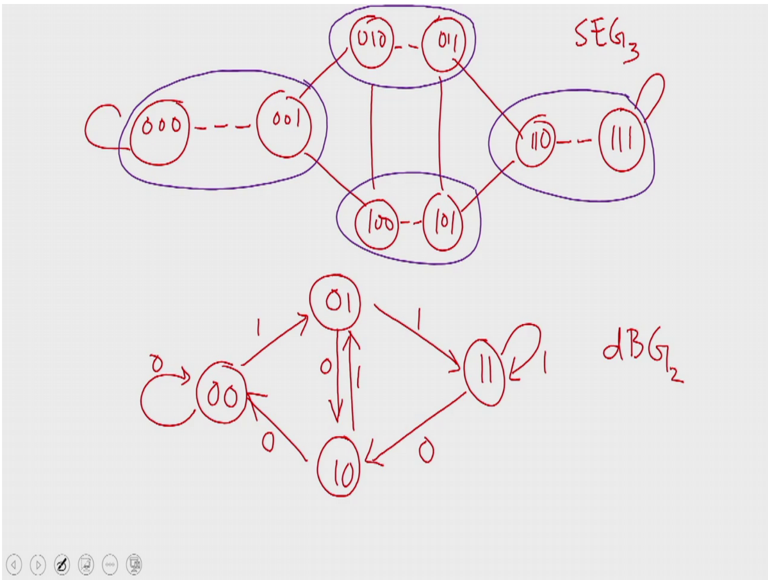

Pictorially, let us take the three-dimensional shuffle exchange graph and then contract the exchange edges. In particular, we are contracting triple 0 and double 0 1 into 1 single vertex. These two vertices agree on the two most significant bits. Therefore, the result of the contraction will be numbered using those two bits.

So, triple 0 and double 0 1 contracted together will form double 0. These two vertice is contracted together and these two vertices are also contracted together, these are contracted together, these are contracted together. Double 0 has an edge to itself 0 1 0 and 0 double 1 contract to give us the vertex 0 1 and 1 double 0 and 1 0 1 contract to give us the vertex 1 0. Double 1 0 and double 1 contract to give us the vertex double 1. The remaining edges can be fitted like this. You can verify that we have these remaining edges and the edge labels would be like this. So, the picture below is de Bruijn graph of dimension 2 and here we have a shuffle exchange graph of dimension three.

So, the relationship between the two networks is readily apparent their shuffle exchange graph becomes a de Bruijn graph if you contract every single exchange edge of the shuffle exchange graph. The main difference is that in de Bruijn graph, the edges are directed and every edge has got a label 2.

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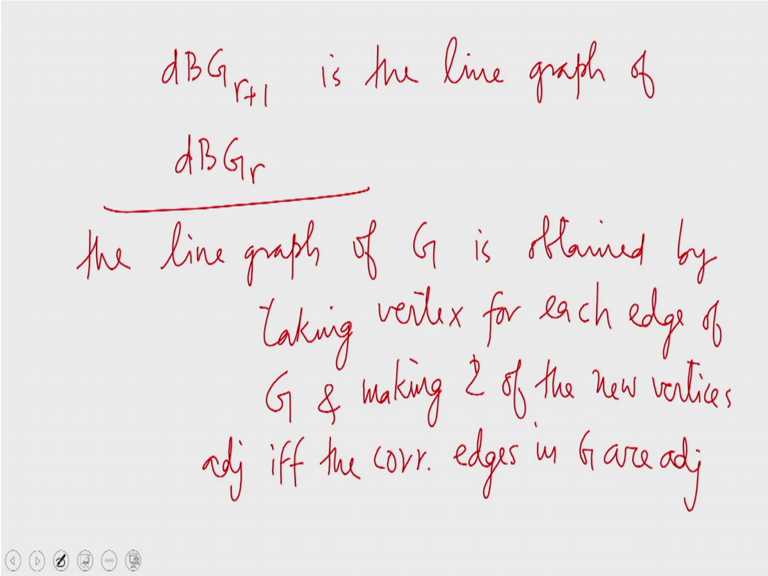

And another nice recursive property that we observe is that a de Bruijn graph of dimension r plus 1 is the line graph of dBG r. The r dimension de Bruijn graph, you would recall from your graph theory that the line graph of G is obtain by taking a vertex for each edge of G.

And making two of the new vertices that is the vertices in the line graph adjacent if and only if the corresponding vertices are adjacent in; the corresponding edges are adjacent in G. Every vertex in the line graph corresponds to an edge in the original graph and two vertices in the line graph are adjacent; if and only if the corresponding edges in G are adjacent. So, I will leave the proof as an exercise for you. So, we will see some simulation results on these two networks in the next class. That is it from this lecture. Hope to see you in the next.

Thank you.