Stochastic Structural Dynamics

Prof. Dr. C. S. Manohar

Department of Civil Engineering

Indian Institute of Science Bangalore

Module No. # 10

Lecture No. # 40

Problem solving session-4

In this lecture we will continue with exercise of solving a few problems. Since this being the last lecture in the series, we will take a look at what we have done in this course and also see what else could be done with this background where, you can move on if you have understood this course.

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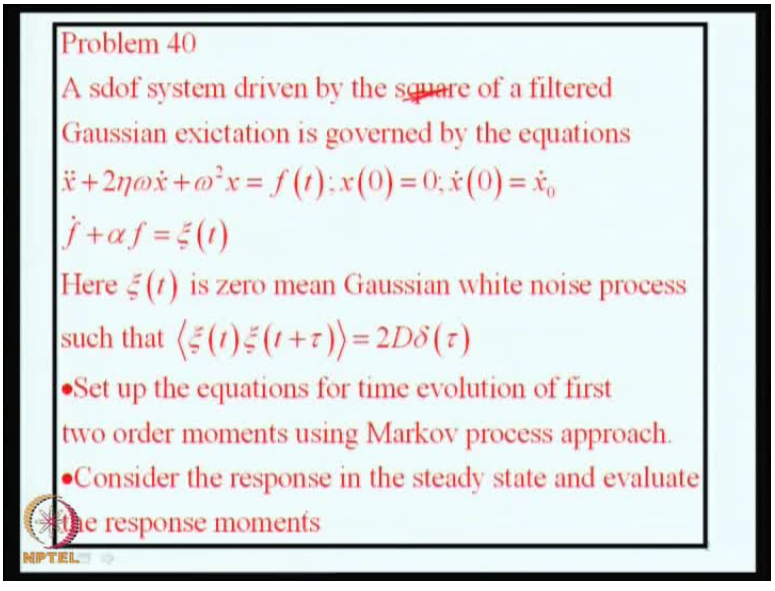

We will begin by considering a problem in random vibration. A single degree of freedom system is driven by a filtered Gaussian excitation and is governed by the equations x double dot 2 eta omega x . + omega square x = f (t) a system starts with some initial conditions and this excitation f (t) is a filtered white noise. That means the white noise x i (t) passes through a first order differential equation filter and this x i (t) 0 mean

Gaussian white noise with covariance given by 2D direct delta of tau where tau is a timed line.

Now, this problem is to analyze the response of the system. But we would like to approach this problem using Markov process theory. The problem is actually to set up the equations for time evolution of first two order moments, using Markov process approach. Then consider the response in the steady state and evaluate the response moments.

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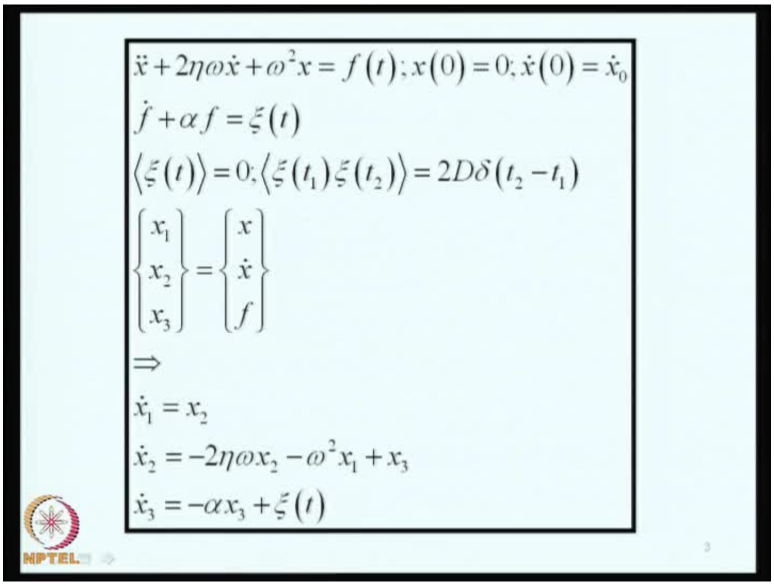

The governing equation is rewritten here. We introduce a state vector x 1, x 2, x 3 where x 1 is x, x 2 is the velocity and x 3 is f. This is the second order differential equation that governs the displacement x and this is the filter equation.

We declare the variables from this system viz. displacement and velocity and this process f (t) as the states. Now we recast this governing differential equation into the state phase form so x 1 dot is x dot; x dot is actually x2. So, the first equation is x1 dot equal to x2.

x2 is velocity; x2 is x dot; Therefore, x2 dot is acceleration. So x2 dot is for acceleration. We go to the governing equation this is minus 2 eta omega x2 minus omega square x plus f (t) and f (t) is x3 and the third equation is x 3 dot is equal to minus alpha x 3 plus x i (t).

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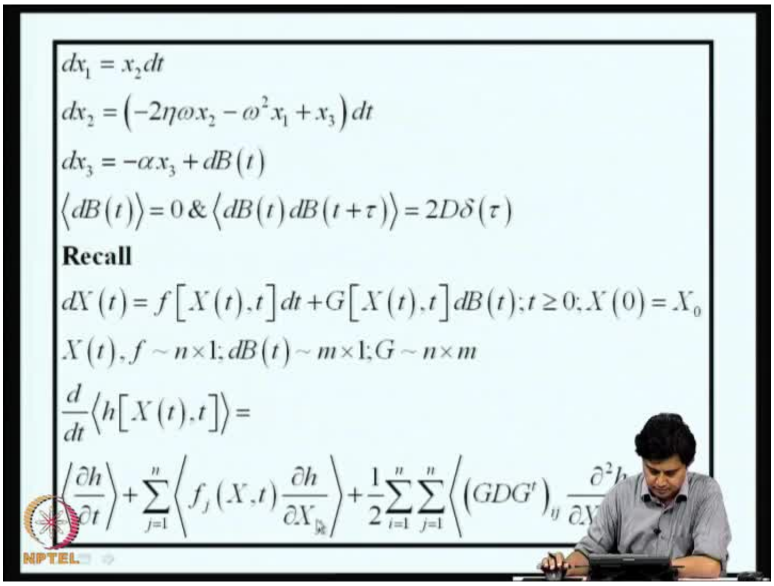

Now, we rewrite this in the form of a (( )) stochastic differential equation. As you have seen this is dx 1 is x 2 dt dx 2 is minus 2 eta omega x 2 minus omega square x 1 plus x 3 dt and dx 3 is minus alpha x 3 dB t.

Where dB t is increment of Brownian motion process with this properties. You may recall that when we discussed the Markov vector approach for solving random vibration problems, we consider the general form of x d e of this form for n dimensional response vector.

And we showed that the time evolution of expectation of a function of h of X comma t a follows this law. So, this we have derived when we discuss the Fokker Planck equation and the backward Kolmogorov equation and the moment equations. Now, we will use this and derive the required equations for the moments.

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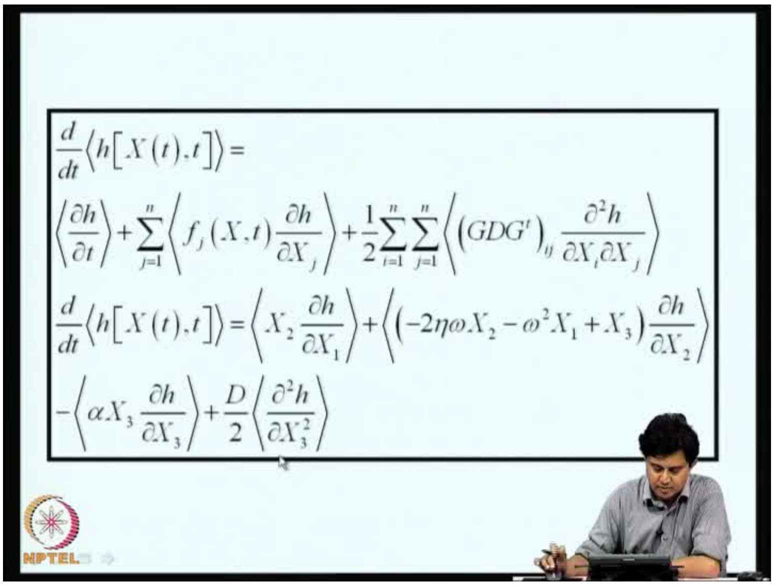

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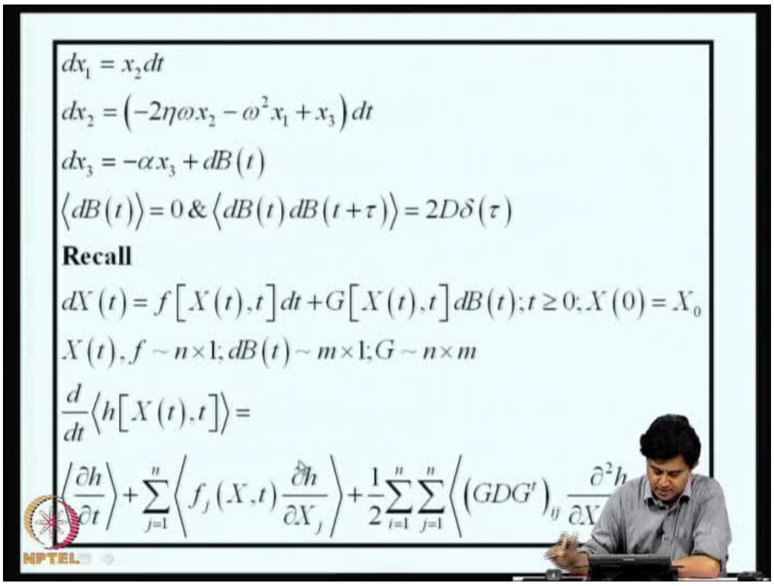

Now, for the particular case, the form of this equation is if we compare this form with standard form, we see that f 1 is x 2, f 2 is minus 2 eta omega x 2 minus omega square x 1 plus x 3, f 3 is minus alpha x 3 and this G into dB t is actually dB t.

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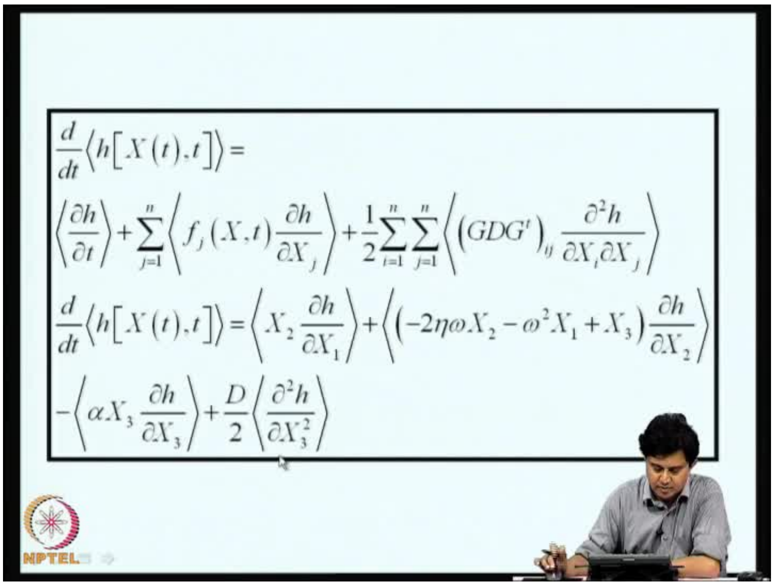

Equipped with that, we can now write the equation for the moment equation where I will actually substitute for f 1, f 2, f 3 and for this GDG transpose ij and if I do this for the system under consideration the moment equation for the expected value of some h of X comma t is given by this

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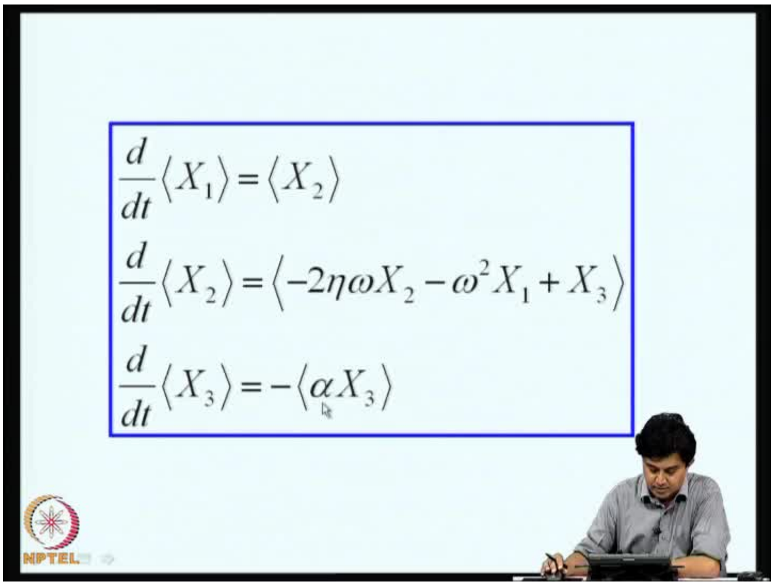

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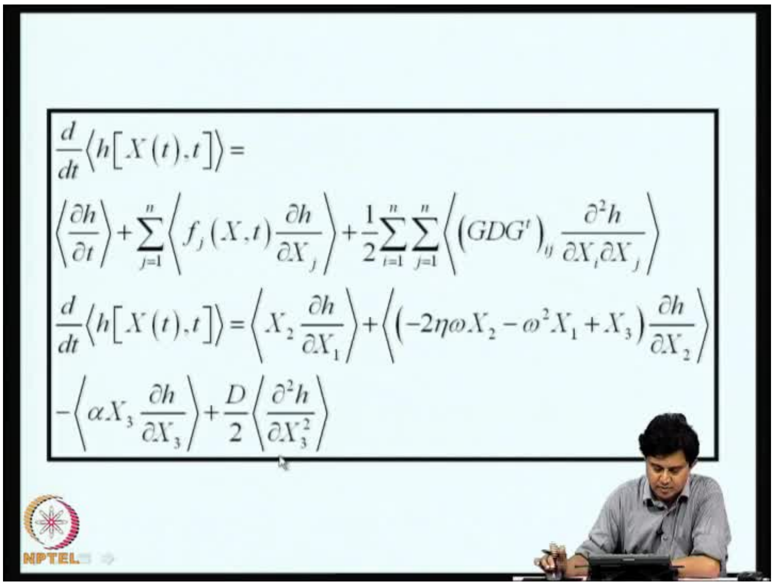

So, if we start by taking h is X 1, expectation of X 1 is if you go back here it is X 2 into dou h by dou X 1 dou h by dou X is 1 it is expectation of X 2. Similarly, if h of X comma t is X 2, I get dou h by dou X 2 is 1. Therefore, I get on the right hand side expectation of minus 2 eta omega X 2 minus omega square X 1 plus X 3.

In the similar manner we can set up the equation for the third element of the state vector. So, these are the equations for the expected values of X 1. You can quickly notice that these equations are closed in themselves. If you want to find out X 1, X 2, X 3, we need to solve only these equations and we can go ahead.

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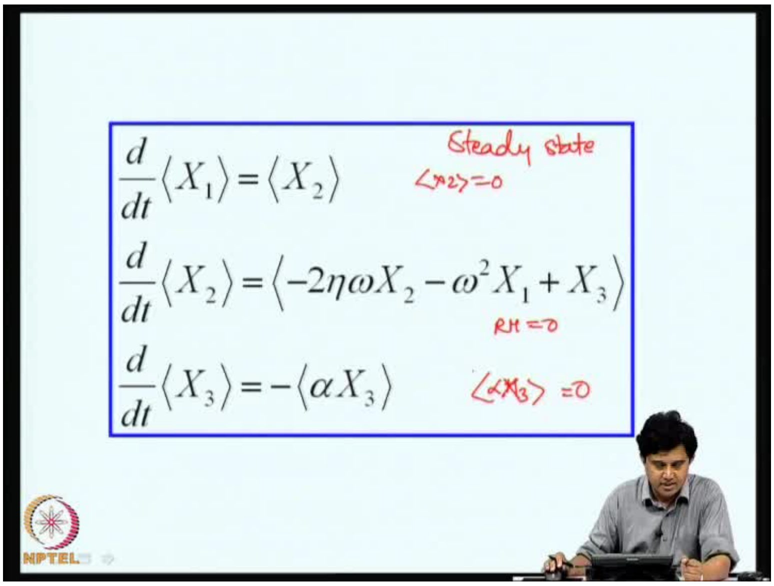

So, if we want to find steady state here in the steady state expected value of X 2 will be 0 and this RHS will be 0 and expected value of X 3 will be 0. So, we can solve these three equations and obtain the steady state values of these moments and you can show that indeed all these three values assuming that initial conditions are deterministic. I mean if vary steady state of course the solutions are independent. It will be possible for us to show that all these three expected values and steady state would be 0.

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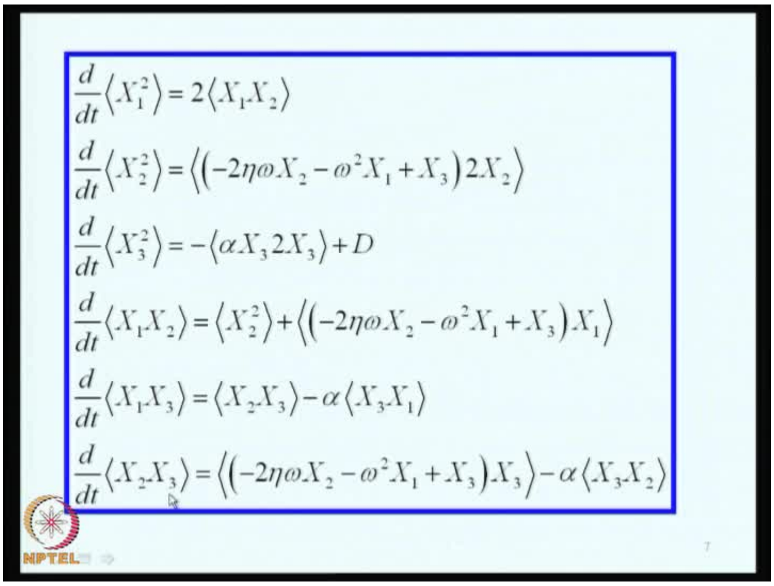

How about the second order moments? So you start with h being X 1 square and X 2 square X 3 square X 1 square X 2 square X 3 square X 1 X 2 X 1 X 3 X 2 X 3 etc

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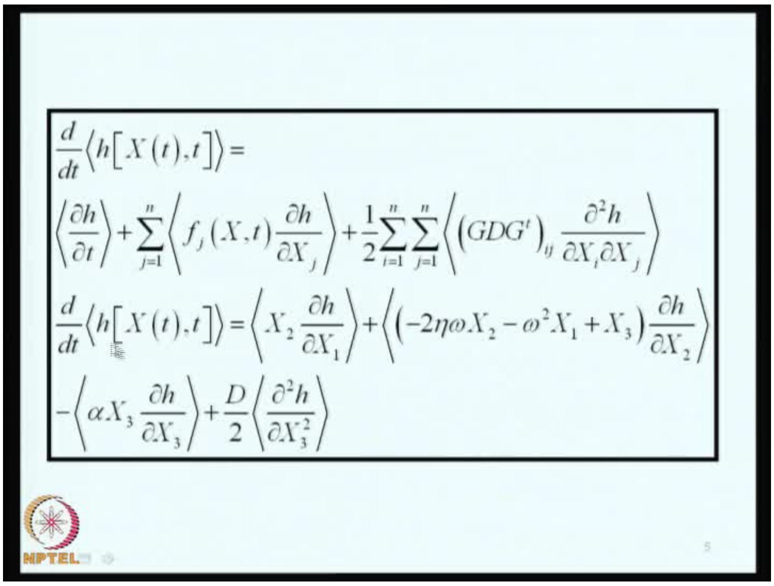

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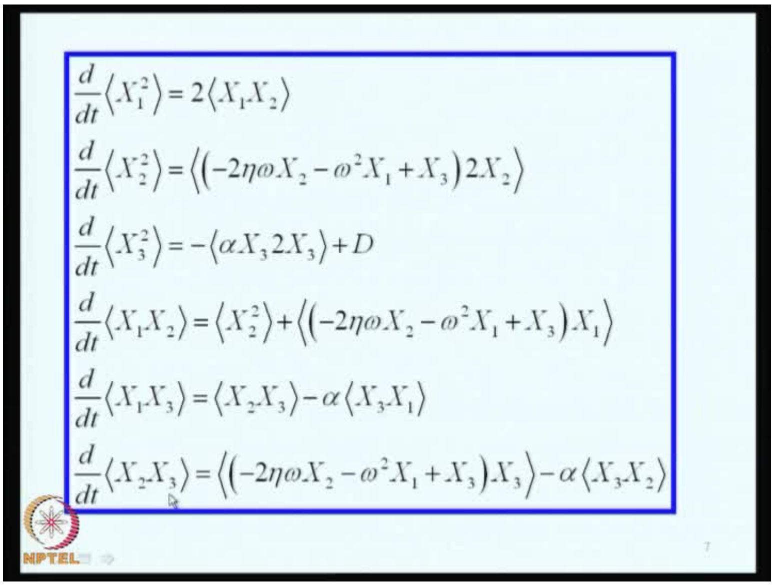

So, we need to go back to the expression for the evolution of h of X comma t according to this equation and substitute.

So, we will be able to derive these equations.

Again these equations involve only moments of the second order. They can be considered separately. So, this also can be solved. We can use something like Runge-Kutta method or some other predictor corrector method to solve these equations if you are interested in time evolution.

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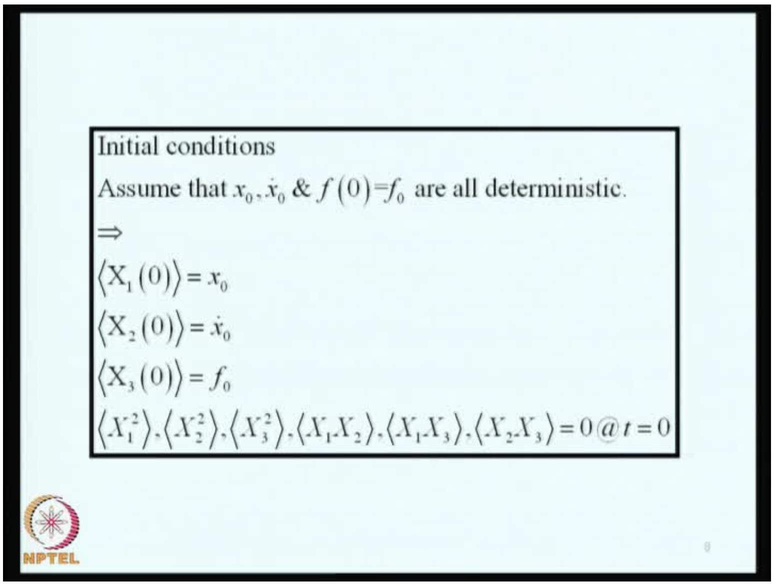

If you are interested in only steady state, if I assume that in the initial conditions on these equations have to be specified, first, let us address that issue. If you assume that x naught x naught dot and f of 0 are all deterministic, then it turns out that the initial conditions for the expected values will be corresponding to this respective initial conditions and then all the second order moments will be 0 at t equal to 0.

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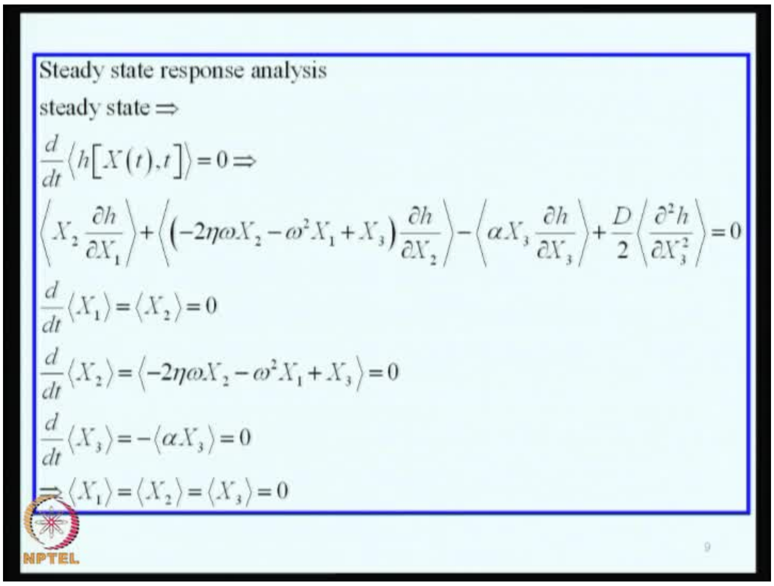

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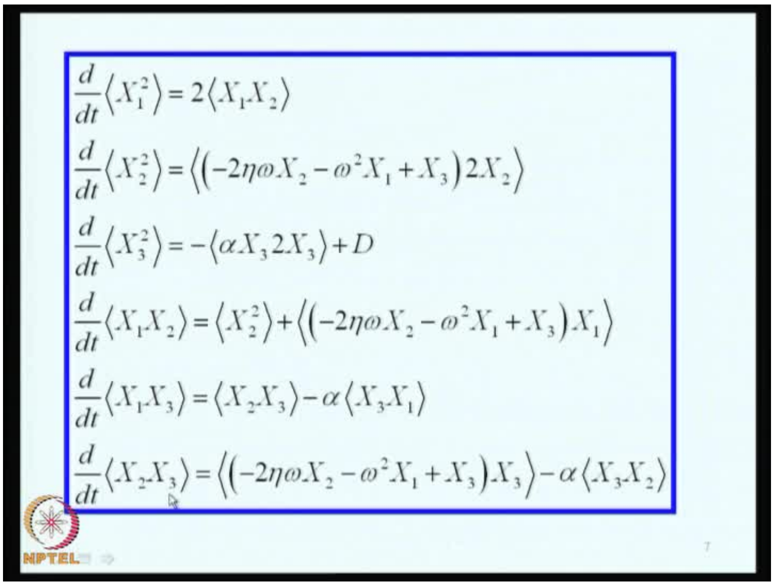

Now, for the steady state response analysis, we put d by dt expected value of h of X comma t to be 0 and this equation which is independent of t now is a equation for steady state response moments. We can write down those equations as I discussed already and we can write all these equations where right hand sides are around 0.

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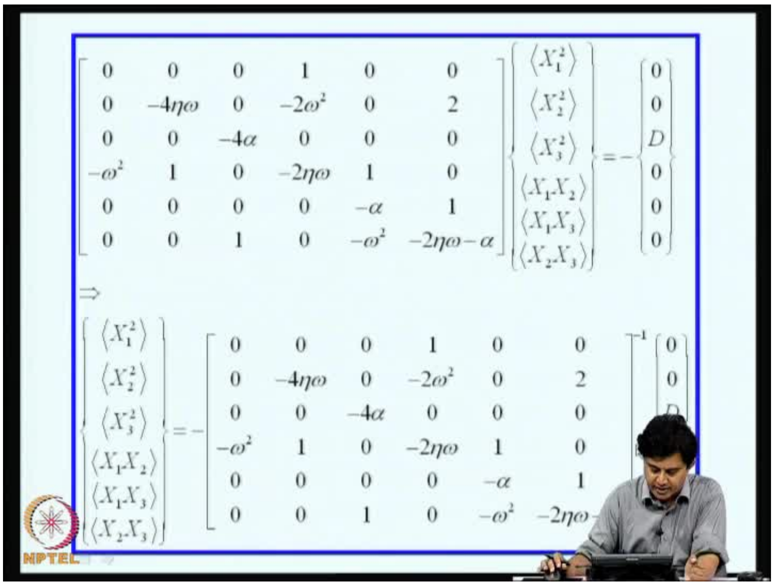

And we can cross this in a metrics form. First, this is the equations for second order moments and we can show that by inverting this. You can get the required values of a steady state response moments mean I have already shown there all 0.

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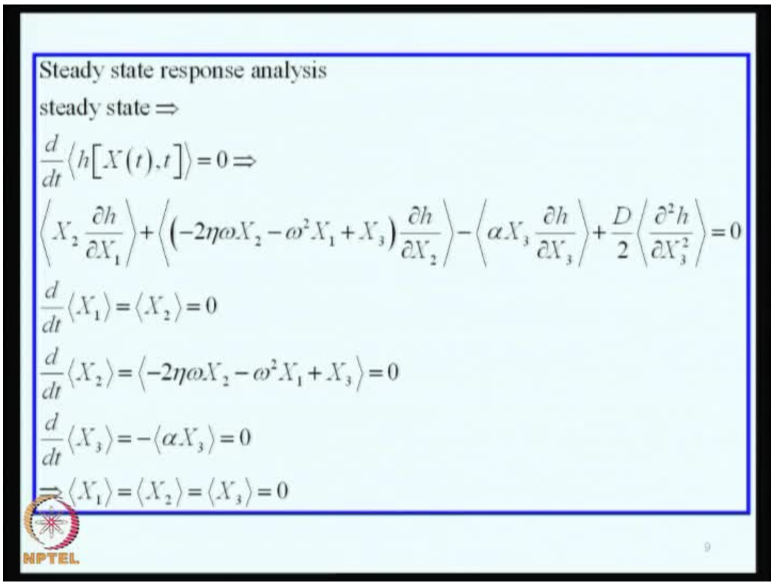

So, the analysis on mean can be done independent of analysis of second order moments. This is the typical property of linear time in variant system driven by random excitations. There is no problem of closing the moments equations for moments are always closed. So they are straight forward to handling.

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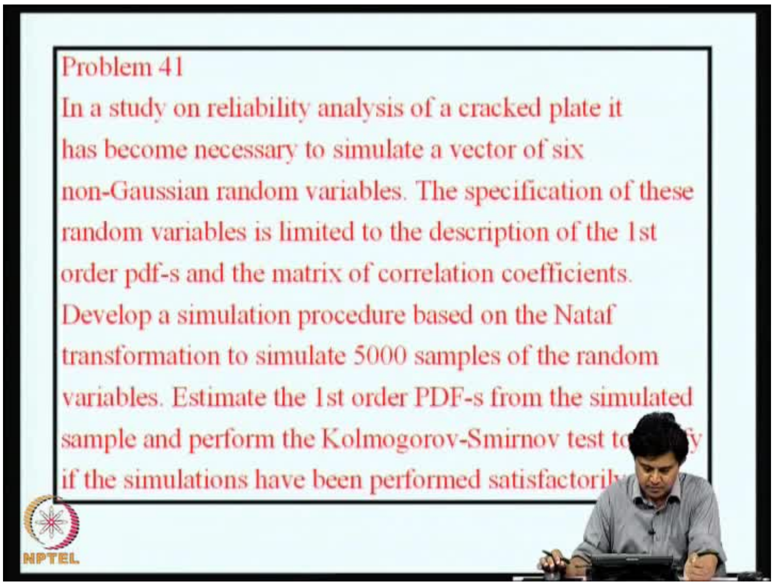

Now, this is the problem on simulation of random variables. We consider a problem in reliability analysis of a cracked plate. We consider a situation where we are asked to simulate a vector of 6 non Gaussian random variables. The specification of these random variables is limited to the description of first order probability density functions and the matrix of correlation coefficients.

Now, the problem is to develop a simulation procedure based on the Nataf transformation to simulate 5000 samples of the random variables. Estimate the first order probability distribution function from the simulated sample and perform the Kolmogorov-Smirnov test verify if the simulations have been performed satisfactorily.

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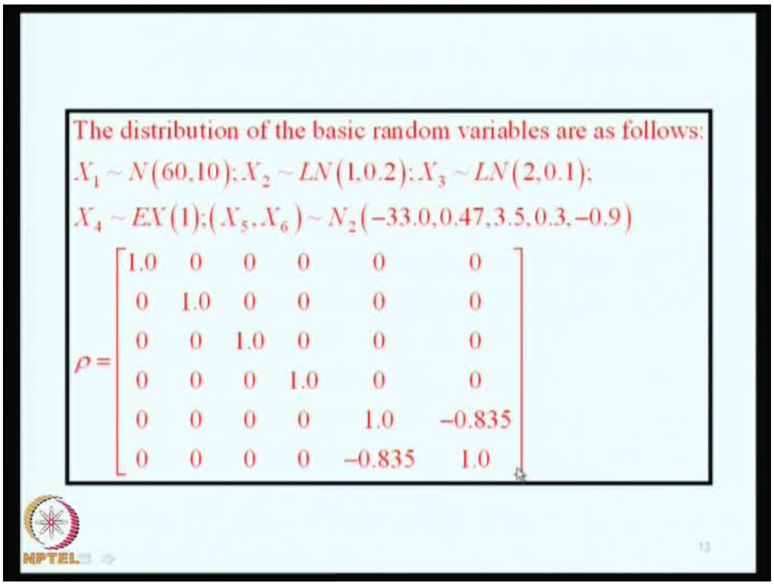

So, this is the problem. the specification of this 5, 6 random variables is X 1 is normal. X 2 is log normal. X 3 is log normal. X 4 is exponential X 5 and X 6 are jointly normal with mean the first two quantities is mean sigma 1 mean sigma 2 and correlation coefficient.

So, the correlation coefficient matrix except for these entries in these rows and columns is diagonal, but rho is still a non-diagonal matrix and also we are complete specification of 6 dimensional non Gaussian random variables involves specification of 6th order join density function. So, you must understand that the problem is now to stimulate samples from this partially specified description of these 6 random variables.

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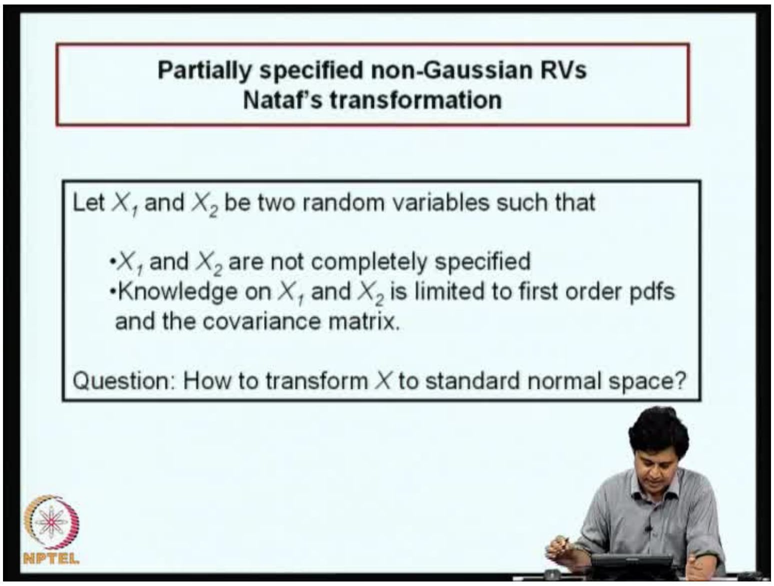

So, if you quickly recall,, this is the reference to the text where we describe the required mathematical tools that is how to simulate non Gaussian random variables using Nataf's transformation tools.

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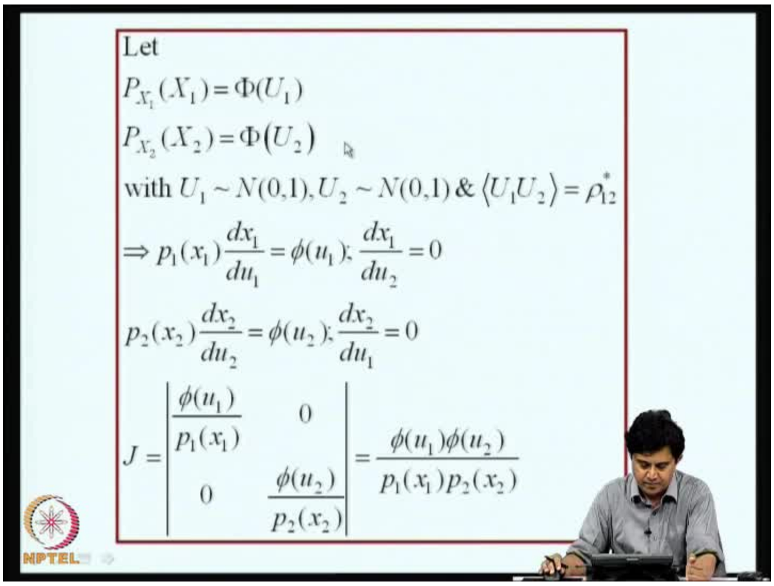

, We have discussed so essentially we introduced two normal random variables. For example, if you are interested in simulating two random variables which are non-Gaussian and which are partially specified through this the these two transformation we introduce 2 Gaussian random variables with an unknown correlation.

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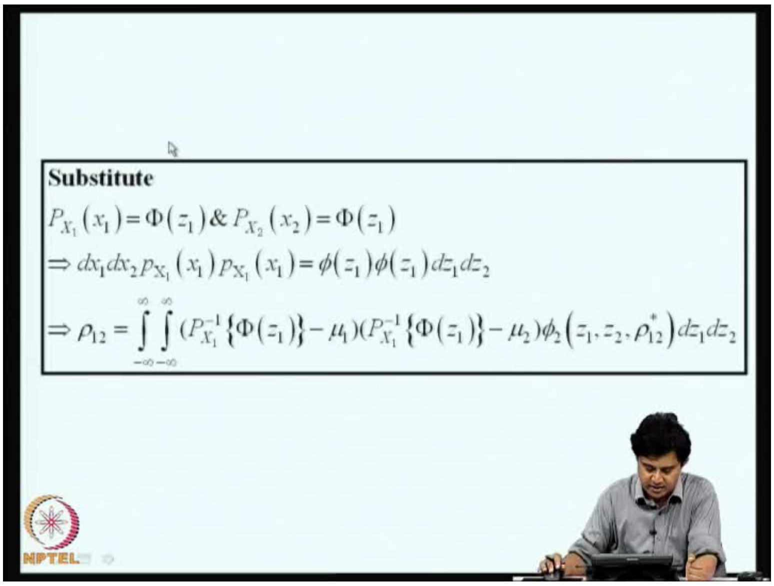

And we will calibrate these correlations to match with the correlation of X 1 and X 2 and as we have seen, we need to solve this integral equation to do that and I have already explained how to handle this computationally.

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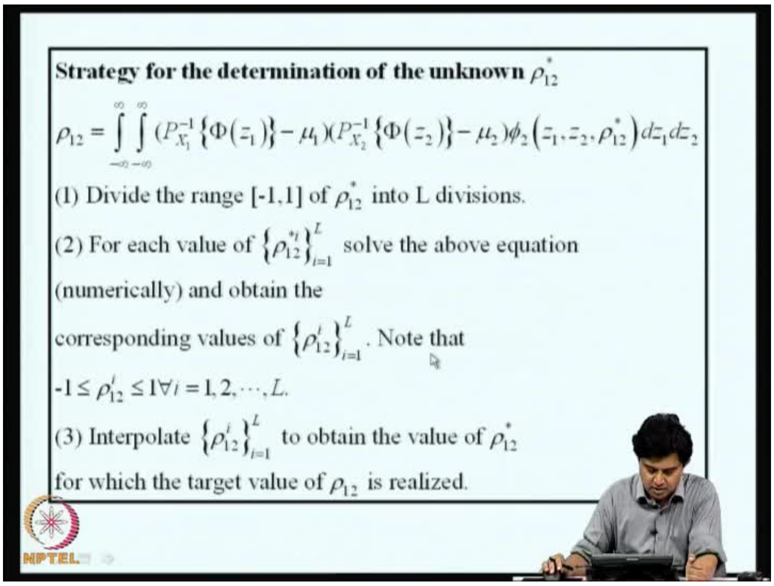

So, the strategy for the determination of unknown correlation coefficient between those two hypothetical Gaussian random variables has been explained. The essential idea here is that the unknown resides inside the integrant and what left hand side is known so what we can start by doing is we know that this rho 12 and rho 12 to star are bounded between minus 1 and plus 1. we can solve this equation for specified values of rho 12 star at certain intervals between minus 1 and plus 1 and get a idea of behavior of rho 12 and from that we can estimate for the required value of rho 12 what should be rho 12 star.

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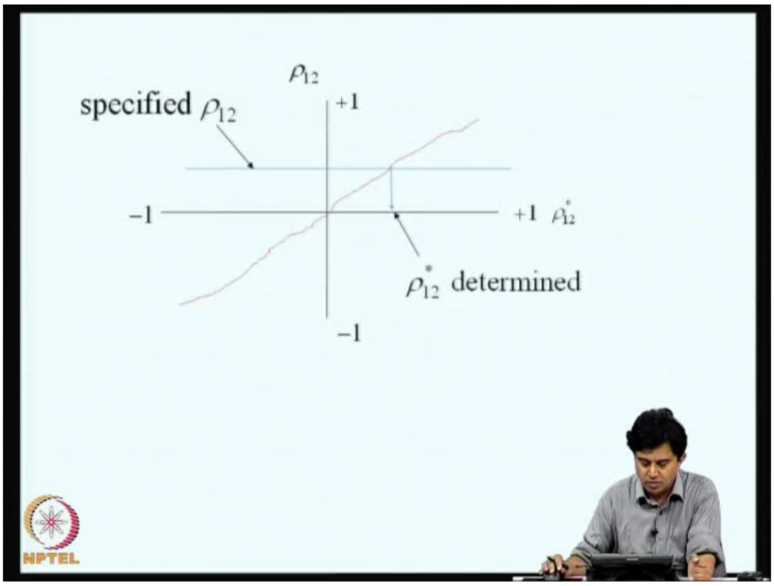

This requires development of a suitable computer program. This is something that cannot be handled on a pen and paper mode. You need to write a computer program to do this and to simulate the required random numbers.

If you are writing your own codes, you should start with simulating uniformly distributed random numbers, apply suitable transformation (( )) or certain transformations and generate the required random numbers.

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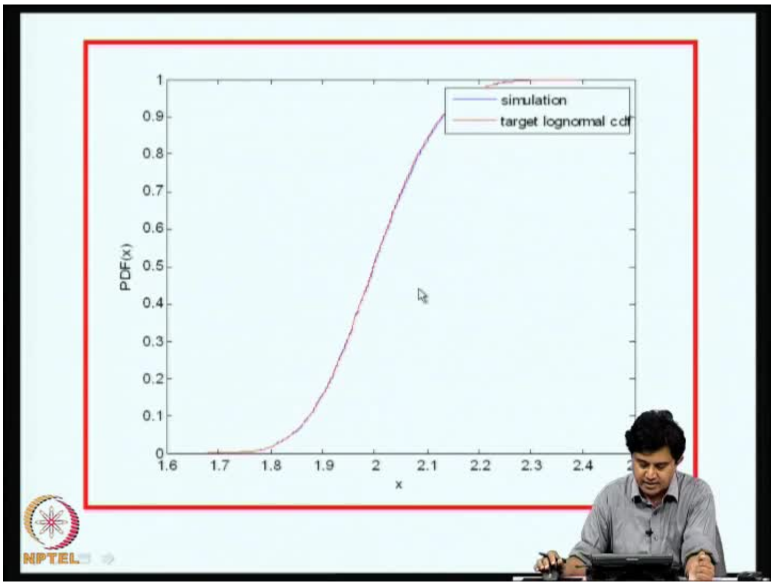

So, such an exercise has indeed been carried out. I am showing some selected results. This is results on the lognormal random variable with a 5000 samples. Blue line is a simulation and red line is a target lognormal cumulative distribution function and they seem to agree quite well. We need to perform the Kolmogorov (( )) test on these two curves using data in these two curves and verify whether we can accept the hypothesis that the data originates from a population whose probability distribution is indeed the target lognormal probability distribution function.

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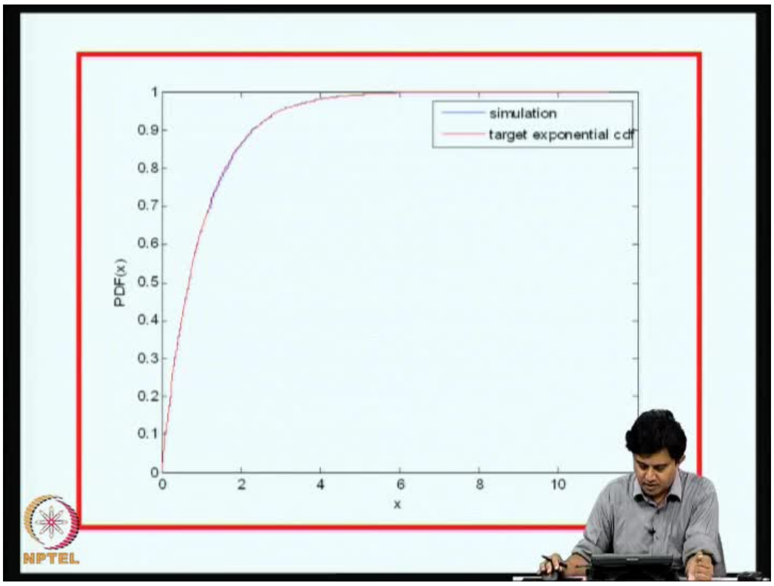

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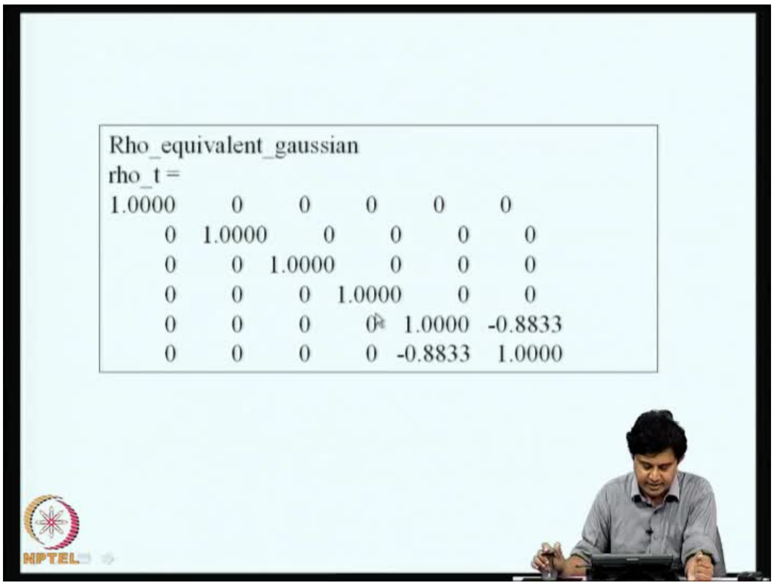

This a similar exercise an exponential distribution. Again to a first look, the simulation looks alright, we can verify this through a proper statistical test.

This is an intermediate data which you may find useful if you would like to reproduce these results where this is the correlation coefficient matrix for the equivalent Gaussian random variables which we need to transform using Nataf's method and this is an intermediate result that you could verify when you implement this procedure.

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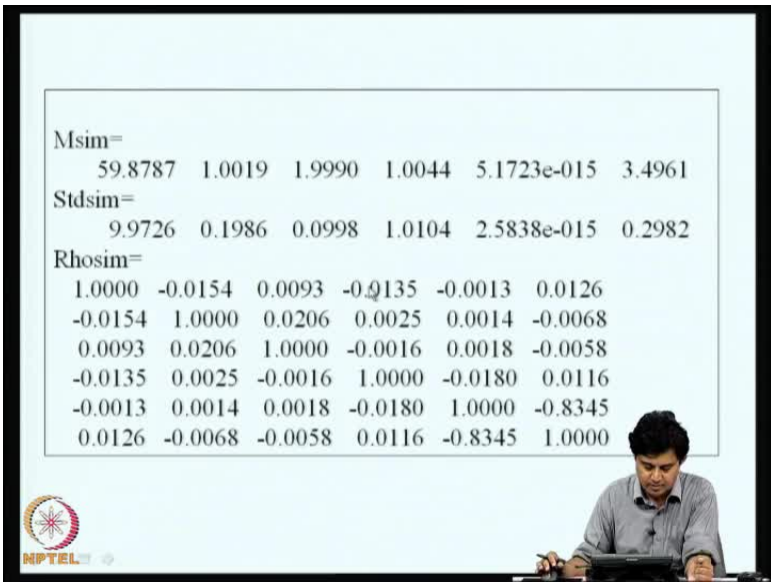

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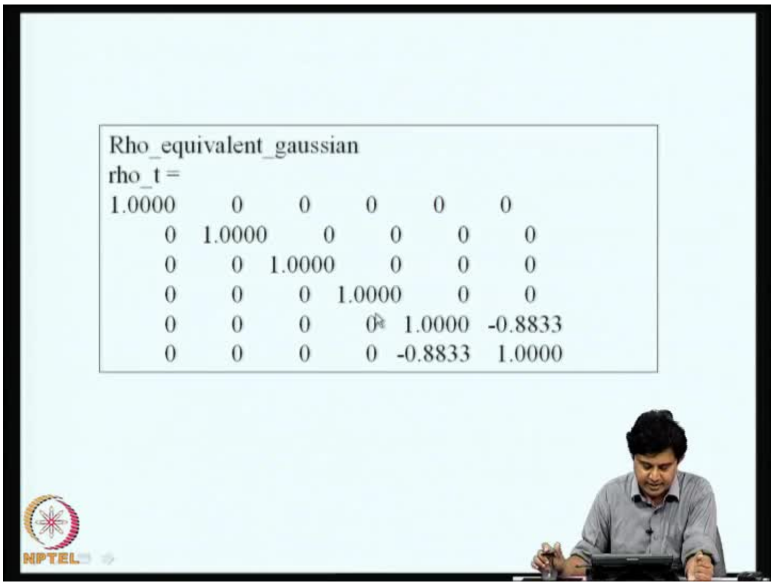

In the actual simulation that was performed in this exercise, the simulated mean vector is as shown here and standard deviations are here and the simulated correlation coefficient matrix it should be a here all these entries should be 0 because these are not strictly 0 because of sampling fluctuations and along the diagonal of course they are 1 and off diagonal here the target value is 0.835 and what has been realized is something pretty much close to that this is equivalent Gaussian and this is 8345 instead of 835 is what we are getting through simulation.

This partial set of results should help you to check if you are doing your calculations right especially this intermediate step of finding equivalent row for the Gaussian random variables.

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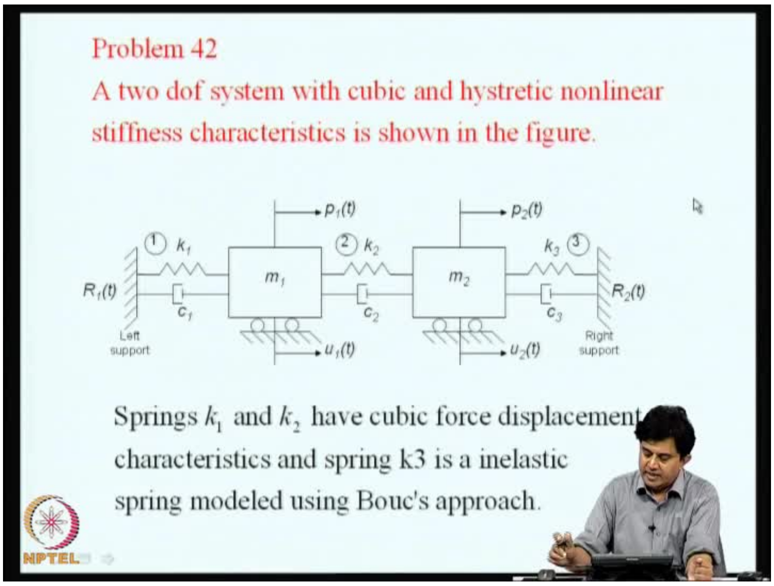

We next consider a fairly complicated problem. This is a problem of a 2 degree freedom system which has both cubic and hysteretic non-linear stiffness characteristics. The springs k 1 and k 2 here have cubic force displacement characteristics. k 3 is an inelastic spring. It has hereditary non-linear characteristics and this is modeled using Bouc's method.

p 1 and p 2 are random excitations and this R 1 and R 2 are the reactions. So, the problem is to formulate the equations of motion and recause the equation of motion into a stochastic differential equation and then numerically simulate samples of response of the system using 1.5 order Taylor's scheme based on theory of stochastic differential equations. I will provide you the intermediate steps this again is an exercise that can only be done through a computer program and you need to develop the program to be able to solve this problem.

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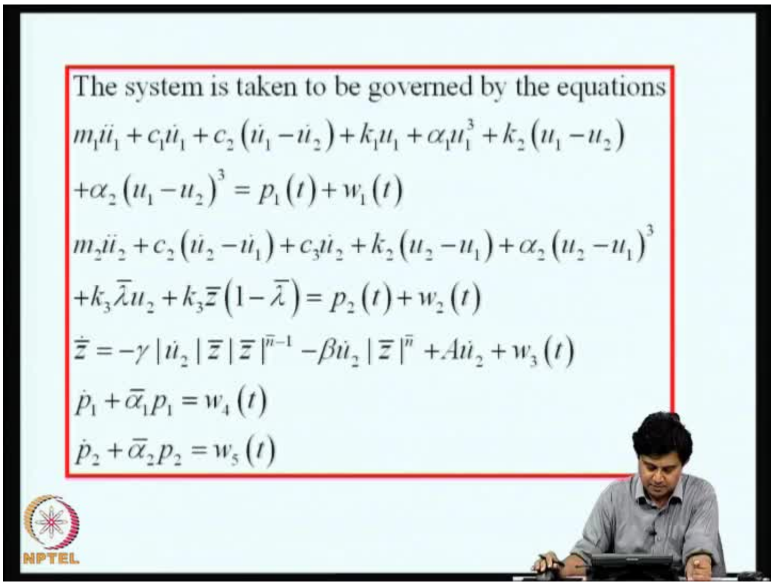

The governing equation here for u 1 and u 2 can be written here as you can see k 1. This k 1 u 1 plus alpha 1 u 1 cube is the cubic stiffness and k 2 is again cubic stiffness, but k 3 the term corresponding to k 3, we introduced an internal variable z bar which is taken to be governed by this equations this is the Bouc's model for yesterday take hysteretic hereditary nonlinearity.

T p 1 and p 2 here are taken to be filtered white noise processes and this we are also adding certain white noise is w 1 w 2 w 3 to these three equations. These three processes can be viewed as modeling errors in developing this governing equations if we have a physical system in mind for which this is an decision idealization. There will be modeling errors and we are notionally representing through these three white noise processes.

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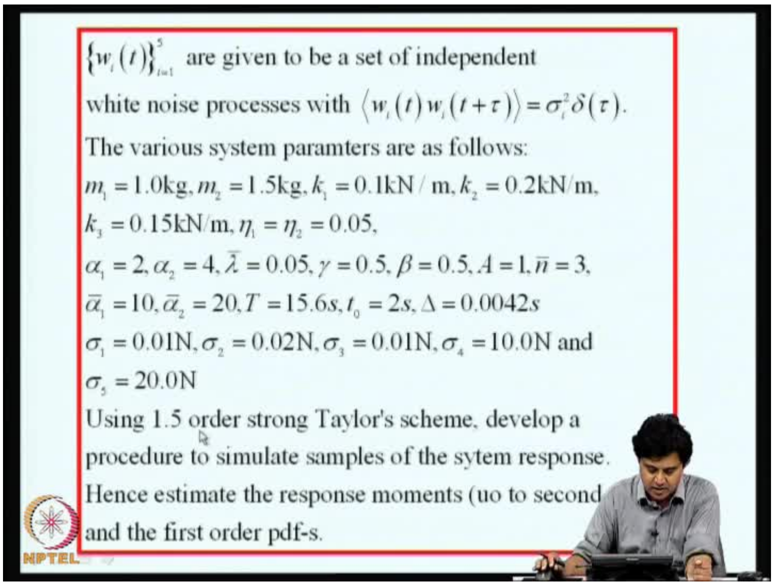

In this problem, there are five white noise processes. We assume that they all independent and therefore we have to recause this into the state space form and there are some details of the system (( )) parameters here numerics which you will need when you proceed with solving this problem.

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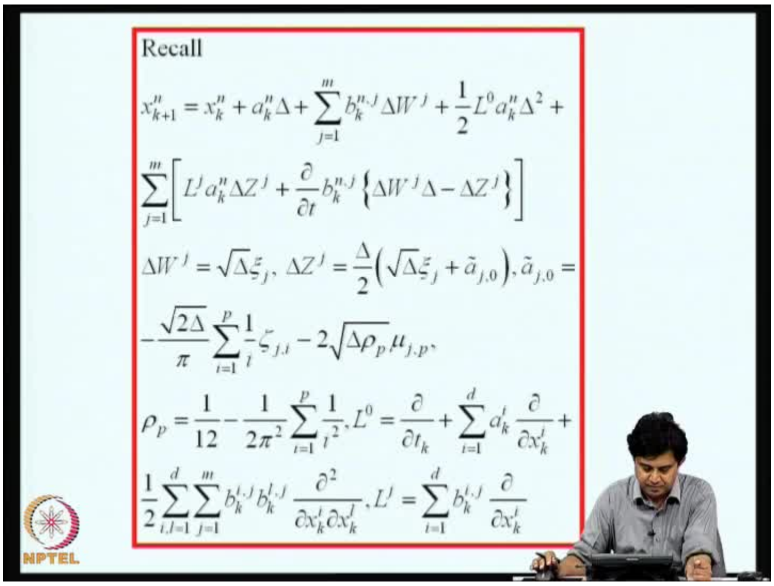

This is thus in a numerical scheme, that is this 1.5 orders strong Taylor's scheme is this I had provided these details earlier. So, you have to implement this on the given problem.

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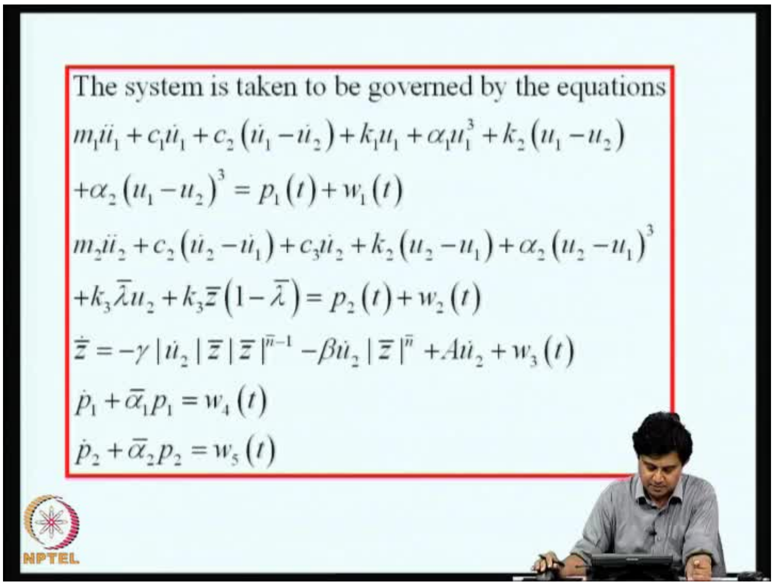

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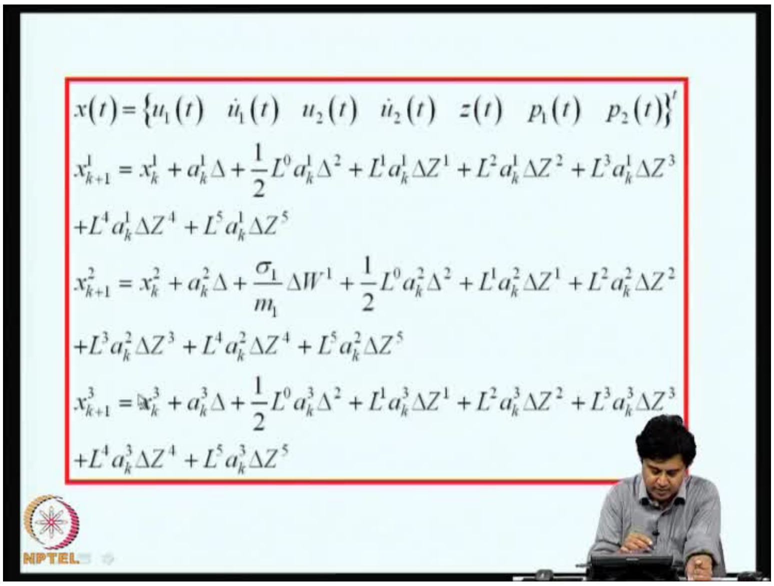

If we do this, you will see that here the number the state space equation will have size 2 plus 2 4, plus 1 5, plus 2 7, so it is a state space with 7 elements. So, you need to formulate the problem and once you do that correctly and use this scheme I have given you the details of the discrete map that you will get when you implement this discretization scheme.

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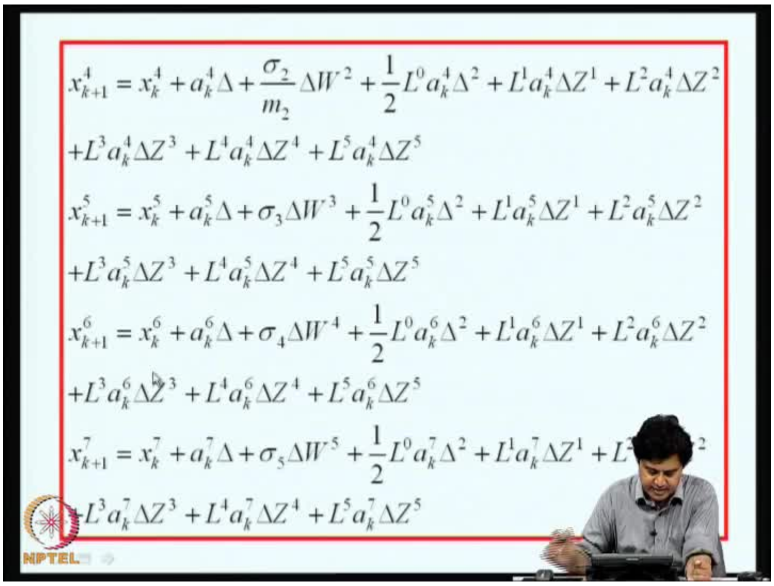

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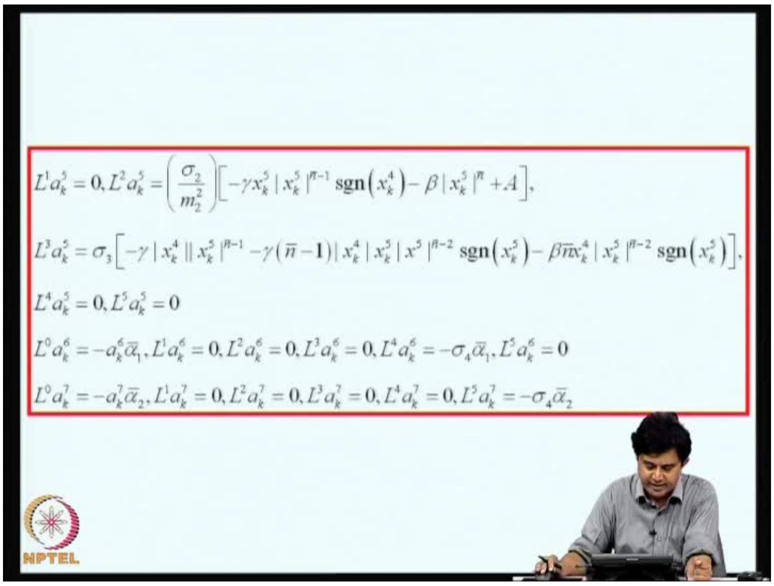

T You can use these details to verify if you are progressing correctly according to the proposed scheme. So, these are all details that you would need when you want to solve this tedious, but conceptually not very difficult to implement.

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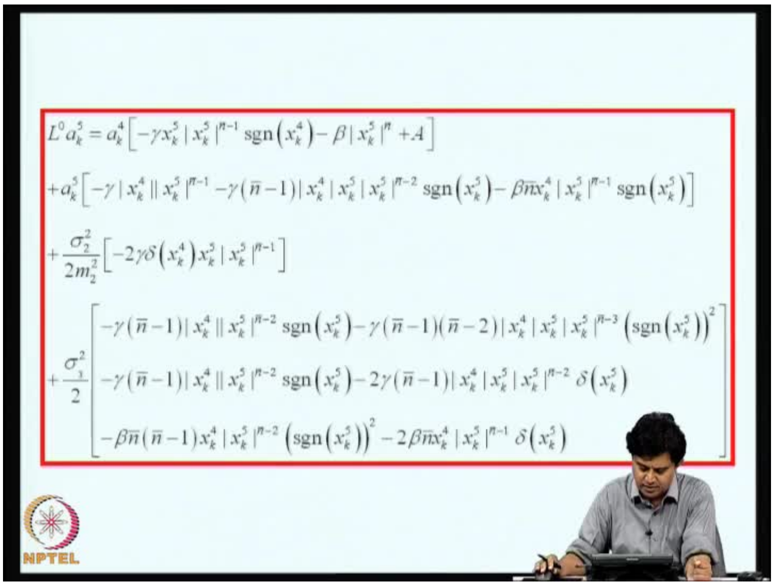

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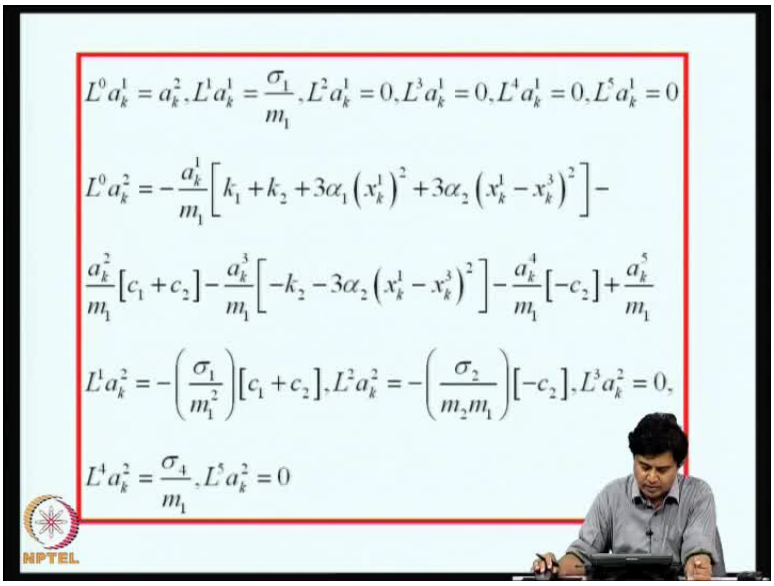

you will see that here we have signum functions and some places we have dirac delta functions on the right hand side. It is not possible to model events that are captured through dirac delta function using the discretization scheme that we are using. One possible approach would be to replace dirac delta function by suitable continuous approximations.

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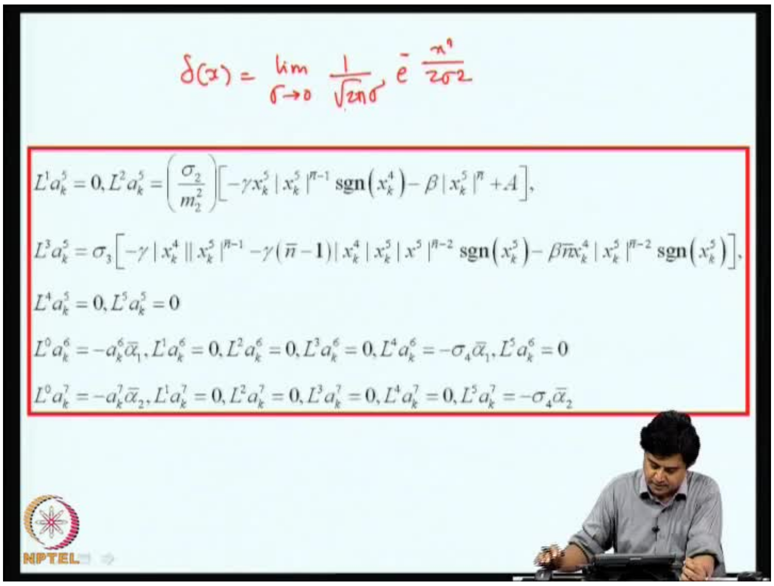

For example, we could replace dirac delta function by a Gaussian density function whose standard deviation goes to 0 and we can use a suitable value for sigma in the numerical calculations and handle this direct delta functions, but we could as well ignore the presence of this directed delta functions in the actual simulations, but you can account for that through this approximate method.

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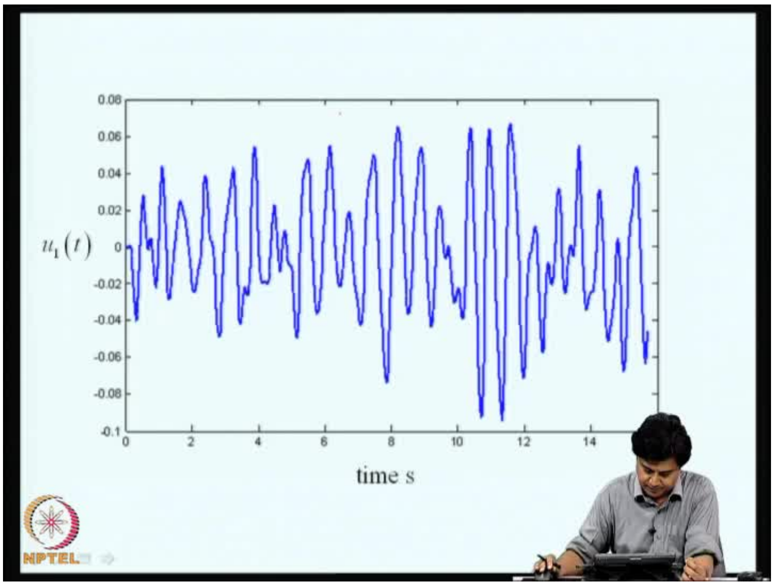

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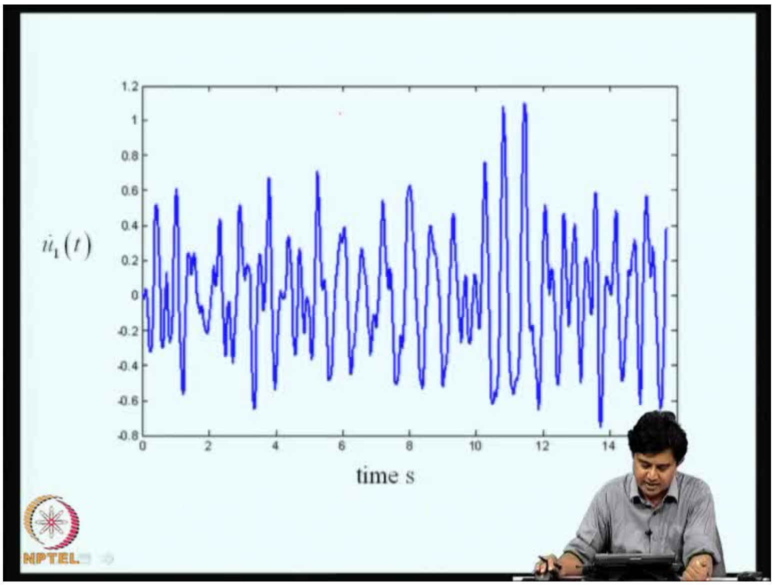

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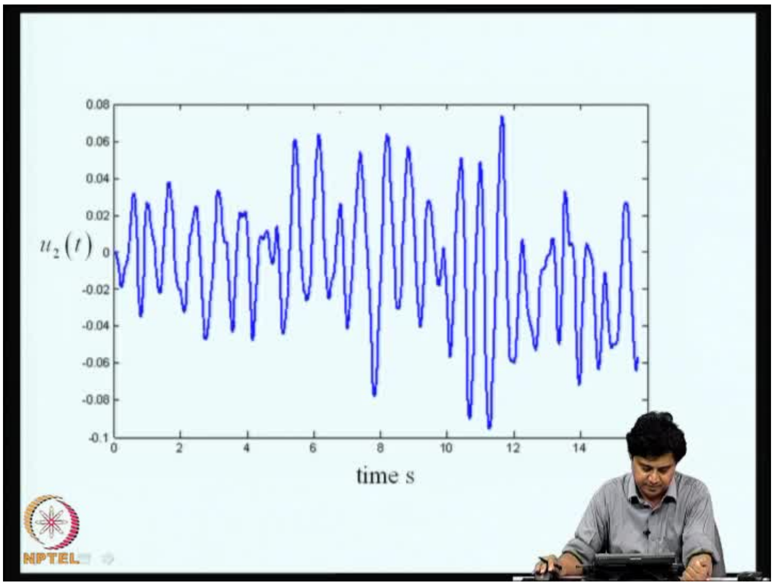

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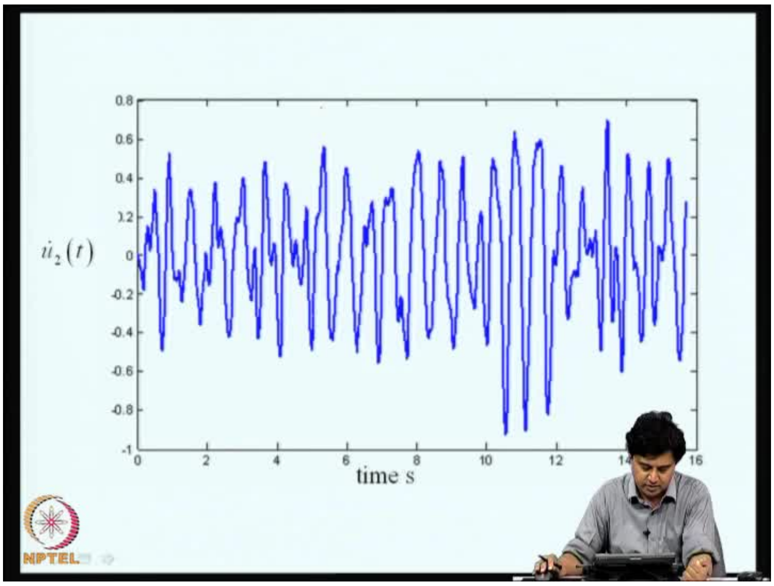

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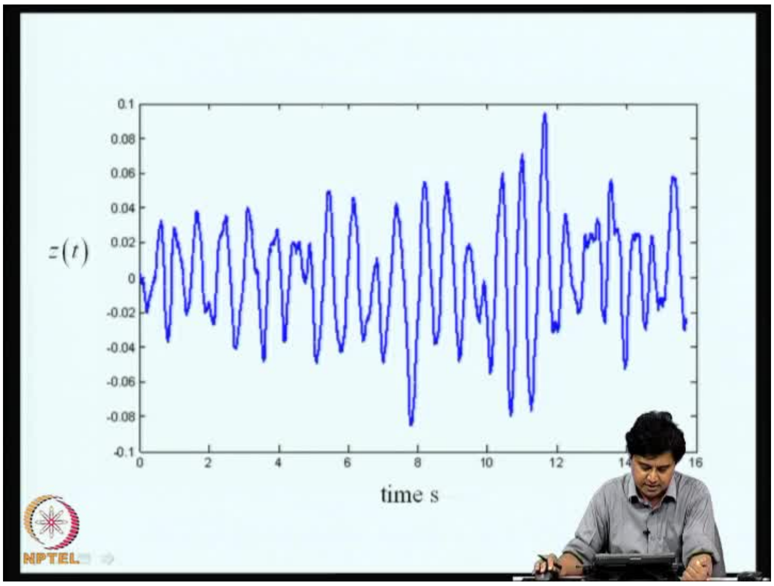

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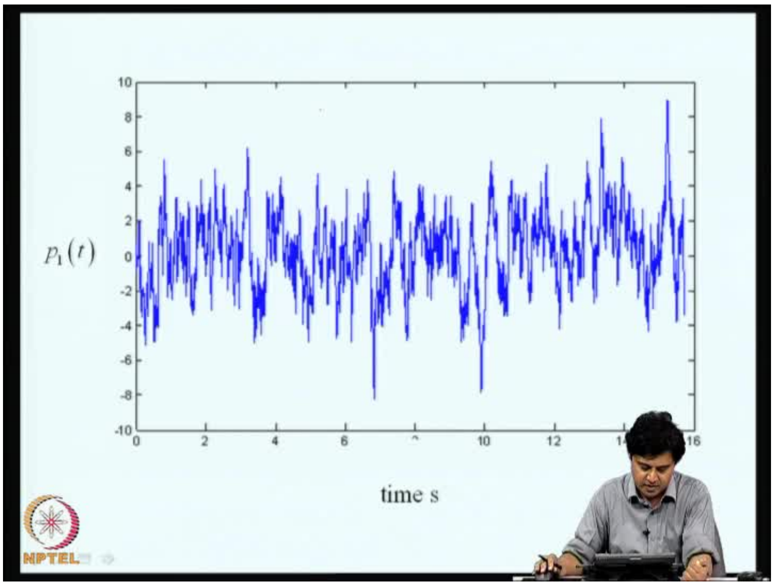

I show some trajectories. These are random samples of random processes. So, when u x implement this solution, you will not exactly get this because of the variations in sampling. So, this approximately provides you an idea how this samples look like.

This is u 1, u 1 dot, u 2, u 2 dot and this is the internal variable z of t s is the sample of p 1 of t and this helps you to see how far your answer should match with. We can check if these are broadly so this is a fairly long exercise that you need to carefully implement and check.

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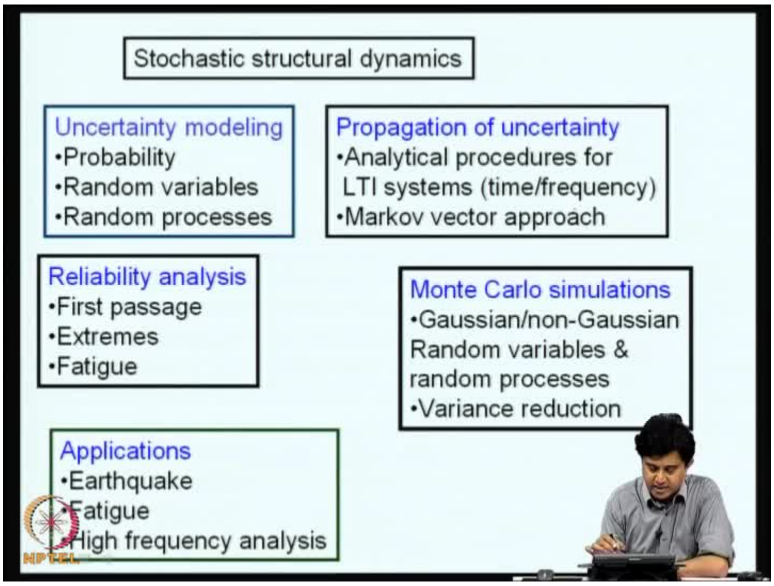

Now, at this conjecture we could probably take a global view of what we have beendoing in this course. This course has been on stochastic structural dynamics. The essence of the subject was, we will model the uncertainties that are in the loads and system properties etc using theory of probability random variables and random processes

and consequently several problems we need to analyze. One is propagation of uncertainty that means if there are uncertainties in system properties and excitations how to characterize the corresponding uncertainties in the system response.

Here, we developed analytical procedures for linear time and variance systems both in time and frequency domains essentially using principle of supper position that is green's function transfer function impulse response function and that type of mathematical tools.

We also develop at another parallel set of tools which are applicable to systems which are driven by white noise or filtered white noise where the response vector can be modeled as a Markov vector and consequently several mathematical tools which are based on theory of Markov processes become applicable to solve the problem.

Thus, we can study the time evaluation of transition probability density function of the response vector using Fokker Planck equation we can solve problems in reliability by using backward Kolmogorov equation.

We can set up time evaluation of response moments equations for that we can set up equations for moments of first persist time so several tools become available when the problem can be modeled using Markov vector approach.

A class of problems are amenable for exact solutions using Markov vector approach. So, Markov vector approach is a source of exact solutions in stochastic structural dynamics. So, it has its own value. Also, it enables us to develop several approximate schemes like a closure approximation schemes and certain other numerical procedures which essentially take off from a Markov vector model for the dynamical system.

We considered response moments like mean auto covariance power spectral density functions etc., We also considered several indices of system performance like level crossing problem, first persist problems, peaks envelopes phase extreme values fractional occupation time etc., and we developed suitable descriptors for response random processes where these quantities where suitably characterized.

For most of these problems, an exact solution was not possible. So, we introduced certain heuristic arguments and develop engineering solutions to the problem on hand.

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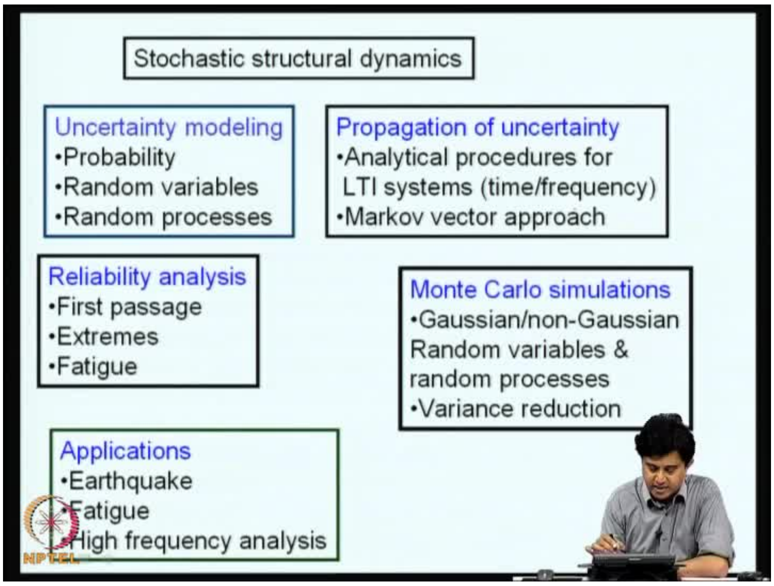

We also considered failures due to first excursion failure, where response crosses a safe limit for the first time. Then, we also considered the highest response in a given duration that also helps to solve the problem of time variant reliability where we would like to find out for a given duration whether response as state within the safe limit or not.

We briefly touched upon failure due to accumulation of damage due to fatigue. We used (( )) minor hypothesis and also very briefly talked about fraction mechanics based approach to treat these problems.

The solution strategy as said involved analytical procedures, but also we spend considerable time developing Monte Carlo simulation tools and we were able to develop procedures to simulate Gaussian or non-Gaussian vector random variables and random processes and completely specified random processes partially specified random processes and so on and so forth.

And we also develop response analysis procedures for simulating samples of responses. We represented in one class of procedures the random processes as mean square periodic. We assume that random processes are mean square periodic and use Fourier representations for samples. In the other approach, we used the Ito Taylor's expansion and discretized numerically the governing stochastic differential equations. I also talked about other alternative represent series representations like (( )) expansion for simulating samples of random processes.

We addressed an important class of problems known as problems of variance reduction that helps us to reduce the variance sampling variance in Monte Carlo simulations without increasing sample size that would typically involve adaptively learning how the system behaves with few simulations and then using that knowledge in finding suitable spaces where we can sample and evaluate quantities of interest.

We considered applications a few majority of the application that we considered where in the area of earthquake engineering and also some applications on fatigue failure was also considered and a brief reference to statistical energy analysis that is a framework for studying high frequency vibrations was also discussed.

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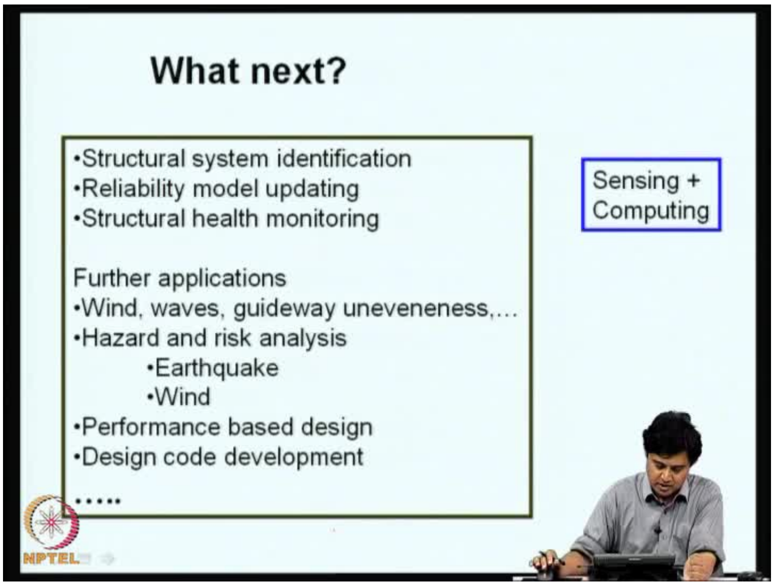

Now, the question that we can ask is what next? What are the other with this preparation what we can do and where do we stand in the current state of knowledge in this broad area of research. Now, a few comments I would like to make in the remaining part of this lecture on this issue.

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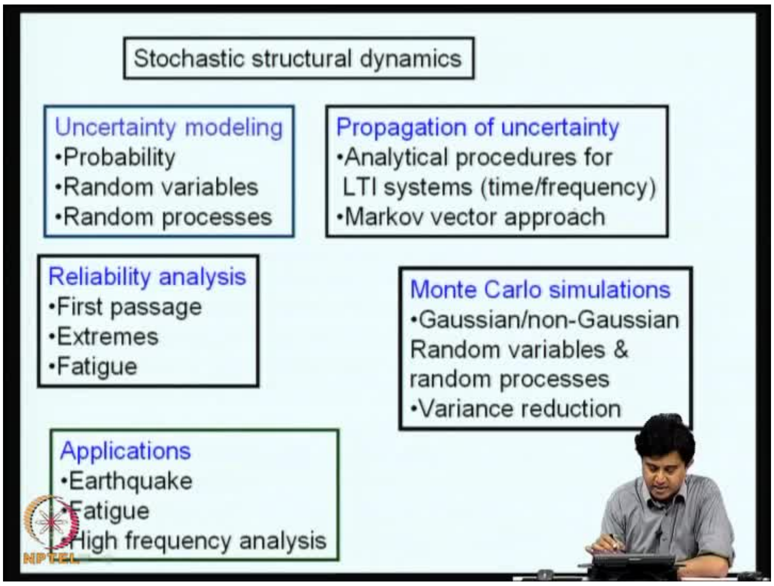

The questions that we have considered so far have dealt with so call forward problems in structural engineering were inputs were specified systems were specified and questions were asked on characterizing the response of the system.

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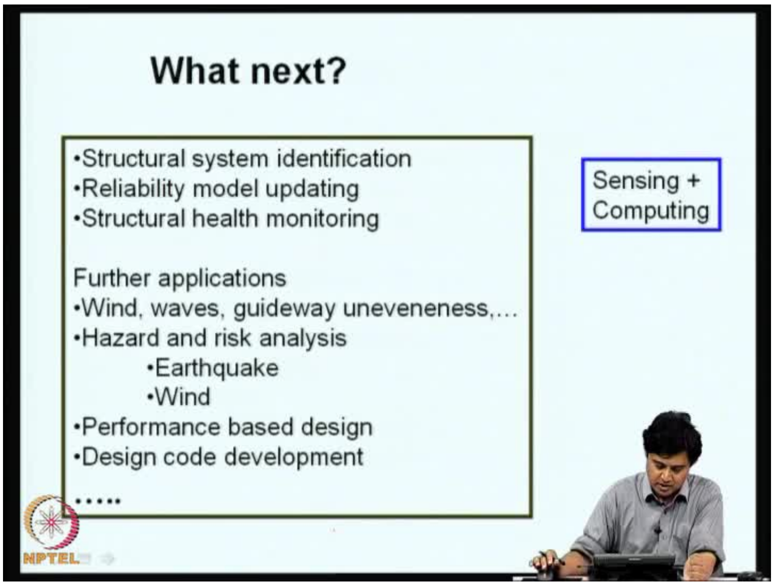

But in modern engineering, there are other concerns like for instance questions on structural system identification has have become important in recent years because of development in sensing and computing technologies now we have instrumented buildings which measure the actions on the structure as well as the corresponding structural responses and we would like to know the condition of the structure based on these measured actions and responses and that subject belongs to study area of structural system identification.

This subject is in the area of structural system identification. We essentially study existing structures an existing structure can be analyzed using mathematical models and also using experimental tools.

So, the prediction from experimental model and mathematical tools often do not agree because of various idealizations for example made in mathematical modeling pertaining to boundary conditions, flexibility of joints constitutive loss damping models etc.,

we in typically mathematically modeling, we make simplifying idealizations in treating these aspects, but in an experimental word, these aspect like boundary condition joint flexibility constitutive laws etc., or depicted correctly there is no idealization there.

So consequently, the experimental measurement that we make becomes useful tool to update the mathematical models. These are updating of mathematical model could be with reference to system parameters such as boundary conditions, stiffness, damping properties, inertial properties etc., It could also be with reference to reliability models. For example, if we have predicted reliability of a structure to be certain number, if we make measurements and understand more about the structure how can that information we assimilated to obtain an updated reliability model. This again is an important question that is being considered in recent years.

An area of engineering known as structural health monitoring is gaining importance. Here, based on our ability to measure the response of the structure in during its operation we addressed questions on accessing for example whether the structure is damaged? Where is the damage? What is the quantum of damage? What is the residual strength or residual life and these types of questions are being addressed in research as well and the subject was stochastic structure dynamics forms a important foundation for studying these subjects.

Apart from for possible Fourier’s in to these areas, there are other issues that we could build up upon. For example, most of the applications that we considered in this course have been on problems of earthquake engineering. Similar studies on wind induced vibrations, offshore structures, under wave loads and guide way on events, auto mobiles taxing on a rough roads or aircraft's taxing on uneven run ways etc., could be studied. The same tools in mathematical tools that have develop become widely applicable here also except that we need to now make suitable models for these actions and interface with them with the suitable mathematical tools.

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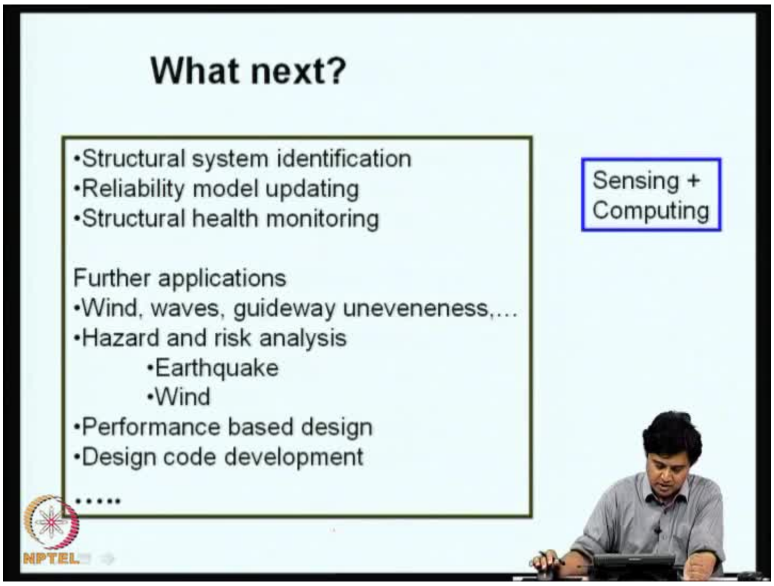

Another area where we have not discussed paid much attention during this course was questions on hazard and risk analysis. So, in all our applications on earthquake engineering, we assume that the earthquake event has already occurred on there is ground motions to which the structure is subjected. But, there is quite a bit of long term in certainty about the very possibility of occurrence of a earthquake event in at a given location during a specified future time interval.

So, if these uncertainties are also model, then we can talk about the seismic hazard and the seismic risk analysis of engineering structures here again the subject of stochastic structural dynamics forms an important component. Studies that I describe for earthquake could also be extend to problems in wind engineering we can talk about hazard and risk analysis of wind load is structures again the tools that we developed in this course become applicable.

We have also not discussed issue related to design. Design is an important engineering activity and the treatment of uncertainty here is again based on probabilistic methods and there are areas known as performance based design where these tools are primarily important.

The traditional structural design code development also needs probabilistic background in calibrating the various partial safety factors. I briefly mentioned that relations between factor of safety and probability of failure. But that line of thinking is to be developed considerably and this discussion takes as into methods of reliability analysis code calibration etc., Some of these areas could be studied based on what we have learnt in this course.

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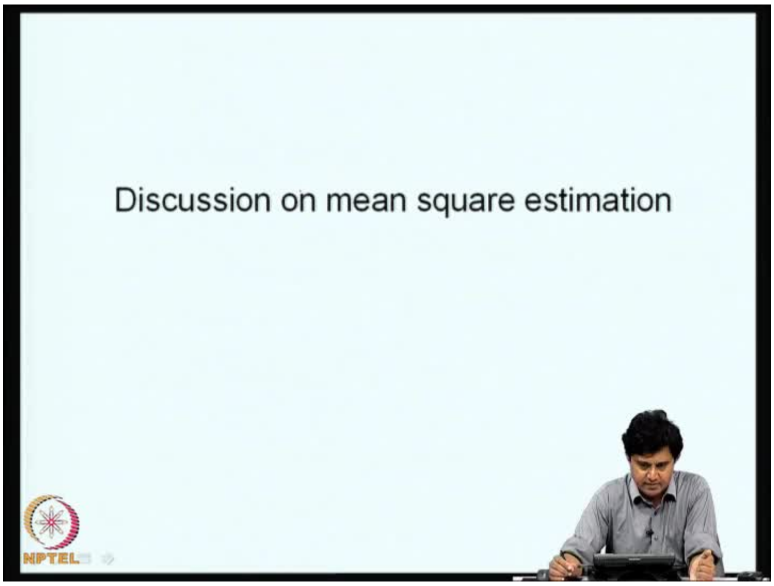

I will briefly now touch upon some issues with background that we have on these are issues related to condition assessment and help monitoring and so on and so forth the basic problem here is we need to estimate certain variables based on knowledge of certain other variables which are correlated with the quantity that is of interest to us.

So, we will consider some of this problems and I will quickly illustrate how based on what we will have learnt we can get a preliminary grasp of the basic questions in this area.

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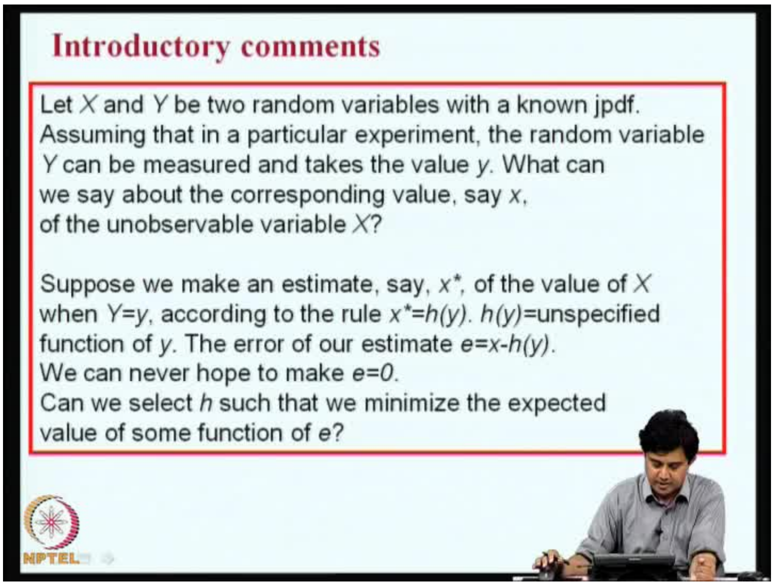

Let us consider a simple problem. Let x and y be two random variables with a known joint probability density function assuming that in a particular experiment, the random variable y can be measured and takes the value small y the lower case y.

The question is, what can we say about the corresponding value says this x of the unobservable variable capital X. Capital X is a hidden variable and we observe y with a primary interest to understand x so the choice of y which quantity to observe must be carefully made and this y should be well correlated with this x then only we can draw suitable inference about x.

Suppose, we make an estimate say x star of the value of x when y equal to y some according to some rule x star is h of y where h of y is unspecified function of y. The error of estimate here is x minus h of y. We can never hope to make e equal to 0. So can we select this function h such that we minimize the expected value of some function of this error. So, this is a basic problem in mean square estimation.

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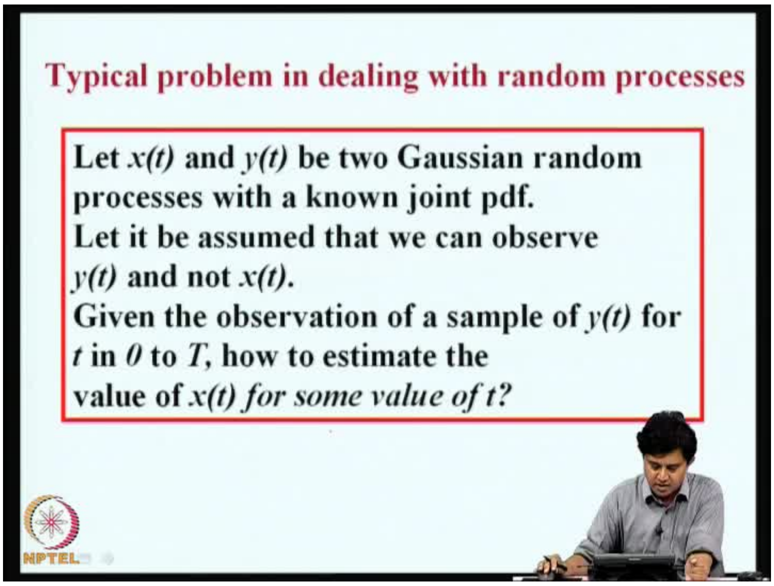

The same problem can be asked in dealing with random processes. For example, let x of t and y of t be two Gaussian random processes with a known joint probability density function. Let it be assumed that we can observe y of t and not x of t. Given the observation of a sample of y of t for t in 0 to capital T, how to estimate value of x of t for some value of t?

This question can be posed in the context of the structural engineering problem. For example, if you are able to measure displacement at a point in the structure when it is acted upon by a load, we may be interested in estimate interest at a point in the structure. The stress itself may not be accessible for measurement at the point where you would like to determine. But we are observing a quantity which is correlated with that so based on that knowledge and based on the knowledge of joint probability density function of a quantity being measured and a quantity that is being sort.

What can we say about the hidden variable? This joint density function that we are mentioning here as known could be known in the form of a mathematical model. It could be a finite element model that relates displacement and stresses and displacement may be measured and we are asking what could be the stress given that there is a finite element model which relates the two. So, the knowledge of joint probability density function can come through an elaborate mathematical model.

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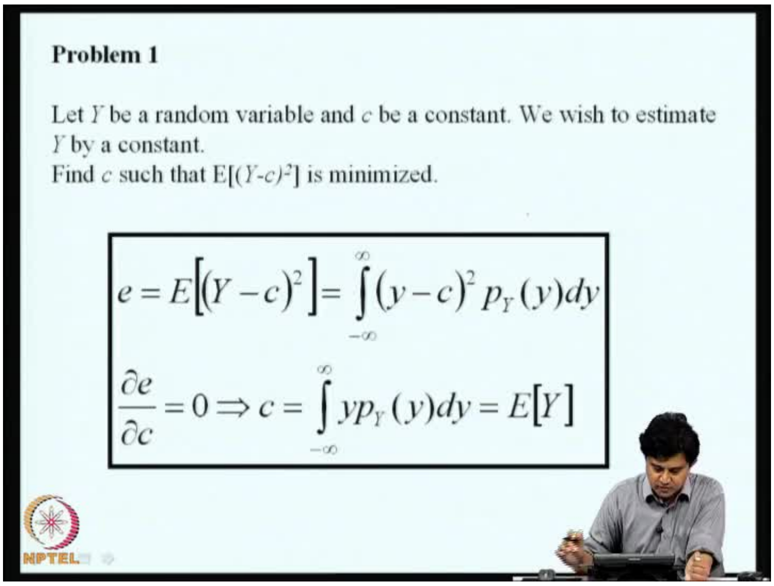

So will consider some simple problems so that we get an understanding of what is the basic mathematically issue here. Let Y be a random variable and c be a constant. We wish to estimate Y by a constant. This constant is what I can offered to observe but I want to know what is Y. Now, the simple thing is we can find c such that the ,expected value of Y minus c which is error and square so that the signs of the errors are given the same importance this is minimized.

So is this which is actually expected value of Y minus c whole square this is Y minus c whole square p y of y dy integral minus infinity to plus infinity. Now, we select this c so that dou e by dou c is 0. If we do that, will get the result that the c is expected value of y. So, you want to replace a random variable you should replace it by its expected value and that value would minimize a mean square error.

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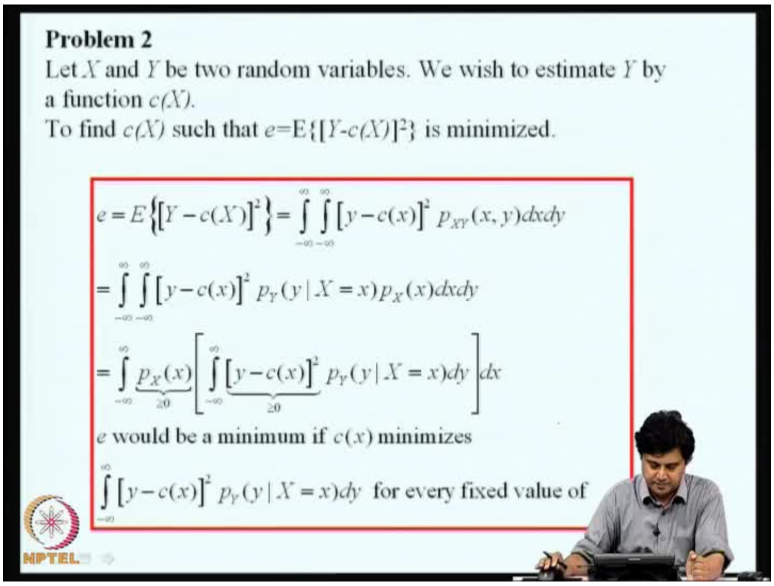

Now, slightly more involved exercise. Let X and Y be two random variables. We wish to estimate Y by a function c of X. Therefore, the problem is to find c of X such that the expected value of Y minus c X whole square is minimized.

So, we can write this expression e is this double integral Y minus c of X whole square p x y x comma y d x d y. Now, p x y x comma y, I write it as product of conditional probability density function of y condition on x equal to x and p x of x d x d y.

Now, I will reorganize this integral. First, I will carry out integration with respect to y and then with respect to x. Our objective is to minimize e. We can see here that p x of x is strictly it is non-negative. Similarly, Y minus c of X whole square is non-negative and consequently this e would be minimum if c of X minimizes this integral because anyway this is strictly non-negative. So, based on that argument, now we would like to select c of X which minimizes this quantity.

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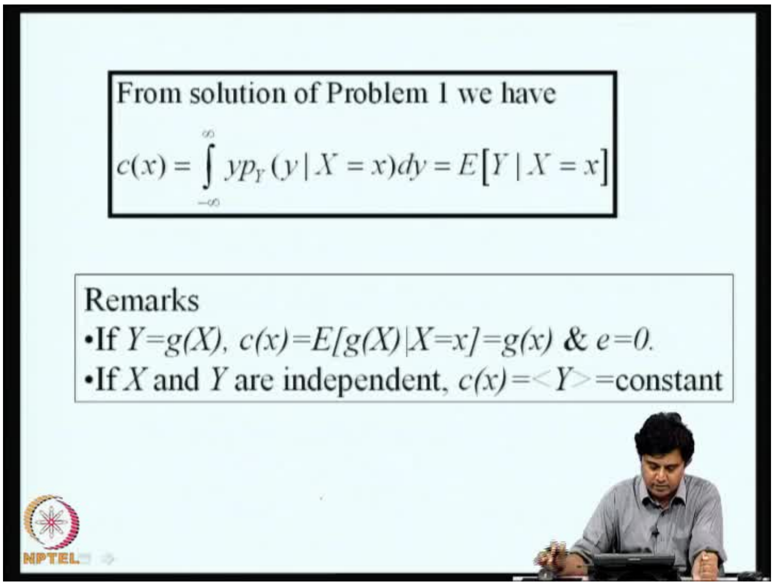

Now, based on the solution to the first problem that we solved where we found out what is the best placement for a random variable, we now get solution to this problem namely c of x is the optimal choice of c of x is conditional expectation of y. It is expected value of y conditioned on x equal to x. This is the optimal choice.

Now, if some remarks can be made if Y is g of X, that means there is a functional relationship between the two. c of x will be expected value of g of X conditioned on X equal to x which is nothing but g of x and e becomes 0 if X and Y are independent c of x is Y which is constant.

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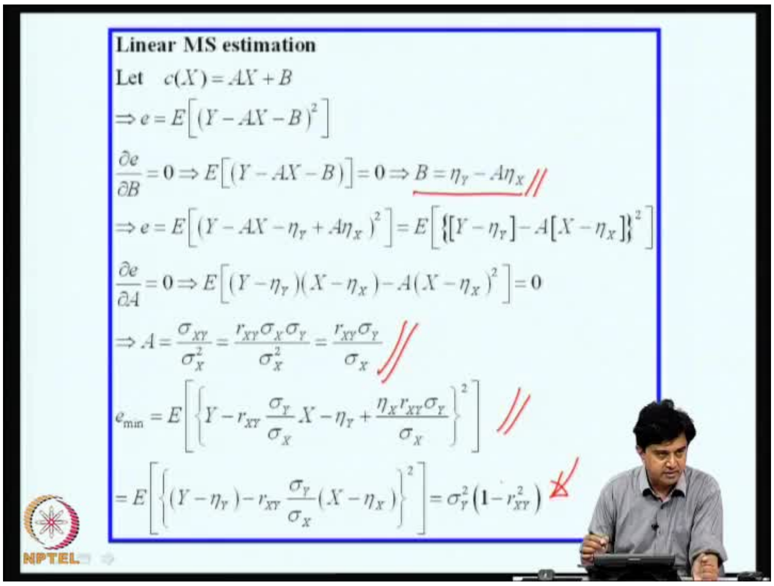

Now, this c of X if we take it to be a linear function, that is AX plus B then we deal with what is known as linear mean square estimation. So, the error of representation is e which is c of X minus AX plus B and mean square error average mean square error is expected value of Y minus AX minus B whole square.

Now, I select A and B so that dou e by dou B is 0 and dou e by dou A is 0. So, if you first do the calculation with respect to B, we get B into B given by eta y minus A eta x and we now substitute that into the expression for e and then implement the minimization with respect to A and we get A to B given by this.

These optimal values of A and B if we now substitute into the expression for the error, this is the error at the optimal point. This error of course is not 0. This is the minimum possible error.

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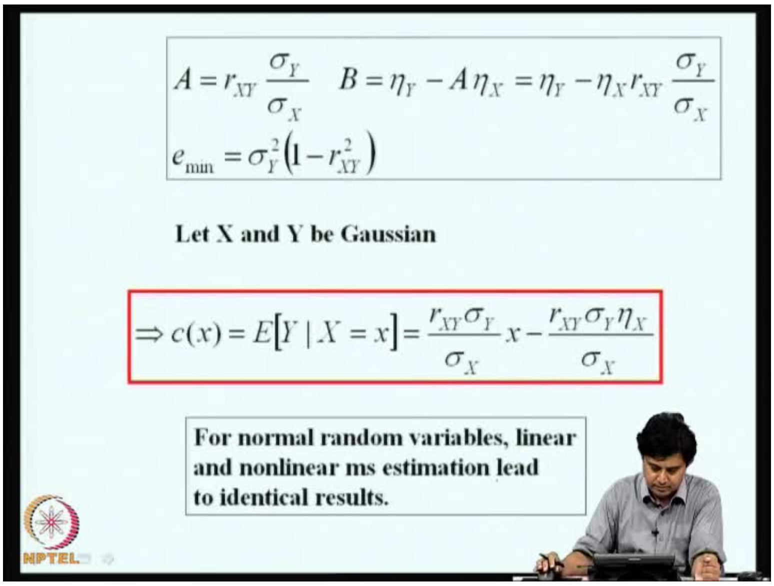

So this is a solution. now if X and Y are Gaussian, I have not made the assumption that they are Gaussian, you can show that c of x is expected value of Y conditional X equal to x transfer to be this. Therefore, for normal random variables, we can see that these two answers match. Therefore, for normal random variables linear and non-linear mean square estimation lead to identical results.

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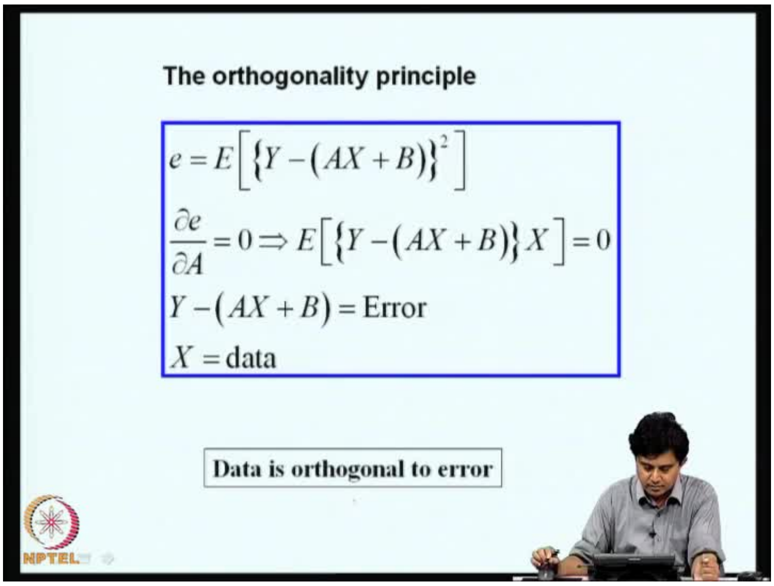

We enunciate what is known as orthogonality principle. This expected value of mean square error is given by this and the condition dou e by dou A equal to 0 would lead us to the condition that expected value of the term inside this brace which is y minus AX plus B into X is 0.

Now, we can see that this term inside the brace here is the error. This is a error of approximation and X is a observation we called it as data. So, what is said is, X is according to this rule this random variable which is Y minus AX plus B is orthogonal to X. Two random variables are set to be orthogonal to each other if the expectation of their product is 0. Now, based on that we say that data is orthogonal to the error.

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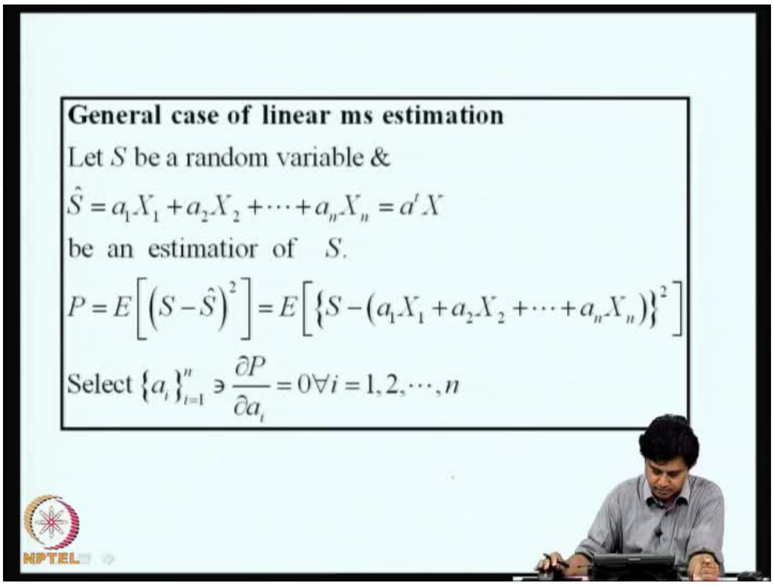

We can generalize the formulation to a set of n random variables. Let S be a random variable and S had b an estimator of S that is a 1 X 1 plus a 2 X 2 and so on and so forth and so. This, I can write it as transpose X.

Now, how to select a 1 a 2 a 3 a n so that this capital P is now the error is minimized with respect to a i’s so how to select a i’s so that dou P by dou a i is zero for i equal to 1 2 n.

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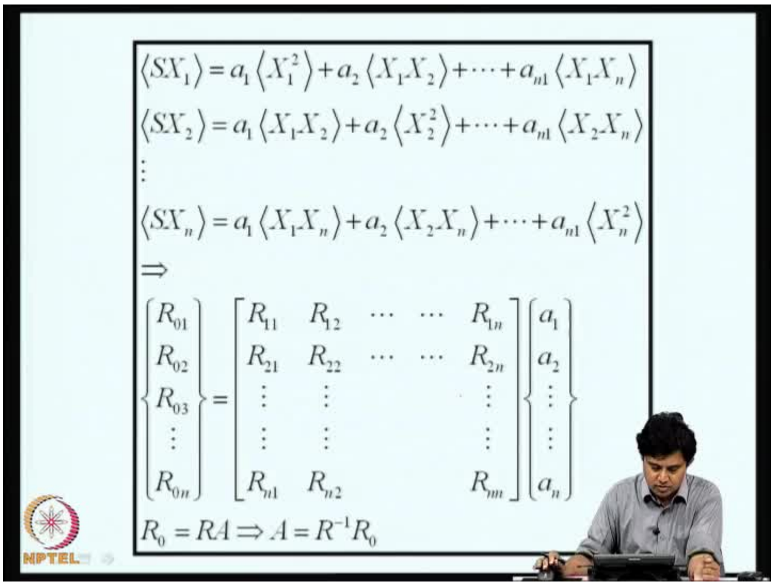

So, this can be done, we have to differentiate with respect to a 1 a 2 a 3 a n separately and put the error to be 0 and this leads to this metrics equation where R i j is expected value of X i into X j. So, based on that, we can get the value of the constant that we are looking for.

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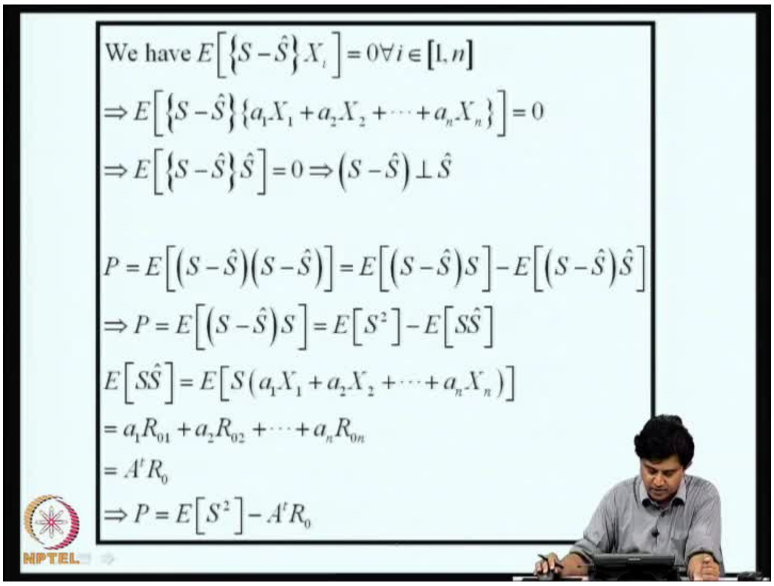

We can make some more observations. We have S minus S hat into X i this is a error into the this data is 0 according to the principle that we talk just now. So, if you simplify this again, we can show that S minus S hat is orthogonal to S hat. Based on that, if we utilize this result, we can actually find out the optimal value of the error and that error turns out to be this quantity. So, this can be verified.

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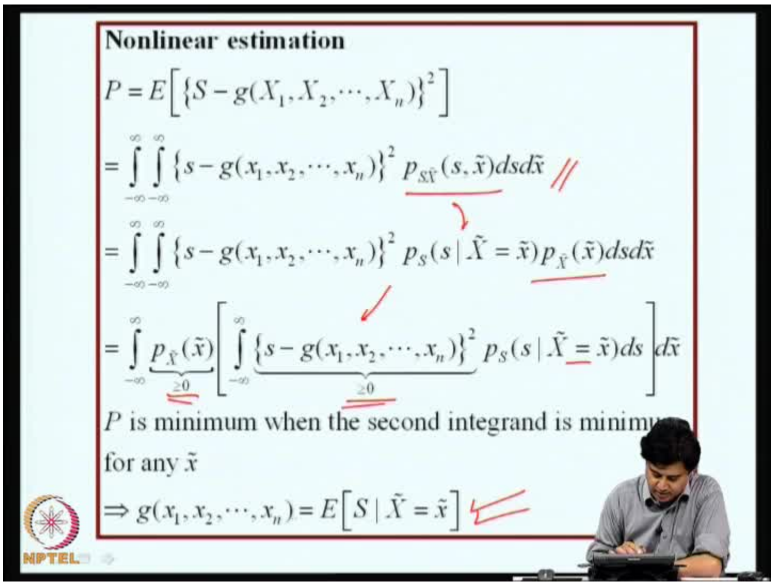

We can extend this for problems of non-linear estimation where we estimate S through a non-linear function of X 1 X 2 X n which is g of X 1 X 2 X n. So, the error of replacement is again s minus g and P is expected value of s minus g whole square which is this and we again split the write this joint density function in in terms of a conditional probability density function and a joint density function of x tilde and we again rewrite this as follows and we notice that and this is positive and this terms inside the brace is positive. Therefore, the way to select g is to minimize this inner integral and based on that, we again get this result that this non-linear estimation also tell us that this should be expected value of s condition on x tilde equal to lower case x tilde.

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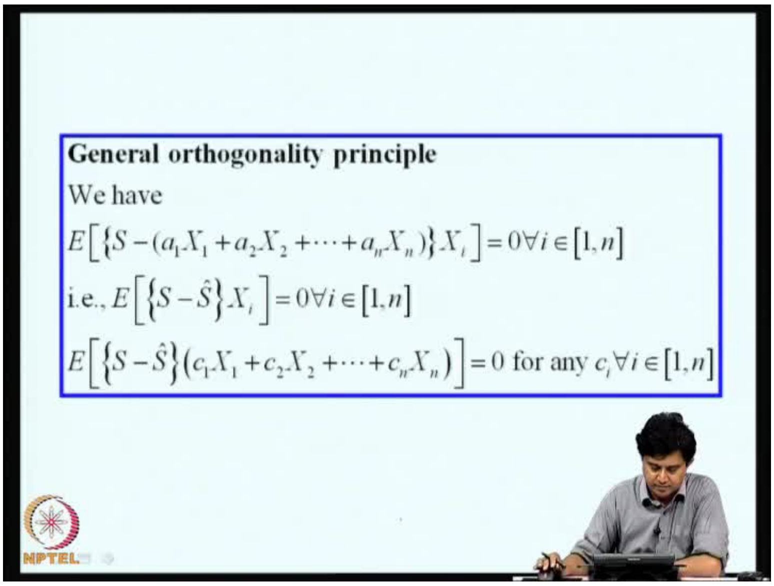

The generalization of orthogonality principle: we showed data is orthogonal to the error we can also show that linear combination of data is orthogonal to the error. This can also be shown using the result that we have.

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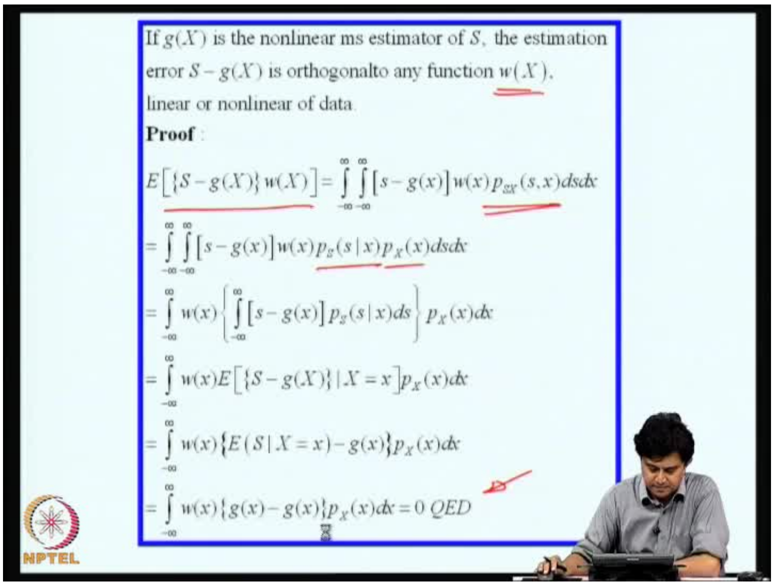

Now, this is a small exercise. If g of X is a non-linear mean square estimation of S, the estimation error S minus of g of x is orthogonal to any function w of x linear or non-linear of the data. So, this proof I have developed here is all again follows the same logic that we have use so far. We write the expression for this expected value of S minus g of

X into the function of this data and rewrite this joint density function in terms of product of a conditional density and a marginal density and again notice certain features of the response and we reach the condition conclusion that S minus g of x is orthogonal to w of x.

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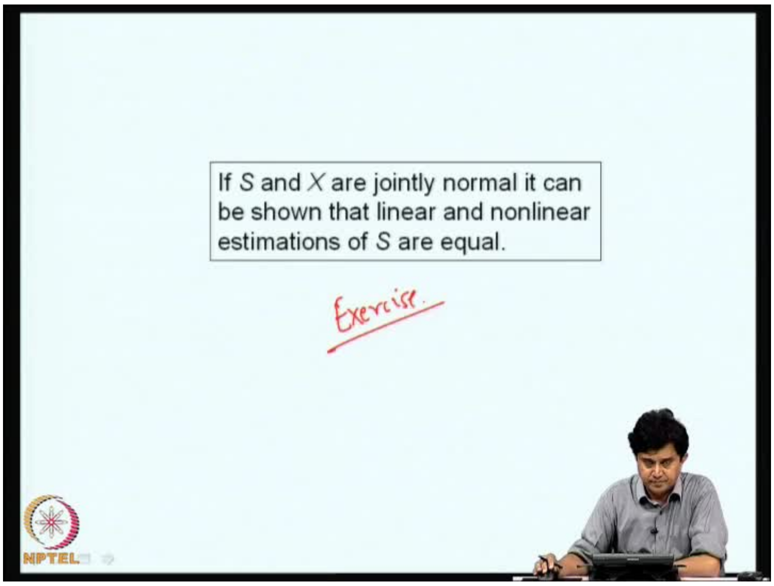

In fact, if S and X are jointly normal, it can be shown that linear and non-linear estimation of S are equal. You can do this as an exercise.

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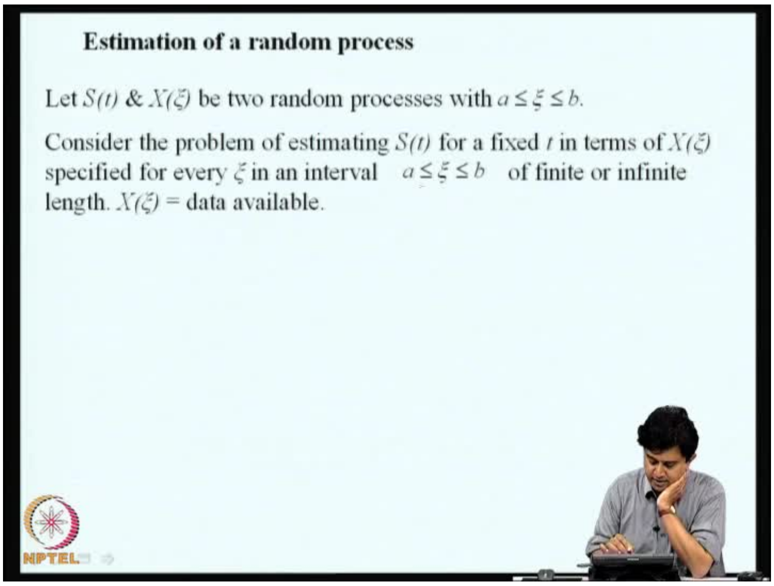

We have talked about random variables. How about random processes? Let S of t and X of psi be two random processes where psi takes values of a to b. We consider the problem of estimating S of t for a fixed value of t in terms of X of psi specified for every value of psi in an interval a to b. That means X of psi is the data available in the example that I mentioned, X of psi is the data on say the strain or the displacement that you have measured and this S of t is a some response quantity of interest which is not measured.

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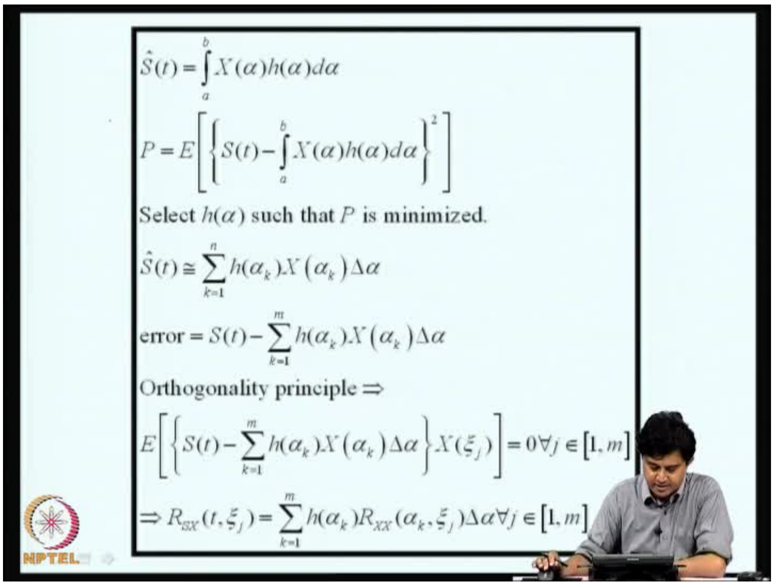

So, what we do is generalize the linear estimation problem we propose a estimator S hat of t is a linear function of the data a to b X of alpha h of alpha into d alpha for this h of alpha is not known.

Now, we need to select h of alpha such that this P is minimum. If we discretize this integral as the summation like a Riemann sum, we have the solution already with us for that. On that, we take the limits. If we do that, we get the equation for h of alpha k which is given here.

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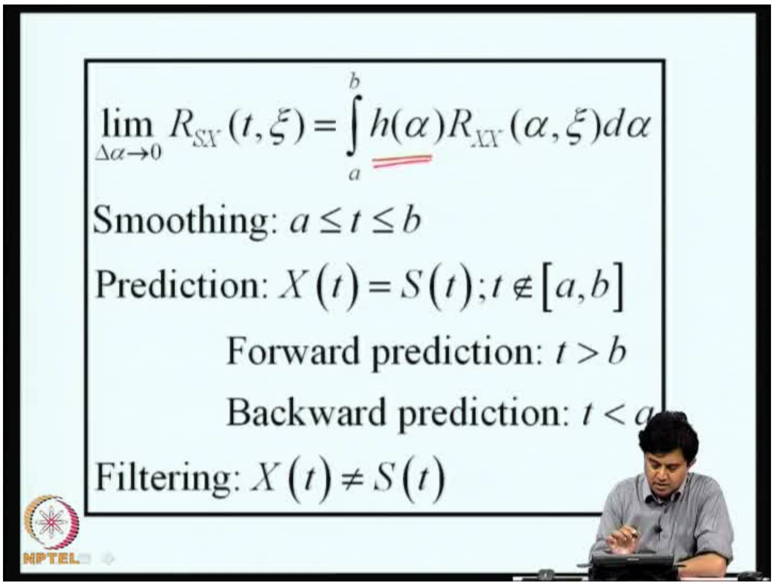

And then, if we take the limit of delta alpha going to 0, we get this integral equation where h of alpha is buried here.

Now, if the time at which we want to find S of t is within the interval a to b, then we call this problem as smoothing. If this t does not belong to the observation time interval, either t is greater than b or t is less than a, we call it as a problem of prediction. Filtering is of course the case where X of t is not equal to S of t. These are the terms that are used in estimation theory.

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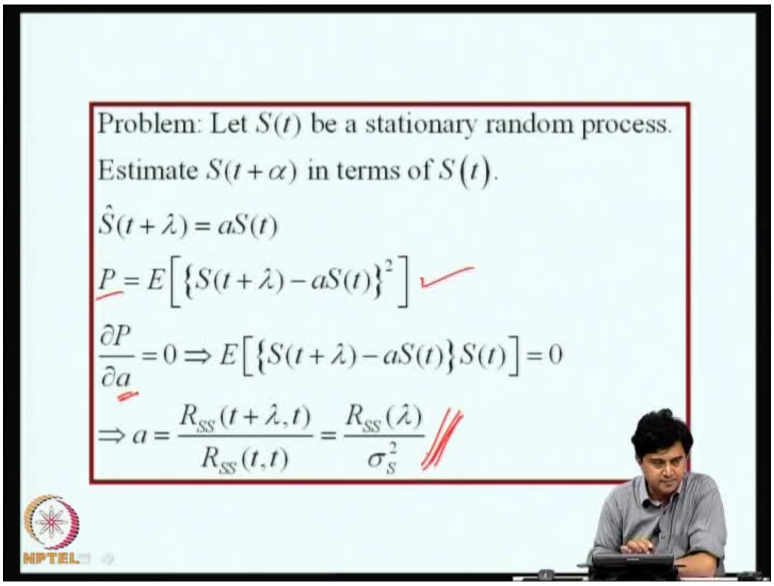

We will consider some simple examples. Let S of t be a stationary random process. Estimate S of t plus alpha in terms of S of t. That is a problem. What I do is,I assume S hat of t plus lambda as a into S of t. So, the error of this representation the expected value of the mean square error is P which is given here and if I now minimize P with respect to a, I get a to b. The optimal estimator for S of t plus lambda is a into S of t where a is given by this.

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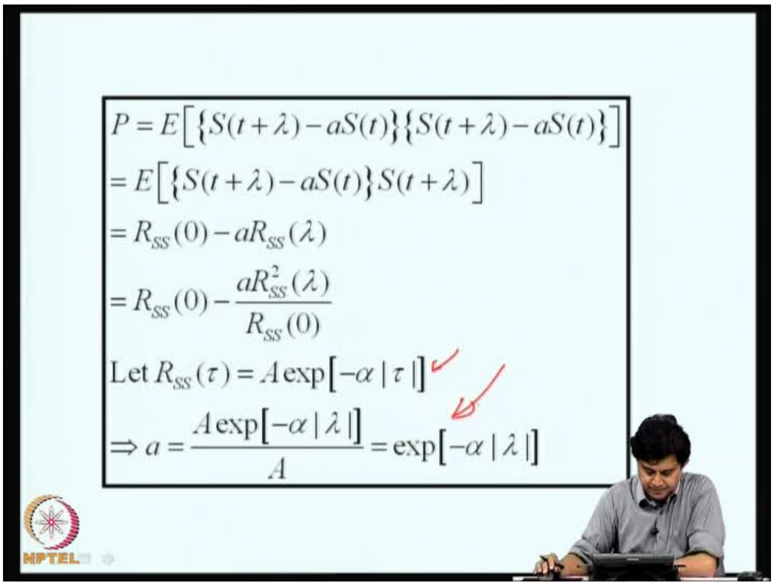

And the corresponding error that is optimal error, you can show that for a specific choice of R ss of tau. It is shown here. You can verify this result.

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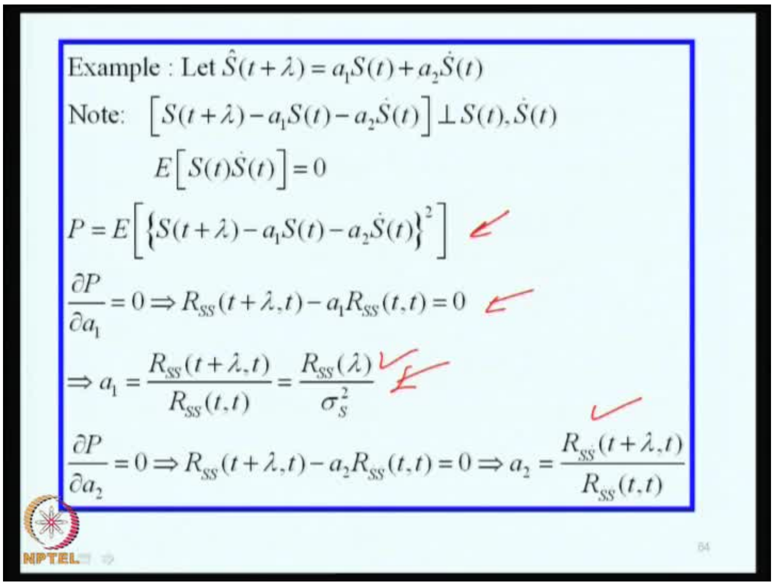

Now, this has slight variation. What we will do here is, we will assume that S hat of t plus lambda is a linear combination of S of t and S dot of t. Suppose, we have made observation on S of t as well as S dot of t it will be a 1 S of t plus a 2 S dot of t. Now we note that, S of t plus lambda minus a 1 S of t minus a 2 S dot of t is orthogonal to S of t and S dot of t and S of t and S dot of t S of t is taken to be stationary. So, the process in this derivative are uncorrelated at the same time instance.

So, again we formulate the mean square expected mean square error minimize with respect to a 1 and a 2 and if we carry out this, we get a 2 and a 1. So, this estimator is something that you have to design and the error actually between two possible estimators, there will be different errors. You can compare them and make further choices.

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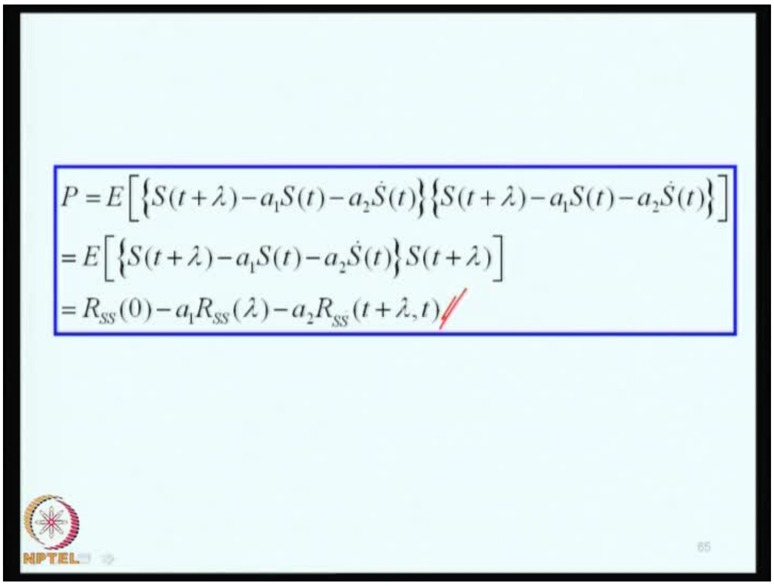

This is the optimal error in this particular case.

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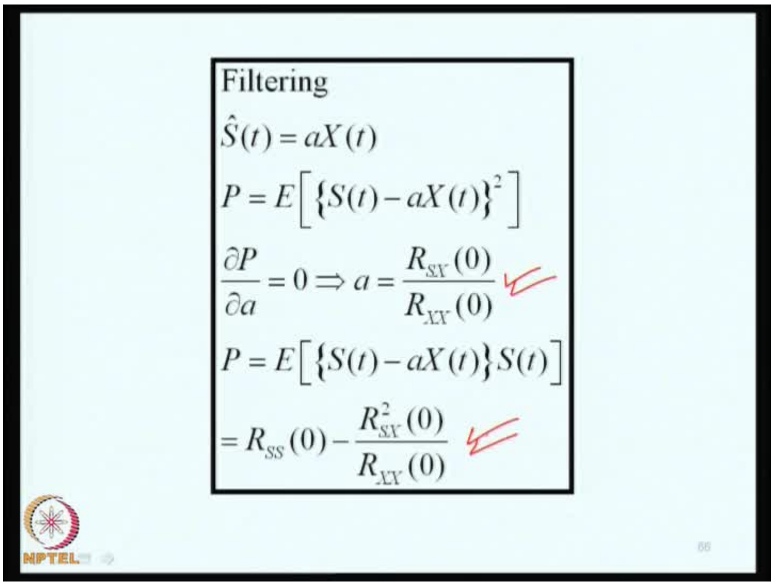

Now, filtering problem: we estimate S of t S hat of t is a into X of t. So, the again P is this minimize with respect to a we get a to be this and this is the optimum error.

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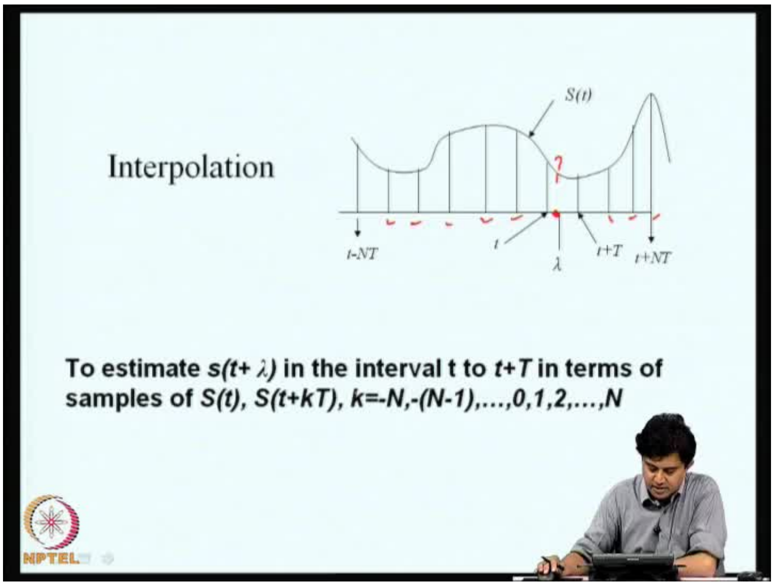

We can consider further problems like problem of interpolation, where the problem is to estimate S of t plus lambda that means somewhere here, this value we do not know. We have not observed in terms of samples of S of t at this discrete time instance.

This problem is used for example in developing structural matrices in stochastic finite element method where these are the nodes and we interpolate the random fields within the nodes using this logic that we are discussing. That is one of the tools that is used in stochastic finite element methods.

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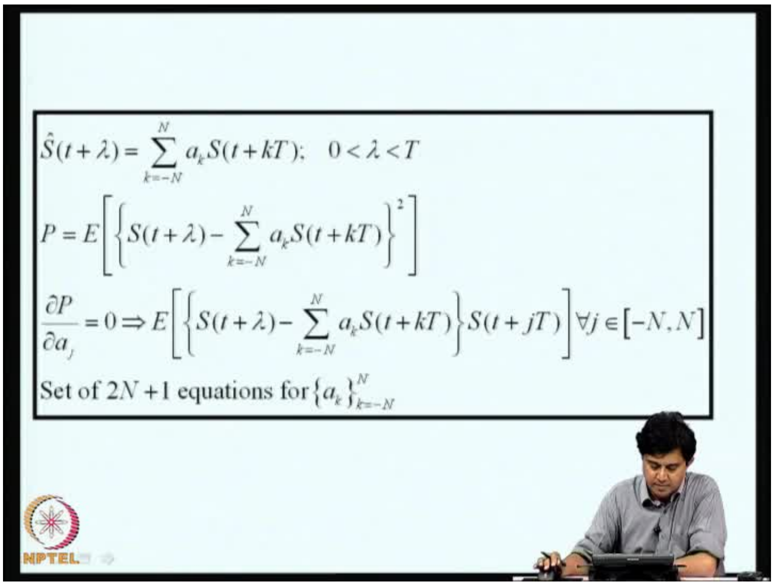

So, what we do is, the estimated S tilde of t plus lambda is taken to be a linear combination of what has been observed and again if we minimize the mean square error of this representation, we get equations for this a k's which are actually a set of 2 n plus 1 equations for these unknown constants and they will be in terms of known properties of S of t and they can be determined.

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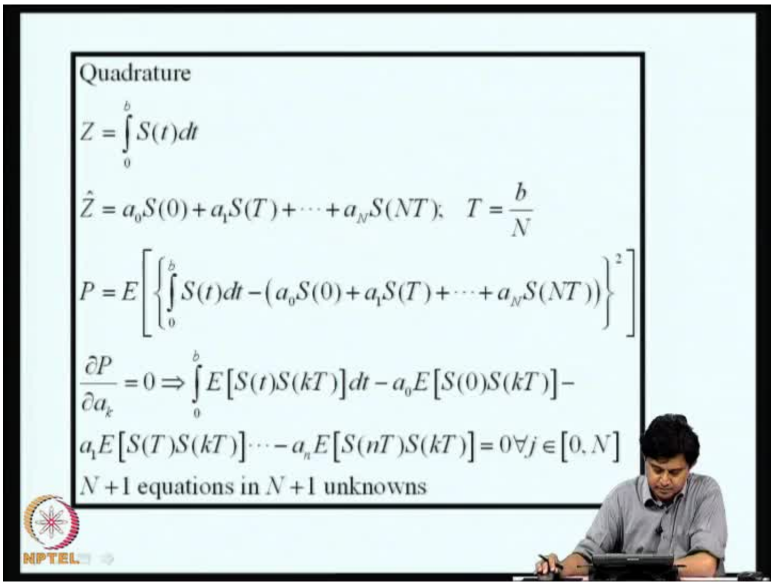

The problem can be extended to problems of quadrature. Suppose, you want to find Z which is 0 to b S of t dt, what I will do in estimator for Z is a linear combination of S of t evaluated at certain time instance 0 capital T 2 T 3 T and capital NT.

We can implement this method and we can derive this a naught a 1 a n and will get n plus 1 equations by minimizing the mean square error with respect to this a k's and this problem also offers a approximate solution.

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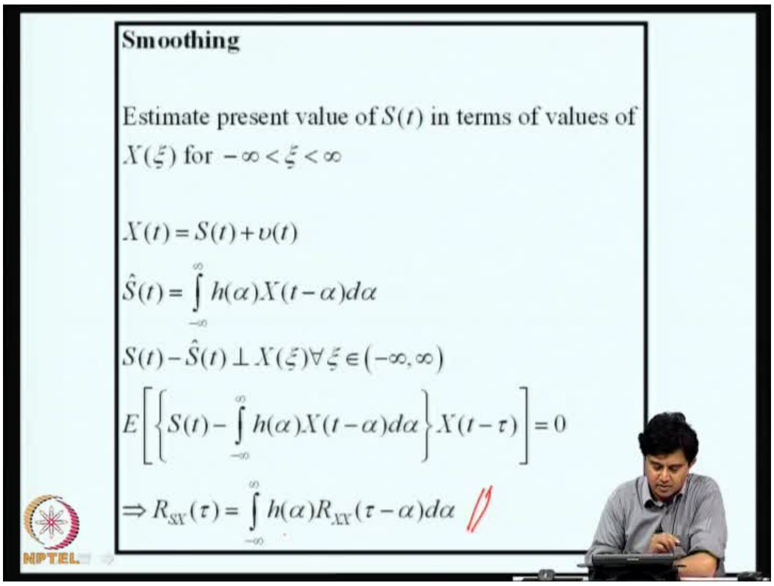

How about a problem is smoothing? We estimate the present value of S of t in terms of values of X of xi for xi varies from minus infinity to plus infinity so X of t is S of t plus some noise. S hat of t is some output of I mean we are convolving X of t with a filter and this filter has to be determined. The transfer function for the filter has to be determined.

And the same the few is now the orthogonality principle which is reflection of the criteria of minimizing mean square error, we get this equation which is an integral equation for this filter which can be used for solving this type of problems.

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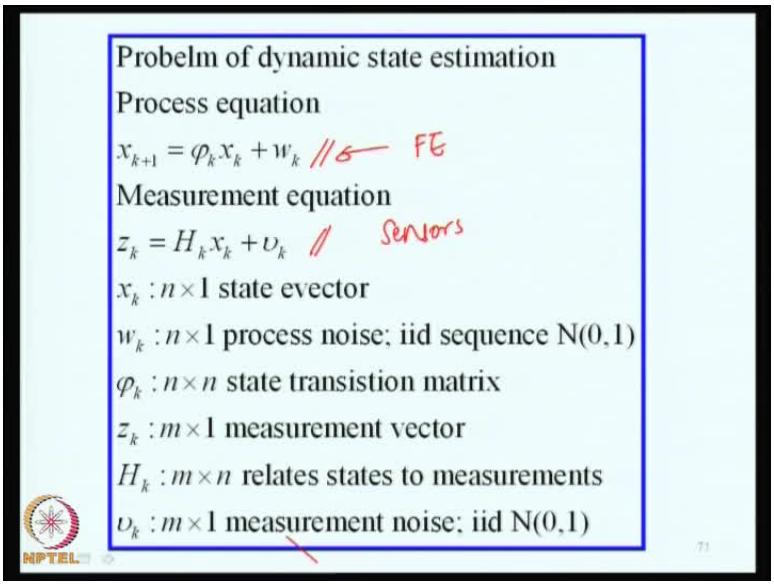

Now, I will close this discussion by making a quick reference to the basic problem of dynamic state estimation. In the discussion that I described, we went through   the joint densities that we are talking about where assume to be known but I already mention that knowledge may come through a mathematical model.

Suppose, we are interested in the process equation which is obtained by a mathematical model x k plus 1 is phi k x k plus w k, then the measurement is an another quantity z k which is measured quantities are linear transformation on the states.

Now, the basic problem here is, this process equation can come through for example finite element modeling in structural engineering application this comes through our sensors.

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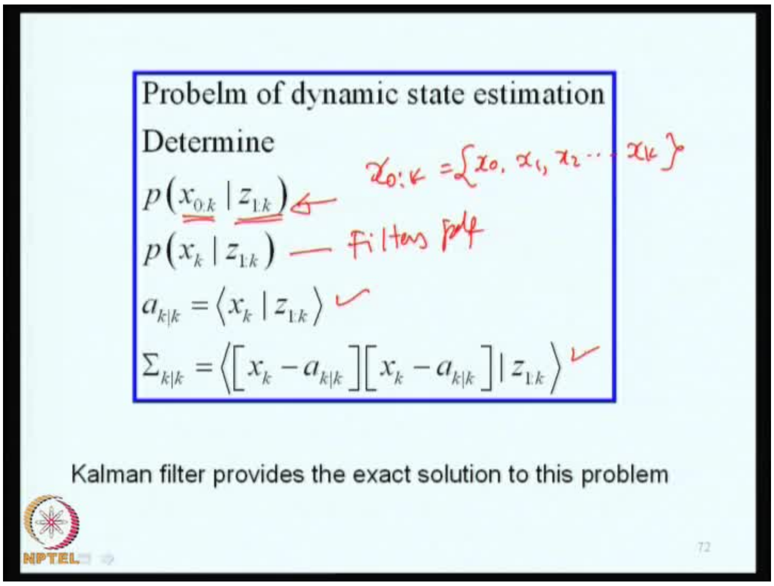

So, the problem on hand is that we need to find out the posterior density function of the response vector given that we have made these measurements. This is joint density function between x 0 k is the vector x 0 x 1 x 2 x k. This is actually the posterior joint density function. We can take the kth instantaneous state and consider probability density of x k conditioned on measurements up to that point and this is known as filtering probability density function.

Alternatively, we can find the expected values, conditional expected values and the conditional covariance. In problems of condition assessment structural systeme identification, the determination of these quantities are of fundamental interest and there are various tools know such as common filters are particle filters etcetera which are designed to answer questions of this kind and that become quite useful for solving as research problem of system identification control etc.,

This gives a glimpse of the application of the topics that we have studied in the course. Based on the topics that we have learnt in this course, you can probably now study say area subjects of structure system identification, dynamical system control theory etc.,

At this juncture, we will close this lecture and this course also closes at this point.