Stochastic Structural Dynamics

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Module No. # 10
Lecture No. # 39
Problem Solving Session-3

So, in this lecture, we will continue with our discussion on solution of few problems, and also introduce a few concepts, and discuss them as well.

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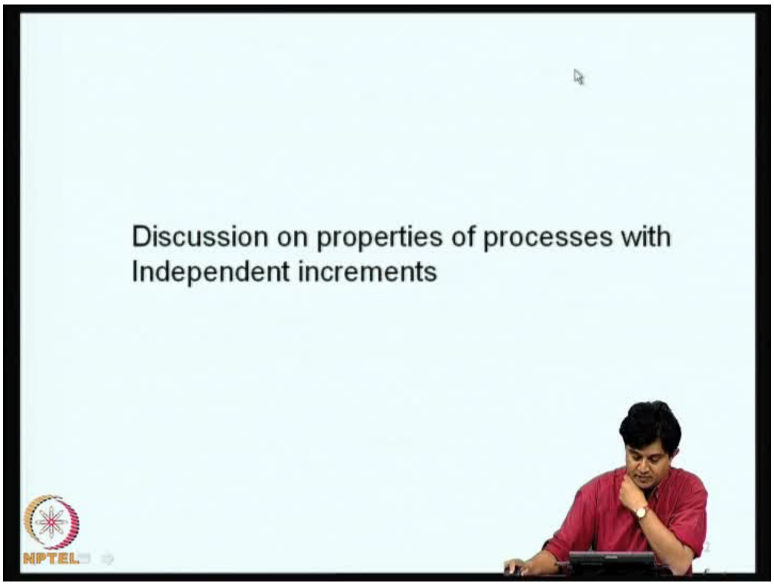

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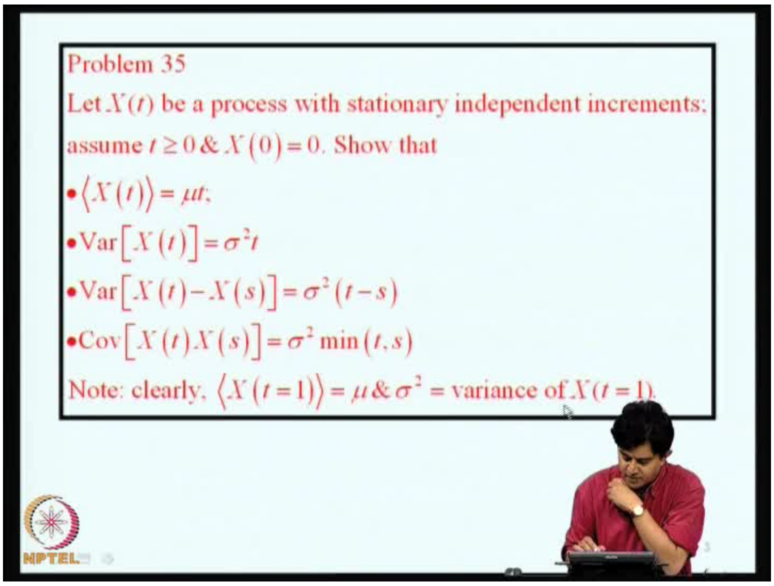

We will begin by discussing properties of processes with independent increments. A problem related to this. We will consider X of t to be a process with stationary independent increments and we assume that t is greater than or equal to 0 and x of 0 is 0. We are asked to show that expected value of X of t is mu t. variance is sigma square t and variance of X of t minus x of s is sigma square t minus s and covariance is given by this. Actually, the data on mean and variance are given at X t equal to one. Expected value of X of t is mu and sigma square is variance of X at t equal to 1.

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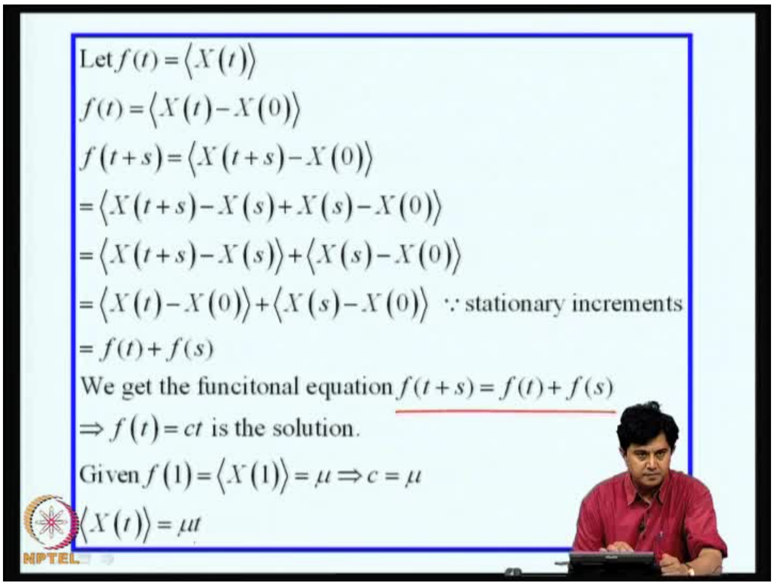

Now, let f of t be equal to expected value of X of t. So, f of t is expected value of X of t minus x of 0. Expected value of X of 0 is taken to be 0. Therefore, f of t plus s is x of t plus s minus x of 0. By adding and subtracting X of s, I can rewrite this as x of t plus s minus X of s plus X of s minus X of 0. The expected value will therefore consists of two terms. First is expected value of X of t plus s minus X of s and the second term is x of s minus x of 0.

Now, if you look at the first terms x of t plus s minus X of s, it is nothing but x of t minus X of 0, because the process is stationary increments so this is f of t and this is f of s. Therefore, we are getting the functional equation f of t plus is t plus s is f o f t plus f of s. Now, a solution is f of t c t for this. We need to find c, but for that we have the value of f of t at t equal to 1 which is expected value of X of 1 which is mu. Therefore, expected value X of t is mu t.

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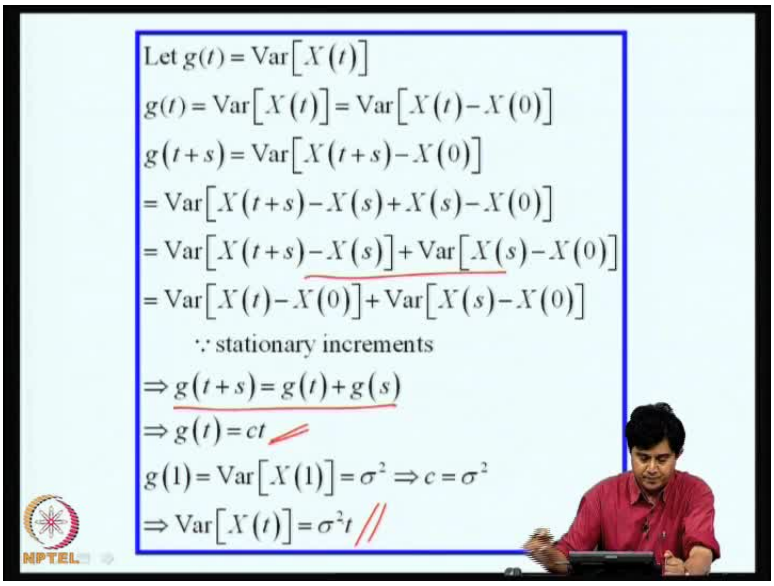

Now, we define g of t as variance of X of t, where g of t is nothing but variance of X of t minus x of 0. I can write it as g of t plus s variance of X of t plus s minus X of 0. Again, I add and subtract x of s and since process increments are independent, I can add the variances in this form here and since the process has stationary increments, the variance of X of t plus s minus x of s is nothing but X of t minus x of 0. That leaves to the functional equation g of t plus s is g of t plus g of s. Again, the solution needs of the forms g of t c t and c transfer to be sigma square and therefore variance is sigma square into t.

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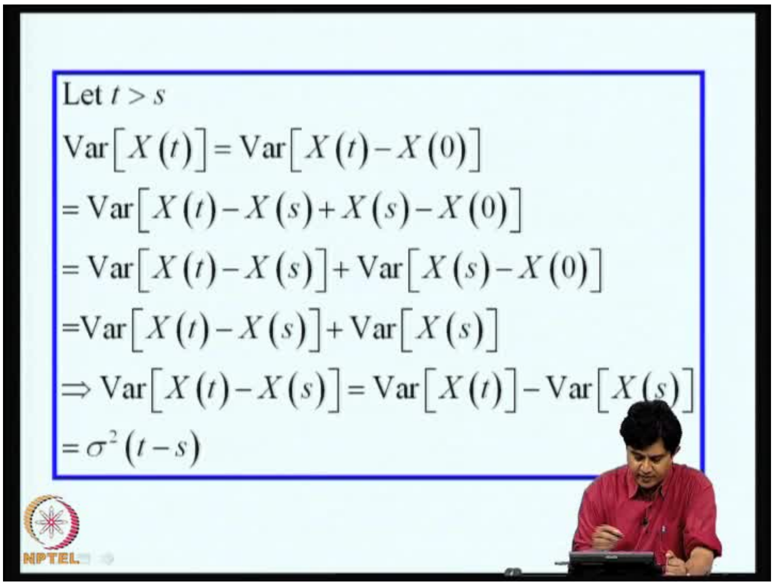

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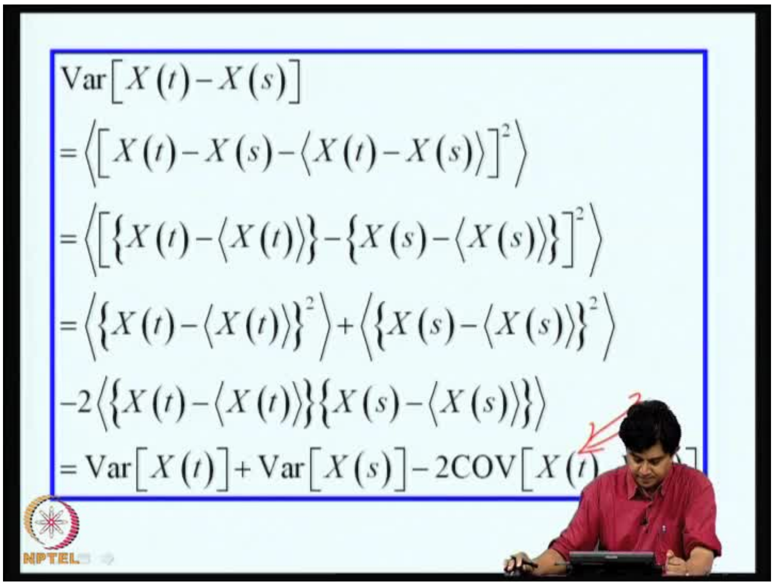

Now, lets begin by assuming t is greater than s and if we find variance X of t, we can carry out this calculation and show that variance of X of t is sigma square t minus s. That is, variance of X of t is that and variance of X of t minus X of s we have to find out that is the auto covariance. If we use square and carry out this, we write this as X of t minus X of s minus mean of that whole square and rearrange this as X of t minus mean of X of t minus X of s mean of X of s and if you carry out this expansion, we can show that variance of X of t minus X of s is given by this.

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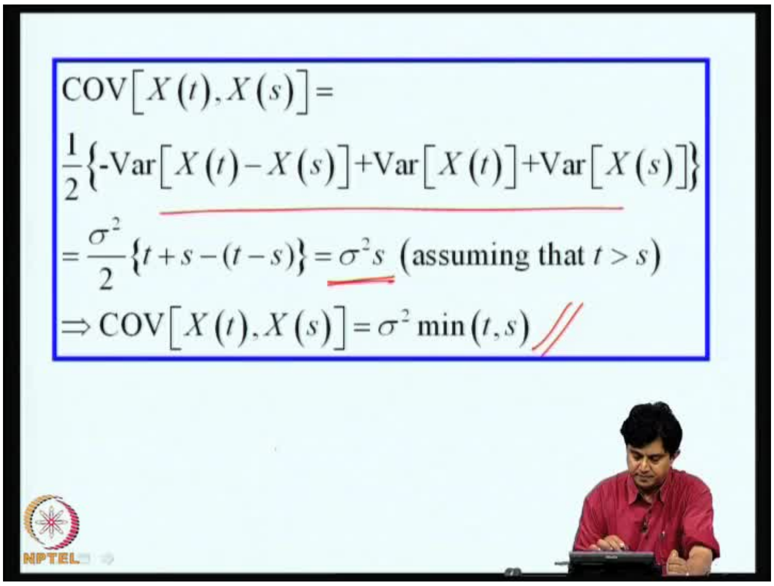

Therefore, covariance of X of t comma X of s we can obtain this expression which simplifies now to be sigma square s and since we assume t greater than s, it follows that covariance of X of t comma X of s is sigma square minimum t comma s.

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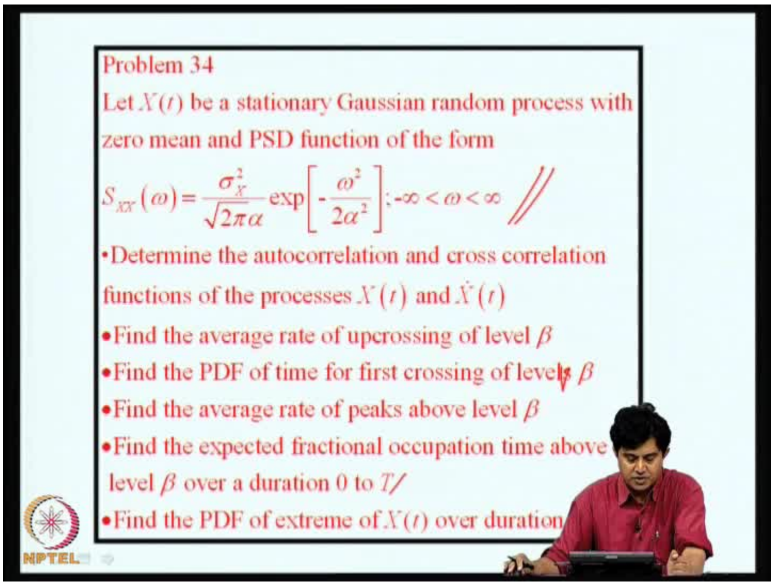

In study of failure of randomly vibrating systems, we consider problem of fist persist time and extremes of random processes over a time duration. Now, we consider a problem related to those developments. We consider X of t to be a stationary Gaussian random processes with 0 mean and power spectral density function which is of this form. That is sigma X square by square root of 2 pi alpha e raise to minus omega square by 2 alpha square where omega is from minus infinity to plus infinity. You can recognize that power spectral density has a form of a Gaussian probability density function.

The problem in hand has several steps. First is determine the auto correlation and cross correlation functions of the processes and its derivative. Find the average rate of up crossing of level beta find the probability distribution function of time for first crossing of level beta. Find the average rate of peaks above level beta. Find the expected fractional occupation time above the level beta where a duration 0 to t. Finally, find the probability distribution of extreme of X of t over a duration 0 to t.

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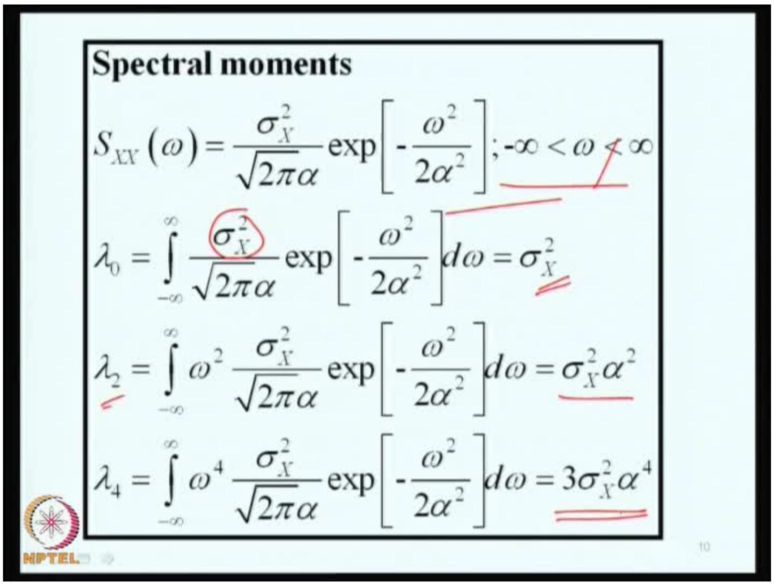

You need to recall that the quantities that we are seeking are closely related to the notion of spectral moments. The process we need is the variance of the process and its first and second derivative to answer some of these questions. The power spectral density function is given to be this. The zeroth order moment is area under the power spectral density function. Now, if you carefully see this excepting term 1 by square root of 2 pi into alpha e raise to minus omega square by 2 alpha square is a valid Gaussian probability density function. Therefore, area under that curve will be 1. Therefore, this integral is sigma X square.

Similarly, lambda 2 will be the second spectral moment which will be proportional to the variance. Now, alpha is a standard deviation for that hypothetical probability density function. Therefore, this is sigma X square alpha square. Fourth moment is actually fourth moment of a Gaussian random variable is 3 into standard deviation to the power of 4 so that is what we get.

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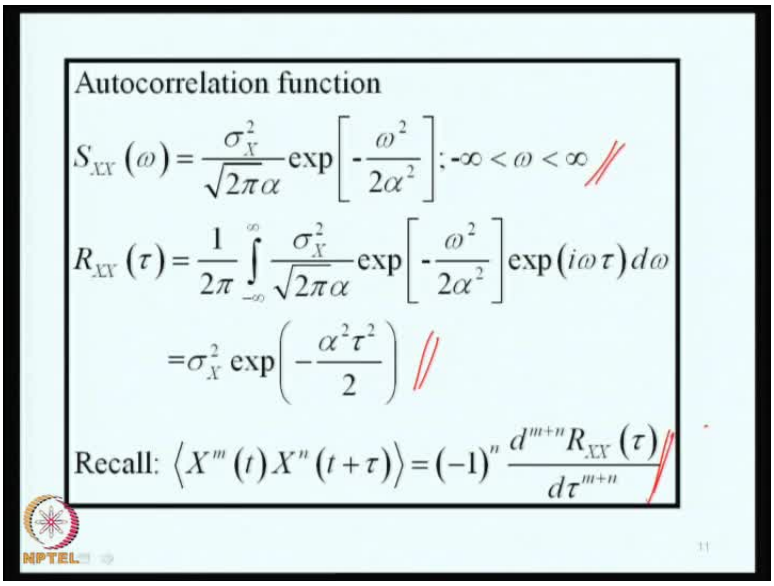

Now, we are asked to find autocorrelation function of the process and it is time durability across it is function etc., So, we have the power spectral density function which is given here and a fourier transform of this will be of the form that is similar to the characteristic function of a Gaussian random variable is 0 mean and that is of this form.

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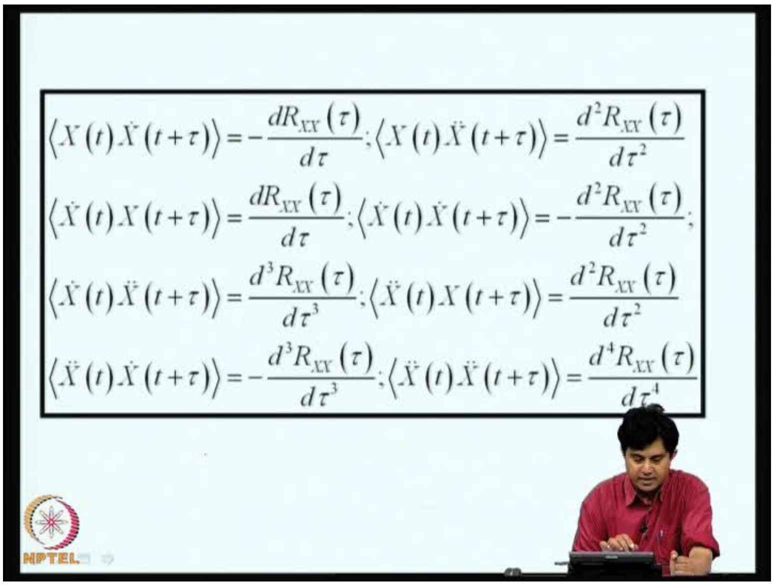

So, R x x of tau is this. Now, we are we are asked to find auto covariance of the derivative process and cross correlation with in process and its derivatives. We need to use this identity process is stationary. Therefore, we need to use this. Expected value of X of t into X dot t plus tau is minus d by d tau of R x x of tau and all other things follow from that identity. We need to apply this and find out all the quantities that are of interest up to the second order derivative.

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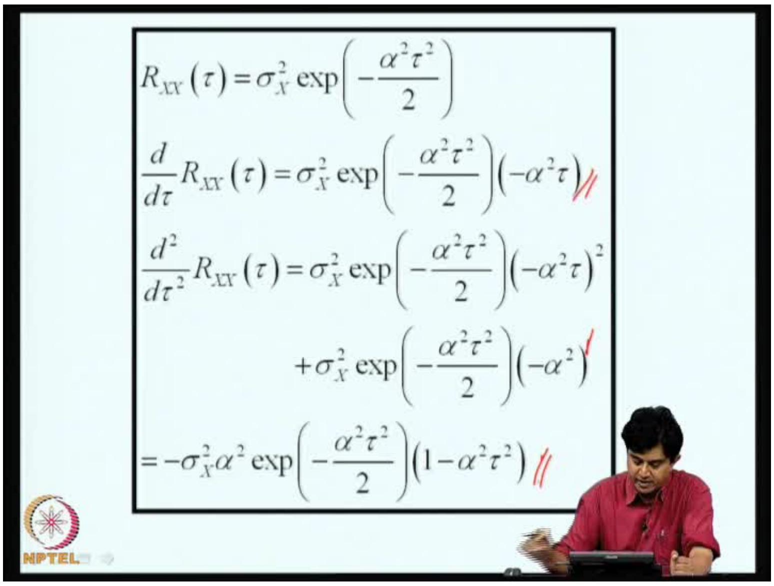

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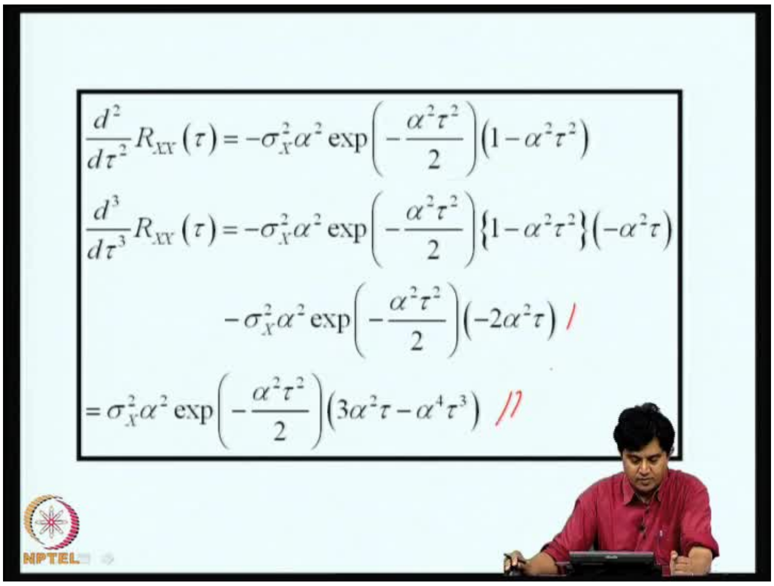

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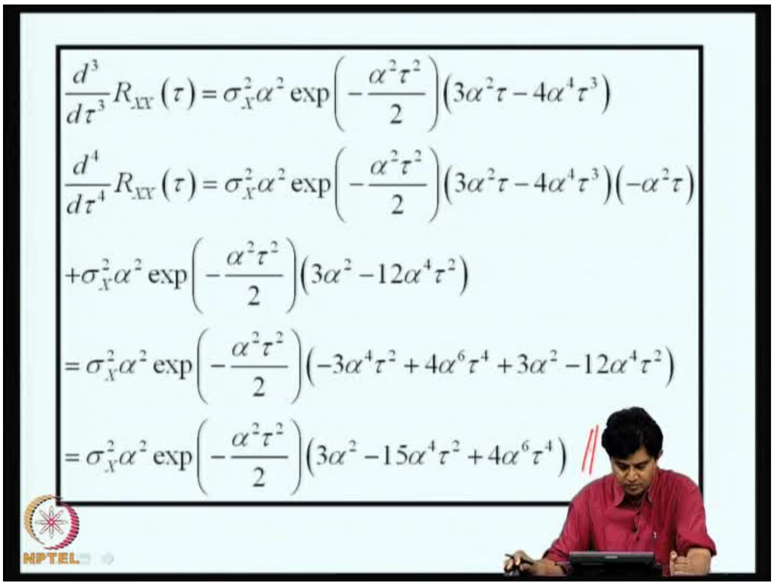

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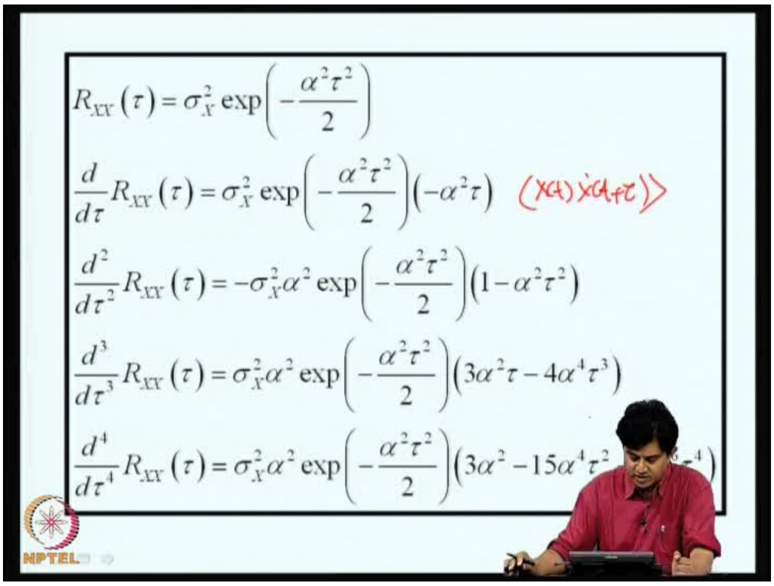

So, if we do that, R x x of tau is this. The first derivative transfer to be this. Second derivative is this. This is the second derivative, third and fourth derivative etc., we can find out. This is the fourth derivative. This is summary of all the calculations and this is auto covariance of R x x of tau. This is auto covariance of x of t x dot of t plus tau. This is the expectation and similarly we can interpret all these quantities.

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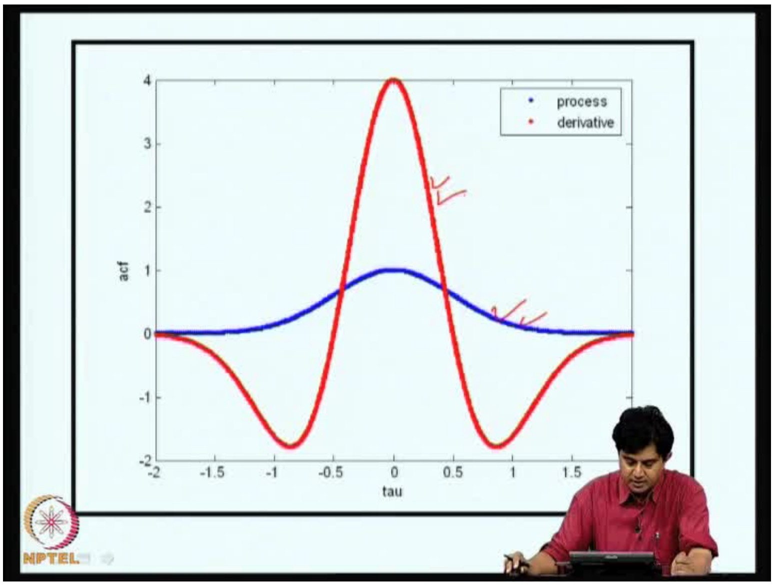

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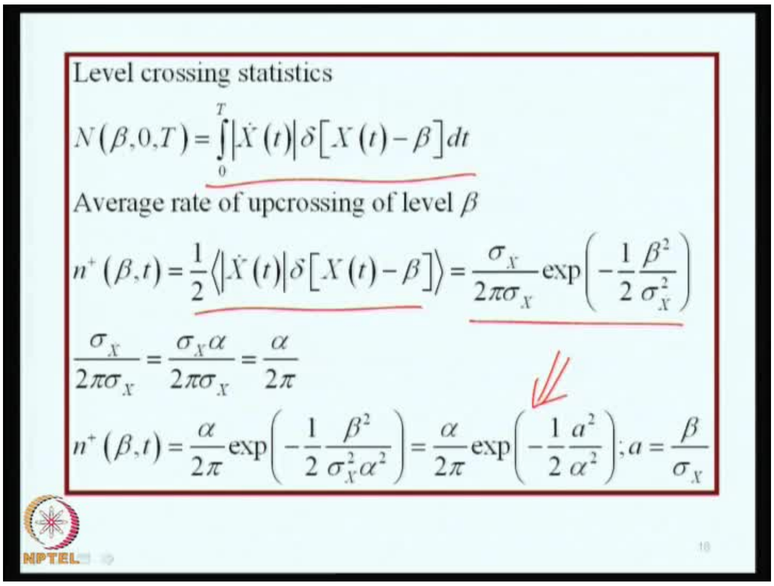

Some of this I have shown here. This is the auto covariance function of the process which is this and its first derivative is this. The part of the answer to the list of questions that were post. Now, the level crossing statistics that number of times the level beta is crossed in that given by this integral 0 to t X dot of t direct delta X of t minus beta d t. So, the average rate of up crossing of level beta is given by this and for a Gaussian random process this is as shown here and we have evaluated now all the spectral moments. So, if we substitute, the answer to be this.

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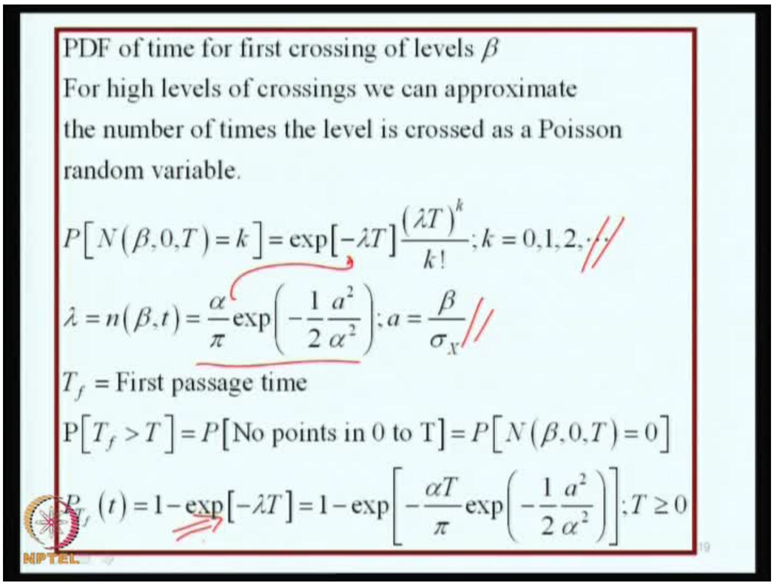

What is the probability distribution function of time for first crossing of level beta? We assume for high levels of crossing, we approximate the number of times the level is crossed as a Poisson random variable and we take therefore probability of N equal to k is given by this and this lambda is the rate of up crossing that we have to determined. This is the expression. We need to substitute this into this and we have the probability distribution of number of times level beta is crossed.

Now, the first persist time t f, if you are interested in probability of t f greater than equal to capital T, that would mean there are no points in 0 to capital T. That mean the probability of n equal to 0 and we have the complete characterization of this Poisson random variable and therefore I can get the first persist time.

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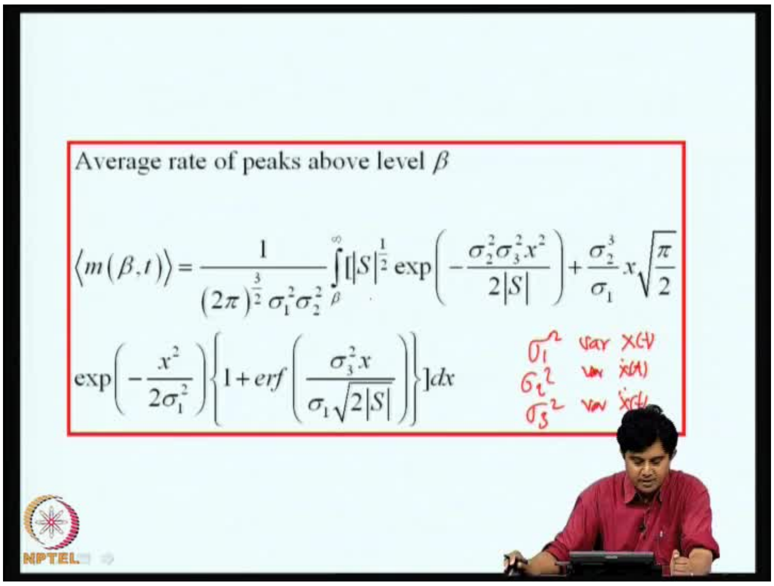

Now, average rate of peaks about level beta, this is the expression that we have derived. You have to go back to the previous lectures. This is for a stationary Gaussian random process. So, the average rate of peaks above level beta has threequantities sigma 1 square sigma 2 square and sigma 3 square. This is the variance of the process X of t. This is the variance of X dot of t and variance of X double dot of t we have determined which are nothing but the 0 which are second and fourth spectral moments we have determined. We need to substitute that into this and get the answer.

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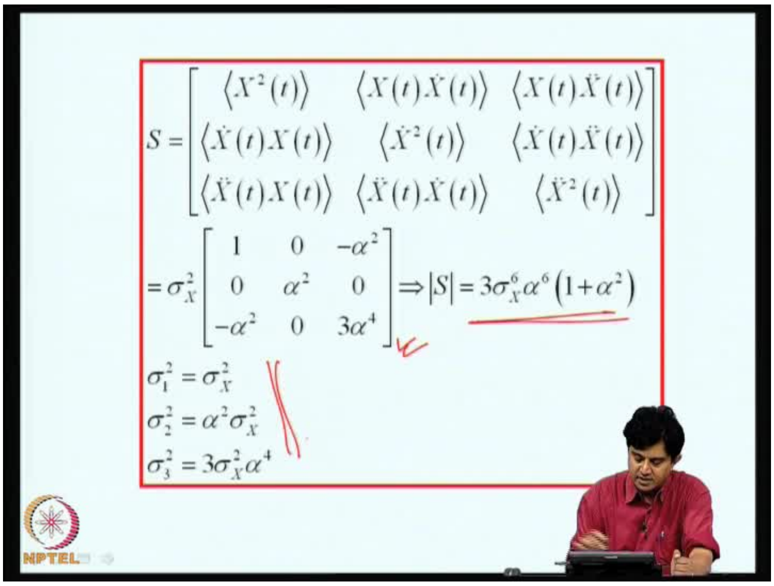

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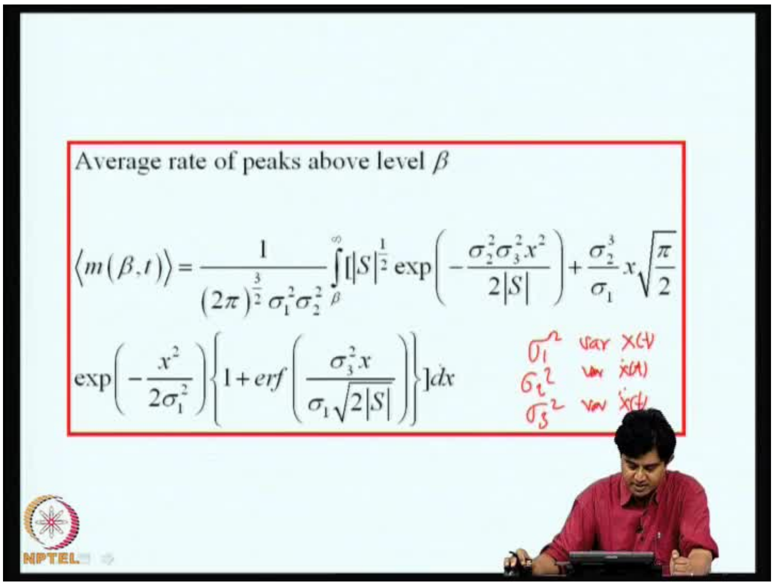

This s is the covariance matrix evaluated at same time t, we get these numbers and this becomes the determinants. We have all the spectral moments and determinant of s. Go back to this expression, we have answer to the question that is post.

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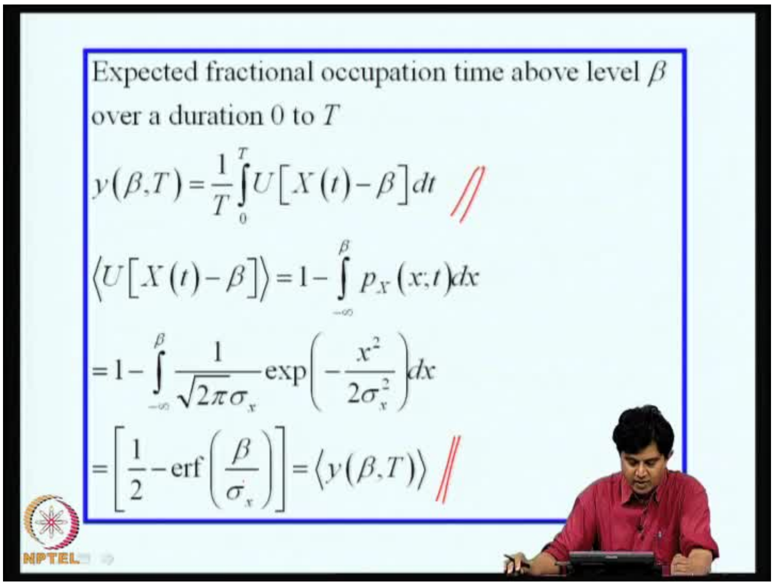

Expected fractional occupation time above level beta over a duration 0 to t is, this is the definition. You can show that the expected value is given by this. This is where only the first spectral moment is required.

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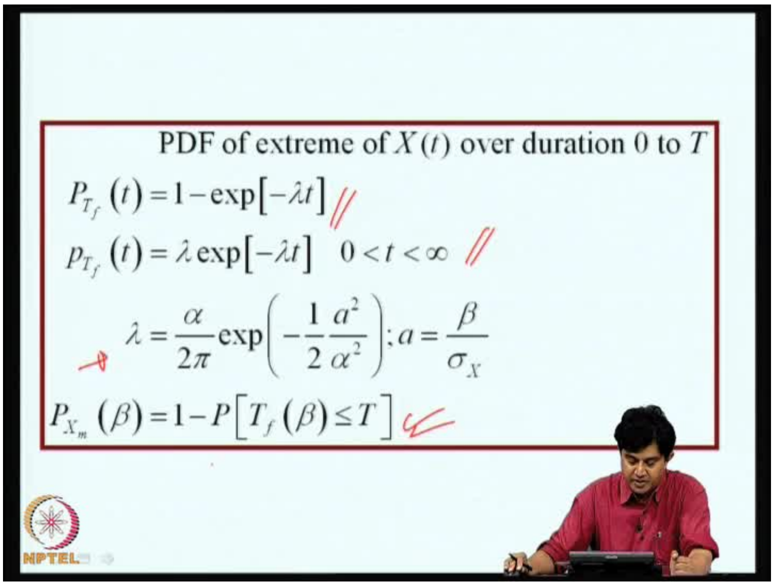

Now, the probability distribution function of extreme of X of t over duration 0 to t. This is related to the first passage time as we have seen earlier. We have already solved the problem of probability distribution and density function of first persist times and this lambda is the parameter we had determined. The probability that x m is less than equal to beta is same as the first persist time is t f of beta is greater than capital T. From that, we get this and this is the answer that we are looking for.

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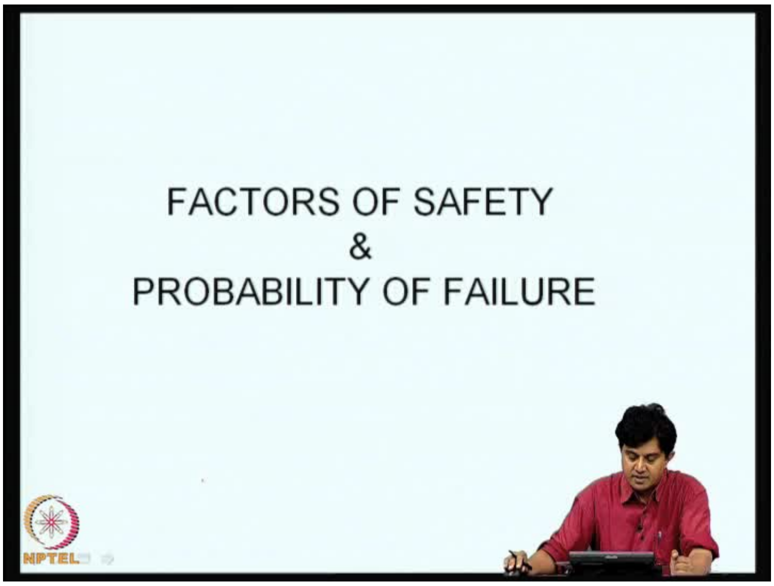

So, this completes the discussion on characterizing properties such as first persist time, level crossing statistics, peaks, occupation time and extremes of a stationary Gaussian random process. One of the questions that often come up in discussions is the possible relationship between factors of safety and probability of failure.

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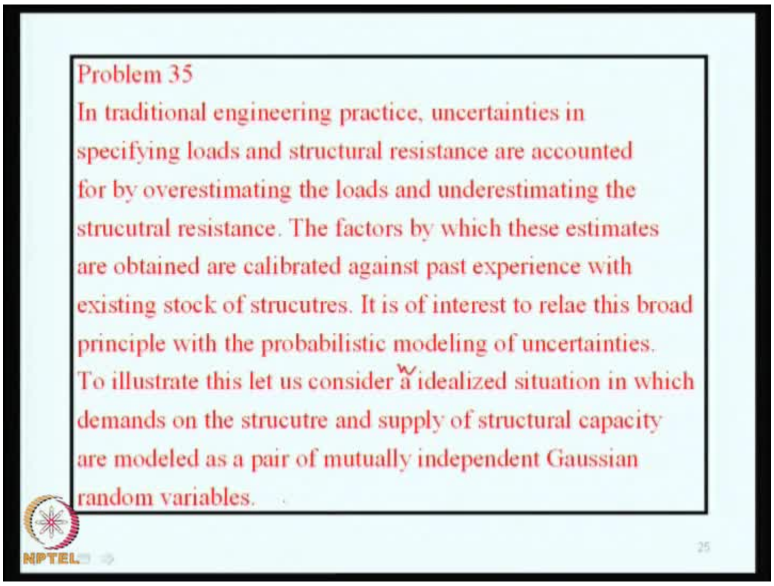

This can be explained through simple logic and that is what I am trying to do here. In this problem, what we do is, we consider the following situation. In traditional engineering practice, uncertainty is in specifying the loads and structural resistance are accounted by overestimating the loads and underestimating the resistance. The factors by which we over estimate loads and the factors by which we underestimate the resistance is calibrated based on experience with performance of existing stock of structures.

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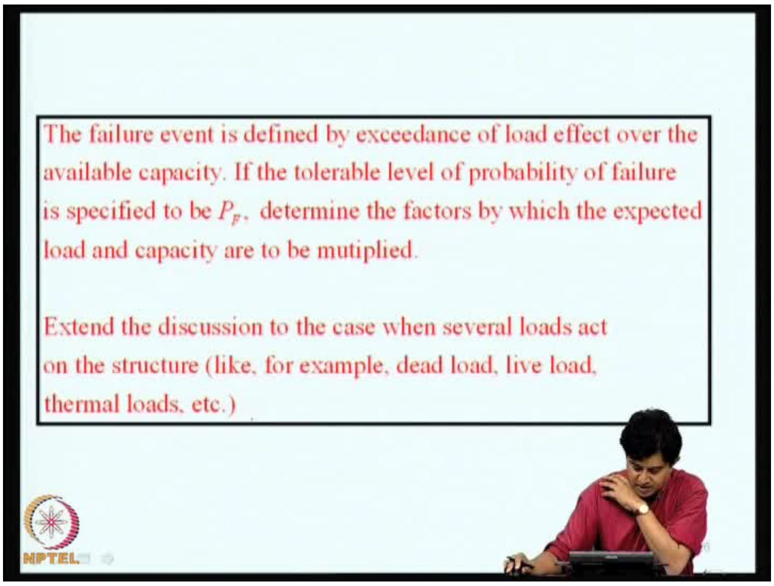

Now, when we look at this problem with a probabilistic framework ,the question that we can ask is, how can we arrive at these factors using theory of a probability? So, what we do is to illustrate this, et us consider an idealized situation in which demands on the structure and supply of structural capacity are modeled as a period of mutually independent Gaussian random variables. The failure event is defined by exceedance of load effect over the available capacity. If the tolerable level of probability of failure is specified to be P F, the question we are asking is: determine the factors by which the expected load and capacity are to be multiplied so that the target probability of failure condition target probability of failure is matter.

Now, we called out the problem is also to extend the discussion on the case when several loads act on the structure, like for example dead load, live load, thermal loads etc., How do we extend the lotion of factor of safety to such situations?

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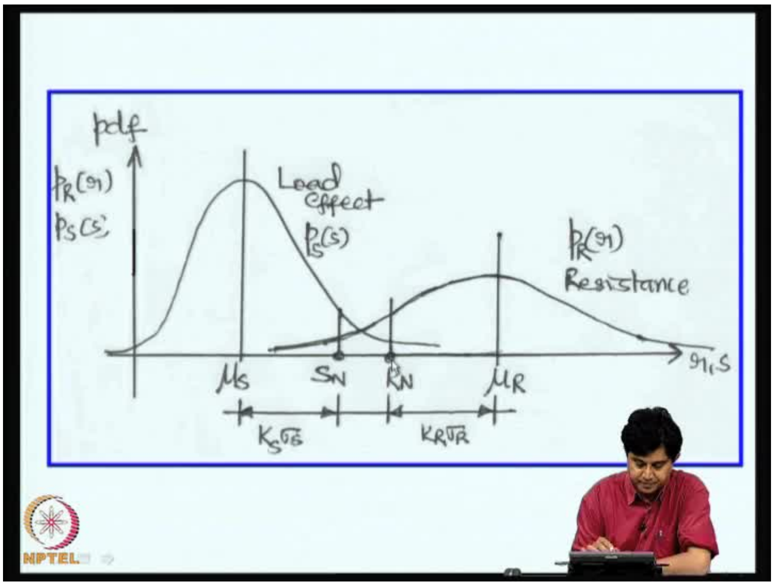

Now, in this view graph, we are showing the probability density function of load effect. This is density function and this is the probability density function of structural resistance. The expected structural resistance is higher than the expected load effect and this discharge between thesetwo vertical lines is some measure of safety margin in the structures. mu S is mean of the load effect, mu R is a mean of resistance, p S of S is the probability density function of S, p R of r is the probability density function of resistance. mu S plus case into sigma s we call it as characteristic value it could be 1 sigma or 2 sigma or some number. Then similarly, mu R minus k R into sigma R is called characteristic value of the resistance. T We will use these two notations for some discussions to follow.

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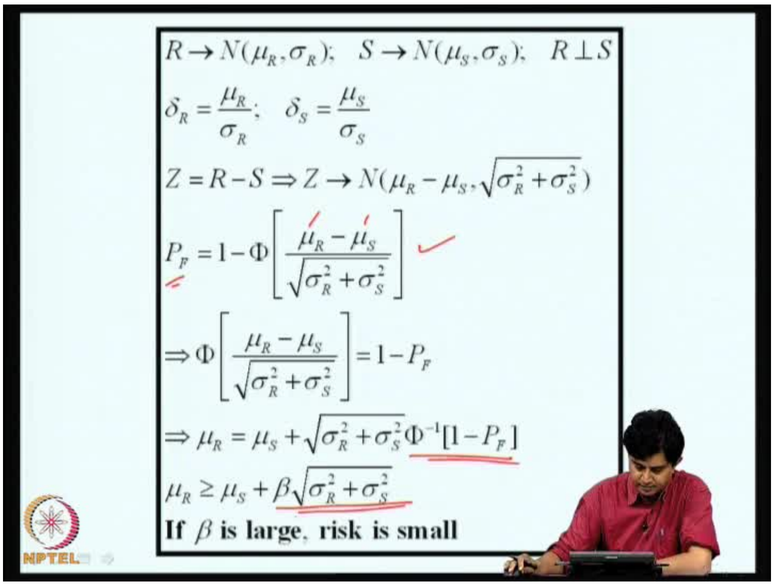

Now, as per the given problem, r is modeled as a normal random variable with mean mu R and standard deviation sigma r. Similarly, s is modeled as a normal random variable with mu S as a mean sigma S as S standard deviation and R and S are independent. We introduce the notation coefficient of variation delta r and delta s which is the ratio of mu R by sigma R mu S by sigma S.

Now Z, we define as R minus S. Z is a safety margin we can say. Z is a random variable. Since R and S are Gaussian random variables, Z is also a Gaussian and we can show that the mean of Z is mu R minus mu S and standard deviation is sigma R square plus sigma S square. So, from this the probability of failure can be evaluated as the event probability Z greater than Z less than 0 so that is safety margin is less than 0. So, the probability of failure can be evaluated and to be shown by given by this.

Now, the question we are asking is,suppose P F is specified, the mean of R in mean of S are buried, the question we are asking is how much we should over estimate the loads and how much we should under estimate the resistance so that this target probability of failure is realized. We rearrange these terms and X plus mu R and mu S plus sigma R square plus sigma X square square root phi inverse this. That means, mu R should be greater than or equal to mu s by this factor if beta is large, the risk is small. That means the risk is loads exceeding the structural capacity.

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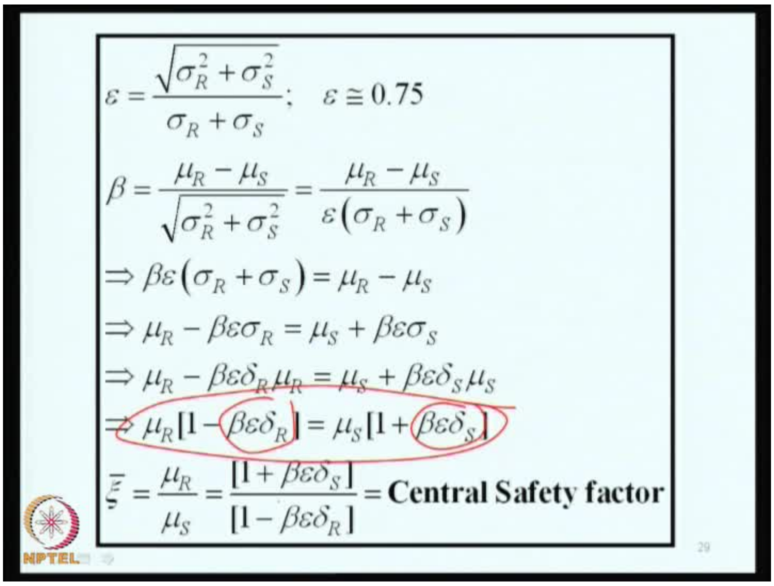

To bring it to the form of the factors that we are looking for, we introduce the variable epsilon which is square root of sigma R square plus sigma s square divided by sigma R sigma S. Typically this number is about point 7 5. It is non-dimensional. So, we write for beta which is mu R minus mu S by square root of sigma R square plus sigma S square in terms for this square root sigma R square plus sigma S square, I write epsilon into sigma r plus sigma S and if I rearrange the terms, I get this particular format where the structural resistance is under estimated by this factor and load is over estimated by this factor and these factors are explicitly related to the target probability of failure and parameters epsilon and delta are related to the uncertainty in variable R and S.

We can define for example, the ratio of mu R to mu S and this we can call it as central safety factor. So, through this analysis we are able to relate factor of safety explicitly to the target failure probability and uncertainties in the loads and structural capacities.

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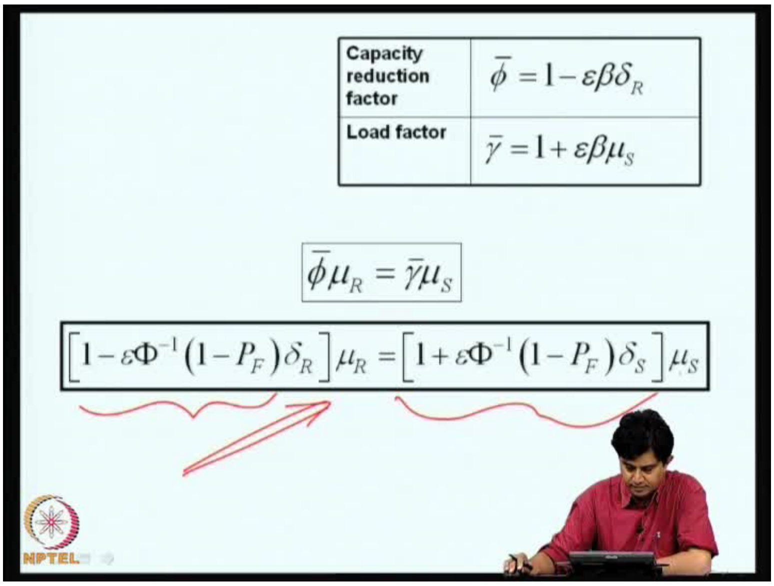

That would mean the format that we have got is in the form phi bar in to mu R must be equal to gamma bar in to mu S. This is the design equation where phi bar is called a capacity reduction factor and gamma bar is called a load factor. The design format that we have to follow is given by this. That means, this reduction factor here in to mu R must be equal to this over estimation factor in to the load.

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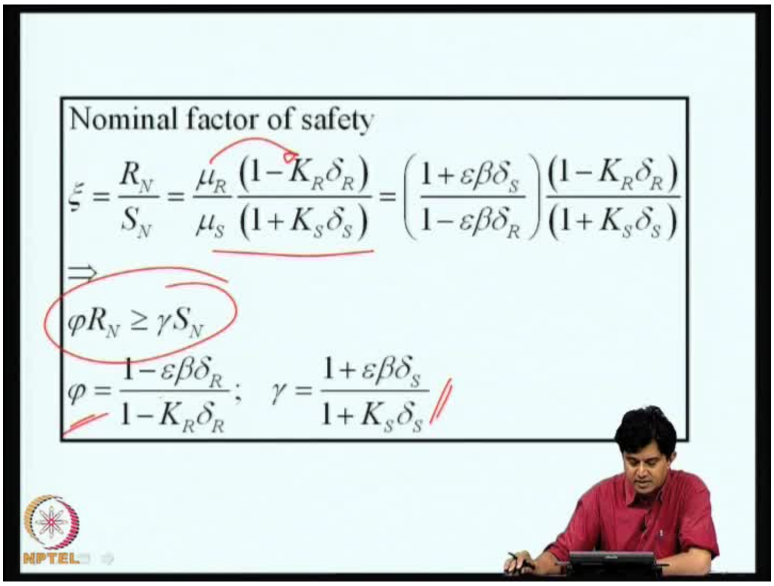

Now, we have defined factor of safety with respect to mean values. We can also define with respect to the characteristics value. The characteristics values are typically used in specifying strength such as for example, concrete and things like that, we use characteristics values so the factor of safety can also be expressed in terms of characteristic value. Now, if we do that, a characteristic value of R is related to expected value of R and standard deviation through this relation and similarly another relation for characteristics value of S and if we rearrange these terms, I get the design format in terms of characteristic equations characteristic values of R and s and here the factors are different from what were applied on the mean and mean values.

So both the formulae, if you use this or the other one with respect to the mean would ensure the same level of probability of failure but they are expressed in terms of different typical values for R and S. In one case it was mean, here it is in terms of characteristic values.

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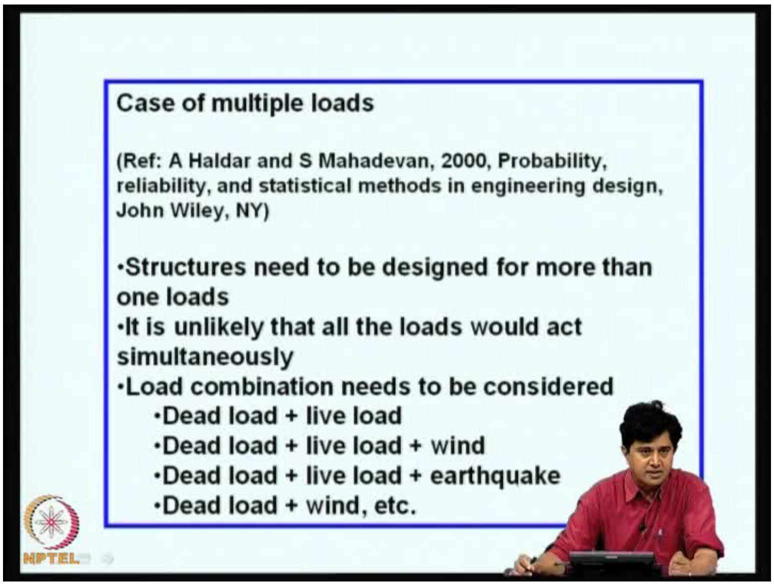

What happens if the structure is now subjected to more than one loads? For example, as structures need to be design for more than one loads, it is unlikely that all the loads would act simultaneously. we may need to consider load combinations like dead load plus live load, dead load plus live load plus wind, dead load plus live load plus earthquake and so on so forth. So, in that situation, how do I over estimate the load effects and under estimate the structural resistance?

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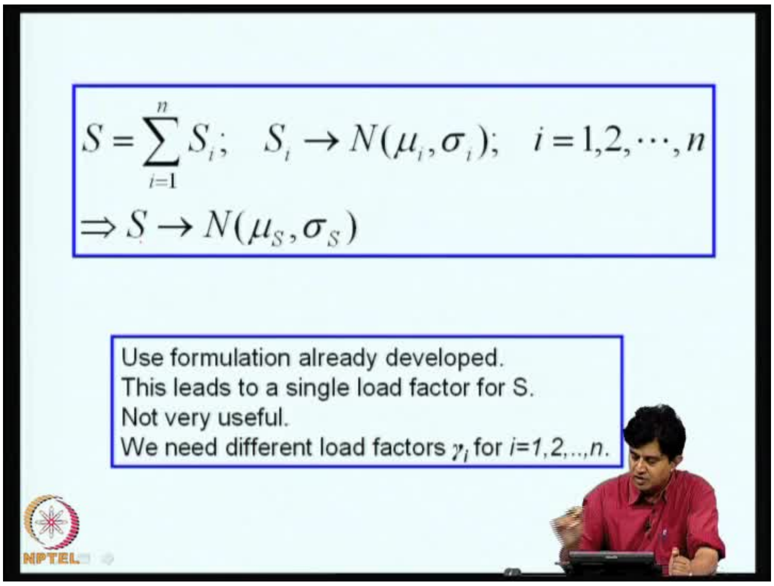

We again consider linear models. We assume that the load combination is linear. So, I can define s as summation i equal to 1 S i where S i let us assume they are all normal distributor independent and S thus becomes a normal random variable. This is the combined effect of all the loads. So, we can directly characterize we can go back to the first formula where the combined effect of all the loads is characterized in terms of single random variable. But this approach is unlikely to be useful because the uncertainty is associated with say different load sources or characteristically different. For example, the uncertainty in dead load and uncertainty in wind or earthquake or traffic on a bridges etc., are quite different. We need to assign the factors individually to each of these load sources. So, how do we proceed?

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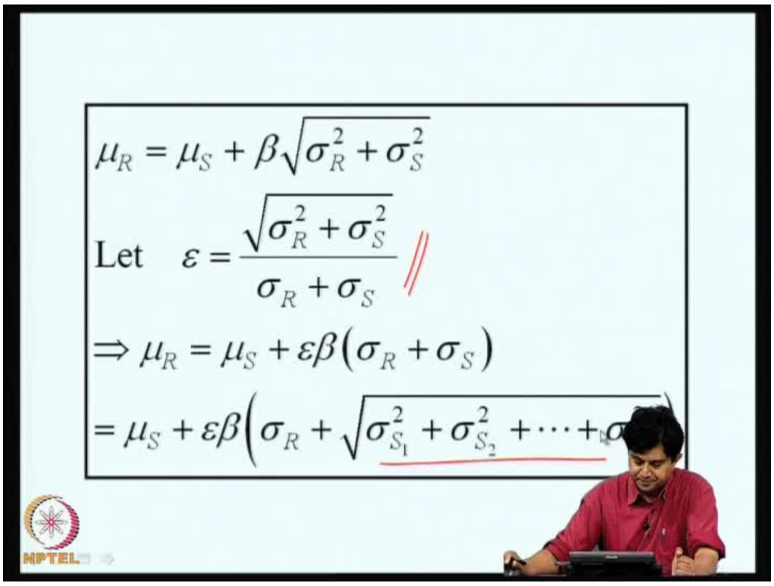

We again begin by the formulation mu R is mu S plus beta sigma square root sigma R square plus sigma S square and we defined this epsilon we retain that. But now, sigma s the standard deviation S now I express in terms of the different sources of loads as shown here.

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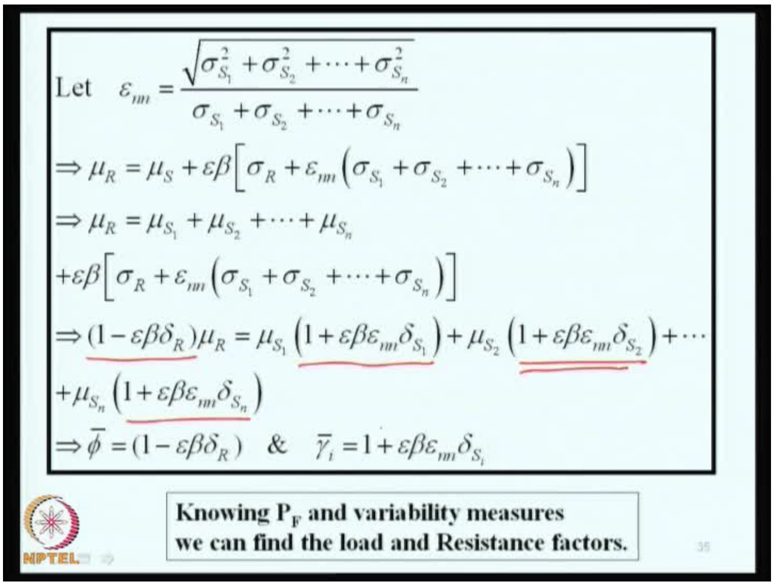

I introduce another quantity epsilon n n as the ratio of square root of the sigma S 1 square sigma S 2 square etc., divided by sum of all these standard deviations and based on that i can write the expression for mu R and mu S through this and if we now rearrange these terms in systematic manner, we get the design equation as shown here. This is the reduction factor on R and this is the over estimation factor on S 1 over estimation factor on S 2 and so on so forth.

So, each of these factors which multiply the mean values of resistance and loads depend upon the target reliability that we are looking for or the target probability of failure that is specified and the uncertainty characteristics of various load sources and structural properties. Now, this is the advantage of using a systematic frame work to characterize uncertainties.

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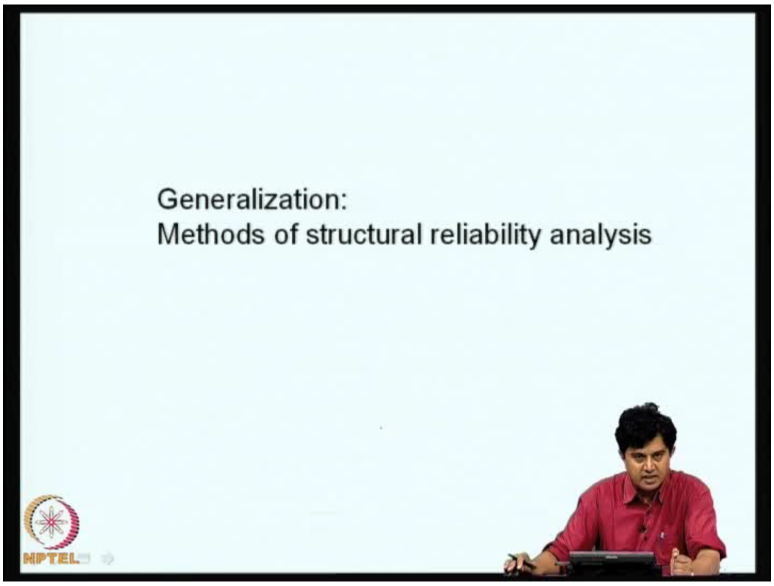

Here, we are able to decide if we need to estimate the factor by which says the load source S 2 has to be over estimated. This formulation clearly tells us what data should be gathering and how to interpret that. Now, the discussion here is for highly idealized situation of linear problems but in reality structural behavior and uncertainties etc., are more complicated. Typically, we get non-linear and non-Gaussian non-linear performance functions and non-Gaussian random variables. So, the theory of structural reliability helps us to tackle those problems. So, generalization to these discussions, if you are interested, you should look into methods of structural reliability analysis.

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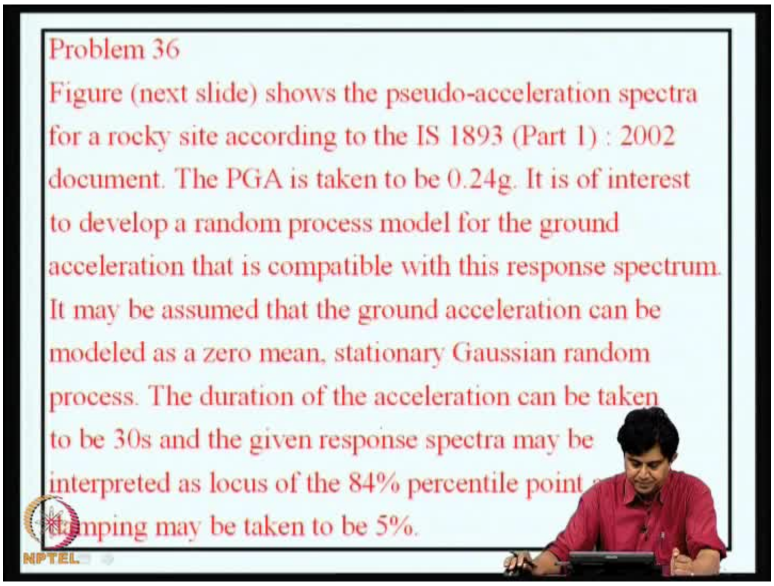

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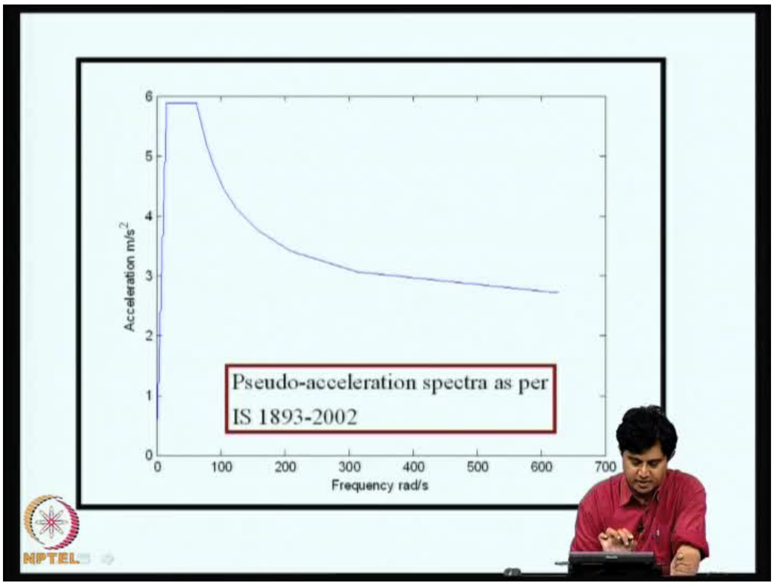

Now ,we return to some problems in theory of random processes. The problem that I am considering here is: we consider the pseudo acceleration response spectra given in the IS code 1893 for a certain value of p ground acceleration and that is given here. The question that we are asking is, how do we simulate, how do we obtain a power spectral density function which is compatible with this pseudo acceleration response spectrum? So, that is the problem. The problem is to determine power spectral density function which is compatible with given response spectrum.

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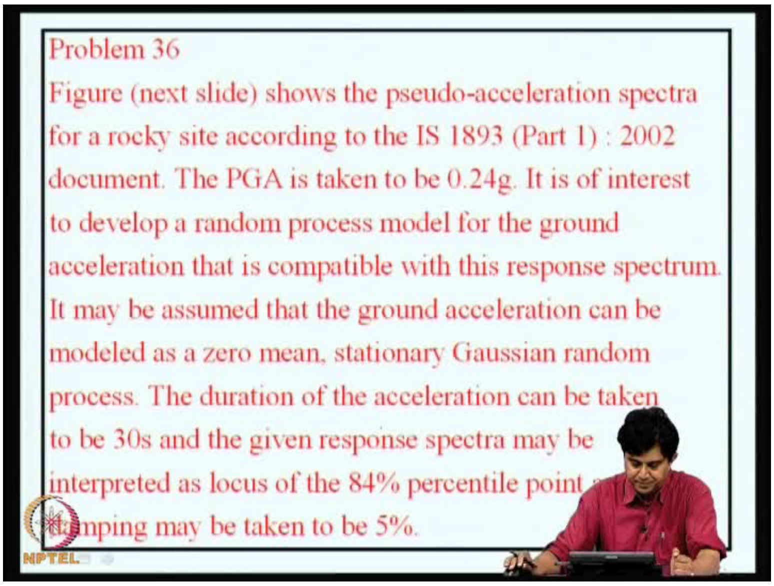

So, underlying this assumption is the logic that the ground acceleration is a Gaussian random process is 0 mean. So, I can read through this. The figure shows the pseudo acceleration spectra for a rocky site according to i s 1893. The p ground acceleration is taken to be point 2 4 g. It is of interest to develop a random process model for the ground acceleration that is compatible with this response spectrum. It may be assumed that the ground acceleration can be modeled as a 0 mean stationary Gaussian random process. The duration of the acceleration can be taken to be 30 seconds and a given response spectra may be interpreted as locus of 84 percentile point and damping may be taken to be 5 percentage.

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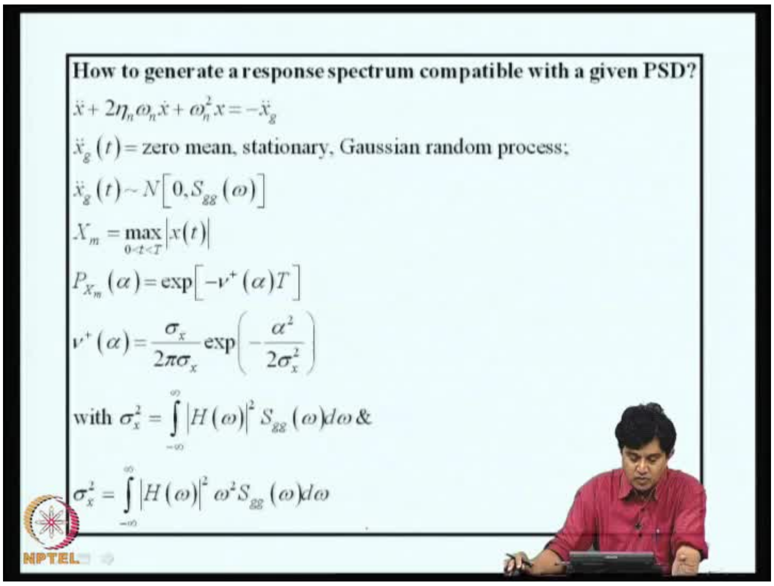

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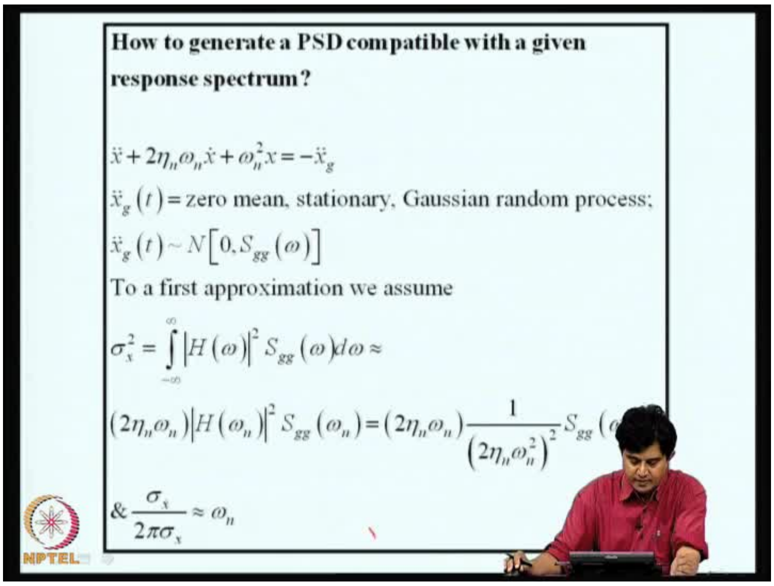

Now, with this data, how do we proceed? So, this we have discussed. We have discussed the problems on how to generate a response spectrum compatible with a given power spectral density function and also we have discussed how to generate a power spectral density function compatible with the a given response spectrum. These two problems we have discussed. This involves the extreme value theory of stationary Gaussian random processes and propagation of uncertainties in single degree of freedom linear system.

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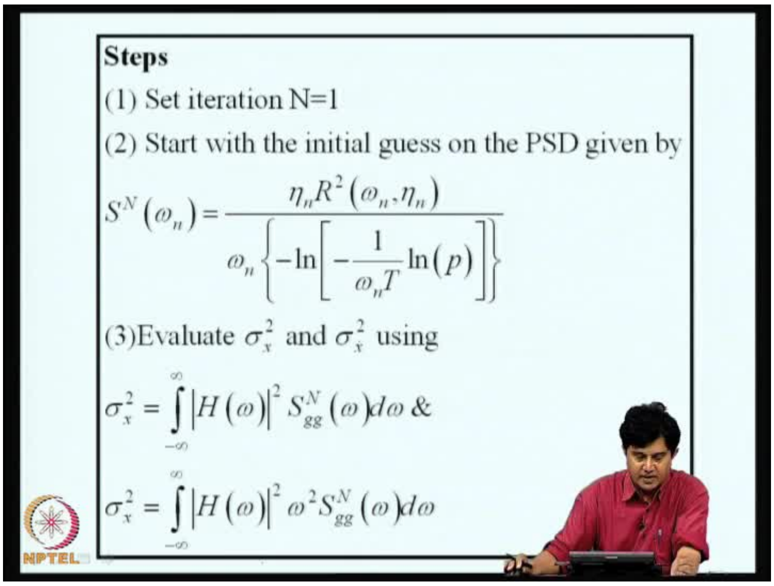

I will not repeat this theoretical formulations but we can quickly recall the steps again I had discussed this in one of this earlier lectures. To solve the given problem, you need to develop a computer program is not something that you can do on pen and paper. There are iterations involved etc., but this is a problem that is worth doing. You will learn several things about extreme values of random processes and definition of power spectral density function.

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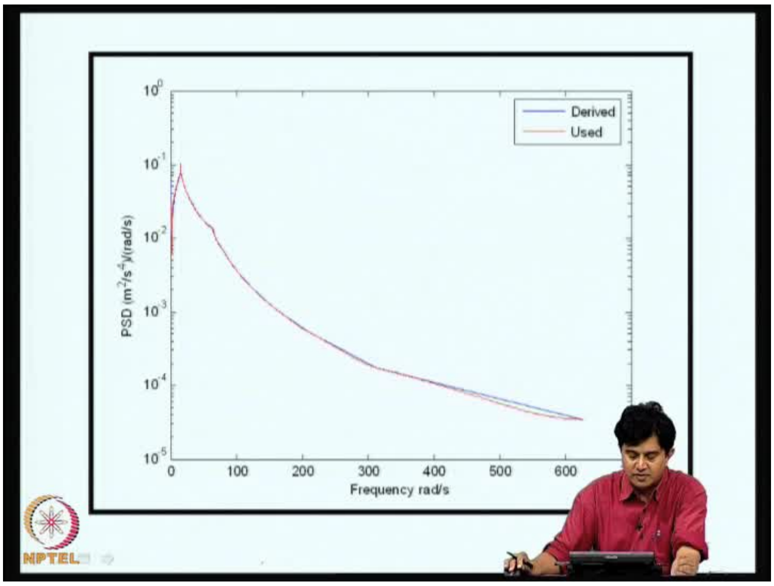

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I will show some of the results that has been obtained through on such software that we have developed. This is the target power spectral density function. What we did is the blue line is the target power spectral density function, the compatible power spectral density function obtain from the given response spectrum. To check whether the formulation is right using this power spectral density function, we rework the response spectrum and check that it is showing the same answer.

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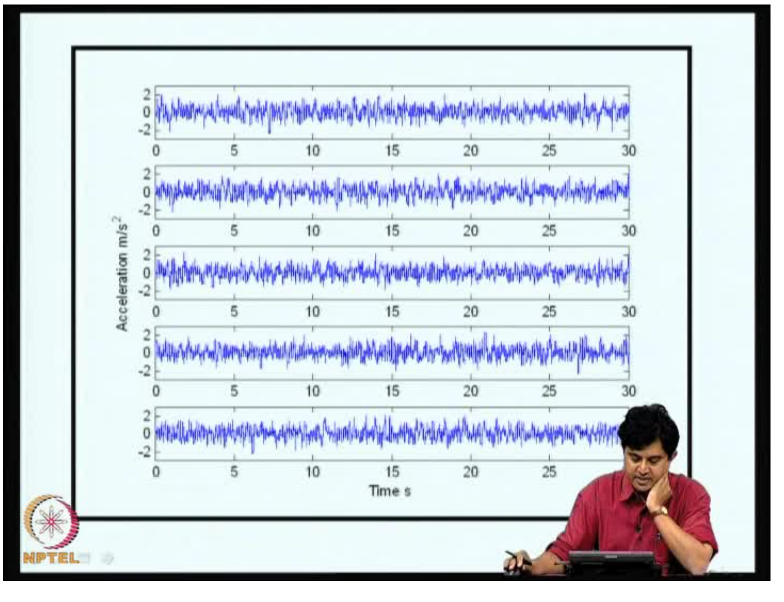

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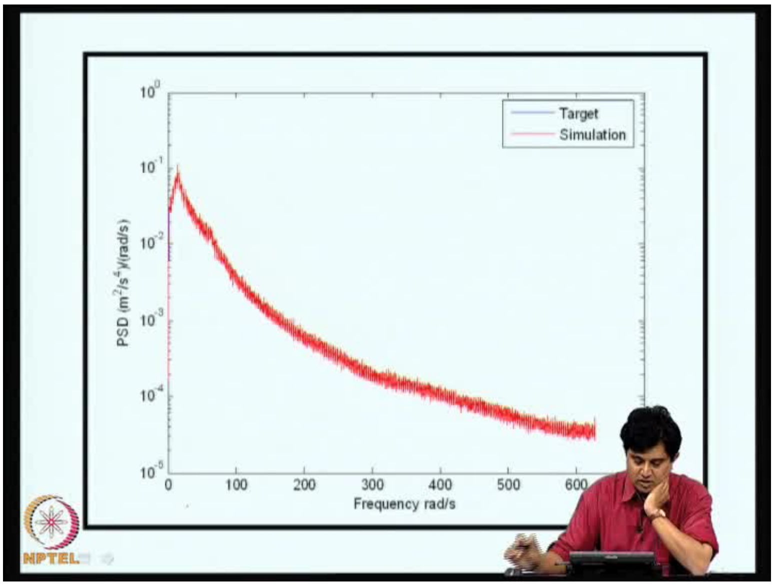

Here the that result of that exercise is shown. This is the red and blue are sitting on each other. So, it is satisfactory. Once the power spectral density function is determined assuming that the random process is mean square periodic and using theory of fourier series representation of time histories of such samples of such random process, we can simulate samples. Some of these samples are shown here for sake of illustration and from this samples, you can use methods of statistical estimation theory and estimate the power spectral density function. That also is shown here. From these 100 samples, we have estimated the power spectral density function and that is comparing the blue line is buried inside this waving z line. This is due to sampling fluctuations as sample size increases this will become a smooth curve.

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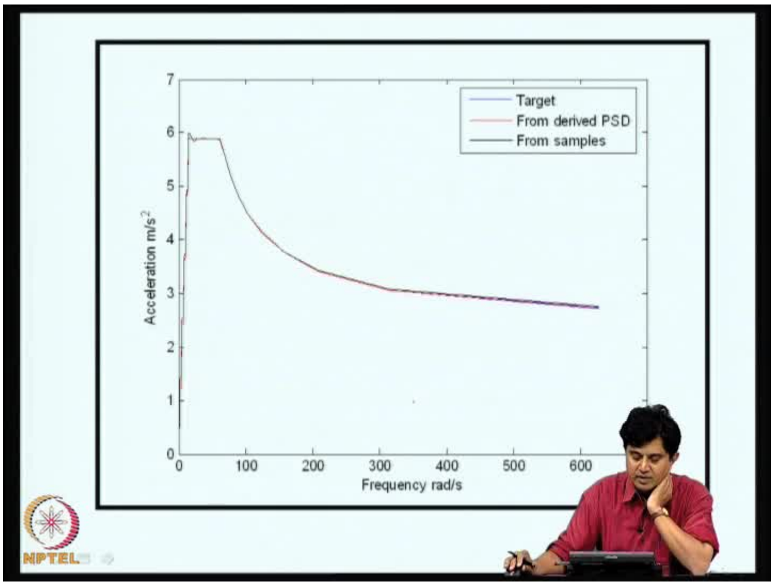

So, from the simulated sample estimated power spectral density function, again we have estimated the flow response spectra, just to make sure the things are all right in estimating the power spectral density. So, this also shows reasonable match.

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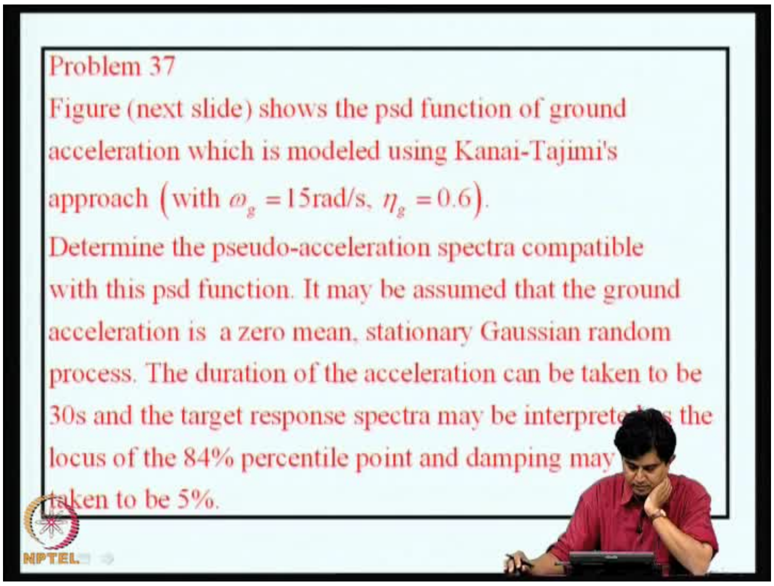

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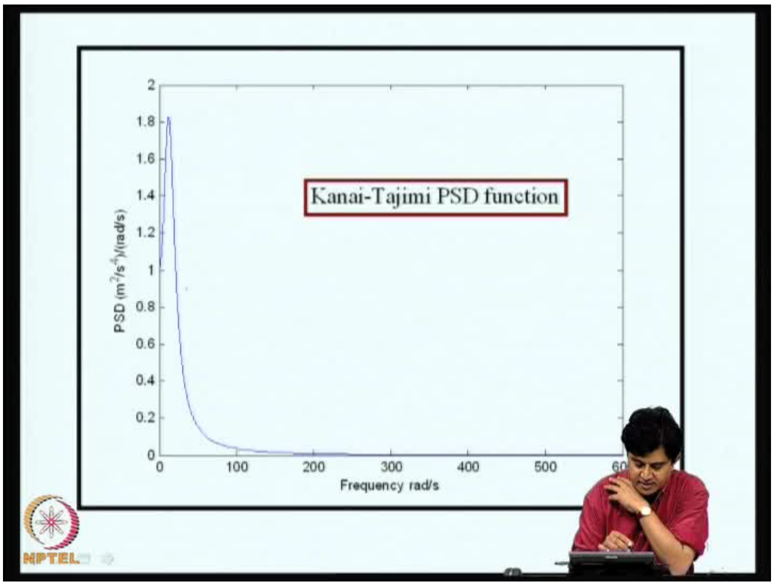

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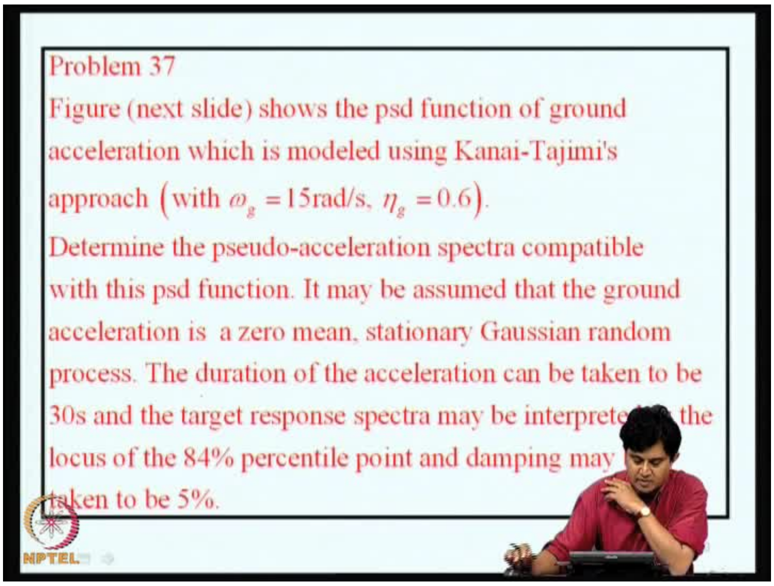

Now, in this example that we concluded discussing just now, we started with a well-known pseudo acceleration response spectrum and derived a compatible power spectral density function. Now, we can do the other exercise. We can start with a well-known power spectral density function model and try to derive the compatible response spectrum. So, this is the exercise that we do here. This figure, we consider a Kanai Tajimi power spectral density function. The problem on hand is, the figure shows the power spectral density function of a ground acceleration which is modeled using Kanai Tajimi's approach with omega g as 15 radian per seconds and etcetera g is point 6.

Determine the pseudo acceleration spectra compatible with this psd function spectral density function. It may be assumed that the ground acceleration is a 0 mean stationary Gaussian random process.

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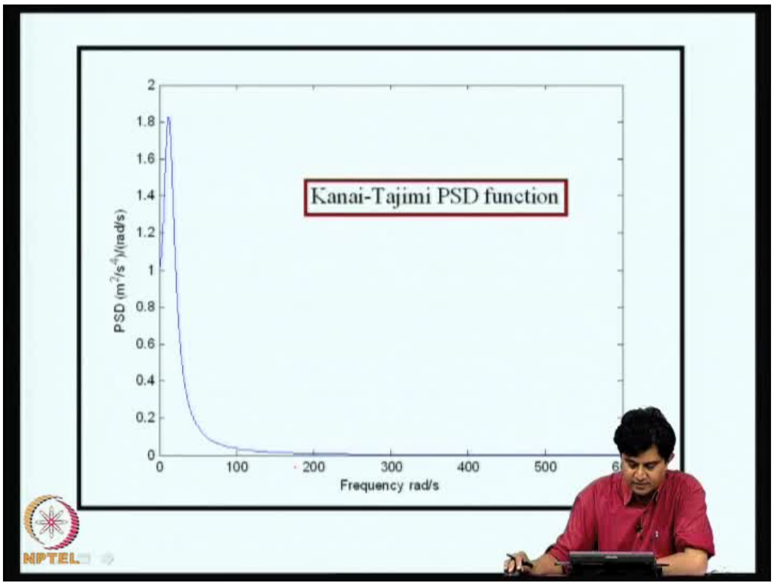

The duration of the acceleration can be taken to be 30 seconds and the target response spectra may be interpreted as a locus of the 84 percentile point and damping may be taken to be 5 percent as in the previous example. Here again, you want to do this, you have to write a program and that procedure is already described in one of the earlier lectures.

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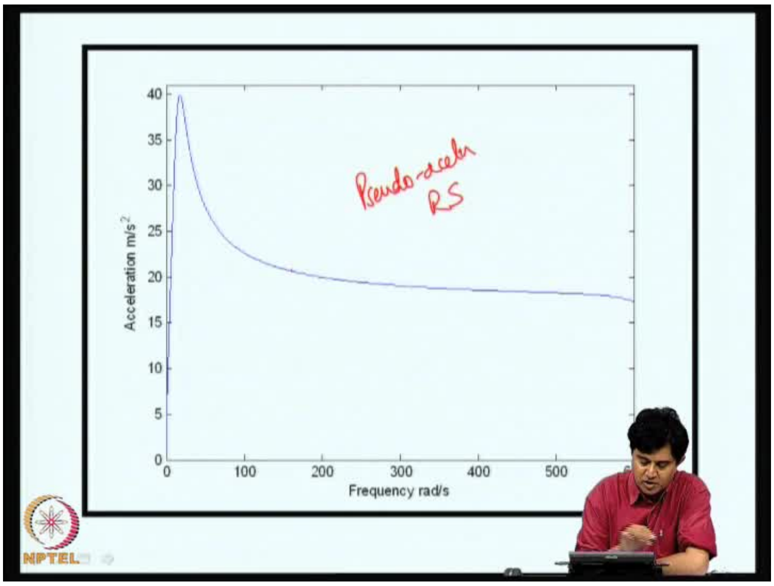

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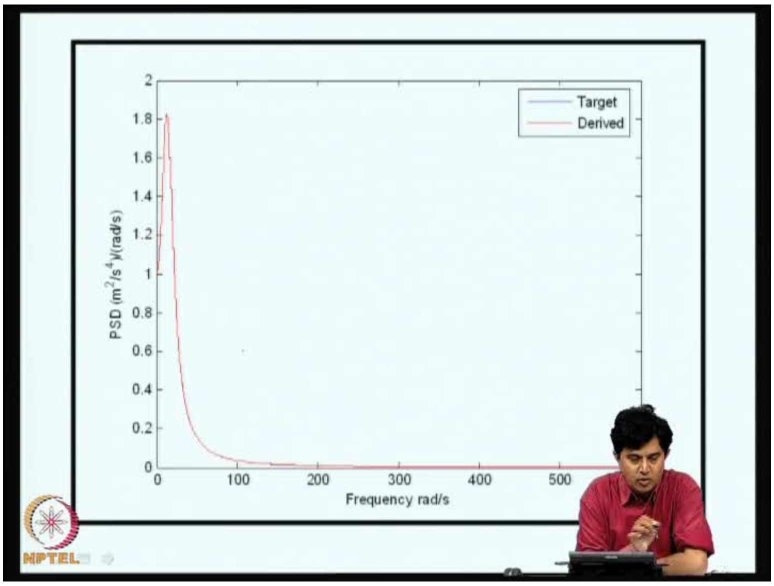

So, compatible with this power spectral density function we obtain this response spectrum. This is the pseudo acceleration response spectrum which is compatible with the given Kanai Tajimi power spectral density function. Here again, after getting this response spectrum, starting with this as the response spectrum, we converted this to the equivalent power spectral density and recovered the target power spectral density function. Here, the two power spectral density functions, one which is the target that is this given Kanai Tajimi power spectral density function and the other one which is derived from the compatible flow response spectrum are superpose and the mutual agreement that we see point towards correctness of the development of the code.

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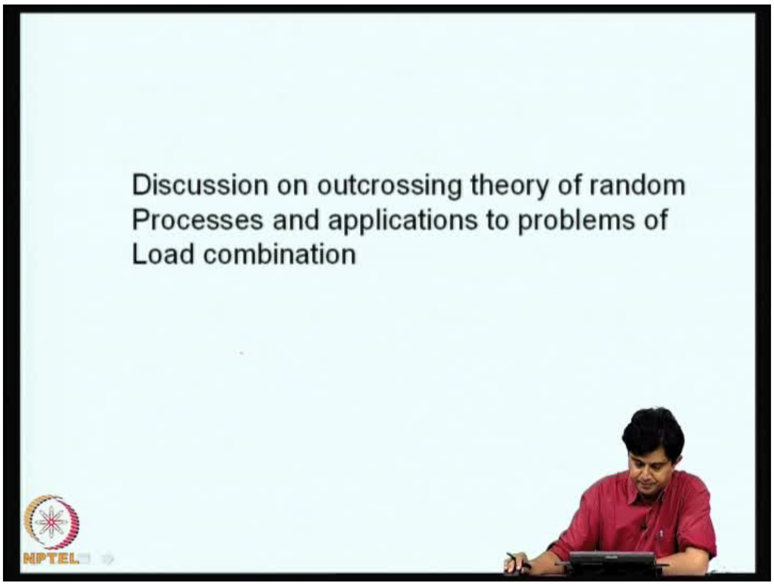

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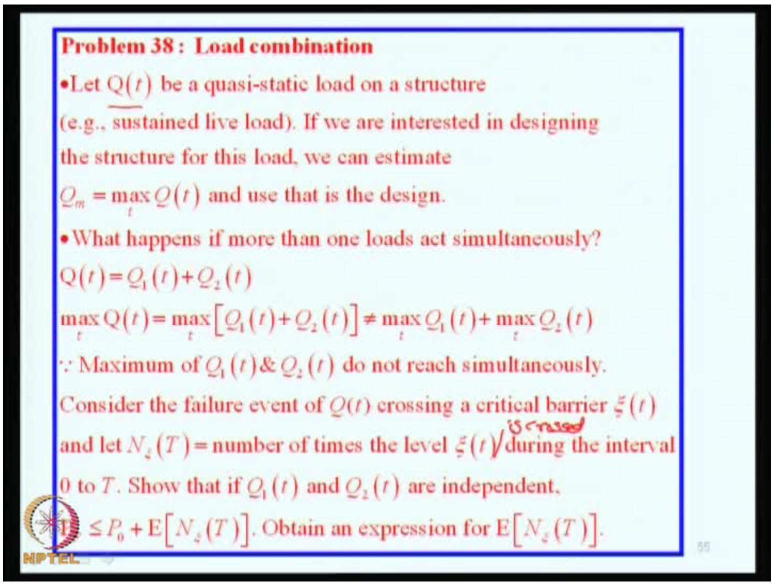

I nowconsider another class of problem. This is discussion on out crossing theory of random processes and application to problems of load combination. We will see what this means. The problem statement is as follows. There is some back ground and then there is a question. Let Q of t be a quasi-static load on a structure, for example sustained live load. If we are interested in designing the structure for this load, we can estimate the maximum value of the live load. For example, Q m is maximum over t Q of t and use that in the design but often this practical situation is more complicated.

We can ask the question: what happens if more than one loads acts simultaneously? For example, Q of t is q 1 of t plus Q 2 of t. Now, the difficulty here is the maximum of Q of t which is maximum of some of Q 1 plus Q 2 is not actually equal to some of maximum values of Q 1 and Q 2 simply because the maxima reset different times.

Now, if we consider the failure event of Q of t crossing a critical barrier psi of t, Q of t can be interpreted as a low defects. It need not to be the load it say we can think them as response due to dead load response due to live load so on and so forth. Therefore, we can talk about a critical barrier and we are considering now the failure event of Q of t crossing a critical barrier psi of t xi of t. Now, notice in earlier formulation when we discussed level crossing problems, this barrier was a constant. It is not time varying. Now, for sake of generality, we are taking this barrier also as a function of time. Now, we define N xi of T as number of times the level xi of Ts is crossed in the interval 0 to T.

Now, the problem on hand is: we have to show that, if Q 1 and Q 2 are independent, the probability are failure we can get a bound which is P naught plus expected value of N psi of t where P naught is failure of the structure at T equal to 0 and also we are asked to obtain the expression for expected value of N psi of T.

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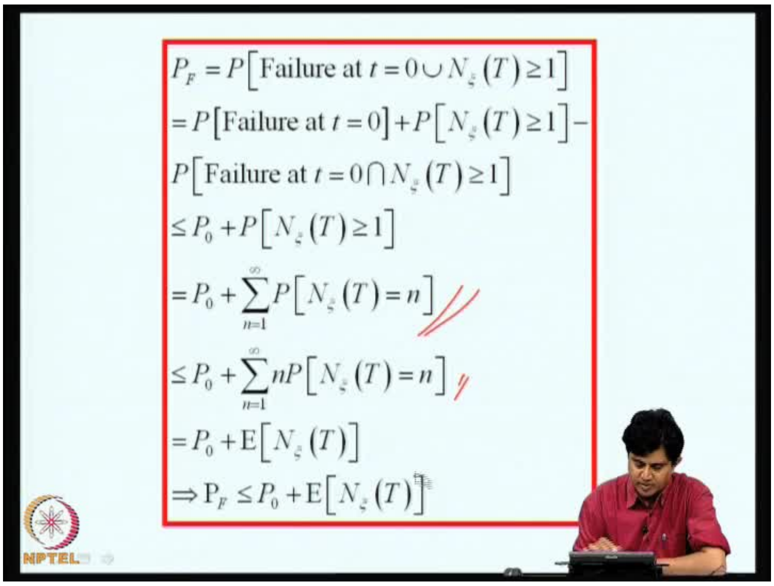

So, how do you proceed? Probability of failure is failure at T equal to 0 union the number of crossing between 0 to T is greater than or equal to one. Now, these two are not mutually exclusive. Therefore, if you use for the axiom, you will get probability of failure at t equal to 0 plus probability that N psi of t is greater than or equal to 1 minus the probability that failure at t equal to 0 intersection N psi of t greater than or equal to 1.

Now, we can place a bound by ignoring the third term. First two terms will be greater than P F will be less than or equal to this. Probability of N psi of T greater than or equal to one given that N xi is a Poisson random process is a counting process we can write in this form.

This itself is less than or equal to here I this is actually summation of probabilities but now I am introducing n here. So, this number will be less than or equal to this. The second term here is nothing but expected value of N xi of T, variable to show that P F is less than or equal to p naught plus expected value of N psi of T. So, that is a first part of the problem.

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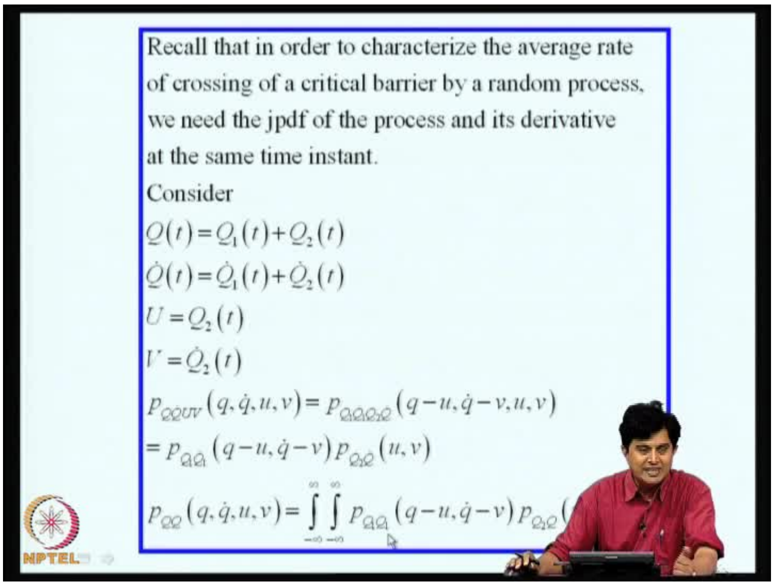

Now, more difficult question is how do you evaluate this expected value? We can recall bit of what we did earlier in order to characterize the average rate of crossing of a critical barrier by a random process, we need the joint probability density function of the process and it is derivative at the same time instant.

Now, we are having the process Q of t. I need joint density of Q of t and Q dot of t at the same time and that involves four random variables Q 1, Q 2, Q 1 dot, Q 2 dot. I need to construct the joint density function of Q Q dot. So, what I do is, I introduce two dummy variables U and V. U is Q 2, V is Q 2 dot because this is two functions of four random variables. I have to make it as a problem in four functions of four random variables. I introduce this dummy variables and first step, I determine joint density of Q Q dot U V and I get this in terms of, you can express the inverse relation and I get this as the relation.

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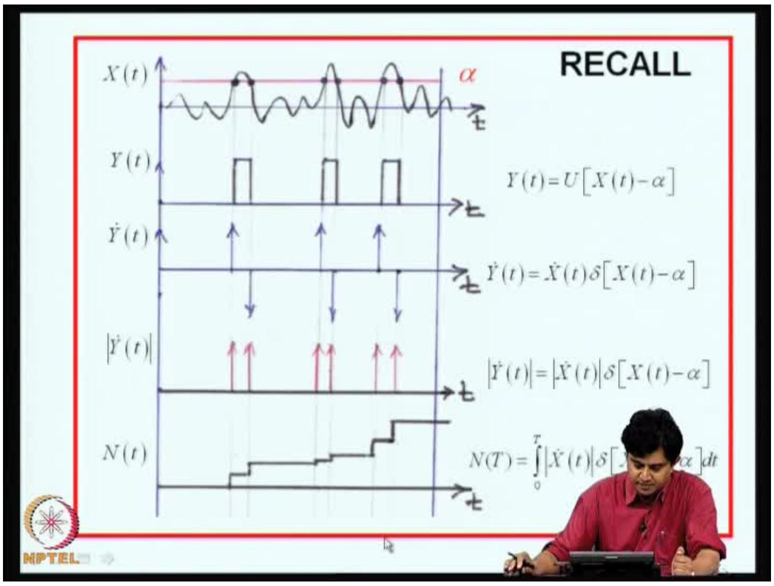

Since Q 1 and Q 2 are independent, I can write in this form and following this, we want the marginal density of Q Q dot, we carry out integration with respect to U y. This is nothing but the convolution of joint density of Q 2 Q dot Q 2 dot with Q 1 Q 1 dot which is not surprising. Now, again we need to recall this is something that we have done for a stationary random process X of t, we have developed the algorithm for finding the number of times a level alpha is crossed and that we have shown to be we first introduced Y of t as the step function u of X of t minus alpha, the differentiation of that and the modulus of that and summing all this spikes, we get N of t as 0 to t mode X dot direct delta X of t minus alpha d t.

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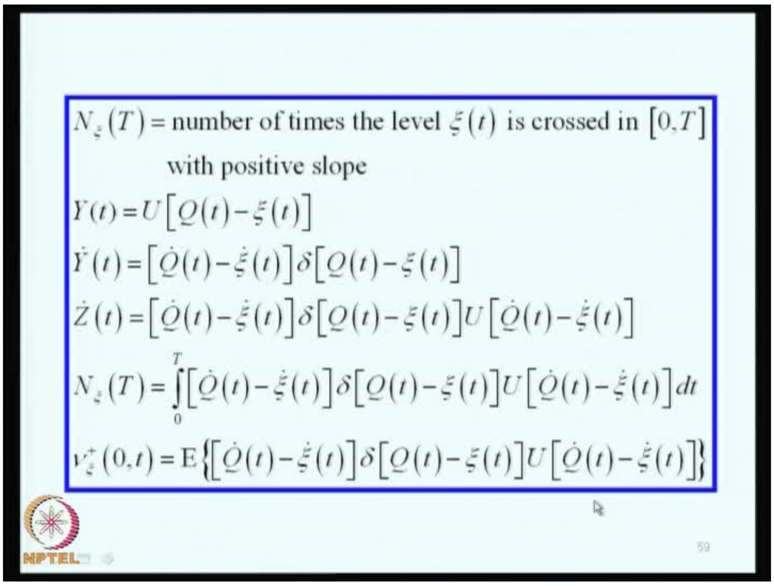

So, we need to exterd this logic for the given problem. N psi of t is number of times the level xi of t is crossed in 0 to capital T with positive slope. Now, I define Y of t is Q of t minus xi of t. This is now a function of time. Earlier, we are taken into the alpha. Now, we have taking into the function of time so that we need to carry forward. Y dot of t is derivative of this which is Q dot minus xi dot direct delta of Q minus xi. Now, I define I want positives crossings. Therefore, i multiply this by the relative gradient to be positive. That is step function of Q dot minus xi of t. Therefore, the required number of evens that we are looking for is, integral of this counter 0 to t Q dot minus xi dot direct delta Q minus xi step function Q dot minus xi dot.

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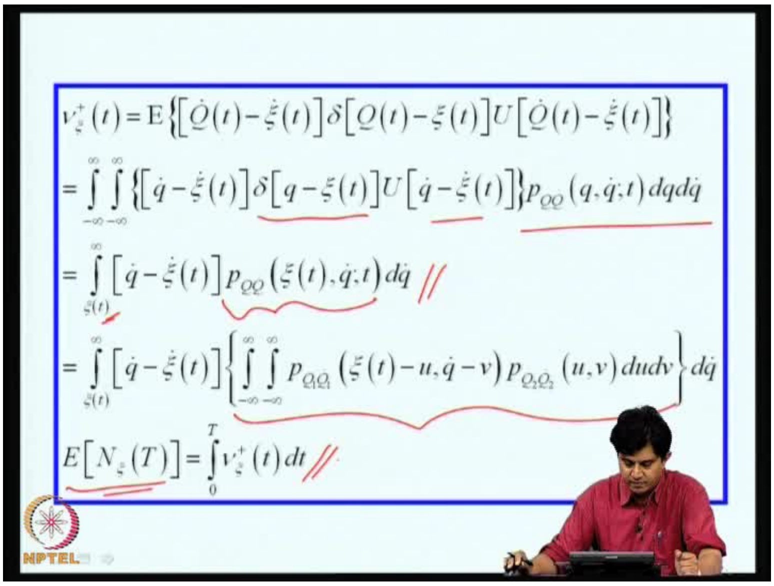

We are interested in the expected value of this integrant that provides the rate at which these events are occurring. The average rate of occurrence of these events. This has the random variables Q and Q dot. This expectation can be written in terms of the joint density of q and q dot and since there is the direct delta function and step function certain simplifications are possible, the direct delta function means integration with respect to q can be performed wherever there is a q replace by xi of t. This step function means the limit from minus infinity to plus infinity will now become limit from xi of t to infinity.

With that in place, I get this expression which is the average rate at with the events are occurring now has the joint density of Q Q dot which we have obtained in terms of convolution of the twosecond order density of Q 1 and Q 2. So, the required expected value is given by this.

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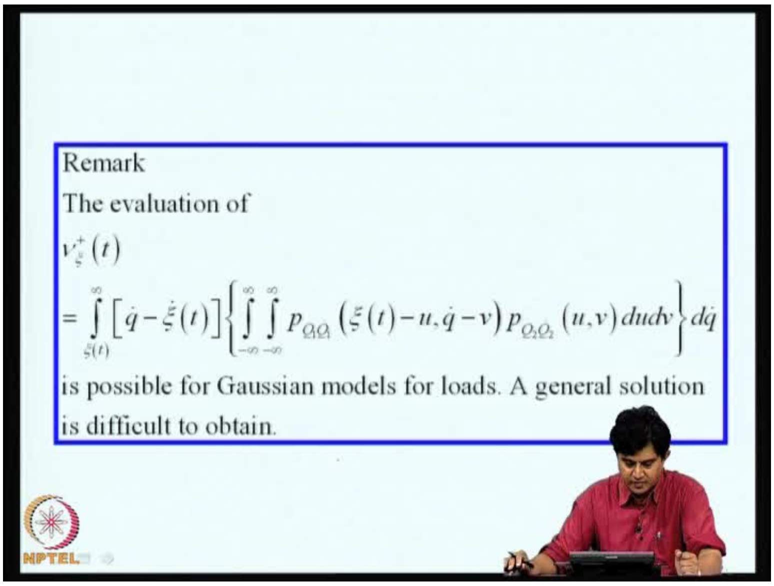

So, the moment we have the characterization of Q 1 and Q 2, we can evaluate this expectation and hence we get a bound on the probability of failure. The actual evaluation of this integral has been reported a literature for certain simple cases. For instance, for Gaussian it is straight forward and for certain other combinations, solutions are available but a general solution is difficult to obtain and certain alternative need to be developed.

Now, the idea of discussing this in the present context is the level crossing theory that we have learnt can be used for situations where there is no dynamics in the sense of inertial effects in structural behavior but still there is a time variation in the loads which cause call for adopting stochastic process models and consequently the questions that result on safety of the system are again related to concepts of level crossing first passes time etc., So, that was a message that I was intended to be convert through this example.

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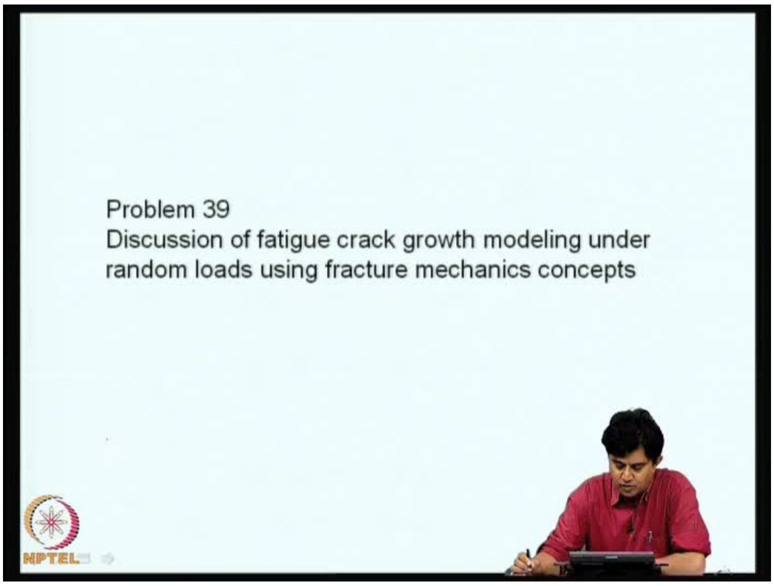

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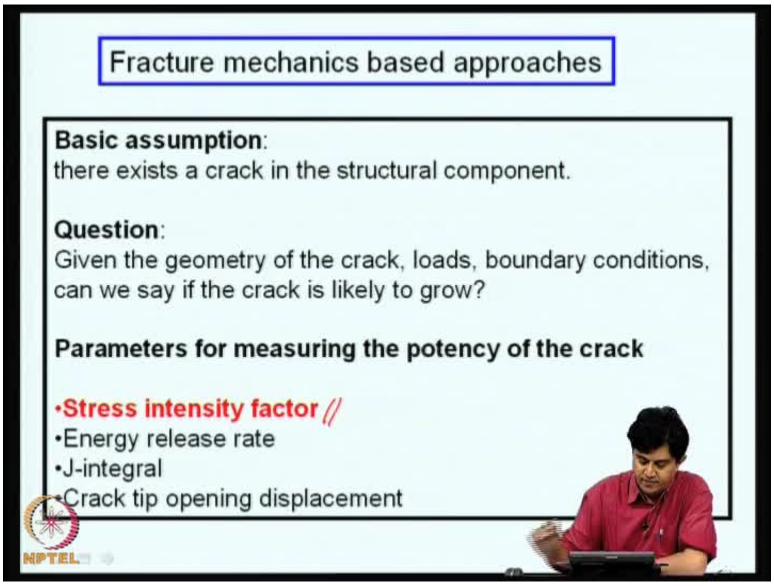

The next example is consider some discussion fatigue crack growth modeling under random loads using fracture mechanics concepts. We have developed the theory for accumulation of damaged under random vibration and we basically adopted formed a minor hypotheses but here we are get trying to look at the problem in a slightly a different way.

Now, the fracture mechanics based approaches the basic assumption is that there exists a crack in the structural component. The question is: given the geometry of the crack, the loads, the boundary conditions, can we say if the crack is likely to grow? This is the basic question that is answered in fracture mechanics. H

What are the parameters that measure the potential of the crack. Several things like stress intensity factor, energy release, j integral, crack tip opening displacement and so on and so forth. We will focus our attention on stress intensity factor.

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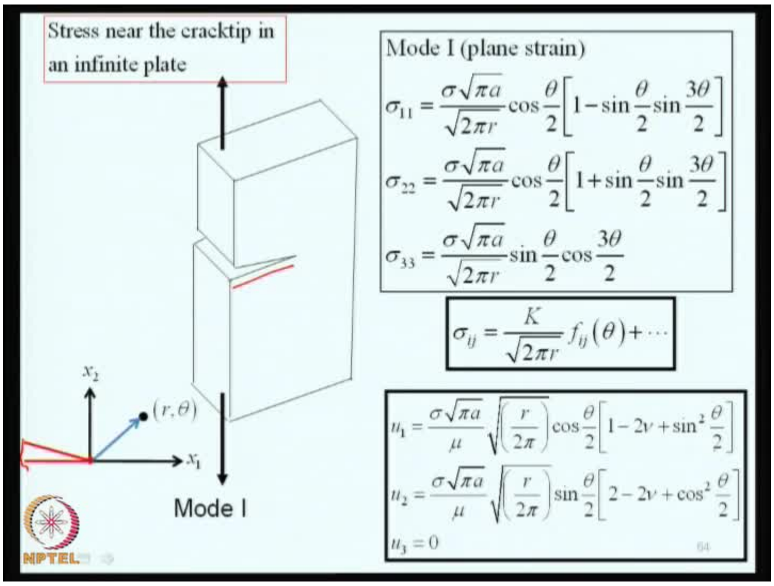

There are 3 different modes of cracking. This is the mode one failure where the specimen is loaded as shown here and this is the crack and we adopt a coordinate system origin at the crack tip. So, this red line is nothing but the crack that we are seeing here and this x 1 x 2 is the courtesan coordinate r theta is the polar coordinate. We can use the theory of solid mechanics and obtain the stress fields assuming say plate strain model near the crack tip and we can show that they are given by the sigma 1 1 2 2 3 3 stress components and what you should noticed is all these three components can be expressed in the form K into square root 2 pi by r into some function of theta.

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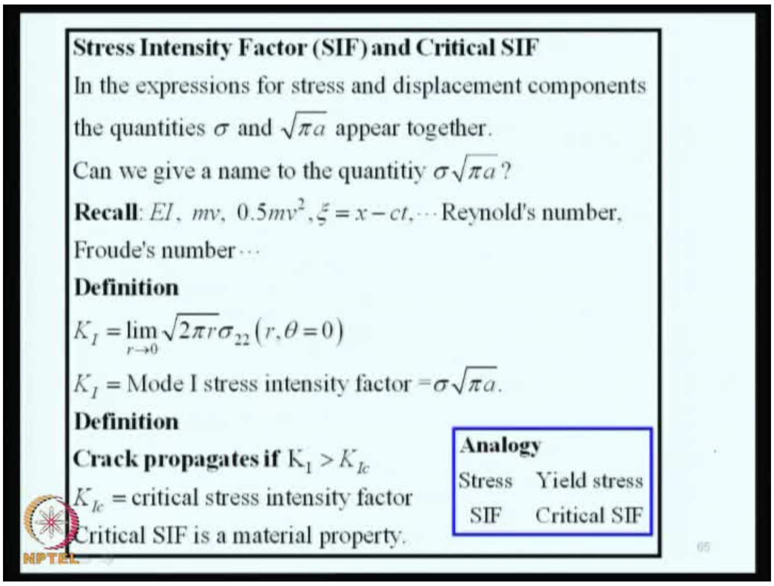

Similarly, displacement field also can be found out. u 1 is again we are having sigma into square root pi a and some function of r and some function of theta. Now, in the expression for stress and displacement component, the quantities sigma and square root of pi a are appearing together. The question is: can we give name for this? We have done this in the past in other branches of mechanics. For example, young's modulus and area moment of inertia get multiplied in Euler Bernouli beam theory and EI we call it as flexural rigidity.

Similarly, mass into velocity, we call as moment mass into velocity square as kinetic energy and so on and so forth. So, what we do is, we call this quantities sigma into square root of pi a as stress intensity factor. So, this is sigma into square root pi a. Now by definition, the crack propagates if K I is greater than a critical stress intensity factor which is the material property. Now, the analogy is like stress and yield stress and SIF and critical SIF.

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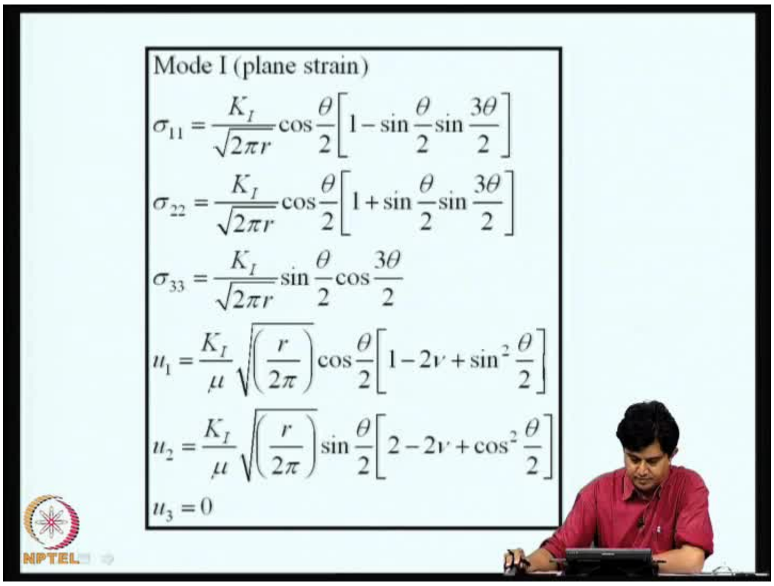

So, if stress exists yield stress, the linear behavior seizes. Similarly, if SIF exceeds critical SIF, the cracks propagates. In terms of stress intensity factor, the stress fields and displacement fields in r theta coordinate can be summarized as shown here.

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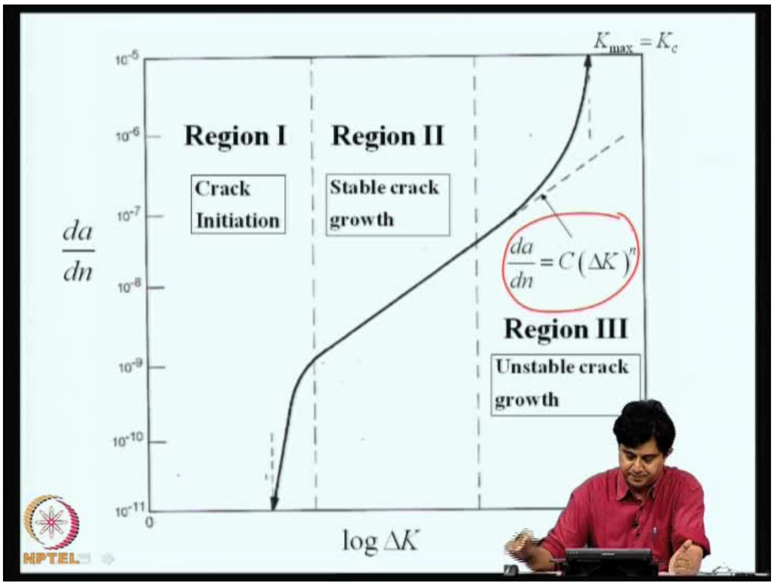

Now, experimentally, it is observed that the rate at which crack propagates, suppose a is a crack length and n is a number of cyclic cycles of loading, d a by d n if you plot as a function of log delta K, delta K is K is the stress intensity factor delta K is a increment.

ICharacteristically three regions are observed. The first region is crack initiation, second one is region of stable crack growth and third one is unstable crack growth.

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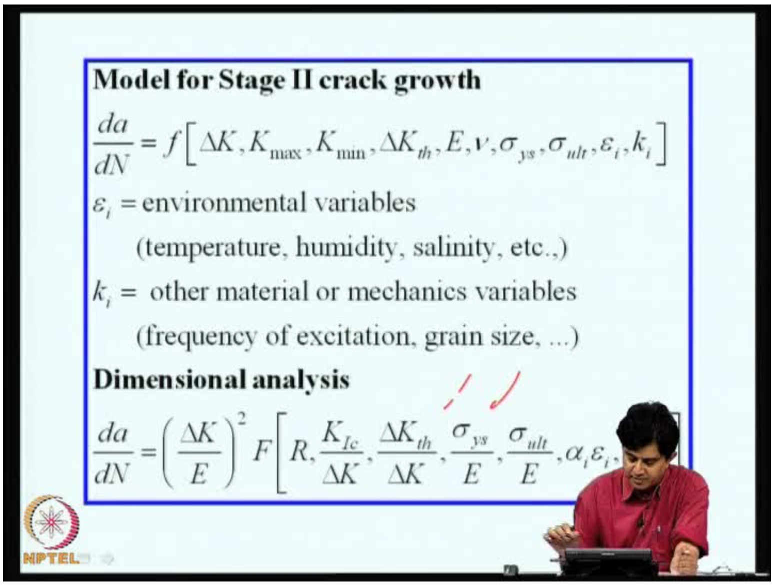

In the region of stable crack growth namely region 2 on a log log scale the relationship between d a by d n and delta K is linear and we postulate a model of the kind d a by d n is C into delta K to the power of n. This d a by d n typically is the function of delta K max K min delta K theoretical, that is this young's modulus Poisson's ratio yields stress ultimate stress strain and so on and so forth. So, this epsilon I are environmental variables like temperature humidity salinity etc.,

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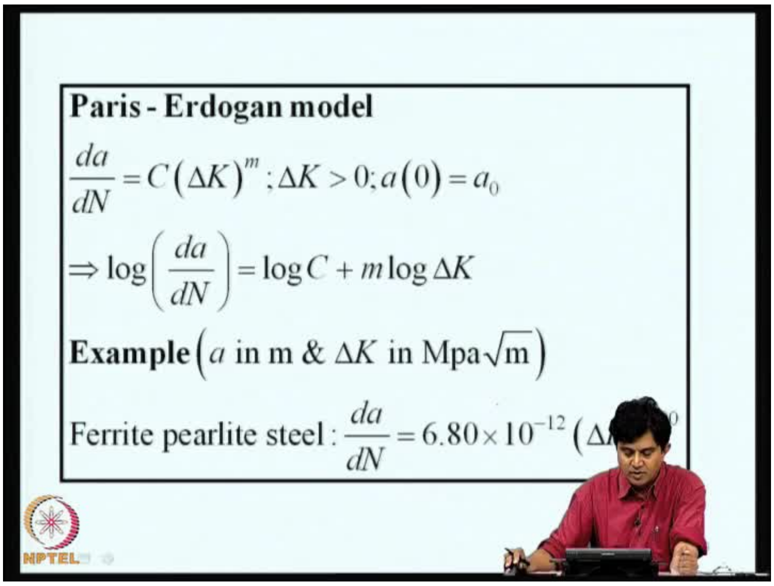

A kind of a dimensional analysis can be performed and the structure of this law can be identified in it is identified in this form a several non dimensional parameters here and one of the model that is popularly used is known as the Paris Erdogan model where we assume that the equation is d a by d N is C delta K to the power of m delta K greater than 0 a of 0 is a naught that is a initial crack length.

Now, as I said, on a log log scale, there is a straight line relationship between d a by d N and delta k. For example, for a particular steel, the parameters a when it is measured in meter delta k in mega pascals into square root m, the coefficients c and m are in this form. So, m is 3 point 0 and C is given in this form 6 point 8 10 to the power 12. So, similar characterization for other material is also available.

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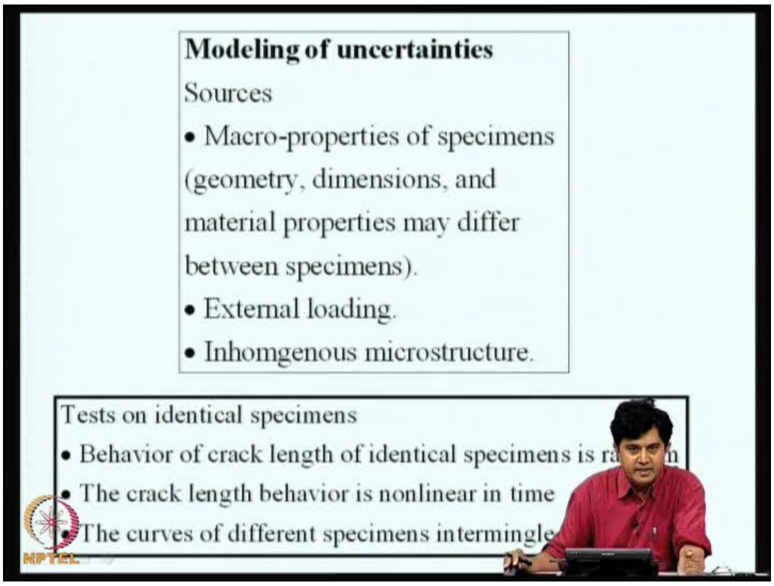

Now, the problem here is the phenomena of crack propagation is highly prone to effects of uncertainties. So, the uncertainties are associated with macro properties of specimens the geometry dimensions material properties they may vary from specimen to specimen. There will be problem with external loadings. There is inhomogeneous micro structure with in a specimen and we test on identical specimens behavior of crack length of identical specimen is random. The crack length behavior is non-linear in time. The curves are different specimens intermingle.

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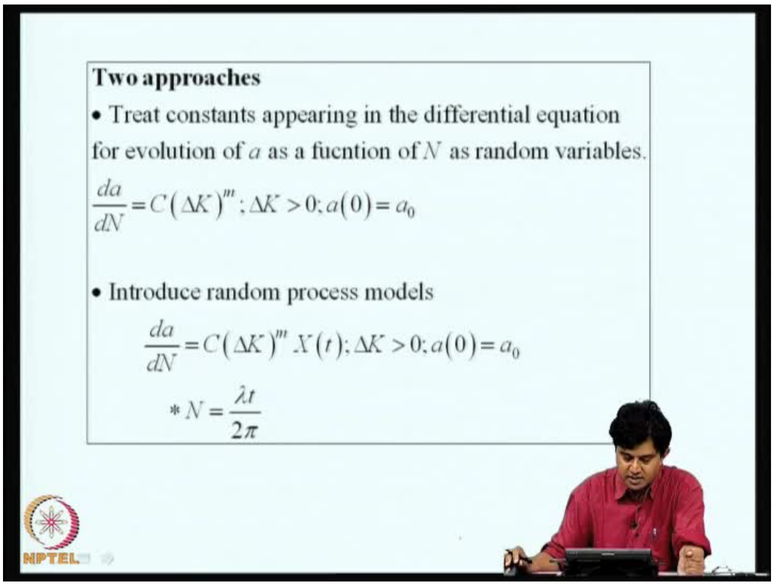

Consequently, we need to use theory of probability and random processes to deal with this situation. There are basically two approaches. In the first approach, we treat constant appearing the differential equation for evaluation of a as a function of N as random variables. So, C and m can be viewed as random variables.

In the other approach, we introduce a random process X of t as d a by d N is c into delta k to the power of m where this n and t are related n is lambda t by 2 pi. So, it is a basically a time evaluation equation but we count time in number of cycles. That is why this n is retained.

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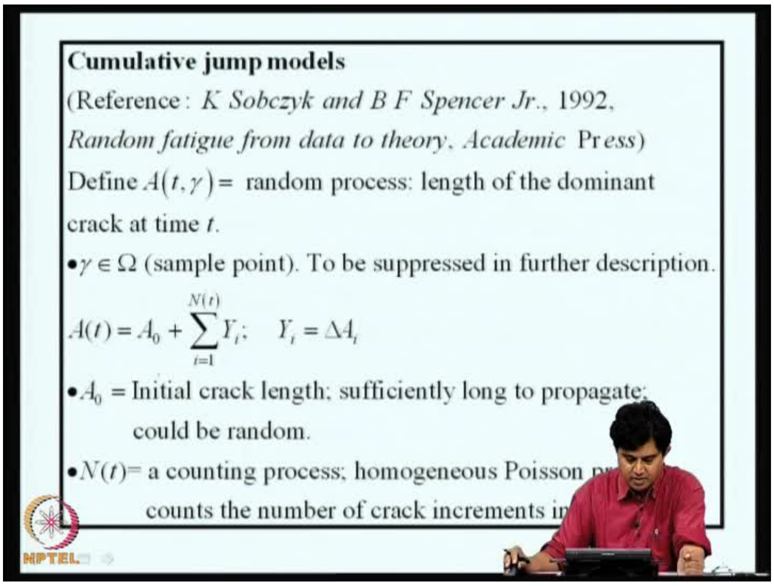

Now, what I would like to discuss is a phenomenon logical model called cumulative jump models that is based on theory of random processes and that I will discuss first and then explain how this can be the model parameters. From this, how they can be related to say the information contained in Paris law and so on and so forth. So, we define A of t comma gamma is a random process which is length of the dominant crack at time t.

Now, this gamma is point is sample point so that will be suppressing in future description. So, A of t, I write as A naught which is a initial crack length plus i equal to one to n of t Y i where Y i is delta a i. So, n of t is a counting process. It is a homogeneous Poisson process that counts a number of crack increments. Every time, crack increments by delta a, the crack propagates and n of t is the Poisson process.

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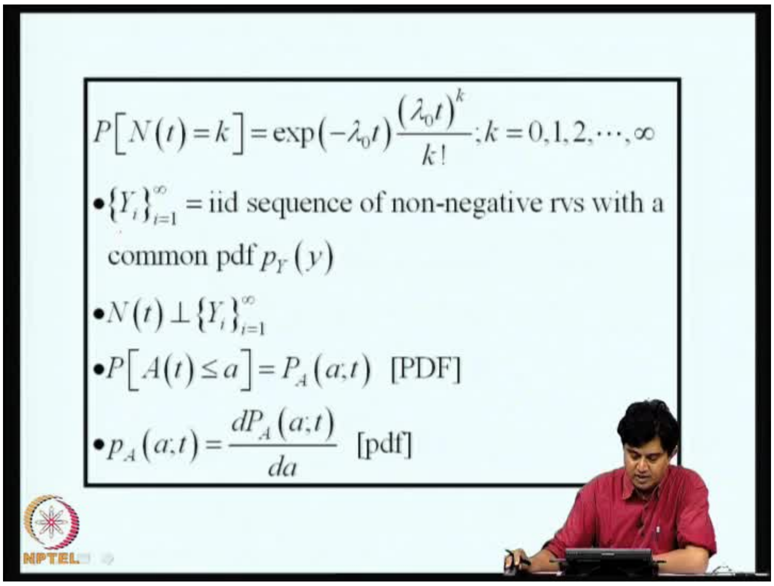

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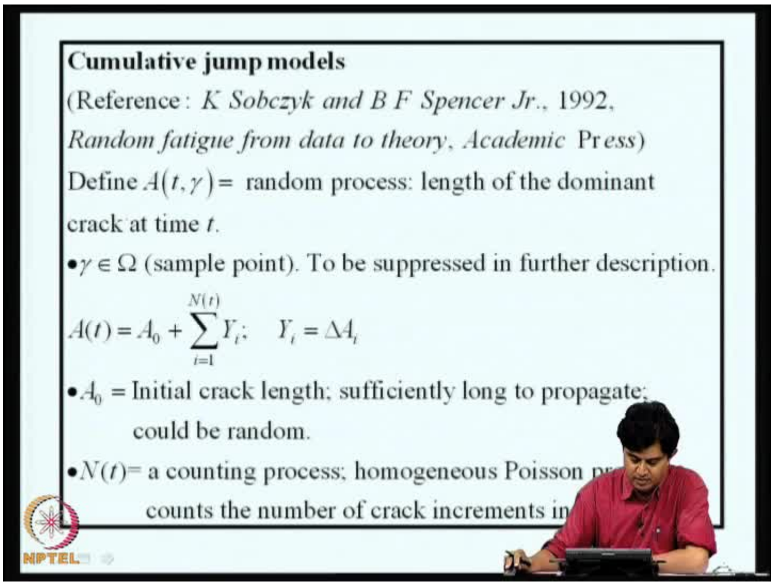

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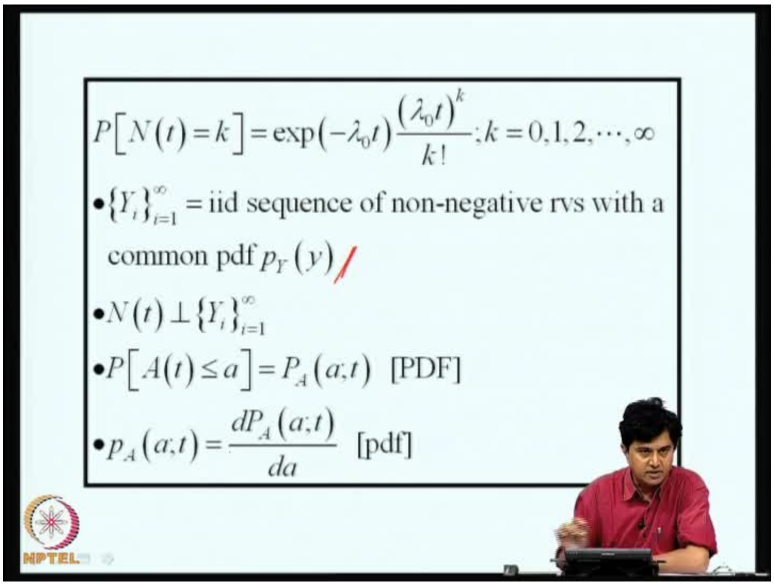

Now, we can characterize this random process with the basic understanding that we have. So, probability of N of t equal to K is e raise to minus lambda naught t lambda naught t to the power of K by k factorial K running from 0 one 2 etc., and we take Y i to be these Y I s which are the increments in crack lengths to be iid sequence of non-negative random variables with a common probability density function Y p Y of Y.

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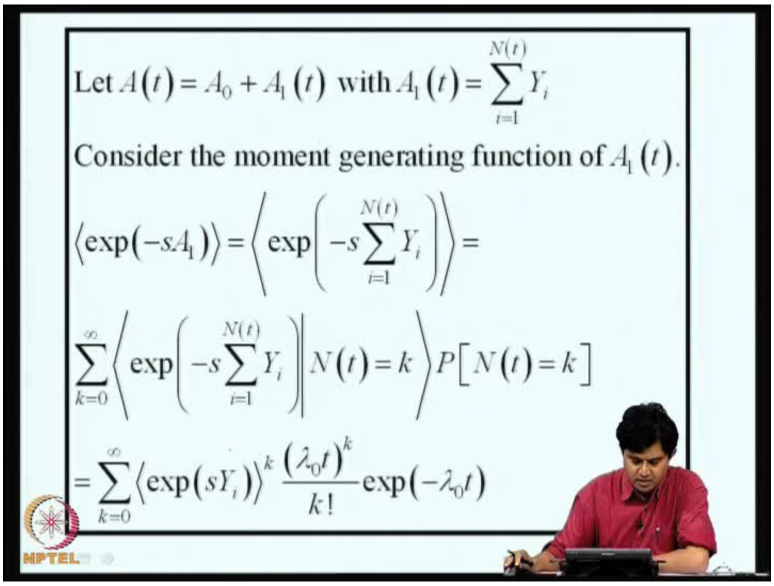

We assume that N of t is independent of this Y I s and we are interested in finding probability of A of t less than or equal to A and associated density function. Now, we define A 1 of t as is summation I equal to one ton of t Y i and we consider the moment generating function which is expectation of minus S A 1 which is given by this here and we first find this expectation by conditioning on N of t the citric that we have been doing and based on that we get this as the characteristic function.

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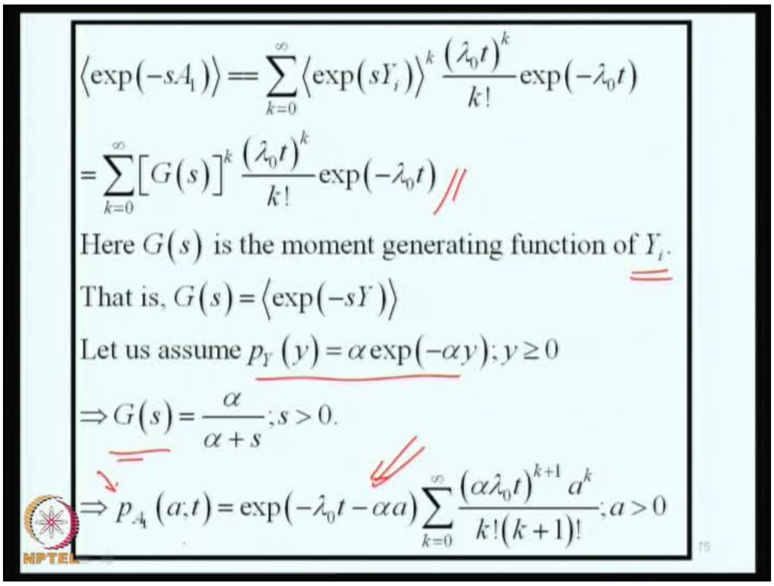

This is the moment generating function of A 1 in terms of the G of s which is the moment generating function of Y i which is identical independent iid sequence with a common G of s which is e raise to minus s Y expected value of e raise to minus s Y.

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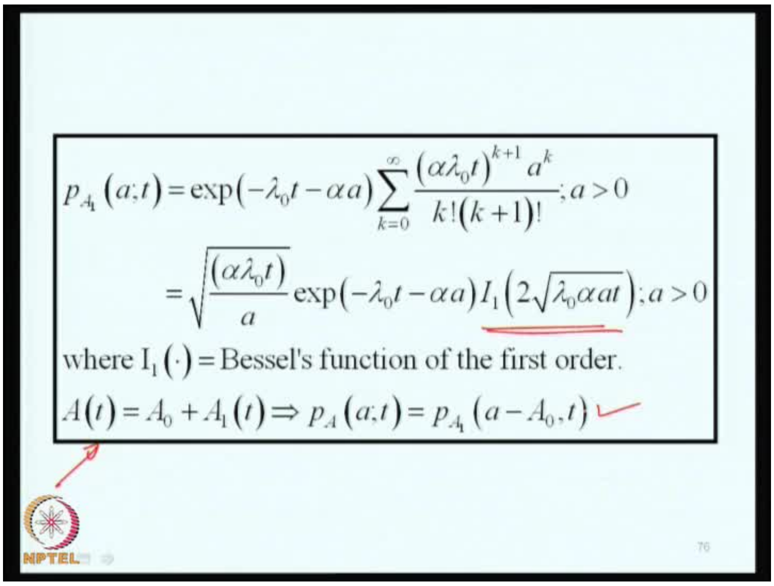

Now, let us assume for the sake of discussion that Y is exponentially distributed. So, the characteristic function here is alpha by alpha plus s and therefore I have now if I do the fourier transform of this, I can get the probability density function of A of t. So, this is which are after some simplification this infinite summation can be shown to be related to the Bessel's function of the first time and therefore p of a 1 is obtained and consequently A of t is which a not plus A one of t can be obtained as shown here.

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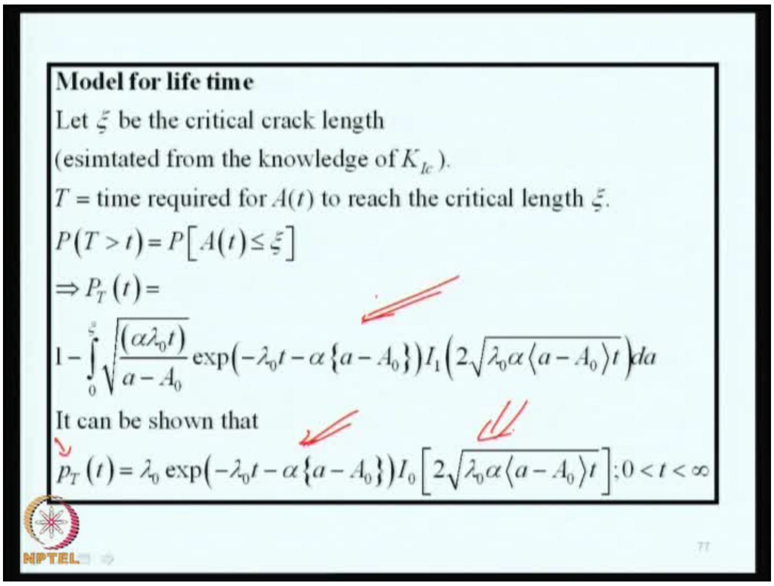

Now, how do it can make the model for the life time. Let xi be the critical crack length estimated from the knowledge of K I C. Now, T be the time period for A of t to reach the critical length xi. So, the first persist time probability T greater than t is same as probability of a of t is less than or equal to psi. So, P T of t therefore can be obtained from the knowledge of probability density function or distribution function of A of t. So, by differentiating this, we can show that or carrying out this integration, we can show that the first persist the time required for A of 2 to reach the critical length is density function is given by this.

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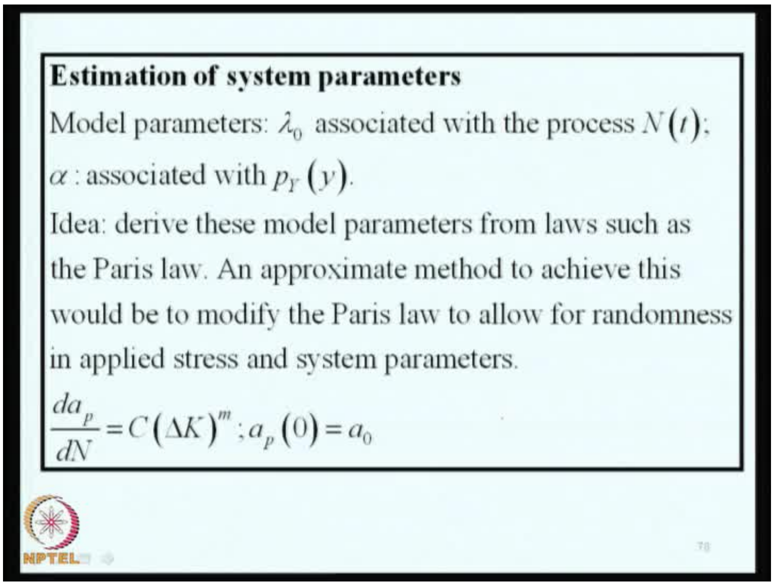

This is fine, but it has several model parameters. How do we relate it to the actual behavior of these specimens. Model parameters are lambda not associated with the process N of t, alpha associated with p Y of y. The basic idea is, we derive these model parameters from laws such as Paris law. An approximate method to achieve this would be to modify the Paris law to allow for randomness in applied stress and system parameters. So, again start d a p by d N is C delta K to the power of m with a p of 0 as a naught.

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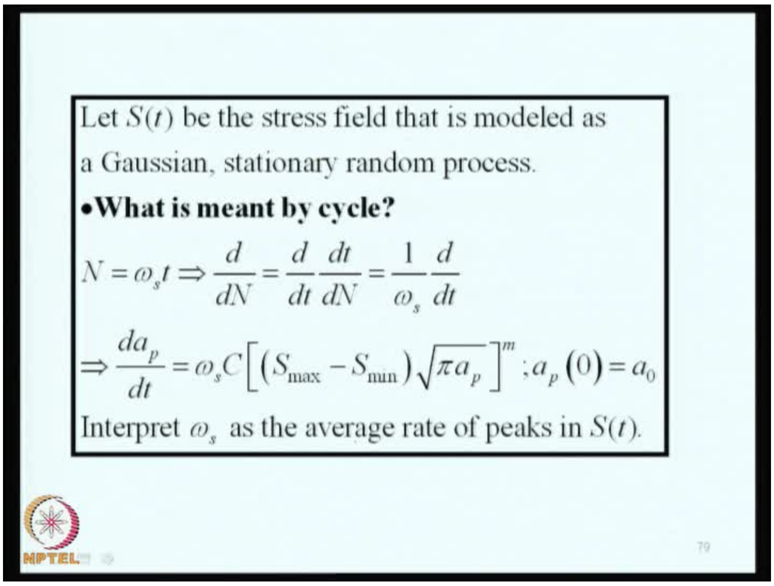

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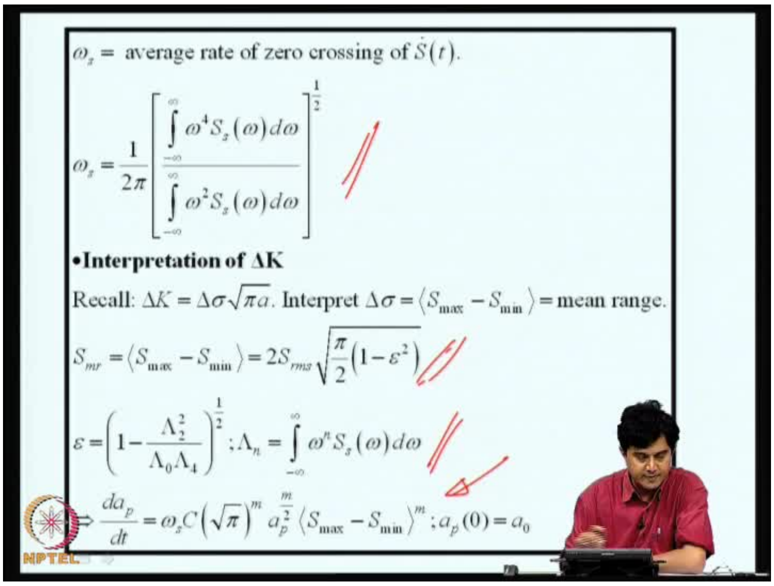

Now, let S of t be the stress field that is modeled as a Gaussian stationary random process and what is meant by cycle N is omega S into t. So, d by d N becomes one by omega s d by d t. This delta K, we replaced by S max minus S min and we get this expression. We interpret omega s which is not known as the average rate of peaks in S of t. So, if S of t as a random process is known, I can find out omega s which is given in terms of the spectral moments for then the second spectral moments.

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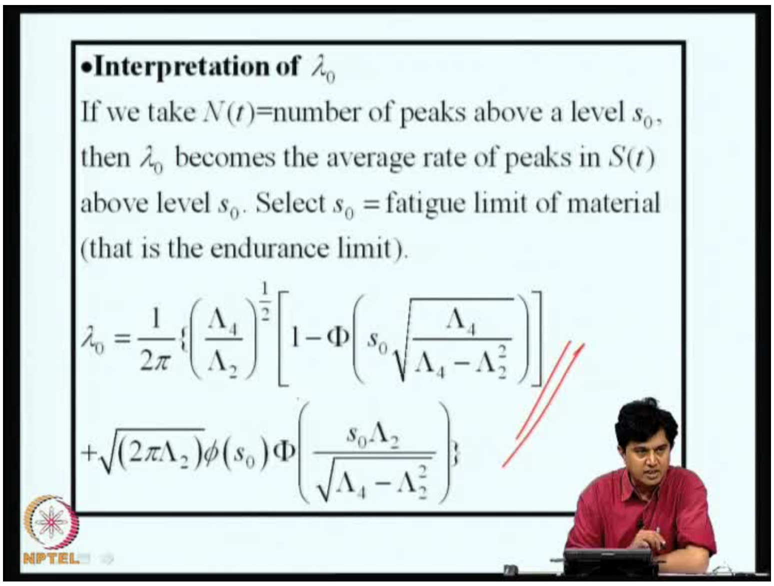

How do we interpret delta K? delta K is delta sigma square root pi a. So, interpret delta sigma as expected value of S max minus S min which is a mean range. So, the mean range, we have now in terms of the peak factors, we can derive that we have derive maximum and minimum. We can also derive the expression for the range which are not been done but you can do it in terms of spectral moments we can obtain that.

Consequently, i have now the law d a p by d t is omega s, C into square root pi to the power of m, a p to the power of m by 2 expected value of S max minus S min to the power of m equal to this.

How do we interpret lambda naught? If we take N of t to a number of peaks above a level s naught, then lambda naught becomes the average rate of peaks in s of t above level S naught. Now, s naught can be taken as fatigue limit of material, that is the endurance limit below which the fatigue damage would not accumulate. So, that lambda can be again evaluated using the theory of stochastic processes and spectral moments and we have this expression.

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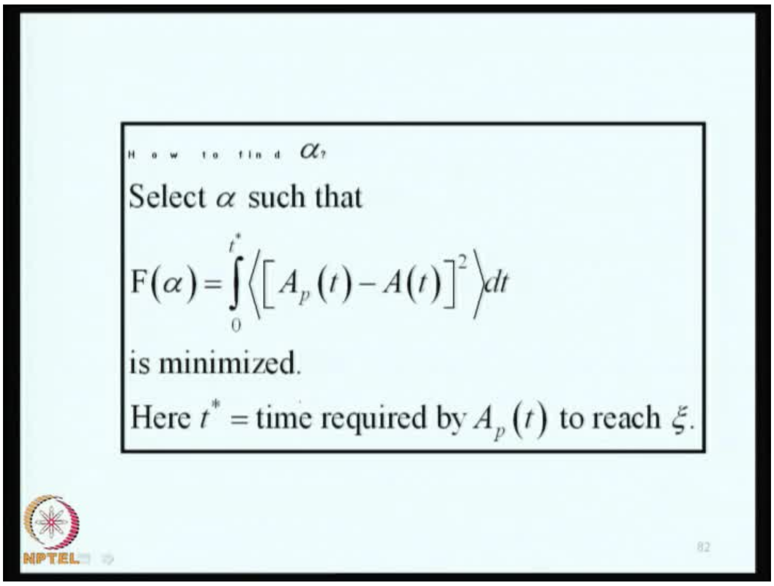

How to select alpha? alpha can be selected by minimizing a F of alpha is a major of mean square error 0 to t star A p minus A whole square d t is minimized. t star is time required by A p of t to reach xi. The point that is made through this example is that by combining theory of random processes and certain empirical laws and concepts of action mechanics, we can get a refine theory for stochastic characterization of accumulated fatigue damage.

We will conclude this lecture at this point. In the next lecture, will continue some more discussions of specific problems of dynamical systems under random excitation. This lecture is concluded at this stage.