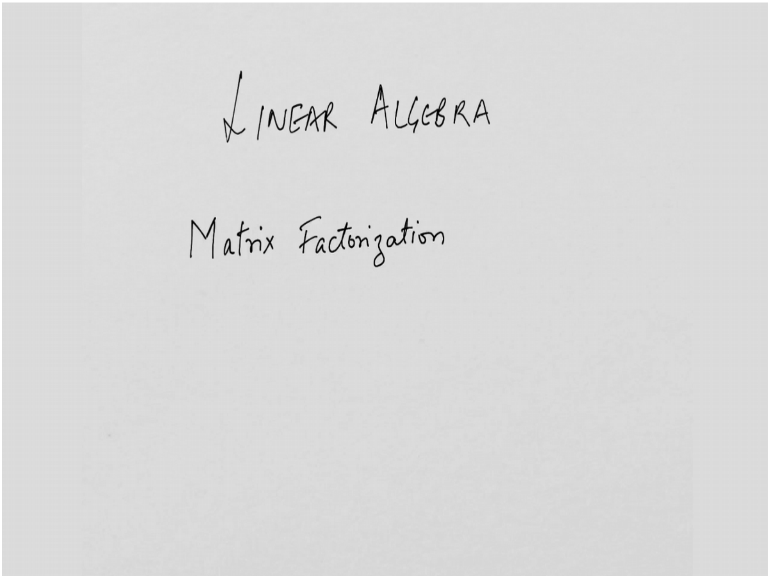
Introduction to Parallel Programming in OpenMP
Dr. Yogish Sabharwal
Department of Computer Science and Engineering
Indian Institute of Technology, Delhi

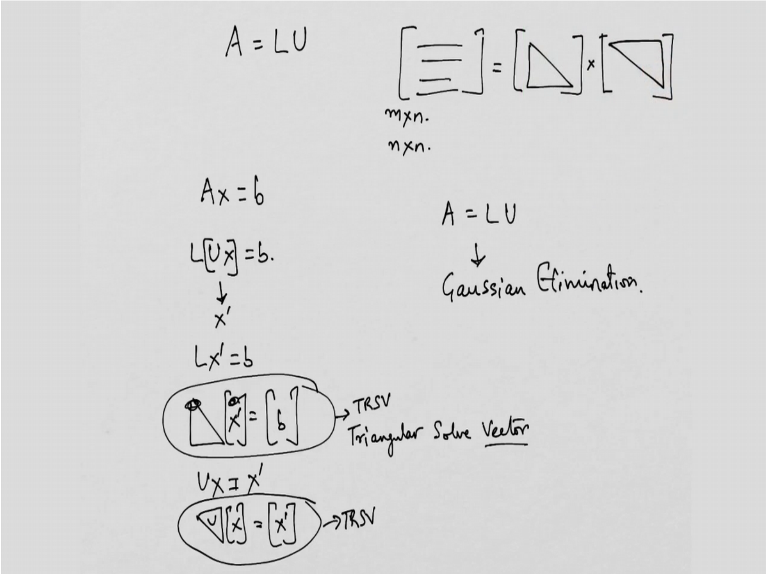
Lecture – 09
Linear Algebra – Matrix Factorization

(Refer Slide Time: 00:29)



We will be doing some more linear algebra. Today in particular, we will be talking about matrix factorization and how to do that in parallel using openMP, there are many different factorizations that you can do for a matrix we will be picking the most simple one which is called the LU factorization.

(Refer Slide Time: 00:42)



So, what happen in the LU factorization? So, you have a matrix a and you factorize it into 2 matrices L and U where A is full matrix, right, L is a lower triangular matrix and U is a upper triangular matrix.

So, in general you can factorize any m cross n matrix this way, but for ease of exposition I will just be talking about a simple m cross m factorization. So, we will consider A to be n cross n, all right. So, why do we do LU factorization? So, one use of the LU factorization is that if you have to solve a system of equations ax equal to b right what you can do is you can factorize A into LU. So, this become the LU x equal to b.

Now, what is x? X can be computed as follows. So, what is U x? U x is a vector U is a matrix; x is a vector. So, matrix vector product is a vector. So, call this x prime. So, this is nothing, but L x prime equal to b. So, you can solve this why because this is a lower triangular matrix times A vector x prime equal to A vector b, right, this is easy to solve back substitution you can straight away read off the value of x 1 prime using these values. So, we will see an example from, right. So, you can do this using back substitution and you get the value of x prime right and once you have the value of x prime now you know that U x was equal to x prime.

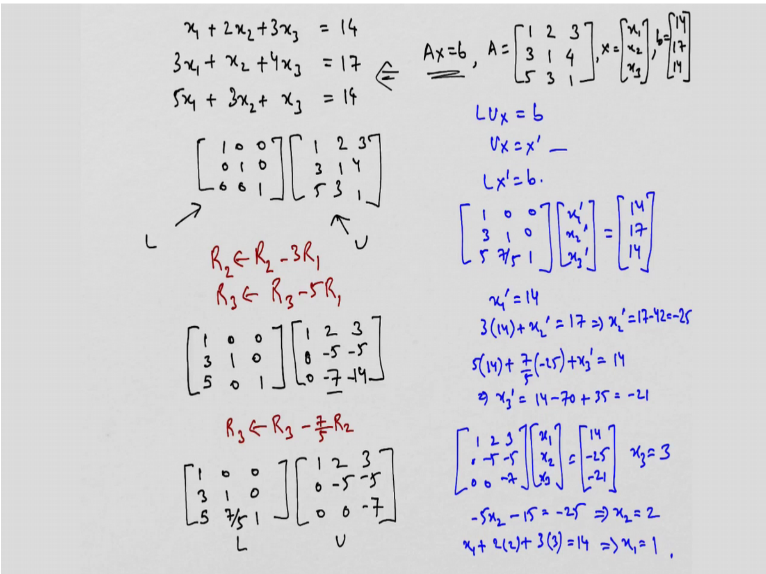
So, now again what is U? U is an upper triangular matrix multiplied by a vector equal to. So, this is x prime you already know x prime; you know U. So, you solve for x again using back substitution. So, this is actually an operation which in linear algebra is referred to as TRS V or triangular solve with a vector solve for a set of variables with the triangular matrix, we have the right hand side is a vector.

So, this is also TRS; we just that instead of a lower triangular we have an upper triangular matrix, right. So, TRS is the name of this routine and most in the most common libraries. So, the most common libraries glass basically in algebra subroutines which will find easily on the net. So, that is the name given to this particular routine.

How do you factorize A into LU how do you get LU out of A? So, this is done using Gaussian elimination, right, something all of you must have done at some point or the other. So, what will do is we will quickly walk through a small example and see how this is solved and then we are going to talk about how to implement that efficiently using blocks right you want an efficient implementation and also how can you paralyze it using openMP.

Let us start with the basic straight. So, I am going to consider a simple set of equations. So, let us say that you are given a system.

(Refer Slide Time: 04:16)



X 1 plus 2 x 2 plus 3 x 3 equal to 14; 3 x 1 plus x 2 plus 4 x 3 equal to 17 and 4 x 1 plus 3 x 2 plus x 3 equal to 14.

So, what do we do we write it in the form ax equal to b and here what is A? A is the matrix 1, 2, 3 that is the coefficients 3; 1 4 4 3 1; right, x is of course, your variables x 1 x 2 x 3, right and what is b? B is the right hand side which is 14 17 14, right. It is clear that A x is equal to b is what this system represents.

So, now, what I want is the LU decomposition of A; right. So, let me forget about x and b for the time being and let me just concentrate on A and I want to factorize z. So, I can write A as; right I am just multiplying by the identity matrix that does not change anything.

So, slowly what I am going to do is I am going to convert this matrix into L and I am going to convert this matrix into U; right I mean this is standard stuff you seen all this before just I am rewriting it in the form of matrices and I just want you to you know be familiar with this. So, that you remember what this used to be so that we can talk about parallelism, right.

So, how do I convert this into an upper triangular matrix? So, Gaussian elimination; so, what do I do? The very first thing I do is I say R 2 goes to R 2 minus 3 R 1; right and R 3 goes to R 3 minus 4; R 1, right what happens is the result of this let me write the right hand side first; the first order means the same, right row 2 becomes. So, this will become 0, right what will happen to this entry R 2 goes to R 2 minus 3 R 1. So, 1 minus 6; this will become minus 4, right and 4 minus 3 times 3; 4 minus 9 minus 4.

Now, let us look at R 3, R 3 goes to R 3 minus 4 R 1. So, this becomes 0 again that was my intention, I wanted to make the first column 0 below the first entry, right what happens second entry?

Student: Minus (Refer Time: 07:10).

3 minus 5 times 2; 3 minus 10 that is minus 7 and 1 minus 5 times 3.

Student: Minus 14.

Minus 14; so, how do I capture this? So, what multiply by this matrix will give me a back says what you have to understand, right. So, suppose I write 1 0 0 over here. So, what does the output matrix going to be when I multiply this with this matrix I mean I have not written out the complete left hand side matrix, but I have just written a part of it, but what do I get on the right hand side when I multiply this?

Student: First row.

First row; right and it is untouched that is what this row is saying that take the first row of the right matrix right add to it 0 times the second row add to it 0 times the third row. So, what happens when you multiply this with the first column right the first entry gets multiplied by the first row entry. The second entry gets multiplied by the second row entry? So, that is the scaling factor of the second row entry and the third element gets multiplied by the third row entry. So, that is a scaling factor of the third row entry and you multiplied this first row with every column of the right hand side producing the first row of the resulting matrix, right.

So, I hope that interpretation is clear all right. So, now, tell me what will be the second row. So, what did I do to get to this matrix R 2 goes to R 2 minus 3 R 1. So, what do I do to the second entry? So, that I get back A.

Student: 3 1 0.

3 1 0, exactly, right. So, what did I do I said R 2 minus 3 R 1 R 2 goes to R 2 minus 3 R 1. So, if I want to go back to A; what do I have to do? I have to say R 2 goes to R 2 plus 3 R 1.

Student: Hum.

So, if I say R 2 goes to R 2 plus 3 R 1; that is what this second row is saying now, right the second row is saying R 2 goes to R 2 plus 3 R 1, all right and what will this entry become 4 0 1 clear and as you can see, right, every time I am going to operate on some row and I am only going to subtract or add to it some factor of some earlier row right that is what I do in Gaussian elimination therefore, this left hand side matrix is going to be lower triangular. So, what do I do next?

So, now, I want to make this entry 0, right. So, how do I do that? So, I am going to say R 3 goes to R 3 minus 7 by 4 times R 2; I just look at the coefficient here I look at the coefficient at the diagonal entry and I subtract right by the factor.

So, what am I going to get now? So, let me write this matrix later, but let me do this. So, 1 2 3 the first row remain untouched, second row remains untouched and what happens to the third row? This becomes 0, this becomes 0 and what happens to this entry; R 3 goes to R 3 minus 7 by 5 R 2. So, minus 14.

Student: (Refer Time: 10:44).

Plus 5 and 5 will cancel plus 7; right. So, minus 7.

Student: Hum.

And what happens to the left hand side matrix. So, this remain untouched this remain untouched and now I have done the operation R 3 goes to R 3 minus 7 by 5; R 2 to get back, I will have to say R 3 goes to R 3 plus 7 by 5, right.

So, spend some time with this at home right you will become familiar with it is not difficult. So, you can multiply these in check it out. So, this is the LU factorization right this is your L and this is your U if I am trying to use this LU factorization to solve a system of equations like what I had in this example; how would I do that. So, as I said right. So, this LU times x is equal to b A x is equal to b that is where I started. So, I am going to take U x to be x prime for the time being, right. So, L x prime equal to b; what does that mean; that means, that 1 0 0 3 1 0 5 7 by 5 1 times x 1 prime x 2 prime x 3 prime that is what x prime is equal to 14 17 14.

So, how do I solve this? So, x 1 prime straight away, I can read it out, right. So, one times x 1 prime is equal to 14 this is back substitution. So, x 1 prime is equal to 14 what about x 2 prime. So, 3 times x 1 prime plus 1 times x 2 prime is equal to 17. So, what does that mean 3 times 14 plus x 2 prime is equal to 17 or x 2 prime is equal to 17 minus 42 which is minus 24.

And finally, you substitute x 1 prime and x 2 prime in the last equality and what do you get 5 times 14 plus 7 by 5 times minus 24 plus x 3 prime is equal to 14; what does this imply? This implies that x 3 prime is equal to 14 minus 70 plus 35; minus 21.

Student: (Refer Time: 13:39)

So, now I have got my x 1 prime, x 2 prime, x 3 prime. So, I have got my x prime and now I will solve for x, right. So, I can say U x is equal to x prime. So, what is U? U is 1 2

3 0 minus 5 minus 5; 0 0 minus 7, right this times x 1, x 2, x 3 is equal to 14 minus 25 minus 21.

So, what does this give me minus 7 x 3 is equal to minus 21. So, x 3 is equal to 3 what does the second equation give me minus 5 x 2 minus 5 x 3 that is minus 15 is equal to minus 25, right. So, that gives me x 2 is equal to. So, this will become minus 10 minus 5 x 2 is equal to minus 10 or x 2 is equal 2 and finally, x 1 plus 2 x 2. So, that is 2 into 2 plus 3 x 3 plus 3 into 3 is equal to 14 and this implies that x 1 is equal to 9 plus 5; 13, 14 minus 13; 1, all right. So, this is your back substitution.