**Stochastic Structural Dynamics**

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**Lecture No. # 07**

**Random Processes-2**

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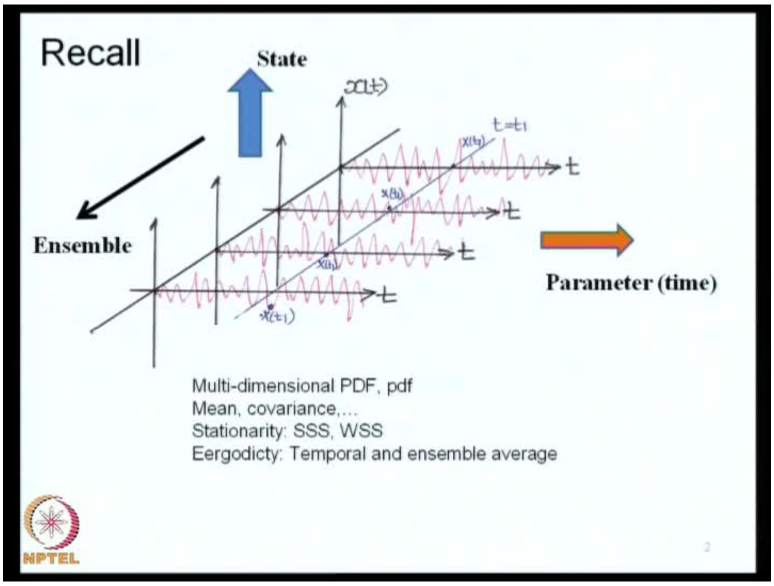


Welcome to the lecture number 7. In this course, in the previous lecture, we have introduced the notion of a random process. So, a random process is a collection of time histories and at any given time t, it is a random variable and we can define a random process; a working definition for a random process could be that it is a random variable that evolves in parameter time t. The description of a random process requires the specification of multi-dimensional probability distribution functions, probability density functions, and so on and so forth. That means, since that every time t - it is a random variable, we have a collection of random variables. And a random process is completely specified, if this collection of random variables is completely specified.

We talked about them mean of a random process, covariance of a random process and other moments. And also we introduce the notion of stationarity, a process is said to be stationary, if one of its characteristics becomes independent of time. If this characteristic is specified in terms of probability density functions or distribution functions, we describe the associated notion of stationarity as strong sense stationarity. On the other hand, if we consider definition of stationarity, in terms of moments like mean, covariance, etcetera. We call this as weak sense stationarity.

A generally understood meaning of stationarity is that - the mean of the random process remains constant with respect to time and covariance between any two random variables here, is a function of the time difference t 1 and between t 1 and t 2. Thus covariance is function of t 2 minus t 1. So, then we say that it is weak sense stationary. The Gaussian random process, as we saw is if it is week sense stationary it is also strong sense stationary.

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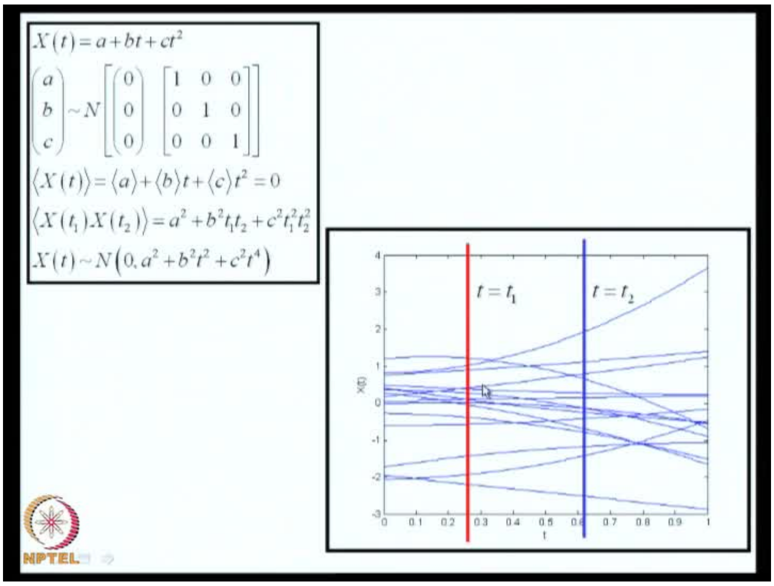


We also discuss, the notion of ergodicty; where we discussed temporal and ensemble average and this is where we will continue with the present talk, before that we can quickly consider as small example, suppose I have a function X of t as a plus bt plus ct square; where the quantities a b c are say random variables, which are normally distributed. Suppose there mean is 0 and standard deviation is 1. And they are independent then if we were to plot the realizations of this process, they appear something like this. The blue lines are various realization of this process X of t along any one realization this random variable a b c assume once specific value they assume in the respective realizations.

So, a t equal to t 1 the number that we get for X of t is given by this set of numbers and t equal t 2 it is at another set of numbers. So, this is a random process in fact it is a non-stationary random process, because its covariance is function of t 1 and t 2 and not t 1 minus t 2. So, its variance is a square plus b square t square plus c square t to the power of 4. So, it is a non-stationary Gaussian random process.

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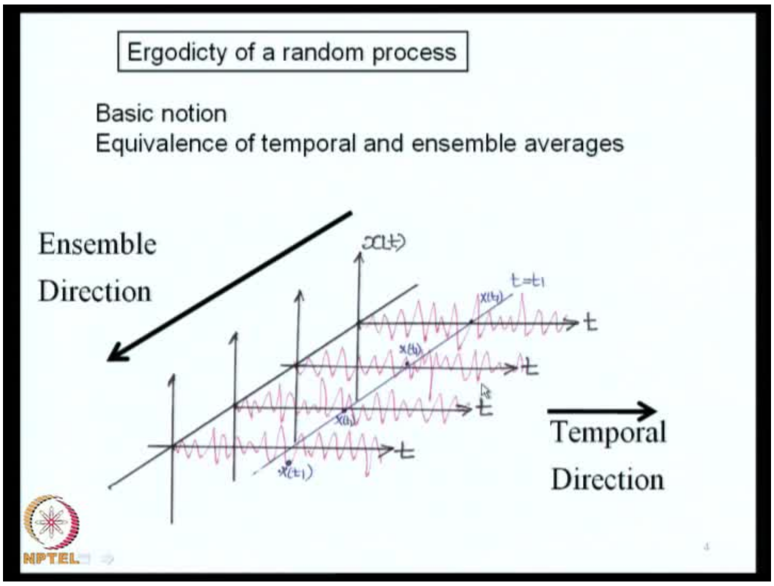


Now, we were discussing the notion of Ergodicty. So, the basic notion here is associated with equivalence of temporal and ensemble averages so this direction we called as ensemble direction and this direction we called as temporal direction so a t equal to t 1, if you will look at realizations of X of t along the ensemble we get X of t 1 here, one number here, one number here, one number here. So, when I take any moment or when I find any average it will be with respect to these realizations.

Now, if I have only a single realization, I can as well consider an average across the single realization that is as I move along this time axis, I start getting different values of X of t in the same sense, as I move along the ensemble direction, I will get different values of X of t. Now, an average along temporal direction, if it is equal in some sense to an average along ensemble direction, we say that the process is Ergodic.

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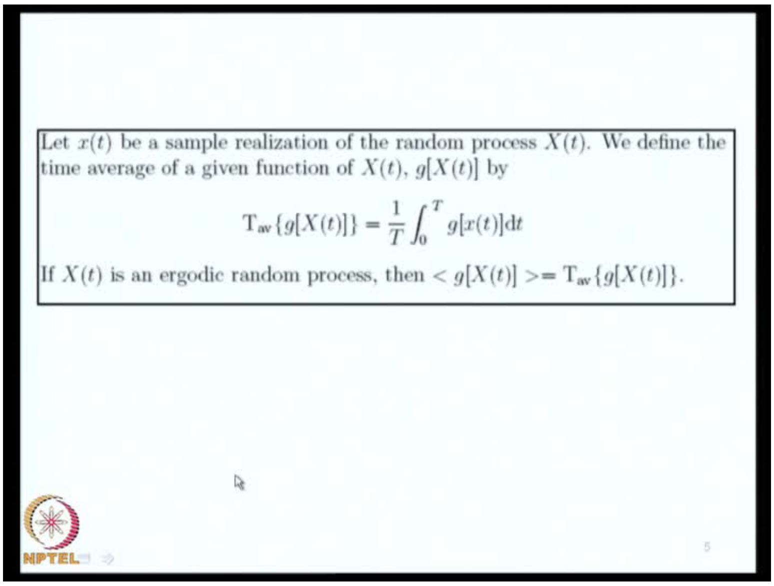


The notion of ergodicty is associated with the characteristic that one has in mind while describing the equivalence of ensemble and temporal averages. So, if we consider if X of t is a sample realization of a random process X of t. We can define a function g of X of t and consider its time average as an integral 1 by T 0 to T g of X of t dt, if X of t is an ergodic random process. The notion is that this ensemble average is equal to this time average, so in what sense it is true, what we should clarify further?

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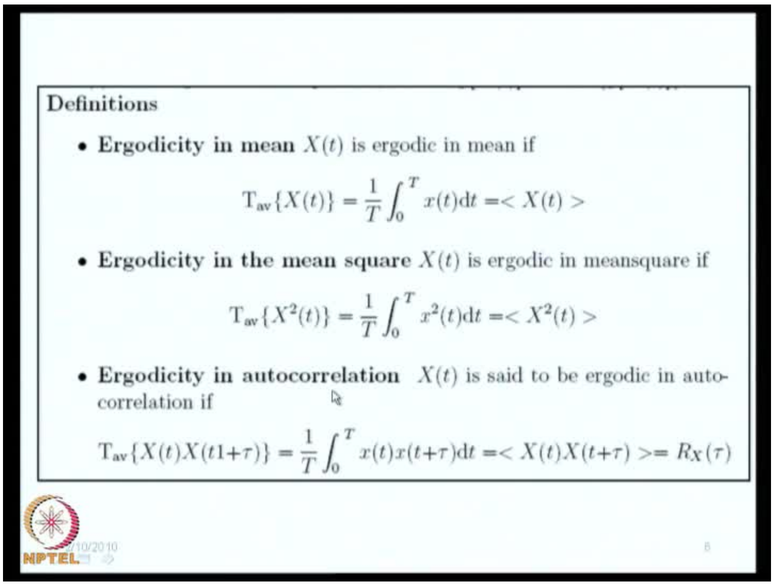

Now, if g of X of t is simply X of t, we talk about time average of X of t being this and this should be equal to the ensemble average that this the expected value X of t. Then we say that the process is ergodic in mean similarly ergodicity in mean square would imply time average of X square of t, which is given by this must be equal to expected value of X square of t. So, one could also discuss, ergodicity in autocorrelation by considering the time average of the product X of t and X of t plus tau, which is this and if this is equal to the expected value of X of t multiplied by X of t plus tau, then we say that X of t is ergodic in autocorrelation.

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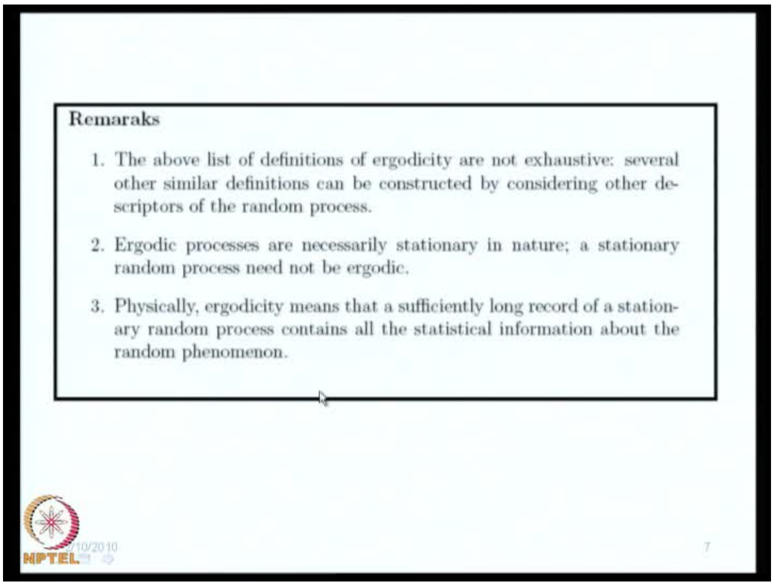


So, what exactly this means? Before, we answer that question, we can consider some remarks. Actually, the three cases of ergodicity I mention are not exhaustive; several other similar definitions can be constructed by considering other descriptors of the random process. For example, one can consider ergodicity in first order probability distribution functions and so on and so forth.

Ergodic process are necessarily stationary in nature; on the other hand, is a stationary random process need not be ergodic is some this is something that you have to think and find acceptable reason for that. Physically, ergodicity means that a sufficiently long record of a stationary random process contains all the statistical information about the random phenomenon. So, this is the physical premises.

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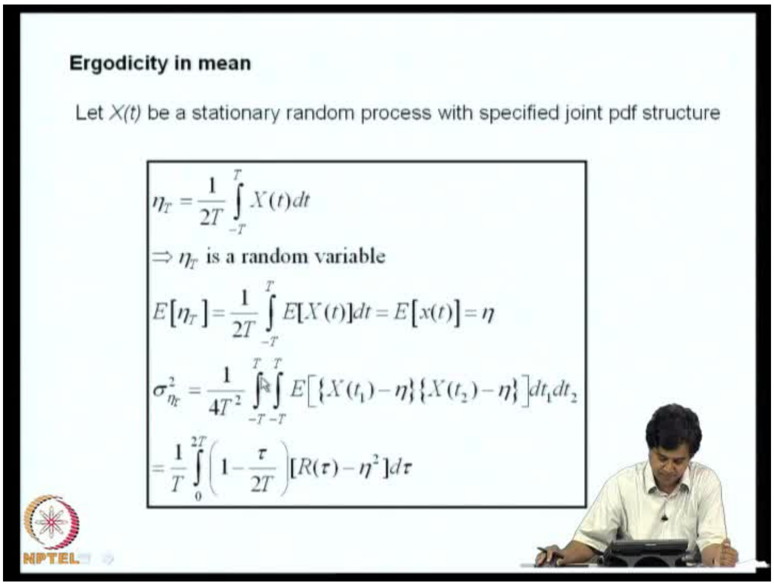


Now, let us take a closer look, what is the exactly meant by ergodicity in mean. Now, let X of t be a stationary random process with a specified joint probability distribution function structure. Now, we consider the transformation 1 by 2T minus T to for plus T X of t dt for certain reasons of simplicity, I am now taking time from minus t to capital T.

Clearly, eta of T can be viewed as a linear transformation of a random process. So, random process is a collection of time histories and for every sample of the time history, I can evaluate this integral and I will get one number. So, if I move across the different realizations of X of t, I will get different numbers for eta of T that therefore, eta of T can be thought of as an outcome of a random experiment and therefore it itself is a random variable.

Now, on the other hand, variant, if I now consider, since this is the random variable, I can consider its expected value and its variance and the expected value is given by this. Now, on the other hand, expected value of the random process itself, you see is a parameter eta. This is a deterministic quantity.

Now, when I say, we are talking about ergodicity, we are trying to equate in some sense a random variable with a deterministic constant that is acceptable, if the variants of this random variable becomes small or goes to 0. When is exactly 0, we can as you know a random variable whose variants is exactly equal to 0 is a deterministic quantity.

So, it is of interest to compute the variants of this quantity and that becomes a double integral and it is given by this integral and which certain simplification. We can reduce to a single integral and this is the expression for the variants.

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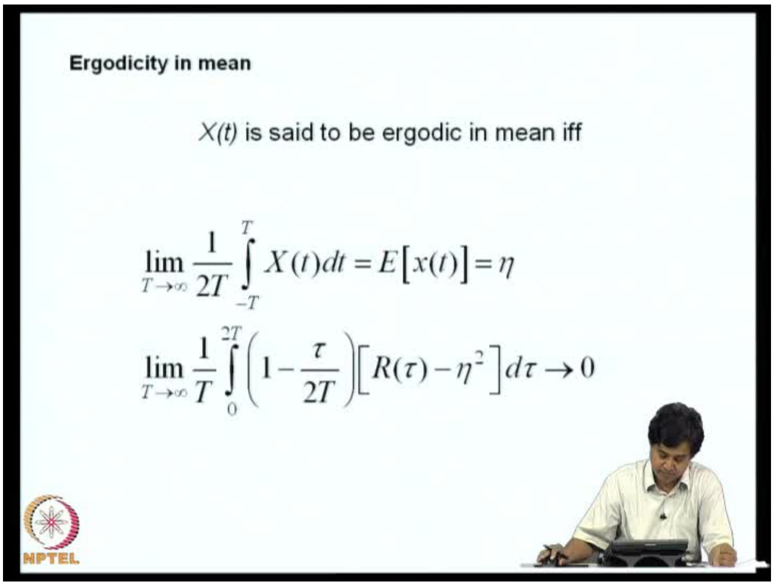


Now, we say that X of t is ergodic in mean, if i f f is only, if and only if, this as limit T tends to infinity. This time average becomes ensemble average eta and not only that the variants of eta T goes to 0. As this length of the record goes to infinity. So, then we say a X of t ergodic in mean that means, if you happen to know all the complete characteristics of X of t these are the conditions under, which we can call X of t as ergodic in mean.

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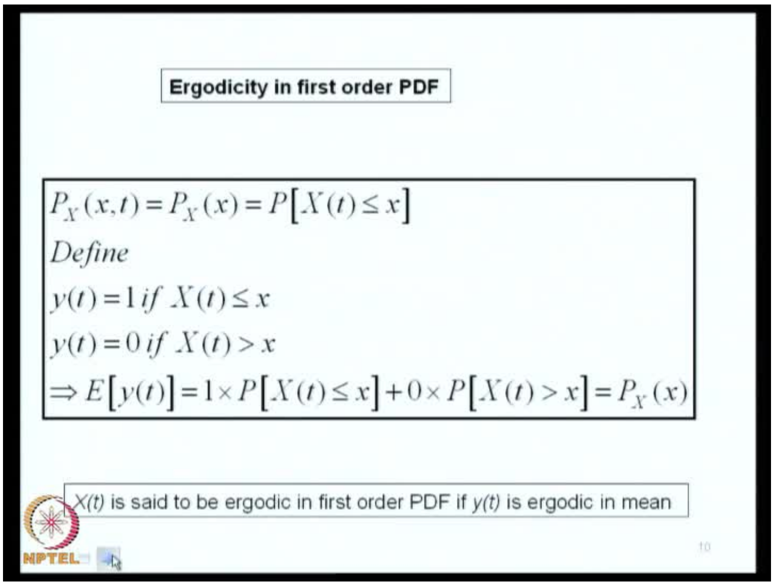


We can consider other notions of ergodicity, for example, we can consider the question, when do we say X of t is ergodic in first order probability distribution function? So, as we know the first order probability distribution function for a stationary random process is independent of time and it is given by the probability X of t less or equal to x.

Now, I define another random process, as follows y of t, I will set to 1 if X of t is less than or equal to x. That means, for a given value of this small x, if a realization of X of t is less than or equal to x, I will call y of t as 1, otherwise I put it as 0.

Now, what is the expected value of y of t itself this is for any given t it is a two state random variable. Therefore, expected value of y of t is 1 into probability of X of t less than or equal to x plus 0 into probability of X of t greater than x. So, this is nothing but P X of x, that means, I have now introduce another random process y of t, whose property is that its mean is the probability distribution function of the process X of t.

Now, since I know what is meant by ergodicity of a random process in the sense of a mean? Now, I can define X of t is said to be ergodic in first order probability distribution function if y of t is ergodic in mean because mean of y of t is the nothing but the probability first order probability distribution of X of t. Therefore, the condition for ergodicity of X of t in the sense of first order probability distribution is equivalent to ergodicity of y of t in the sense of its mean.

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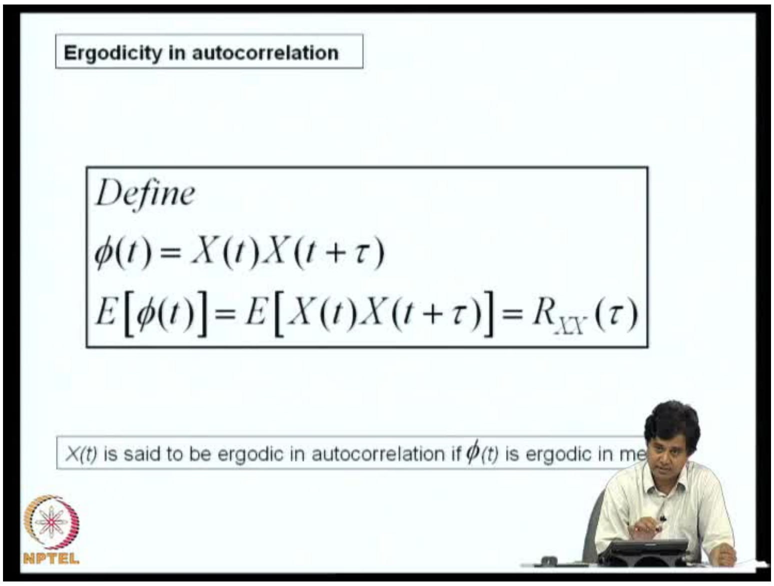


We can extend this notion for instance, if you want to now discuss ergodicity in the sense of autocorrelation. We can introduce now, another random process phi of t, which is the product X of t into X of t plus tau. What is the expected value of phi of t, it is expected value of X of t into X of t plus tau, which by definition is autocorrelation of the random process X of t.

Now, extending the logic that we just now describe, we will say that X of t is ergodic in autocorrelation function, if phi of t is ergodic in mean. So, the conditions for a random process to be ergodic in mean or already spelt out. So, those conditions now, I have to work out in terms of this quantity. Again, let me emphasize this conditions for ergodicity, I am describing can only be verified, if you if you happen to know the complete description of X of t, these conditions are how to verify if you are dealing with data, because you will simply have a single record and you have to go ahead with the pragmatic assumption that X of t is ergodic.

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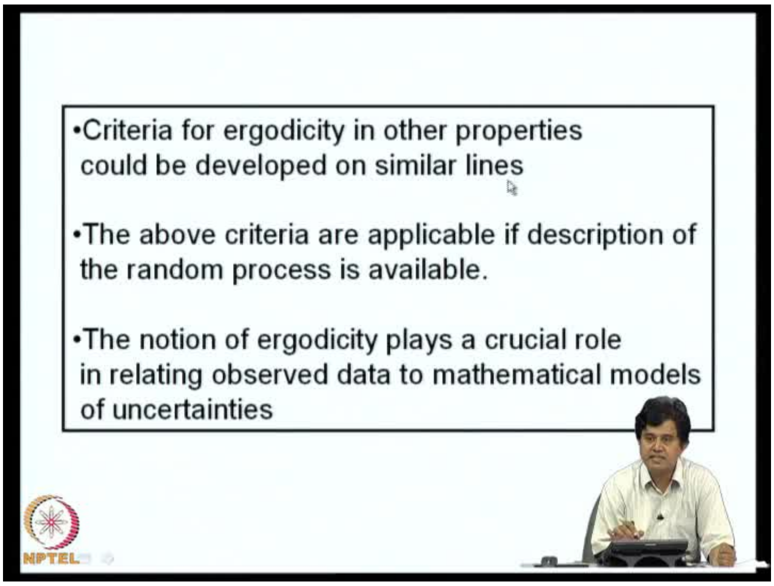


So, the criteria for ergodicity in other properties could be developed on similar manner. Now, these criteria are applicable if description of the random process is already available to us, the notion of ergodicity plays a crucial role in relating observed data to mathematical models of uncertainties. So, it really relates data to mathematical models. Therefore, it is a very important concept in the context of engineering applications.

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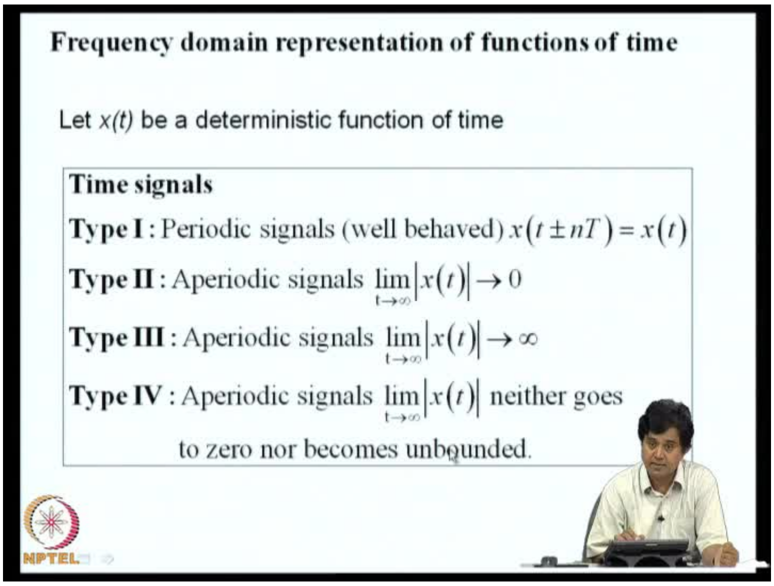


So, this completes a discussion on stationarity and ergodicity. Now, we move on to an another important description of random processes namely, frequency domain representations of random process, as we all know in deterministic time history study of deterministic time histories the notion of frequency plays a crucial role. So, the question now that we are asking is how we can extend this notions to the case of random processes to begin with what you will do is we simply consider deterministic functions.

So, let X of t be a deterministic function of time for the purpose of this discussion we can classify X of t into 4 types of signals, I will call time history of the signals, we say that X of t is periodic, if x of t plus n t is equal to x of t that means for some for some value of capital T the signals simply repeats, then we say that signal is periodic. Any signal which is not periodic is called aperiodic. Among aperiodic signals, we can further classify them as those signals, which decay to 0 as time becomes large and those which below to infinity as time becomes large, they are called type three signals. The other signals are as time becomes large the signal neither goes to 0 nor becomes unbounded.

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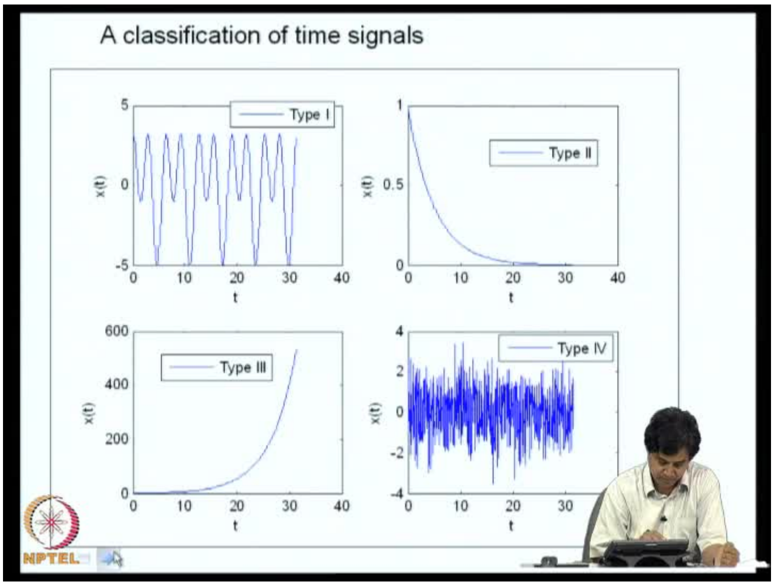


So, let us now consider, whatever the frequency domain representations possible for these types of signals. So, here I have shown a few signals. This signal belongs to type 1, this continues all though the graph as stopped, here it continues. This is periodic signals, this signal is clearly aperiodic and as t becoming t becomes large. This signal is going to 0, therefore it is a type two signals, whereas this signal is again periodic, but as times becomes large it is growing. So, this is a type 3 signal. This signal on the other hand, as time becomes large, it neither decaying to 0 nor growing to infinity. So, these are type 4 signals.

This type 4 signal is of fundamental interest to as in the context of description of random processes. Especially the stationary random processes, because samples of stationary random processes belong to type 4; for type 3 signals there is no hope of any time kind of frequency domain representations. So, you have to deal with these problems exclusively time domain for example, there is an explosion or a blast that kind of situations we are dealing with type 3 type of time histories and is the only the time domain which helps you to tackle these problems.

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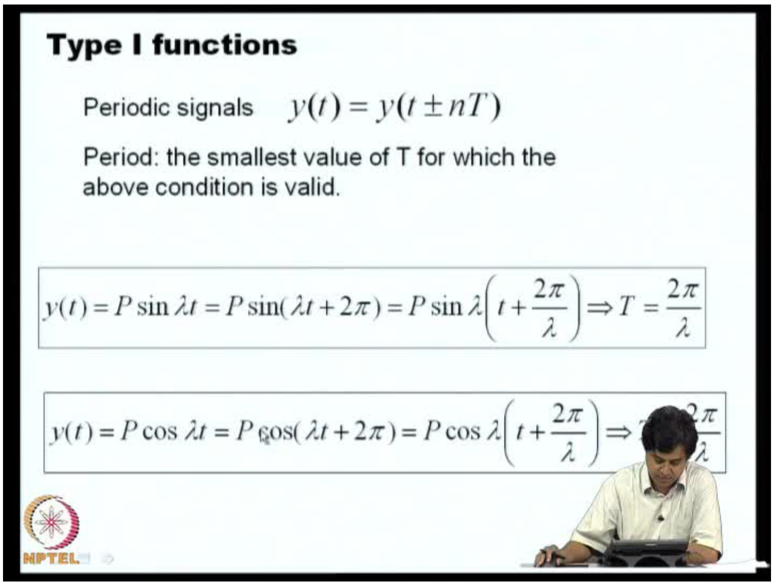


So, let us now go through each of these types of functions, one by one. Let us consider a signal y of t and we say that the signal is periodic, if y of t is equal to y of t plus some n capital T the smallest value of this capital T for which the each above condition is valid is called the period of y of t.

Now, suppose if you consider, now y of t is P sin lambda t; we know sin and cosin functions are periodic with period two pi. So, if I increment this argument by 2 pi the values should remind the same. So, this should be P sin lambda t plus 2 pi. So, did implies and increment into 2 pi lambda in t. So, after every 2 pi by lambda this signal repeats. Therefore, P sin lambda t is a periodic signal with period 2 pi by lambda. Similarly, P cos lambda t is also periodic with period 2 pi by lambda.

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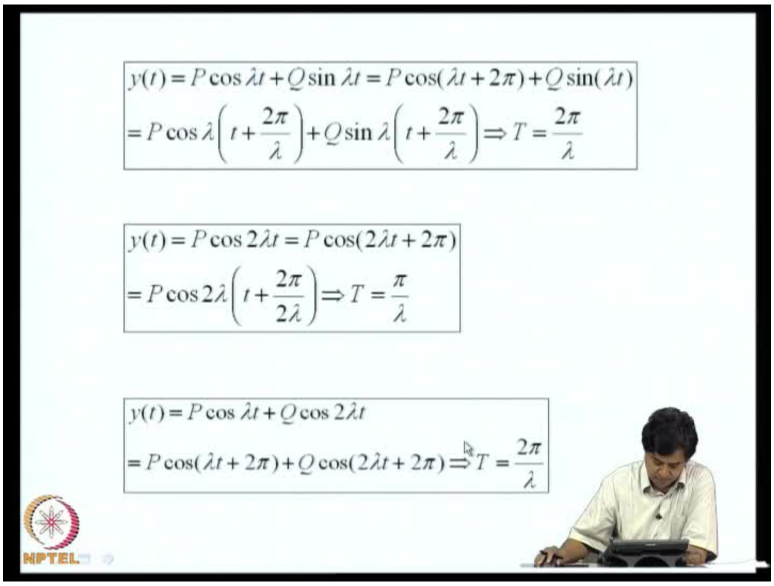


If you add now, P cos lambda t plus Q sin lambda t here, again if I know increment lambda t by 2 pi both these functions return to their values of cos lambda t and sin lambda t here again the period is 2 pi by lambda how about P cos 2 lambda t P cos 2 lambda t plus 2 pi. Therefore, what is the increment in t it is actually pi by lambda; so, P cos 2 lambda t is periodic with period pi by lambda. So, if I know add P cos lambda t plus q cos 2 lambda t what will be the period it will be 2 pi by lambda.

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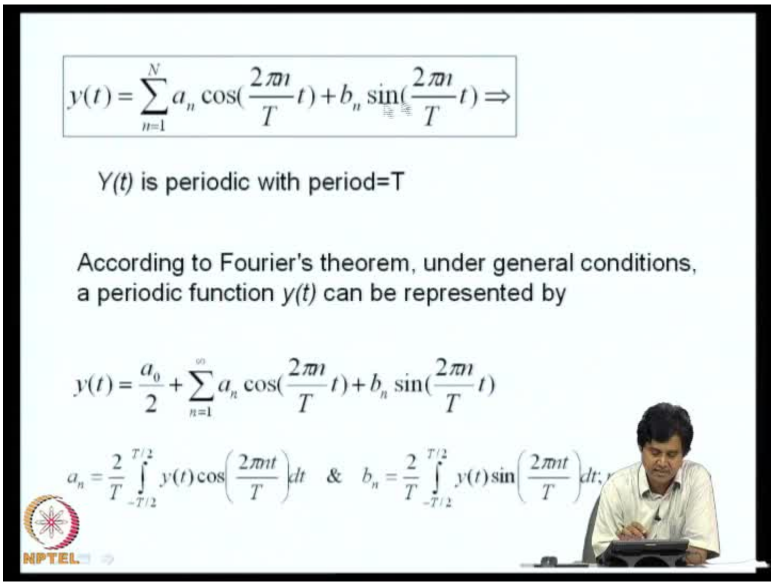

Now, if you similarly extend these arguments and consider a sums of sine and cosine functions multiplied by a n and b n, you can show that this function is periodic with period capital T; that means, if you add sin and cosin functions in this manner the resulting signal is periodic.

Now, what Fourier said was something more (( )). According to Fourier's theorem under certain general conditions a periodic function y of t can be represented in this manner, we already shown that if you add sine and cosine the functions is periodic, but now I am saying that if you have a periodic signal it can always be expressed in this form provided that signal satisfies certain conditions (( )) general conditions. Now, this a n and b n are known as Fourier coefficients and they can be evaluated using these relations.

So, instead of specifying y as a function of time, we can also specify values of these a ns and b ns that also would help us to completely define y of t. So, such a representation that means describing y of t in terms of a ns and b ns as function of n is known as frequency domain representation of y of t.

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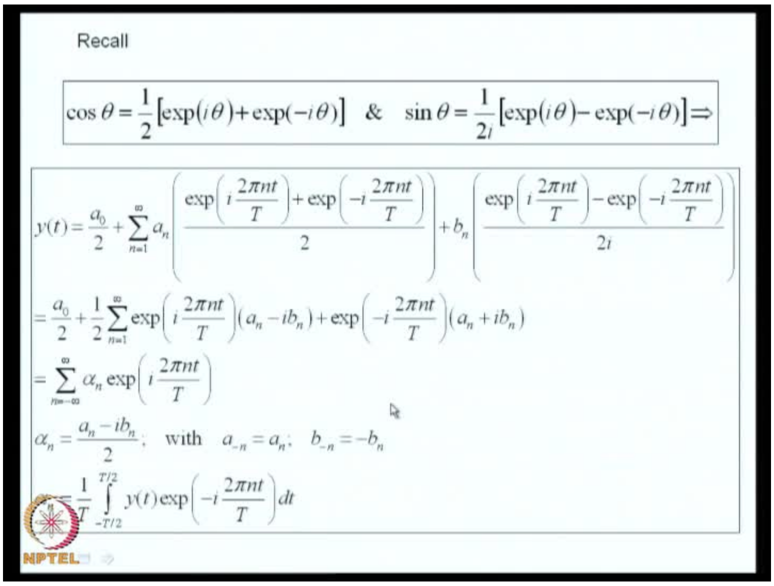


Here, we have used a sine and cosine functions. Since, we know that sine and cosine functions are related to complex exponentials through these relations. So, for sin and cosin functions I can make these substitutions for cos of this argument, I am replacing this and for sin theta I am replacing this and I get the expression for Fourier series in this form, we can do slight rearrangement of these terms and express y of t in terms of complex exponentials and complex coefficients alpha n. In this case, is related to a n and b n through this relation and it is complex valued and it is given by this and this also is equivalent Fourier series representation of y of t, but it is not now in terms of sin and cosin functions but in terms of complex exponentials.

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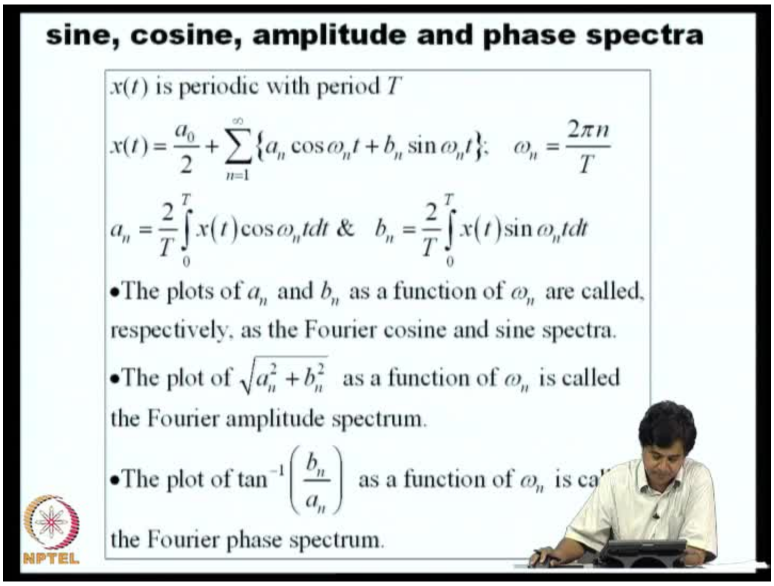


So, we use the phrases like sine, cosine, amplitude and phase spectrum. The word spectrum means on x-axis. We have a frequency parameter. So, x of t is periodic. Let x of t be periodic with period capital T. Therefore, I can write it in a Fourier series as shown here and a n and b n can be evaluated using this formula and the plots of a n and b n. As a function of omega n are called respectively the Fourier cosine and sine spectrum. Similarly, if you plot square root of a n square plus b n square as a function of omega n. This is called the Fourier amplitude spectrum and the plot of tan inverse b n by a n, as a function of omega n is called the Fourier phase spectrum.

So, these amplitude, sine spectrum, cosine spectrum, amplitude spectrum, and phase spectrum are alternate representations of x of t. This is very important in the analysis of dynamical systems, as you would see later. When we start discussing, problems of random vibration.

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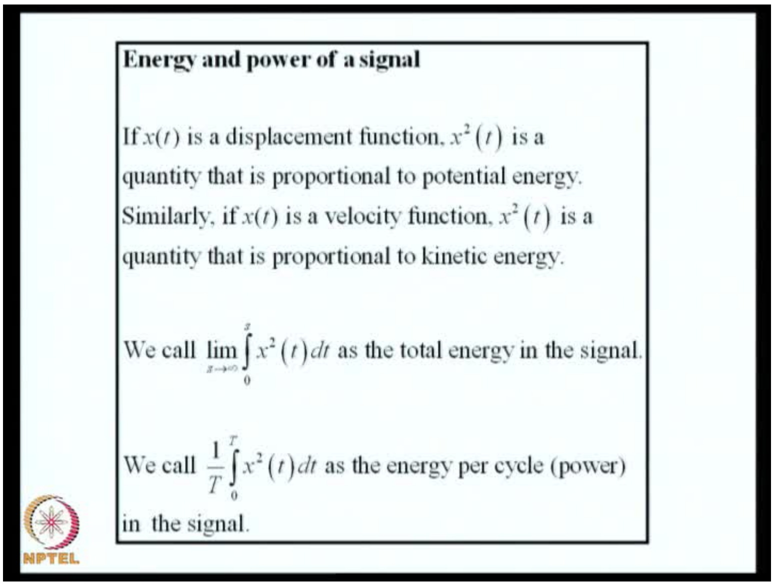


In describing time histories, we talk about energy and power of signals. Now, if x of t is a displacement function then x square of t is a quantity that is proportional to potential, for example, if you have a mass spring system the potential energy is half k x square. Therefore, the energy is proportional to x square of t. Similarly, if x of t is a velocity then x square of t will be a function that is proportional to kinetic energy, because half m x dot square is kinetic energy. Therefore, x square of t is proportional to a kinetic energy, we call this integral 0 to infinity S tends to infinity x square of dt as a total energy in the signal and similarly, if you compute energy per cycle, we call it as power in the signal. So, we talk of energy and power, which seems to be a mechanical quantities even for signals, which may not have this meaning of being displacement or velocity x of t could be for example, role roughness or wave heights and thinks like that where t of course will not be time; the even then we call these quantities. As total energy and total power all though the notion of energy and power is not immediately apparent, if you talk about role roughness or rain fall or temperature and things like that. This is the matter of nomenclature that we have to accept.

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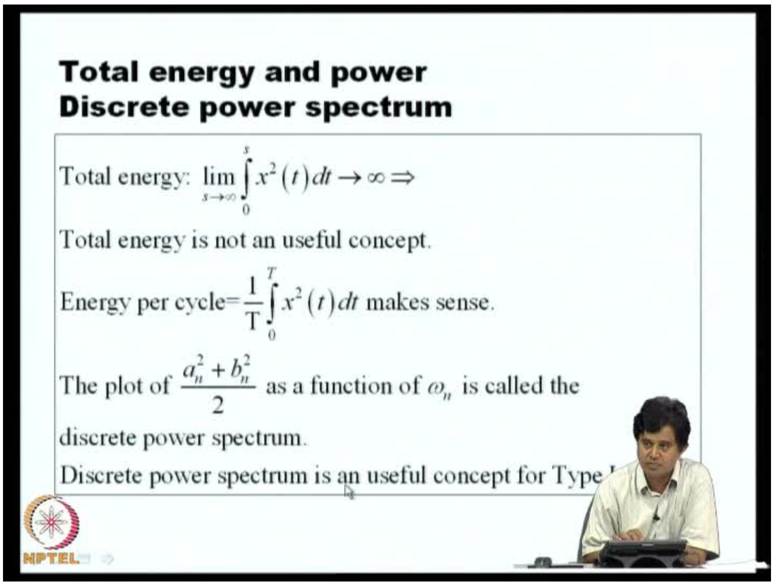


Now, let us consider for periodic signal, what happens to total energy and power. Now, this total energy 0 to as x square of dt, it will become unbounded, because a function is periodic as if you square it and go and finding the area under the curve for increasing values of S that integral. We will go on becoming large and it eventually diverges. So, this is not a very useful concept in describing x of t; in this sense, if there are two periodic signals say x of t and y of t for both these two signals the total energy as S becomes to large we will go to infinity and this number is no longer able to resolve differences between x of t and y of t. Therefore, in that sense is not a useful concept.

On the other hand, if you consider energy per cycle this make sense is neither 0 nor infinity it has. Therefore, certain information x of t and if there are two such signals this quantity will the able to differentiate between properties of the two signals in that sense it is a useful quantity. So, this plot of a n square plus b n square by 2 as a function of omega n is called the discrete power spectrum. Therefore, the conclusion here is discrete power spectrum is an useful concept for describing type 1 signals, whereas energy spectrum if you can think of is not useful because this quantity becomes unbounded. So, for type 1 signals, we have various descriptors, sine transform, cosine transform, amplitude spectrum, phase spectrum and now the discrete power spectrum.

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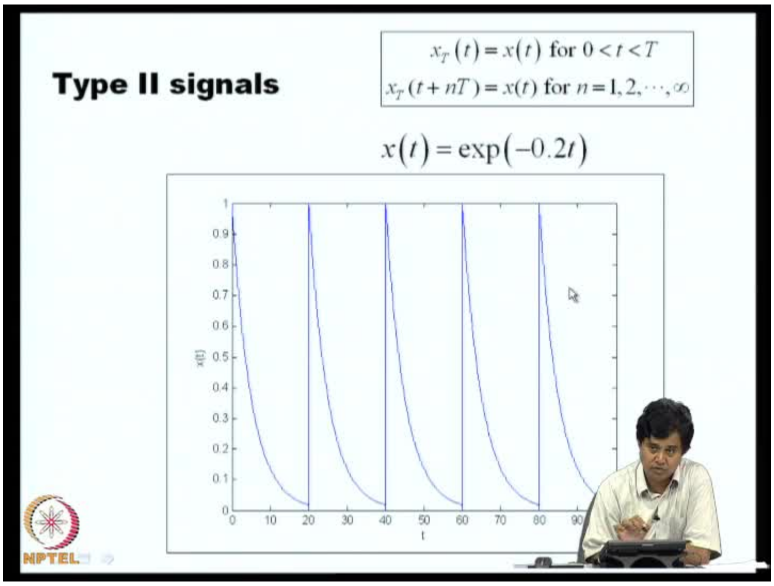


Now, let us move on to type 2 signals, here as t becomes large the functions these are an example, of a type two function as t becomes large exponential of minus point of 2t goes to 0 as t tends to infinity it is clearly aperiodic.

But what I do now is I introduce a new function x subscript t of t which is defined as x of t as long as t is between 0 to capital T capital T is the number that we arbitrarily specify and after words for t greater than capital T it repeats; that means, this is the function that we are looking at as for example, at capital T equal to 20, I will note continue this function it would have go on to 0, but this x T of t will stop here and between 20 and 40 this segment repeats.

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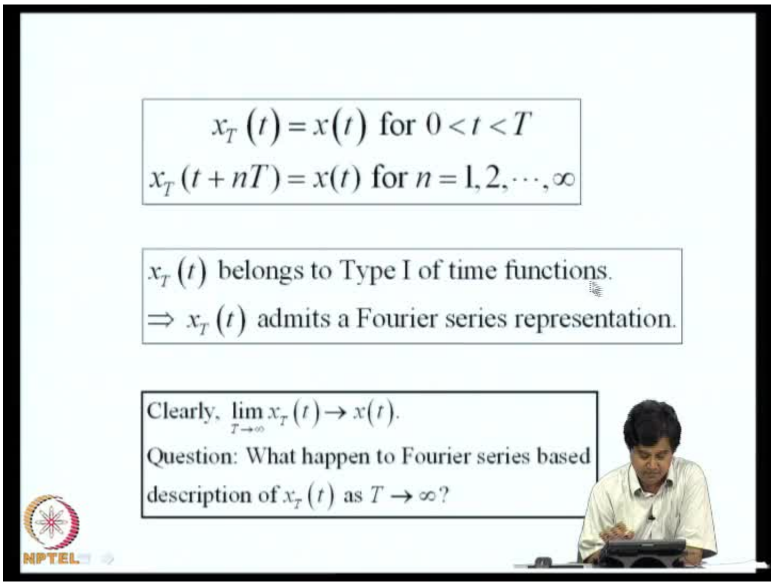


So, now, x T of t that is this function belongs to category type 1. Therefore, this would admit a Fourier series representation. So, x T of t belongs to type 1 of time functions. Therefore, it admits a Fourier series representation. Now, clearly as capital T tends to infinity that difference between x T of t and x of t vanishes.

Now, the question therefore, we should ask is we can develop a Fourier series representation for x T of t, what happens to this Fourier series representation, as capital T becomes large if it leads to meaningful interpretations then the Fourier representation for x of t is meaningful, so let us see what is possible.

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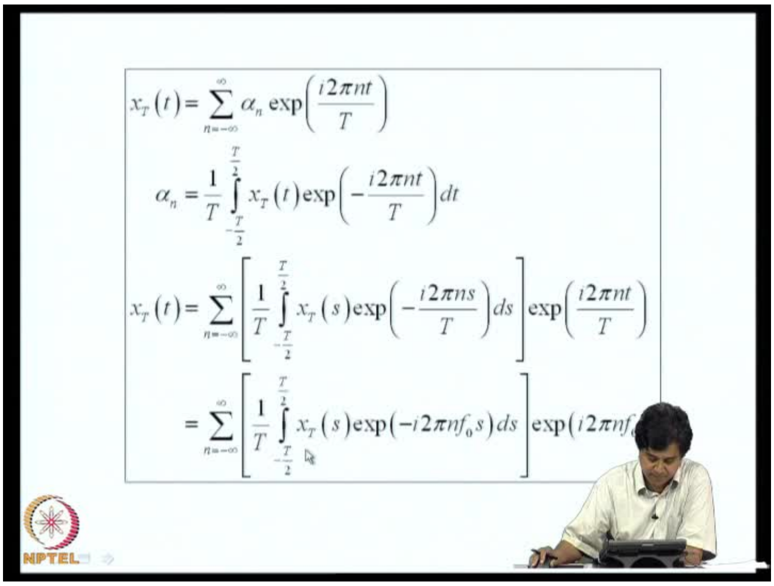


Now, x T of t therefore, can be represented in a Fourier series in terms of complex exponential. In this form, where alpha n are the Fourier coefficients complex valued Fourier coefficients, it is given by this. Now, if I substitute this into this integral for alpha n here I will use this and write this, it is just rewriting of this equation in terms of this alpha n.

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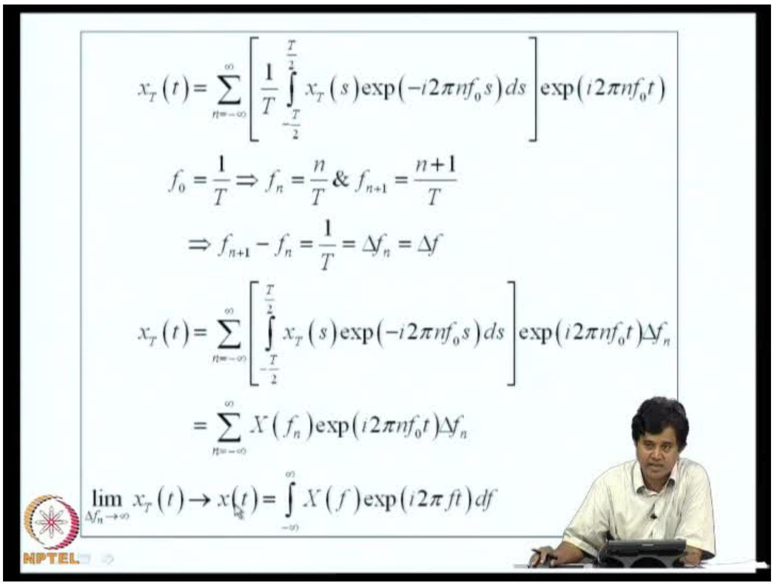


Now, we will slightly rearrange I here 1 by T I am calling as f naught is the 1 by T is the frequency in Hertz, I will call it as f naught and if we now consider f naught is 1 by T.

Therefore, f n is n by T and f n plus 1 is n plus 1 by T. so, f n plus 1 minus f n will be 1 by T which I will call it delta f n and if these frequencies are uniformly spaced as is implicit in the we have written n f naught. This will become delta f, therefore, for this 1 by T I can write it as delta f. so that the delta f n I am writing here this 1 by T is now removed in this expression. So, this integral remains the same except for this change and I rewrite this with this change.

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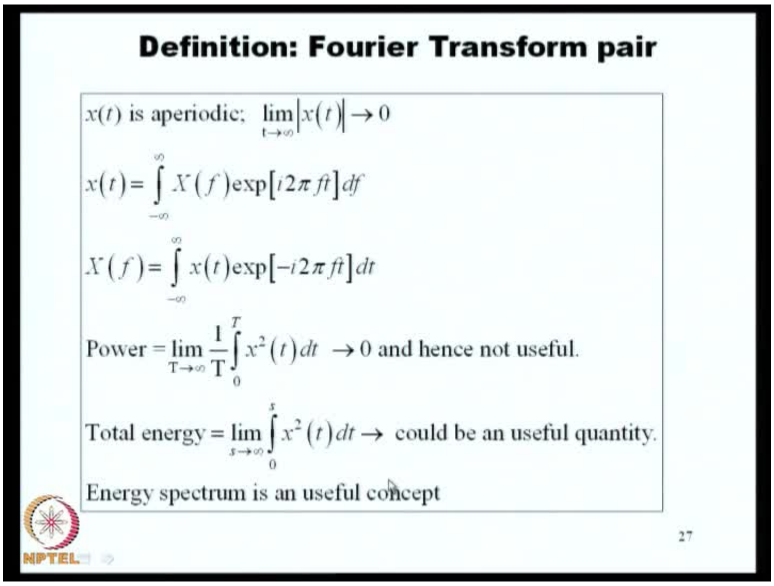


Now, the quantity inside this bracket I call it as capital X of f n. Now, as delta f n goes to infinity that means what is delta f n 1 by T as T tends to as delta f n goes to 0. This is a mistake here, this should be delta f n goes to 0 as capital T tends to infinity x T of t goes to x of t and this summation becomes an integral. So, this is the Fourier domain representation of x of t that means the summation in Fourier series. Now, becomes an integral and this leads to a notion of what is known as a Fourier transform pair. So, x of t is given by X of f exponential this and X of f itself is derived as shown here. So, we say that x of t and X of f form a Fourier transform pair.

Now, let us consider, what happens to power? Now, the signal is periodic. Therefore, if I consider 1 by T 0 to T x square of t dt, what happens this integral 0 to T x square of t dt as capital T becomes large will go to 0, because x of t is decaying to 0 under certain conditions. Therefore, a finite number is divided by denominator, which is capital T which is becoming unbounded. So, this quantity goes to 0 that means the notion of power is not useful for a periodic signals; simply, because if you have two aperiodic signals, which decade to 0 as T tends to infinity the power would be 0, no matter what are the details of x of t and y of t. So, the differences between x of t and y of t cannot be resolved using power as a descriptors, but other hand, the total energy this could be an useful quantity that is 0 to infinity x square of t dt, if it is finite then the description of x square of t that is energy spectrum is a useful concept for aperiodic signals. So, for aperiodic signals I have a Fourier transform and the notion of an energy spectrum, which are useful.

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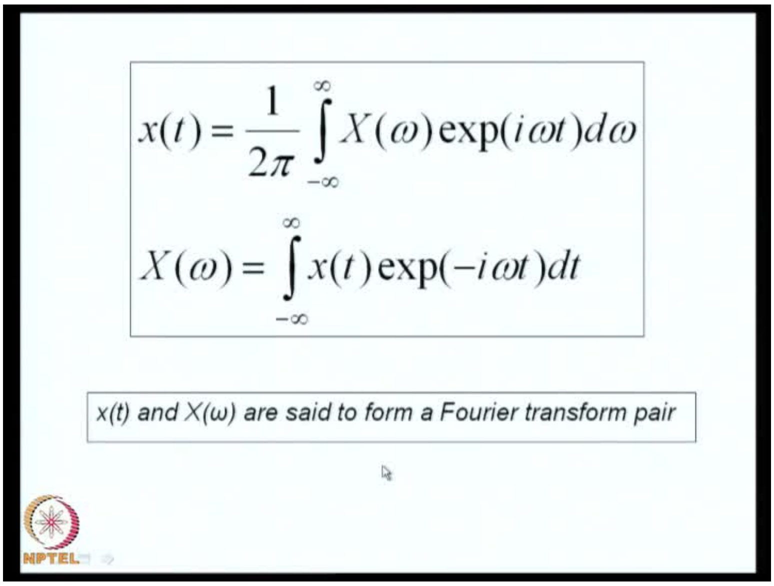


So, we write in terms of frequency in radiant per second, the relationship between x of t and X of omega is displayed here and we say that x of t and X of omega they form a Fourier transform pair; knowing X of omega, you can always evaluate x of t and knowing x of t you can always evaluate X of omega, therefore, their equivalent in that sense.

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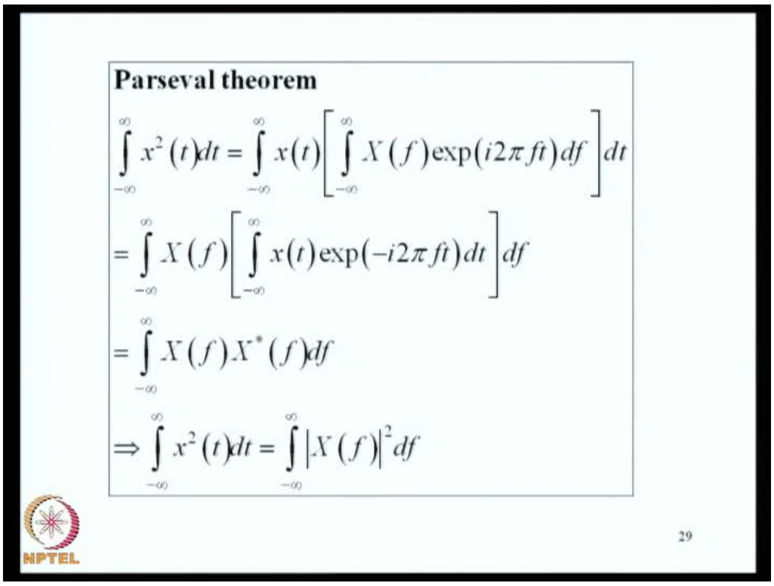


This is an important theorem in signal processing, which will be of value to us that is known as Parseval theorem. Now, suppose if you consider total energy in the signal and evaluate it as a minus infinity plus infinity x square of t dt. This can be written as integral minus infinity to plus infinity x of t into x of t for the second x of t, I will write the Fourier representation and I will interchange the order of integration. So, I will push this x of t and perform integration with respect to time first, you will see that this is nothing but the conjugate of the Fourier transform. So, I get this X of f into X star of f df that is this is nothing but modules of x of f whole square df.

So, therefore, according to Parseval theorem the total energy can be computed either in time domain or in frequency domain both are equal. So, depending on the context, you can use either this approach or this approach and it will lead to the same answer that is what Parseval theorem says.

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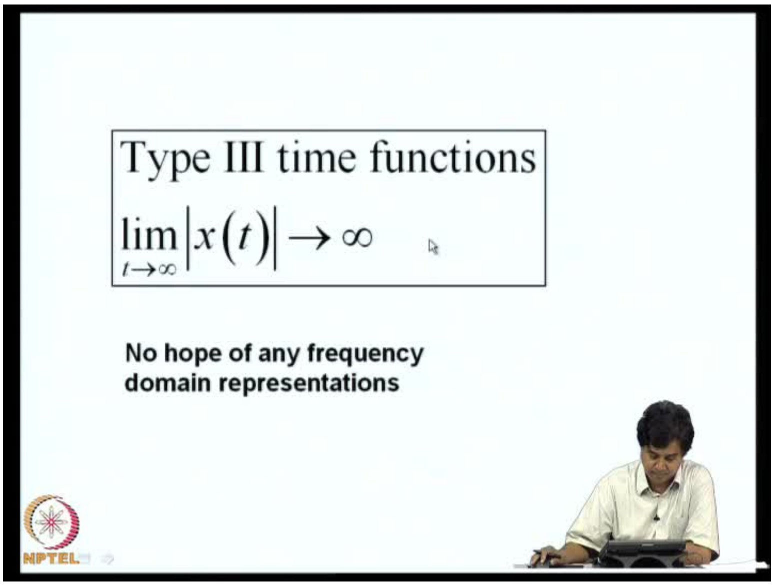


So, we have completed the discussion of type two functions. For type three time functions, as I mentioned already the functions which become unbounded as time becomes large there is no hope of any frequency domain representations. We cannot write a Fourier series, we cannot write a Fourier transform. So, these problems, this types of problems can only be handle in time domain.

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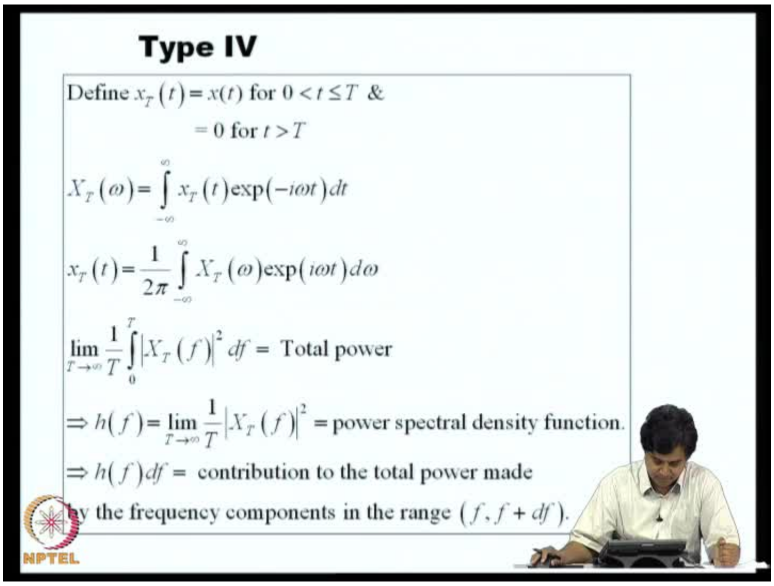

Now, we move on to type 4 signals, what are they, the signals which neither decade to 0 nor go to infinity as time becomes large, here I can introduce x T of t equal to x of t as I did for a aperiodic signals, I can still define x T of t.

Now, we can define the Fourier transform of x T of t as in terms of these integrals, if you now consider total power that means 0 to T x T of f whole square df as t tends to infinity is a total power. This makes sense this ratio could make sense, because the function is not decaying to 0 nor going to infinity. Therefore, this ratio could make sense and we will see under what condition it could make sense and this integral integrand h I call it as h of f which is limit of T tend to infinity 1 by T the Fourier transform x T of f its modulus whole square, we define as power spectral density function.

Why the word density? This quantity h of f df is can be viewed as contribution to the total power made by frequency components lying in range f to f plus df. So, in that sense it is a density. So, h of f is a very useful descriptor. For type 4 signals, you can verify that energy would not be useful. In this case I leave it as an exercise.

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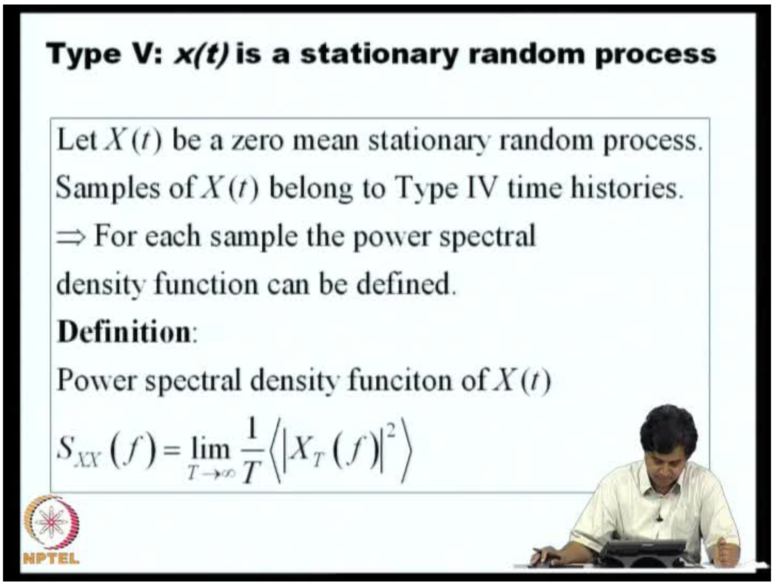


Now, let us consider a new type of functions, I have introduce only 4. Now, I introduce the fifth type, here X of t is a stationary random process. That means, X of t is a collection of time histories. Let it have zero mean and if you consider samples of X of t they belong to type 4 type of time histories, because samples of stationary random process neither go to 0 nor go to infinity as time becomes large. So, since each realization of X of t belongs to type 4 of time histories for each sample I can define a power spectral density function as I did for a single type 4 functions. Now, this leads to the notion of power spectral density function for a random process. This may define as S XX of f as limit T tend to infinity 1 by T. This is the ensemble average, across ensemble of realizations of power spectral density functions of a corresponding to individual realizations of X of t. So, this is an ensemble average. So, this quantity is defined as a power spectral density function of X of t.

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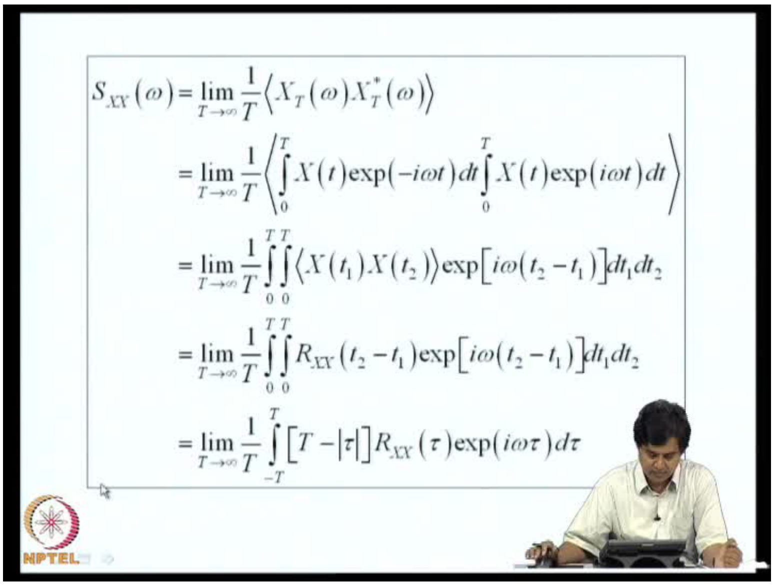


Let us see, what we can learn from this. So, we will now work in frequency in radiant per second. So, S XX of omega would be this now, for X T of omega I will write the Fourier representation since I am considering the time interval to be only between 0 to capital T there is no need for me to distinguish between X T of t and X of t, because both are equal. So, this subscript capital T I am omitting here. So, this is X T of omega is conjugate I will replace minus I by I and this is the conjugate. So, this becomes a double integral and if you assume, that interchanging of integration and this expectation are permissible we can rewrite this in this form.

This is nothing but autocorrelation function of X of t. This expected value and since mean of X of t is taken to be 0 this is also the covariance function of auto covariance X of t. So, this S XX of omega is now this double integral. Now, using certain transformations, we can reduce this double integral to a single integral and that is given here.

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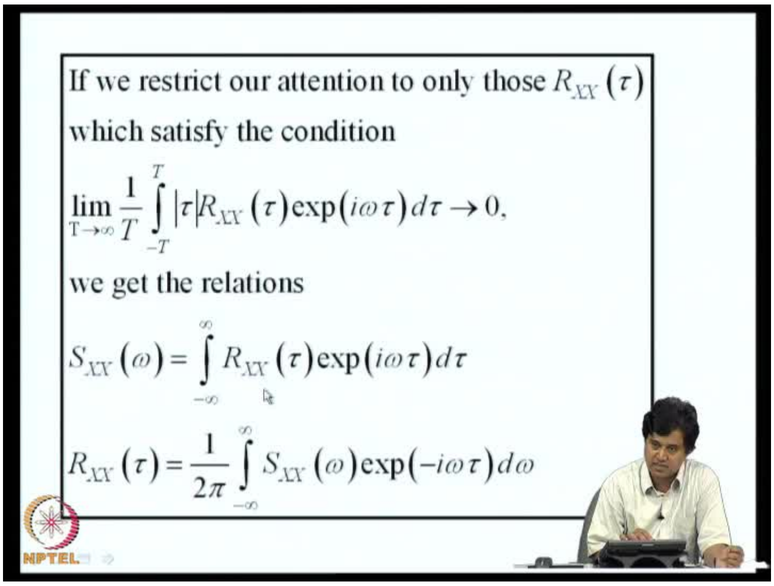


Now, if you look at the details of this integrant, if we now restrict our attention to only those R XX of tau which satisfy the condition the t tending to infinity 1 by T modulus of tau R XX of tau exponential i omega tau d tau goes to 0. We now get the relation S XX of omega is given by this integral and conversely R XX of tau is related to S XX of omega by this that means, where if these conditions are satisfied the auto power spectral density function of X of t and the auto covariance function of X of t for a Fourier transform there.

So, it would mean that for a stationary random process, the power spectral density function happens to be the frequency domain description of the stationary random process that is the Fourier transform of the time domain descriptor namely the auto covariance functions.

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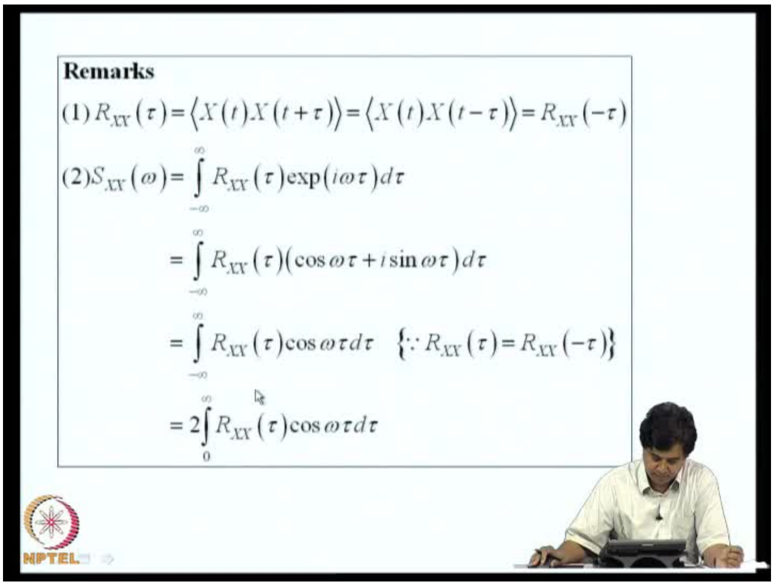


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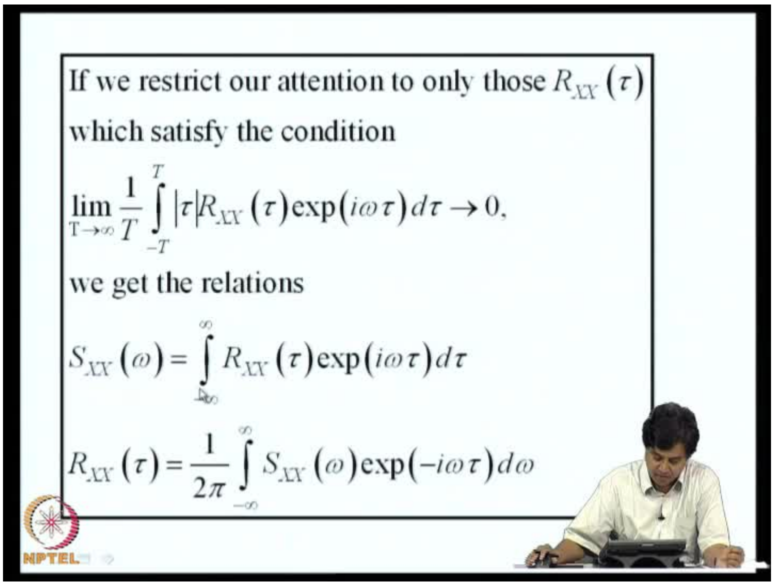


Now, we can make some observations on the nature of this auto covariance function. So, R XX of tau is expected value of X of t multiplied by X of t plus tau. So, since the process is stationary R XX of tau is same as R XX of minus tau now S XX omega is this integral according to a definition for exponential i omega tau, I will write cos omega tau plus i sin omega tau. Now, R XX of tau as we have seen here is an even function and this integral is between symmetric limits sin omega tau we know is an odd function. So, an even function multiplied by odd function is an odd function; so integral of odd function between symmetric limits is 0. So, this integral now simply become this. So, this the imaginary part of this is 0. So, S XX of omega is indeed a real valued function.

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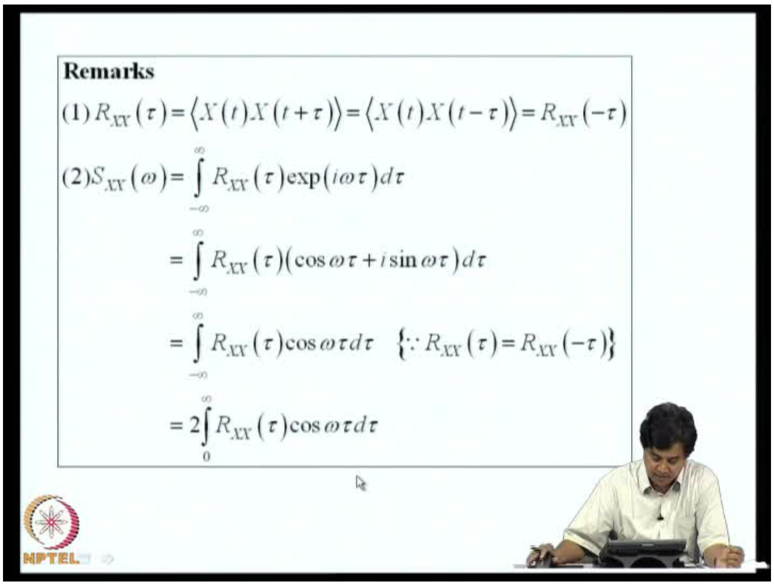


Again, since this is an even function and this also even function. Instead of integrating from minus infinity to plus infinity, I can integrate from 0 to infinity and multiply this by 2. So, the relation that we saw here that is S XX of omega R XX of tau in terms of complex exponential. Now, can be expressed in terms of cosine functions.in this form.

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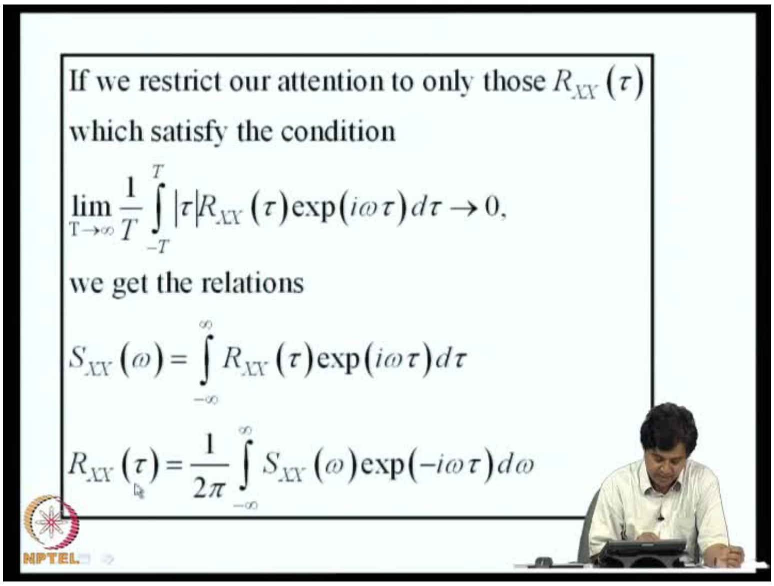

Now, what will be the value of in this integral, if I put tau equal to 0, what I get 1 by 2 pi minus to plus infinity S XX of omega this is 1 d omega.

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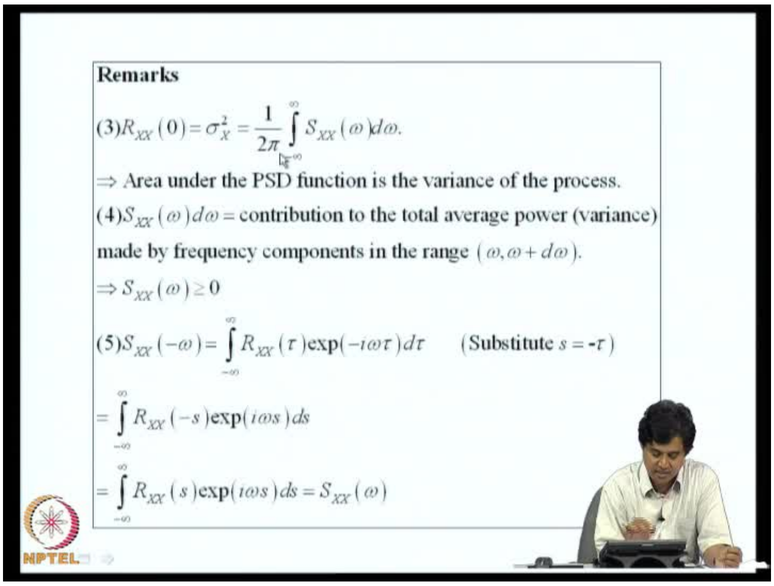


What is R XX of 0? It is nothing but the variance of the process. Since process is stationary it is independent of time. So, I get an important relation that the area under power spectral density function divided by 2 pi is equal to the variance of the random process, this is the very important property.

Now, we will now look at the meaning of the quantity S XX of omega into d omega, I am talking about density functions, so what it means? We can interpret this as contribution due to the total average power. Now, the total average power is now variance variance as now another interpretation that it is now the total average power in the signal S XX of omega into d omega can be viewed as contribution to the variance of the process made by frequency components in the range omega to omega plus d omega. Since variance is positive is non-negative S XX of omega is also nonnegative, so the power spectral density function; therefore, is symmetric in omega it is non-negative and area under the power spectral density is equal to the variants. So, this is the symmetry argument it is based symmetry of R XX; so, S XX of minus omega if I right I will write minus infinity to plus infinity R XX of tau exponential this. And now I will make the substitution is s equal to minus tau and further simplification shows that S XX of minus omega is same as S XX of omega. So, S XX of omega symmetric non-negative area under the function is equal to the variance area under the function divided by 2 pi is the variance.

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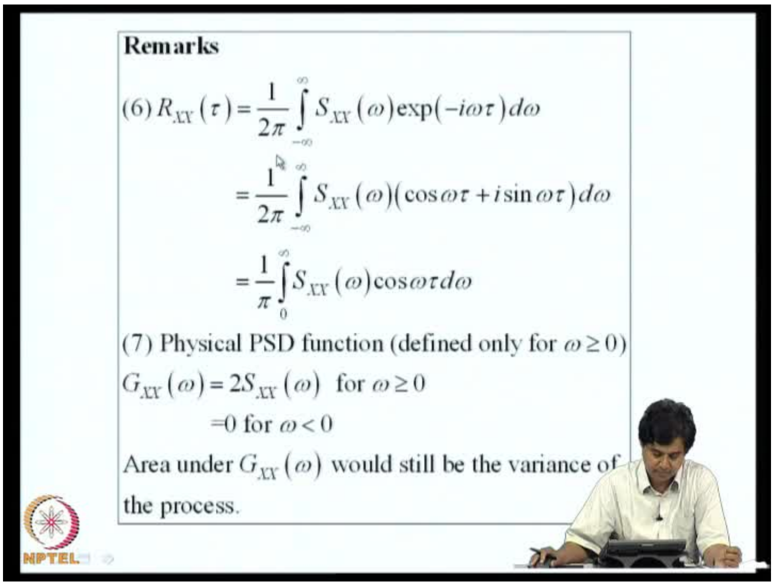


We can get alternate representation of R XX of tau by slightly rearranging this complex exponential term. Now, for complex exponential, if I write cos omega tau plus i sin omega tau; again using the logic that sin omega tau is an odd function the integration with respect omega now, sin omega tau is odd function with respect to omega S XX of omega as symmetric. Therefore, even function therefore an even function multiplied by odd function is odd and area under that curve under symmetric limit is 0. So, sin omega tau will not contribute and R XX of tau would be real valued as it should be.

Now, if I replace tau by minus tau you can simplify. See here, the cos omega tau will be replaced by cos of minus omega tau which is same as cos omega tau so the symmetry of R XX of tau is retained.

In the definition of S XX of omega we are taking limits from minus infinity to plus infinity, but in engineering practice we often treat frequency to be positive, especially vibration related problems. So, we want frequency to be strictly positive in that case non-negative in that case we introduce a function known as physical power spectral density function, which is defined as 0 for negative frequency and it is twice the mathematical power spectral density for omega non-zero; clearly area under G XX of omega d omega from 0 to infinity will be equal to the variance of the process, because this will be two integral of 0 to infinity S XX of omega divide by 2 pi would be half the variance. So, this will also be that property that the area under the power spectral density equal to the variance is still retain.

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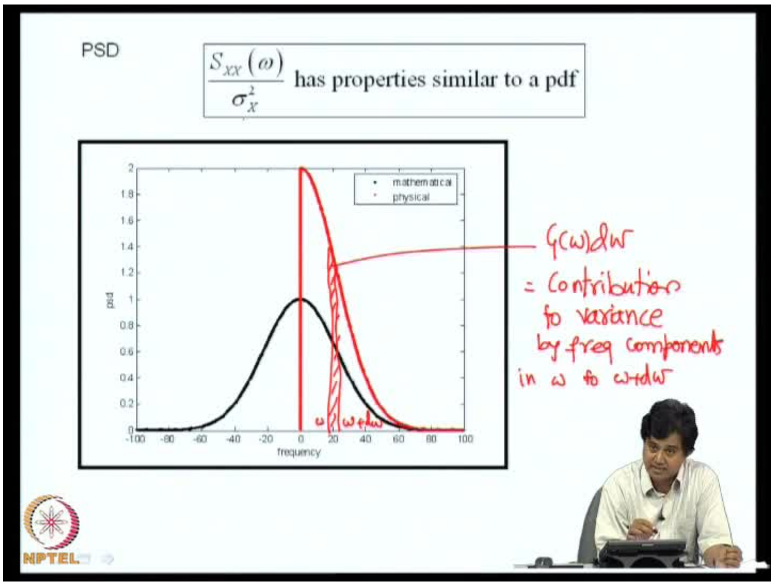


So, this is what is shown here, this black line here is the mathematical power spectral density function and the red line is the physical power spectral density function. Now, if you consider this say omega to omega plus d omega and take this area. So, this is G of omega into d omega, this will be the contribution to variance, which is nothing but the total average power by frequency components in omega to omega plus d omega in that sense it is a density function. So, density means you multiply by frequency you get the variance contribution to variance in that range of frequency.

Now, if you divide the power spectral density by the variance of the process, what happens - area under this function will become 1 it is non-negative. So, it has properties similar to that of a probability density function of a random variable. So, just as we defined moments of a random variable. Later in this course, we will define moments of this power spectral density and we will see that it has certain useful interpretations.

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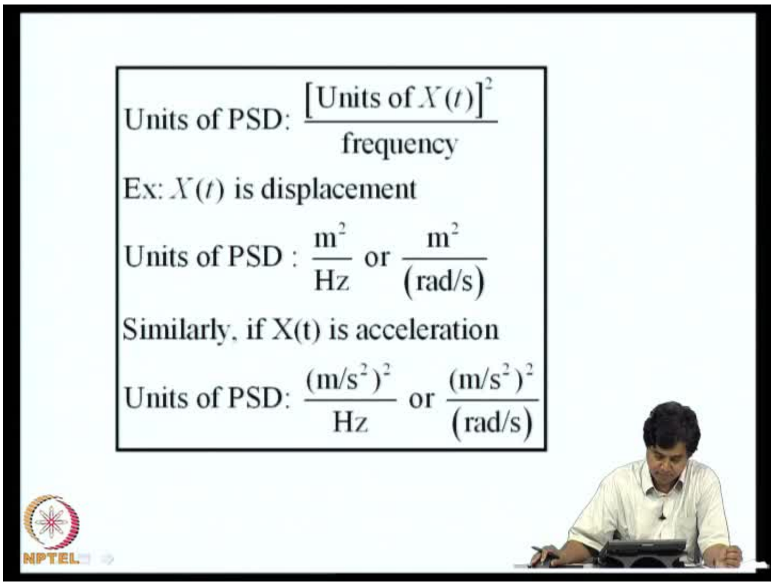


What will be the units of power spectral density function? So, since the area under power spectral density has units of variance of the random process. Therefore, units of PSD should be units of X of t whole square divided by frequency; this will be the units of the variance so if X of t is displacement the PSD will be meter square by Hertz or meter square per radian per second. Similarly, if X of t is acceleration this will be meter per second square whole square per Hertz or meter per second square whole square divided by radian per second. So, the units of power spectral density follows this I mean this will be according to this.

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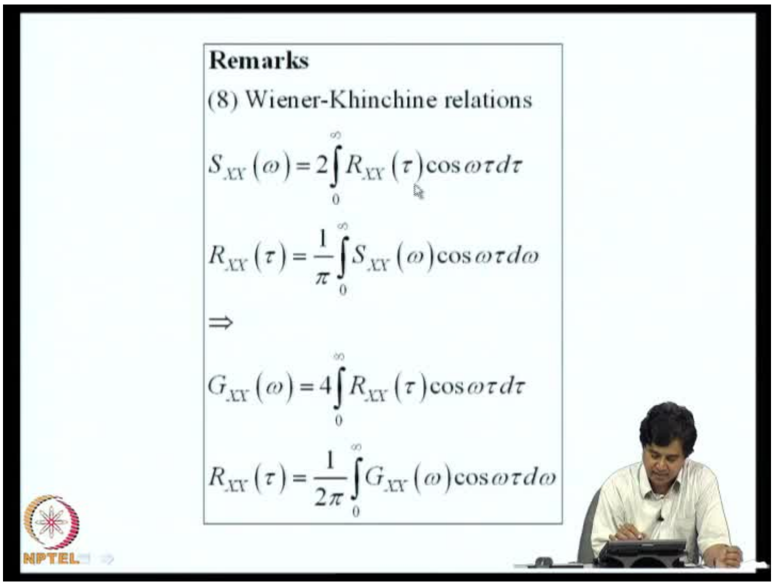

Now, if you read text books on random processes. There are various versions of representations of power spectral density and auto covariance there is question of frequency being negative or not and question of exploiting the symmetry of S XX and R XX in displaying this relation so this relation that S XX of omega is 0 to infinity R XX of tau cos omega tau and this in terms of power spectral density is known as Wiener-Khinchine relations.

So, in terms of physical power spectral density, which is 2 of S XX of omega the relationship between auto covariance and physical power spectral density can be shown to be given by this. So, one has to carefully handle these issues in any modeling exercise, you must be clear whether you are handling a physical power spectral density function. Whether frequency is from minus infinity to plus infinity; frequencies is in Hertz or radian per second, these questions have to be clearly answered at the outside, otherwise numerically you will have considerable difficulty in interpreting the results.

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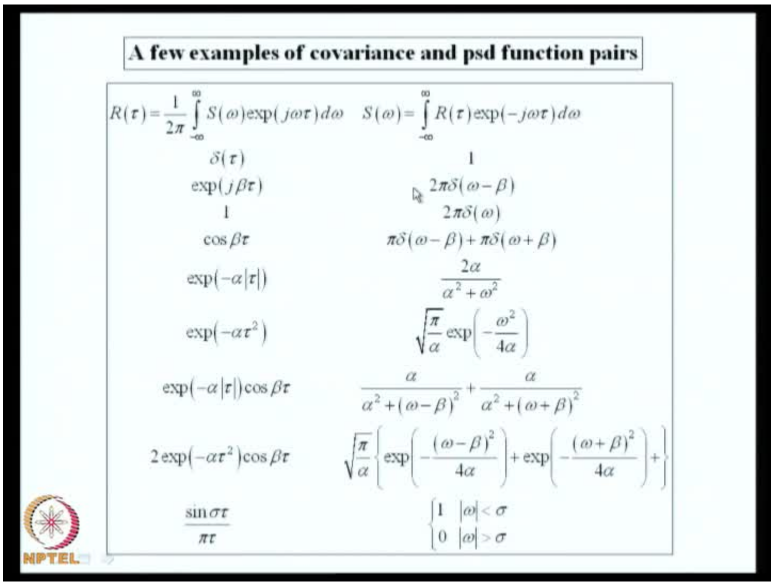


So, here I have tabulated some relationship between a few auto covariance functions and their Fourier transforms. So, if auto covariance function is the Dirac's delta function, the power spectral density will be a constant such a process is known as white noise, we will see more of it in due course.

If auto covariance is a complex exponential a power spectral density will be a direct delta function, if auto covariance is a constant; again we get a direct delta function, if it is harmonic, I get a pair of direct delta functions, if it is exponential minus alpha mod tau I get 2 alpha by alpha square plus omega square so on. So, if it is like a Gaussian density type function this will be this, if it is a modulated harmonic, this will be the power spectral density. So, you can easily derive this there is a catalog of such Fourier transform relations available and computationally of course there is first Fourier transform algorithm available to move from time to frequency and frequency to time. Therefore, this representation can be derived even, if the auto covariance and power spectral density functions have more complicated descriptions.

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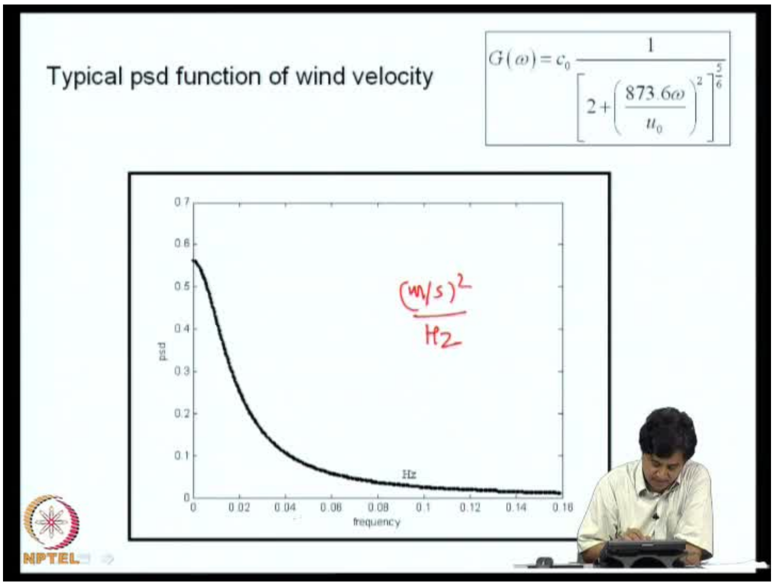


Here, I am displaying a few typical power spectral density functions in random vibration analysis, as we would see subsequently the power spectral density function becomes a very valuable modeling tool, so for wind velocity the load is described in terms of power spectral density function, if it has a nonzero mean, the mean has to be separately describe. This is the stationary component of wind velocity and a typical plot of this function is shown here and the units of power spectral density will be velocity square divided by frequency, which is in Hertz. So, this will be meter per second whole square divided by Hertz.  This will be the units of the power spectral density function in this case.

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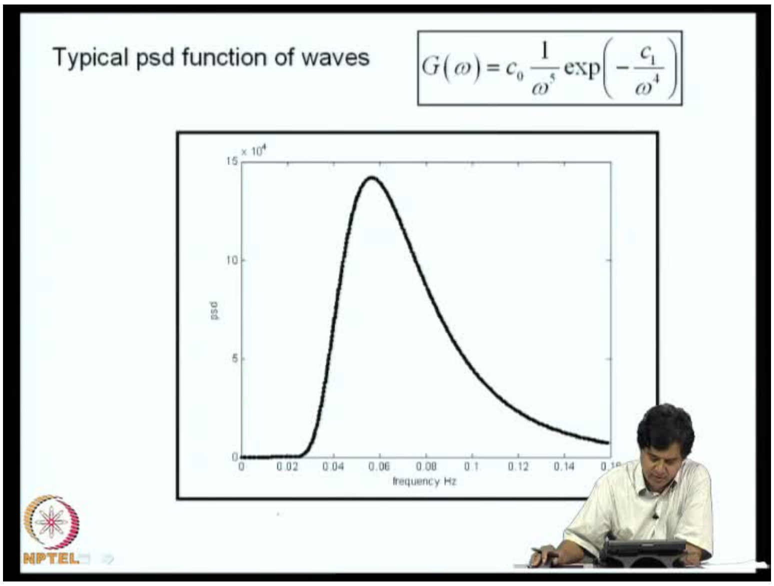


We will see that wind velocity typically has significant power in low frequency regions. Similarly, the wave heights you know they have again low frequency power in low frequencies; and this is a typical power spectral density function for wave heights and this is again around 0.06 to 0.1 Hertz in this region.

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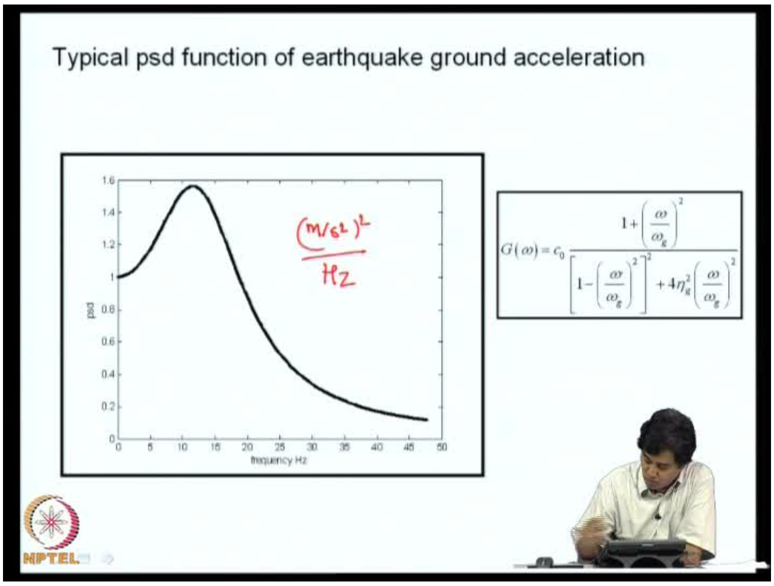


If we look at earthquake loads, this typical power spectral density functions. The frequency is now between about 0 to 30 Hertz. So, the peak here is around 10 Hertz in this example and the units in this case would be meter per seconds square whole square divided by Hertz. This is a form of the power spectral density function later in the course will actually, see how this form is arrived at based on certain physical arguments.

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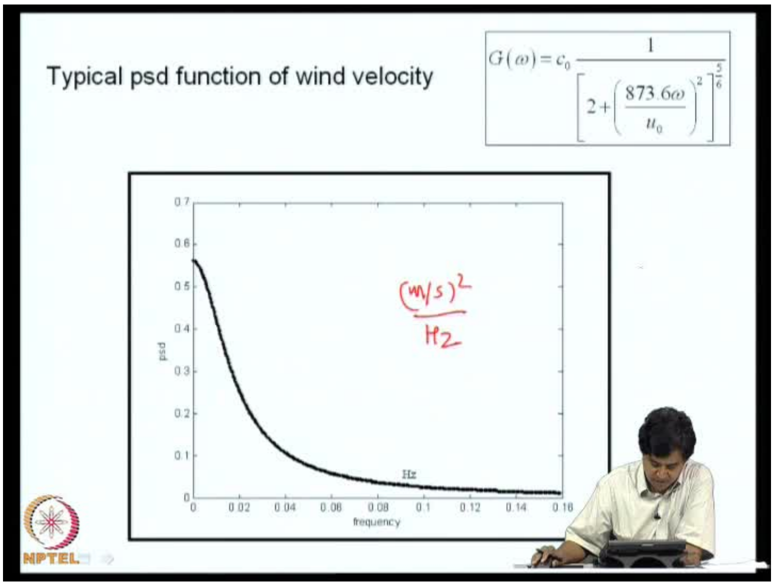


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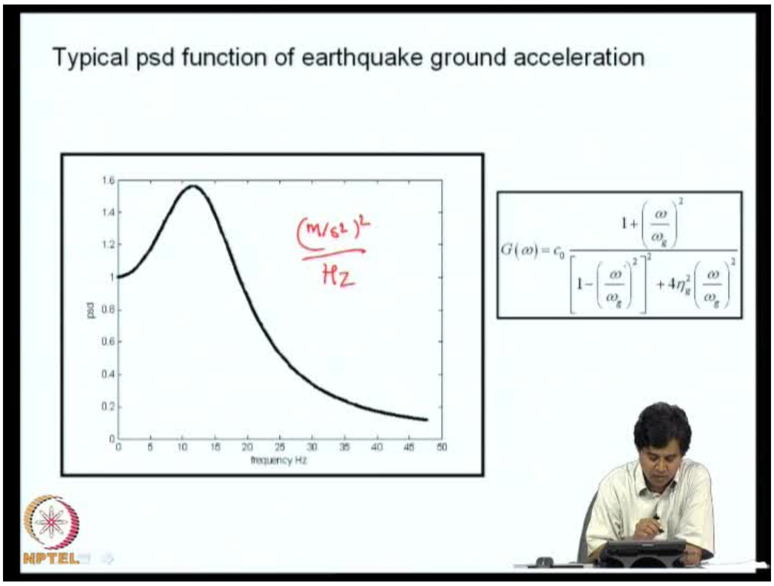


If you look at these three functions together, you will see that if you are looking at engineering structures, whose natural frequencies our in this regions say 5 to 10 Hertz or 12 Hertz, they are prone to the dynamic effect of earthquake loads, there is a possibility of resonance because the input signal has significant power in frequency ranges where we expect this structures to have their natural frequencies.

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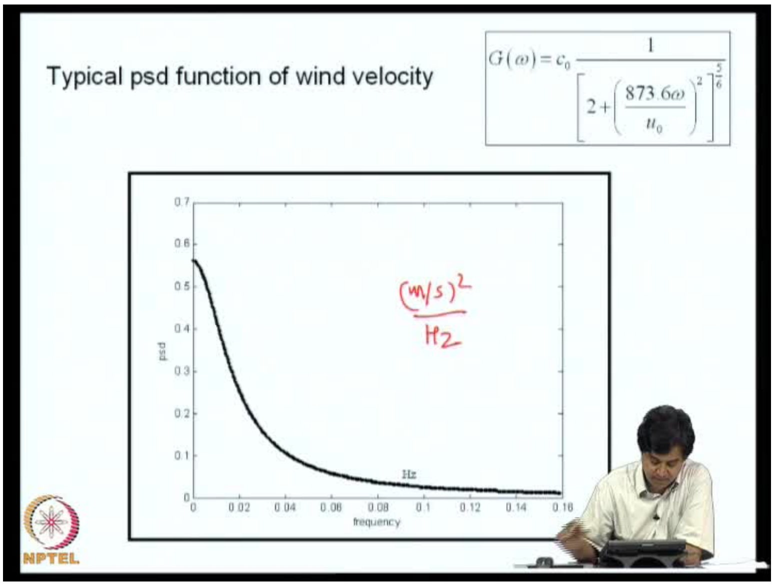


So, earthquake loads are going to affect though wide range of structures, whose natural frequencies lie in this region. Similarly, if you are looking at structures, tall structures like chimneys, long span, bridges, etcetera. There frequencies are likely to be less the fundamental modes are going to have low frequencies. So, consequently they are brown to the dynamic effect of wind loads.

So, we can think of the notion of resonance that we have talked in the context of harmonically driven linear systems. We can extend that notion for resonance in randomly driven systems, where we now look at the power distribution as a function of frequency and infer on possibility of occurrence of resonance.

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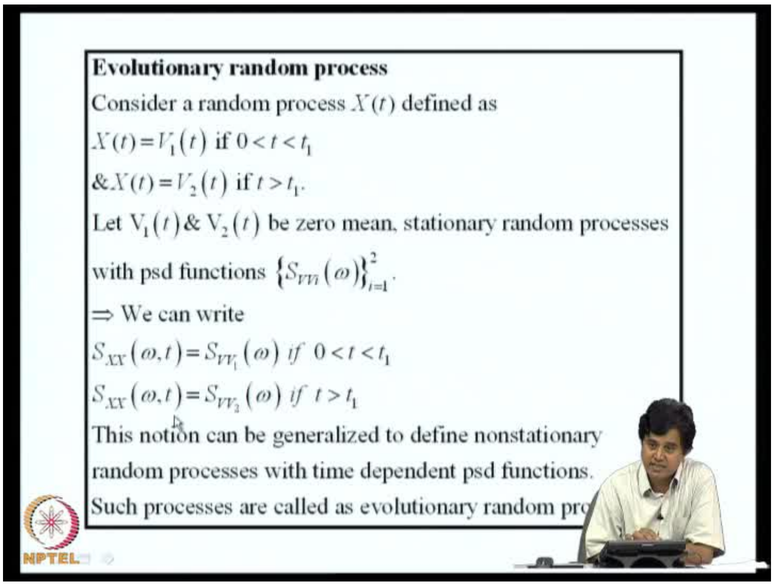


A last category of random processes that we will discuss in this talk is what are known as evolutionary random processes? Suppose, if you define X of t as V 1 of t for t equal to t 1 and V 2 of t for t greater than t 1, where V 1 V 2 are zero mean, stationary random processes with different power spectral density function. So, I can rate the power spectral density of X of t to be V V 1 of omega, if we are in this time region or it is this, if you are in this time region. So, generalizing this notion we can introduce the notion of an evolutionary random process where the power spectral density functions. Now, also is a function of time. This time dependency in some sense captures the non stationarity in frequency content of the signal that we will see later, how it can be used to model earthquake loads.

So, with this will conclude this lecture.