Stochastic Structural Dynamics
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Module No. # 10
Lecture No. # 38
Problem Solving Session-2

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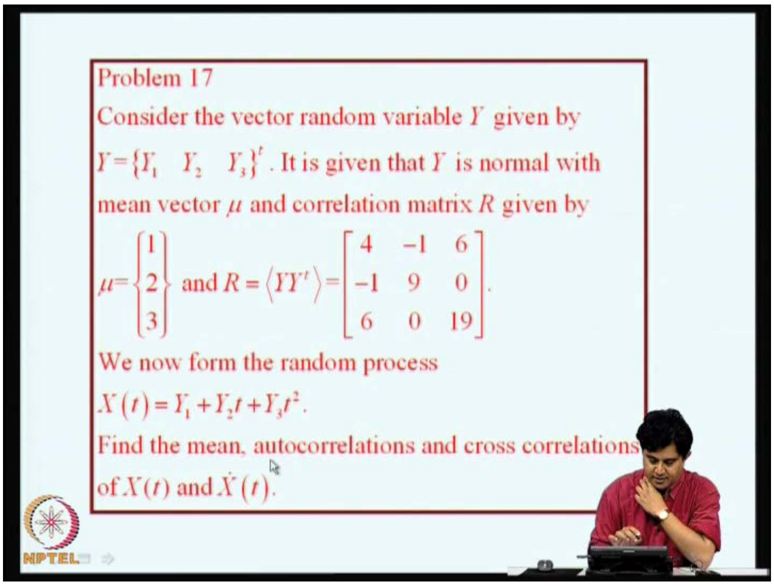

So, in the previous lecture, we solved some problems covering topics in theory of probability; we will continue with that; we will discuss some problems related to theory of random processes. So, again there are set of problems, that I will be discussing in this lecture, they are not particularly ordered in any sequence of increasing complexity or any such scheme, they just go by topics. So, we will now start with the first problem, the problem is as follows.

Consider the vector random variable, Y given by Y 1, Y 2, Y 3; it is 3 cross 1 random vector; it is given that Y is normal with mean vector mu and correlation matrix r given as shown here; mean is 1 2 3 and correlation is matrix is this; we now found the random process X of t is Y 1 plus Y 2 t plus Y 3 t square.

So, the problem is to find the mean autocorrelations and cross correlations of X of t and X dot of t. Y is a 3 cross 1 vector; so mean is 3 cross 1 and correlation matrix is 3 by 3; it is symmetric.

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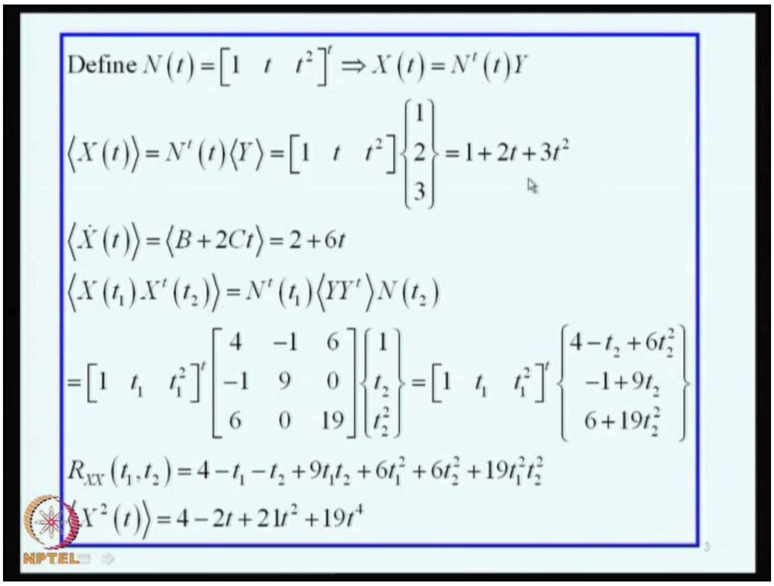

Now, we define a vector N of t as 1 t t square, so that X of t can be return as n transpose Y. Thus, now expected value of X of t will be n transpose into expected value of Y, which is 1 t t square and this is 1 2 3; this will be 1 2 t 3 t square. Now, we can differentiate the mean and get the mean of the derivative; in this particular case, it will be B plus 2 C t, expected value of B plus 2 C t, which is 2 plus 6 t.

Now, this matches with the derivative here, 2 plus 6 t so that is find as we expect. Now, the expected value of X of t 1 into X transpose t 2 will be N transpose t 1 Y Y transpose N transpose t 2. So, with that, if we write now N transpose, this is 1 t 1 t 1 square transpose and this matrix Y Y transpose and N t 2 is 1 t 2 t 2 square. So, if we carry out this multiplication, we get this expression for R X X of (t 1, t 2) and for t 1 equal to t 2 equal to t, we get 4 minus 2 t plus 21 t square plus 19 t 2 the power of 4.

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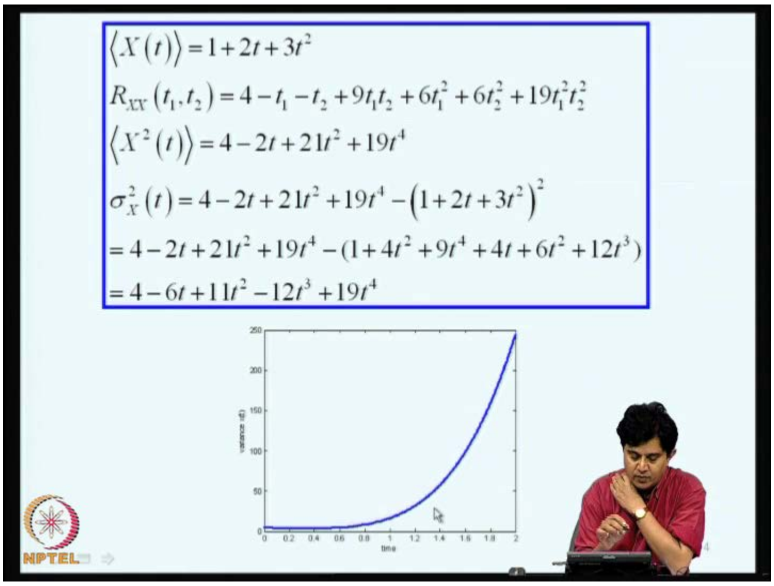

So, now, we have expected value as given here, 1 plus 2 t plus 3 t square and correlation function autocorrelation function is as shown here and from that, I got the mean square value and I can now get the variance as mean square value minus square of the mean. If I do that, I get this expression and that I have shown here just to make sure that calculations are right one check is that, variance is positive and it is to that extend, the answers are reasonable.

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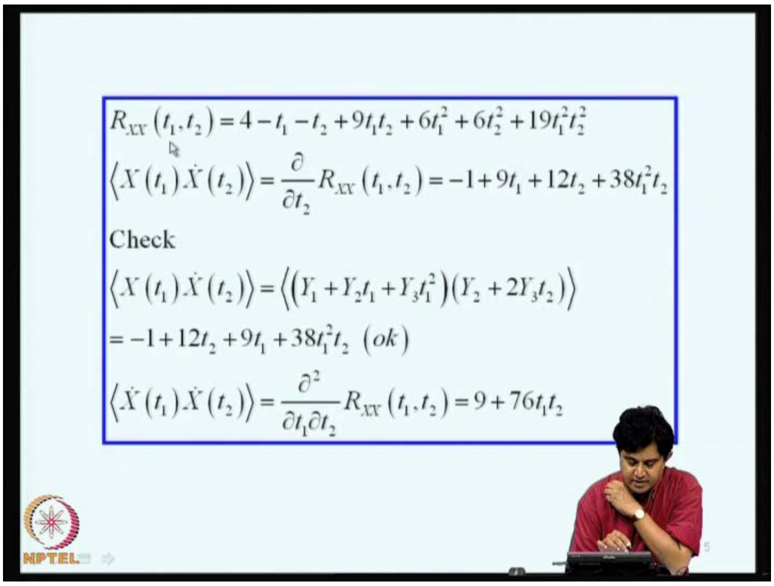

Now, this is the autocorrelation function for X of t. Now, if we want now cross correlation between X of t 1 and X dot of t 2, we can begin by differentiating this autocorrelation function and we get this expression minus 1 plus 9 t 1 plus 12 t 2 plus 38 t 1 square t 2. We could check this by directly evaluating expected value of X of t 1 into X dot of t 2; X of t 1 is Y 1 plus Y 2 t 1 plus Y 3 t 1 square; X dot of t 2 is Y 2 plus 2 Y 3 t 2 and if we carry out this expectation operation, we get what we got earlier by differentiating the autocorrelation function of the parent process.

Similarly, we can find out now the autocorrelation of the derivative process by differentiating the autocorrelation function of X of t with respect to t 1 and t 2 and we get this function. So, this completes the solution to the problem; this helps you to manipulate simple properties of a random processes first and second order properties.

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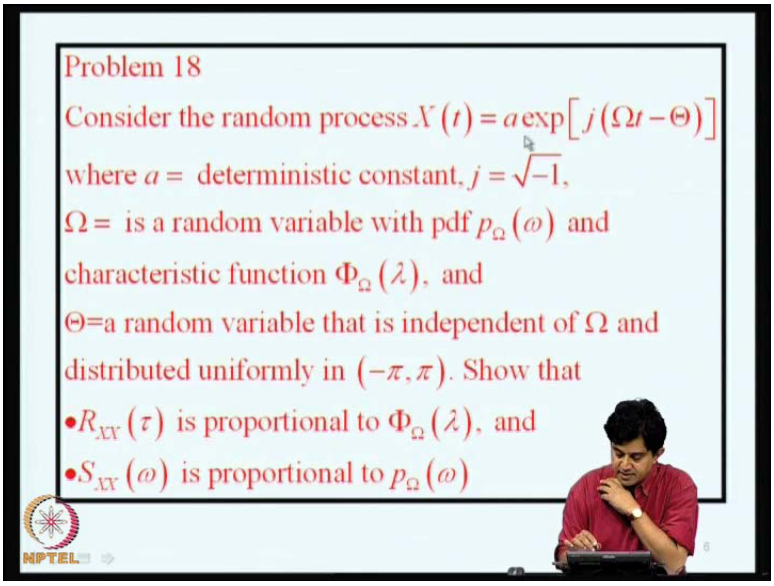

In the next problem, we consider a random process X of t, which is a into exponential j omega t minus theta; a is a deterministic constant; j is square of root minus 1 complex number; capital omega is a random variable with probability density function p omega of omega and characteristic function phi omega of lambda; theta is a random variable that is independent of omega and distributed uniformly in minus pi to plus pi. So, we are asked to determine autocorrelation and power spectral density function of X of t and show that they have certain properties.

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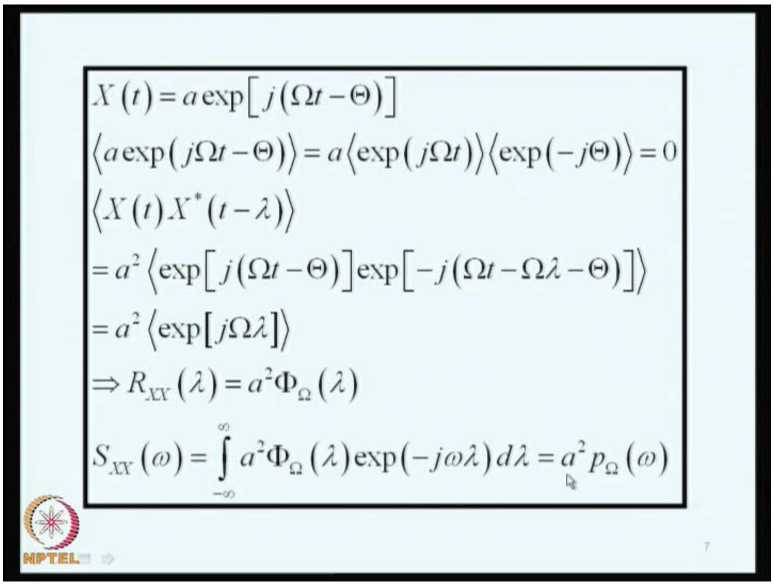

Now, X of t is a into exponential j omega t minus theta; therefore, expected value of this is a, is a constant. So, we get expected value of j omega t into expected value of minus j theta, because omega and theta are independent, I can multiply these expectations since theta is uniformly distributed between minus pi and or 0 to 2 pi, e raise to j minus theta is cos theta minus j sin theta and average over 0 to 2 pi that would be 0 therefore, the mean would be 0.

Now, the expected values of X of t into X conjugate t minus lambda; X of t is a complex valued random process. So, when we find autocorrelation, we have to take the conjugation; if we do that, we get this to be a square into expected value of j omega lambda; this is nothing but if you look at the expectation here, it is nothing but the characteristic function of random variable capital omega. So, that would mean R x x of lambda is a square into characteristic function of lambda omega.

Now, the Fourier transform of this if we take, that is S x omega will be the Fourier transform of this function since this is the characteristic function of or proportional to the characteristic function of capital omega the, Fourier transform of this will be the probability density function.

So that is what we are asked to show that, autocorrelation function is proportional to the characteristic function of omega and power spectral density function is proportional to probability density function of omega. We will return to a similar problem slightly later and make some more observations on this solution.

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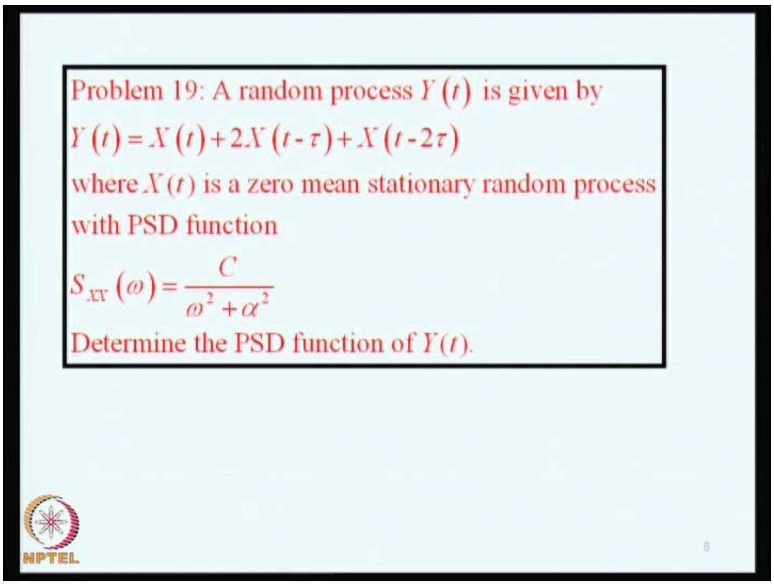

Another exercise in manipulating simple random processes, we define a random process Y of t as X of t plus 2 X t minus tau plus X of t minus 2 tau, where X of t is a 0 mean stationary random process with power spectral density function; S x x omega is C divided by omega square plus lambda square alpha square. The problem on hand is to determine the power spectral density function of Y of t.

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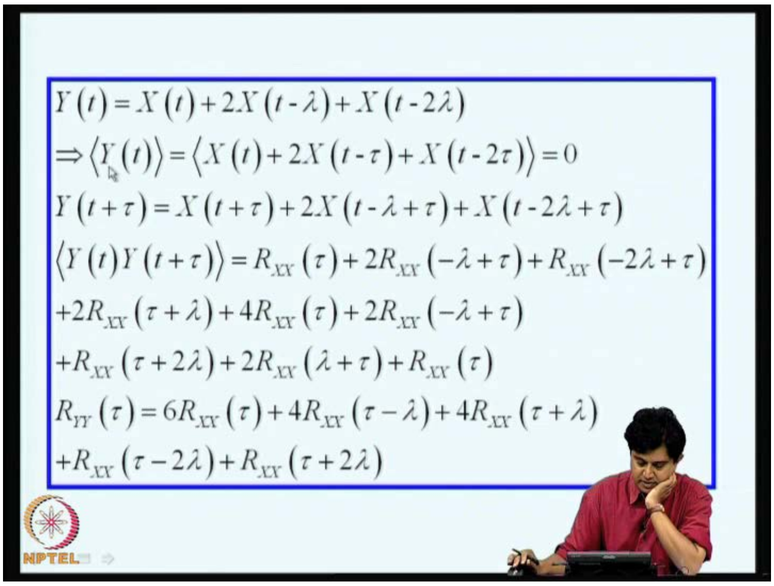

So, we need to evaluate the mean and auto covariance of Y of t and take the Fourier transform. We could do it in alternate way, but this is what we are trying to do now. So, if Y of t is X of t into 2 X t minus lambda plus X of t minus 2 lambda, the expected value of Y would be 0, because X of t has 0 mean.

Now, I can write an expression for Y of t plus tau by replacing t by t plus tau and I get this expression. Now, if I multiply this with this and take an expectation, we can show that we will get somewhat long looking expression, which involves for instance, X of t into X of t plus tau is R x x of tau; X of t into 2 X of t into 2 X t minus lambda plus tau is this so on and so forth. So, we have to simply this, to get the required expression.

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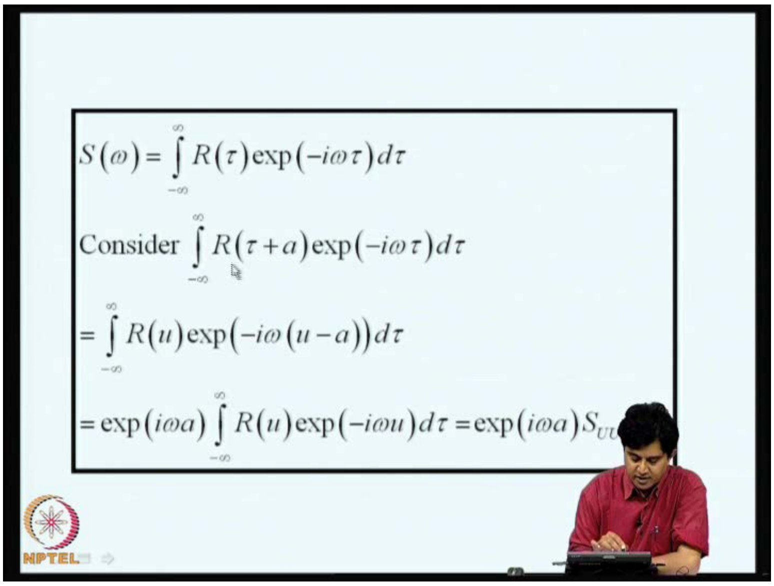

Now, we have now determined the auto correlation of Y of t, which is the auto covariance also since mean is 0. Now, we need to find the power spectral density function now, before we do that, we can quickly do a small calculation. If S of omega is the Fourier transform R of tau, if you now consider Fourier transform of R of tau plus a, it will be R of tau plus a exponential E raise to minus omega tau d tau. And if I now substitute tau plus a as u, I will get this expression from which it follows that the Fourier transform of R of tau plus a will be exponential i omega a into S u u of omega; S u u of omega is a Fourier transform of R of tau.

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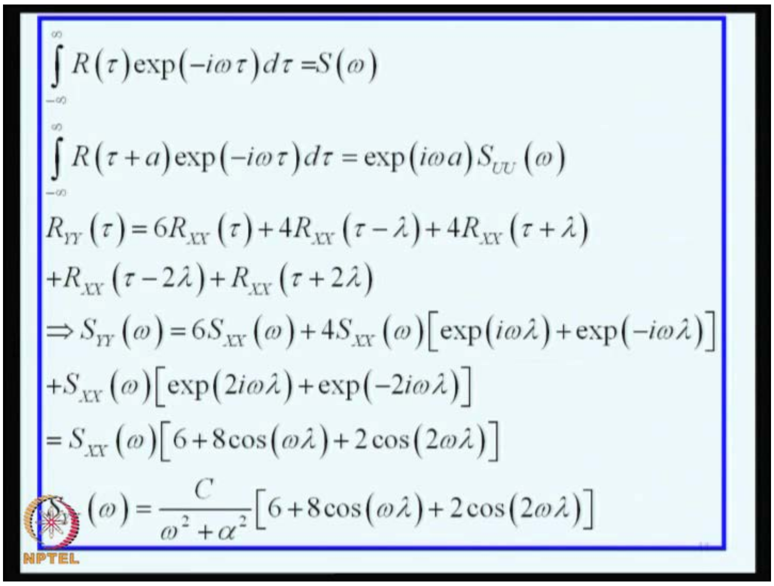

So, the time delay or a time shift in R of tau leads to a multiplier of the kind E raise to i omega a in the frequency domain. Now, using that, we can now write the expression for quantities like R x x of tau minus lambda, tau plus lambda, etcetera we can now take the Fourier transform and use the result that we just now obtained. We get the power spectral density function of Y in terms of power spectral density of X along with these exponents and if we rearrange that and use the definition of exponential function E raise to i theta is cos theta plus i sin theta, we can show that the power spectral density function is given by this expression.

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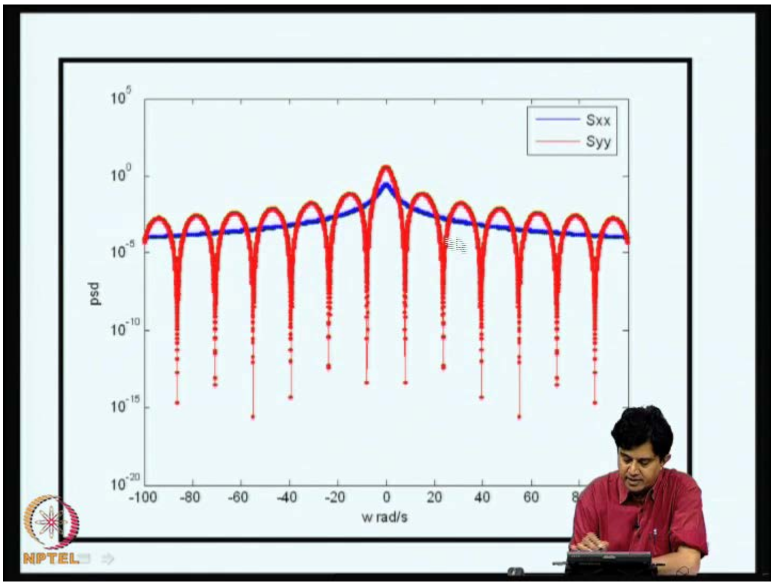

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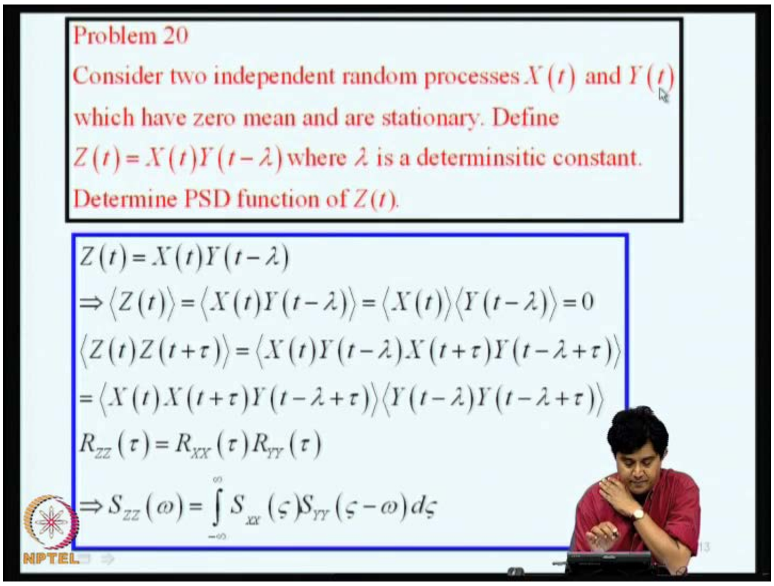

A plot of this is shown here; the blue line is the power spectral density function of X of t and this red line is the power spectral density of Y of t. A similar exercise, here we considered two independent random processes X of t and Y of t which have 0 mean and are jointly stationary; they are independent therefore, individually individual stationarity implies joint stationary. Define a new process Z of t which is X of t into Y of t minus lambda, where lambda is the deterministic constant.

So, the problem is to find out the power spectral density function of Z of t. Now, Z of t is X of t into Y of t minus lambda; so expected value of Z of t is expected value of this product, but since X of t and Y of t are independent, I can multiply the expectations and since X and Y have 0 mean, the mean of Z of t becomes 0.

Now, we will construct the product Z of t into Z of t plus tau so that will be X of t into Y of t minus lambda X of t plus tau Y of t minus lambda plus tau. So, if we complete this calculation, we can show that the auto covariance of Z of t is obtained as product of auto covariance of X of t and Y of t.

Now, we are asked to find the power spectral density function. So, multiplication in time domain is the convolution in frequency domain. So, to obtain the power spectral density of Z of t, we need to convolve the power spectrum power spectral density function X of t with power spectral density of Y of t. So, this is a required answer; a slightly different kind of problem.

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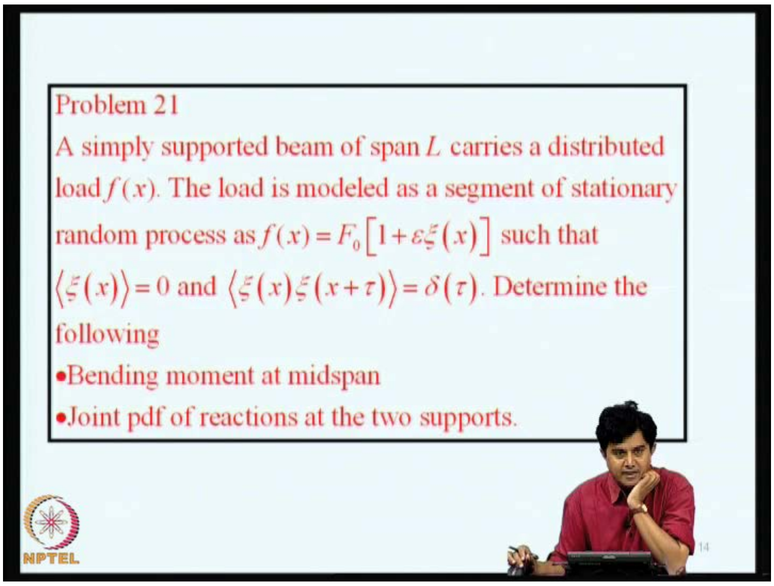

In next couple of problem, we will consider statically loaded beam structures, where load is a random field evolving in space that is a kind of problem that we will be considering. So, to start with, we will consider a simply supported beam of span L, which carries a distributed load f of x. The load is modeled as a segment of a stationary random process as f of x is equal to F not into 1 plus epsilon xi of X such that this X i of X has 0 mean and it is a white noise with unit strength.

So, we are asked to find the bending moment at the mid span and joint probability density function of reactions at the two supports; this is the problem on hand.

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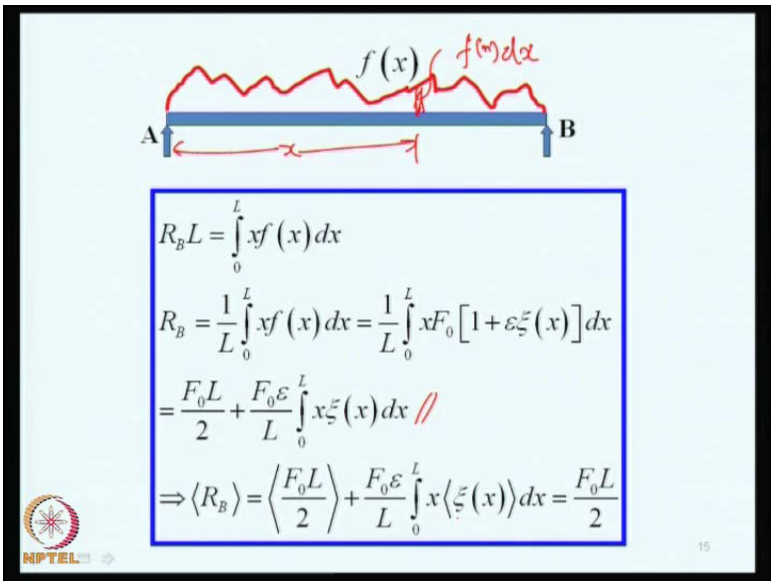

So, we can begin by considering the reaction; this is the beam and this is the load; this is notionally shown f of x is given to be a white noise, so we cannot really the white noise is not physically realizable, but this is schematically shown here. So, we take moments of forces about point a, I get reaction B into L is f of x d x into x, that is the moment due to…

So, we consider an elementary strip this and this load is f of x d x. So, this is the moment; from this, I get R B as 1 by L x into f not 1 plus epsilon psi of x d x or in a slightly simplified form we get this. Now, we take the expectation, first term is deterministic; it stays as it is; the next one is F not epsilon by L x into expected value of X i of X; X i of X is the 0 mean therefore, the mean of a reaction is F not L by 2.

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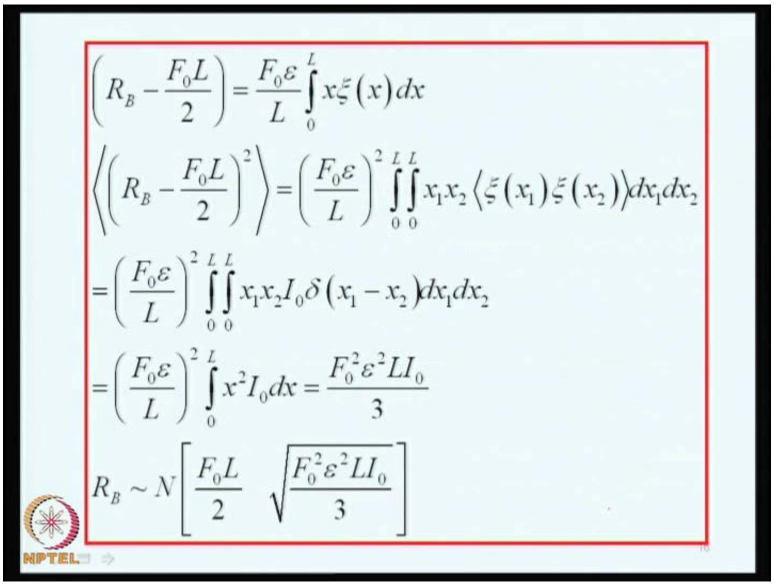

Now, I will consider the variable F R B minus F not of L; this is another random variable, but now it will have 0 mean and it is given by F not epsilon divided by L into 0 to L x psi of x d x, the mean of this is 0. Now, if you find the mean square value, we have to square this; so a single integral become double integral and if you take expectation, we have inside these integral expected value of X i of X 1 into X i of X 2, which is given to be white noise since psi of x is the white noise, this will be a direct delta function and one of the integration can be done easily and this is followed by the next integration, because this is the simple function, I get the variance as this.

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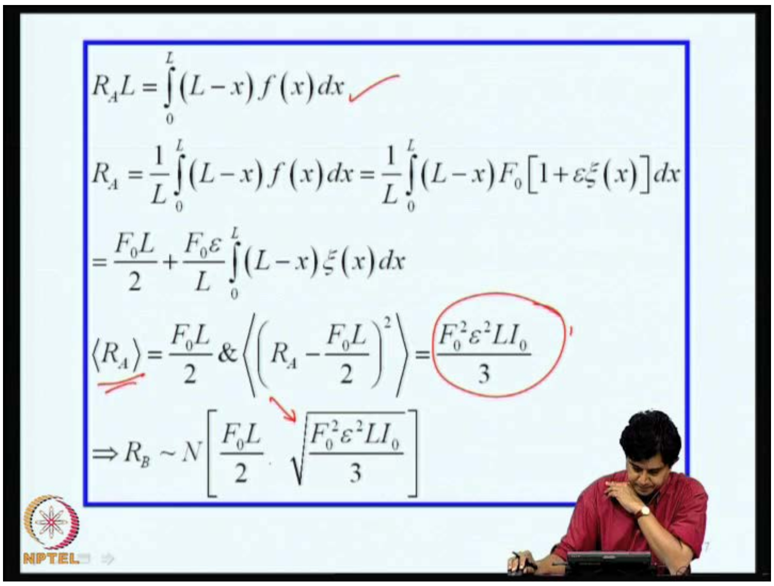

So, what we have shown is reaction B is normally distributed with mean this and standard deviation this. Now, similarly, we can show the expression for reaction A; so here we take moment about B and the expression will be this and if you simplify and follow the steps that we are just now outlined for finding reaction B, we will find that mean of reaction A is again this number and standard division will be this. So, as you can expect, since structure is symmetric, loading is symmetric, the two reactions are identically distributed.

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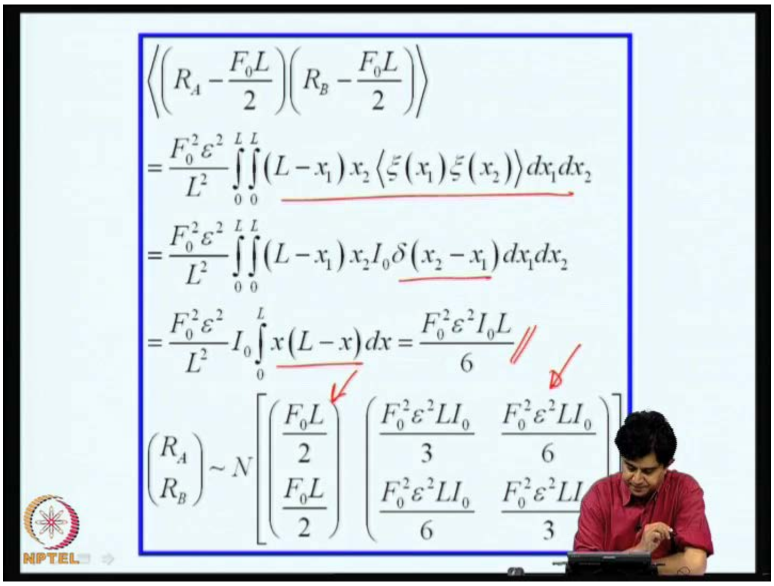

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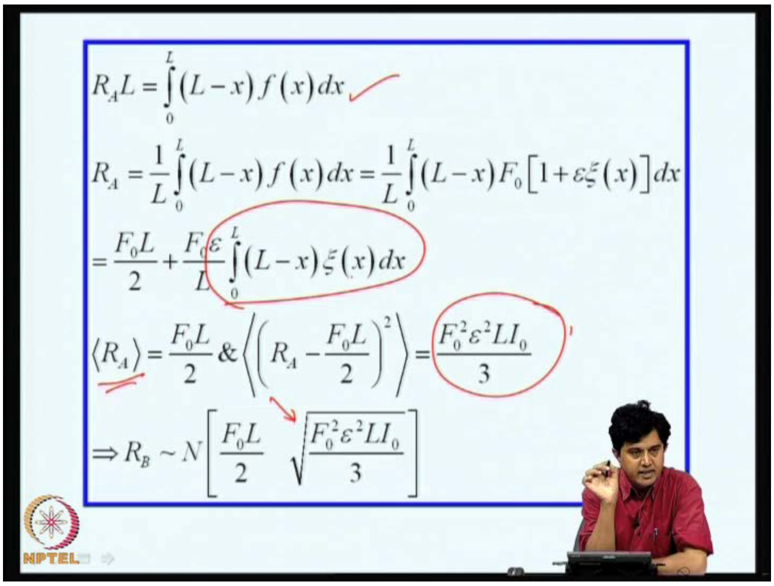

Now, how about the correlation covariance of reaction A and reaction B? If you do that exercise, to do that exercise, you have to take the expected value of R A minus mean of R A into R B into minus mean of R B. So, if you multiply this, each one is single integrals so the product will be the double integral and we get this and we can now carry out integration with respect to X 1 and X 2. We again note that, expected value of X i of X 1 into X i of X 2 is direct delta function. So, one integration can be done quickly; the other integration is also straight forward, we get this as the covariance.

Since R A and R B are linear functions of X i of X, we are looking at quantities like integrals of Gaussian random processes. So, linear operation on Gaussian random processes preserve the Gaussian property therefore, R A and R B will be Gaussian and in fact, they are jointly Gaussian and this is the mean and this is the covariance function.

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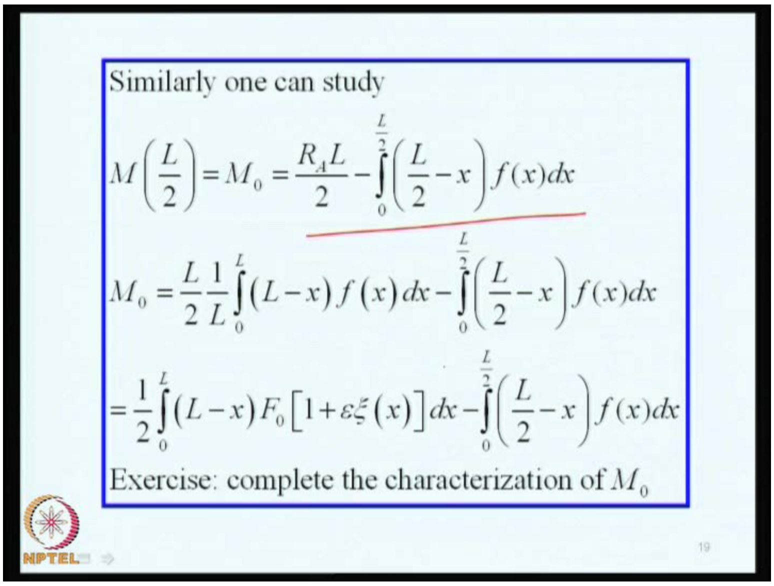

Now, similarly, we can find out the bending moment at the mid span that is one of the part of our question so that is given by this expression and we can manipulate this and I have not filled up these details; I will leave it as an exercise to complete the calculations. So, first you find the mean and then find the subtract the mean from M and find the expected value of this square of the difference, you get the variance and that is what you need to do by following the steps, which we have done for finding reactions at A and B.

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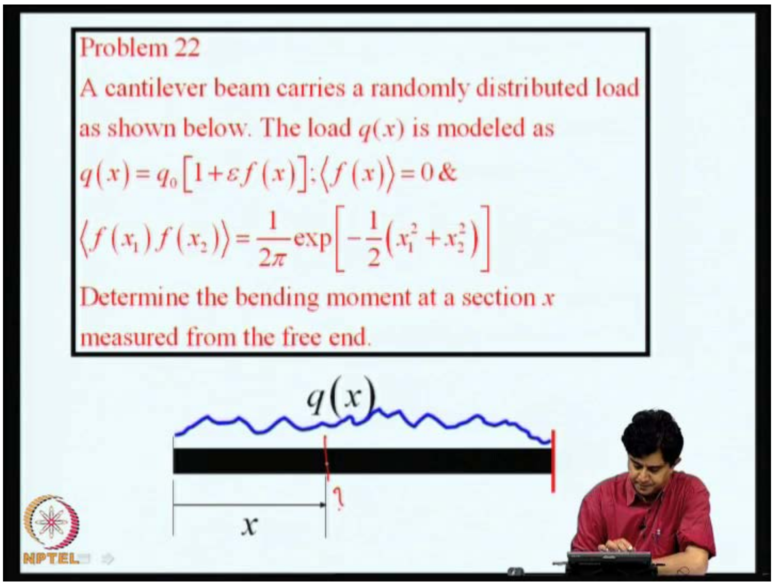

Now, a similar problem, now we have a cantilever beam, it carries a randomly distributed load as shown here, q of x is the randomly distributed load; the load q of x is modeled as q of x into q not 1 plus epsilon f of x; f of x is the 0 mean that expected value of f of x 1 into f of x 2 is not a function of X 1 minus X 2, therefore, the process is not stationary; this random field is not stationary. The question that is being asked is determine the bending moment at a section x measured from the free end, that means, what is the bending moment at this section.

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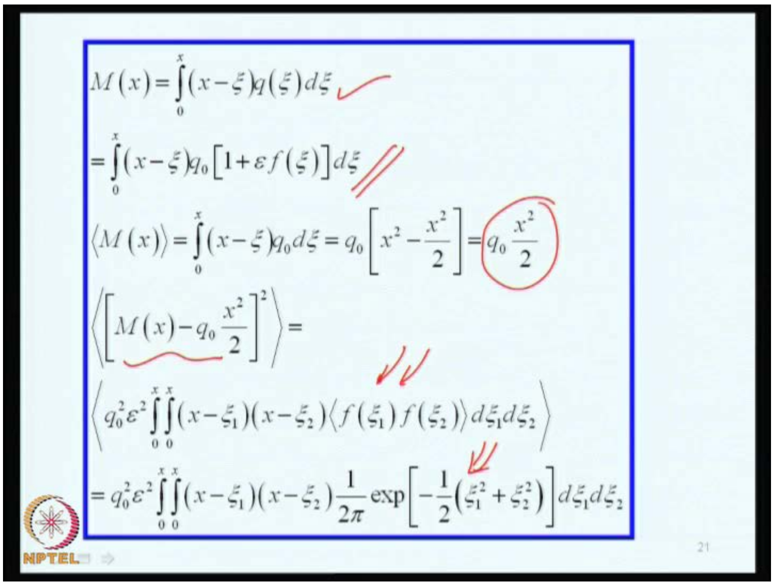

So, bending moment at that section can be easily found out. You find out the bending moment due to an incremental load q of psi into d psi, take moment about that section we get this and then, integrate from 0 to x. Now, you manipulate this expression; we get the bending expression for bending moment to be this. Now, mean of M of x is given by this; we have we have told that the expected value of F of psi is 0 therefore, the mean value will be q not x square by 2.

Now, I will detect from M of x, q not X square by 2 as here and square it and take expected value, I get the variance of bending moment at x and that this is the expression that we have to deal with. Now, the auto covariance of F of psi is given to be of this form; now you can recognize that this is as the form of the Gaussian probability density function; so you can quickly see that we are writing expression for expected value of Gaussian quantities therefore, evaluation can follow simple rules.

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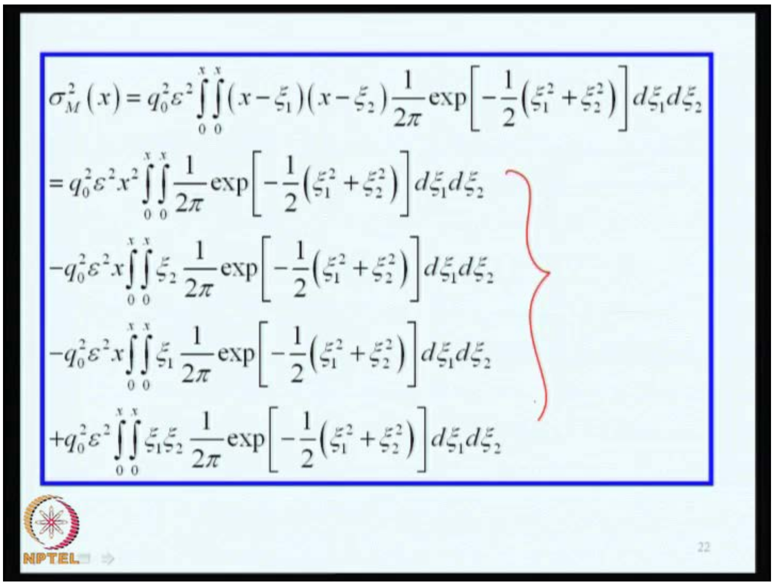

So, following that, but limits are from 0 to X that has to be bound in mind. So, if we do this, there are various terms; we will get these four different terms; this is straight forward, you can check if these are right.

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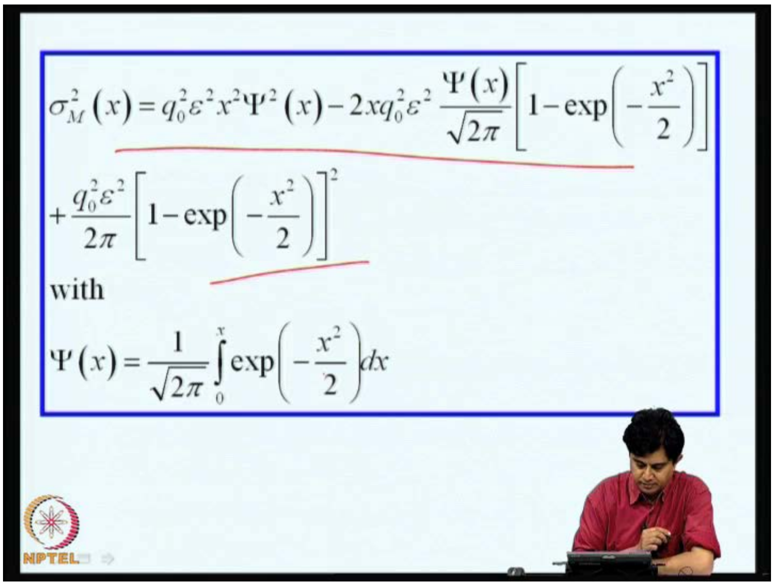

Then we get the expression, if you simplify, variance of bending moment is given through this expression, where this capital psi of x is the Gaussian integral, 1 by square root of 2 pi 0 to x e raise to minus x square by 2 d x. So, the applied load is a non-stationary random process, we are able to get the variance of the bending moment at any point x.

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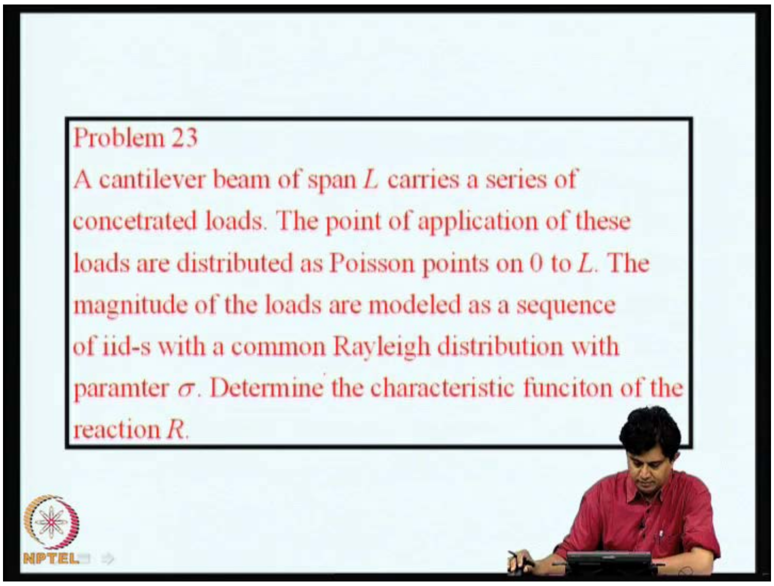

So, the next problem is again on a cantilever beam. Here first go through the problem; a cantilever beam of span L carries a series of concentrated loads. The point of application of these loads is distributed as Poisson points on 0 to L. The magnitude of the loads is modeled as a sequence of ii-ds with a common Rayleigh distribution with parameter sigma. So, we are asked to determine the characteristic function of the reaction at the support.

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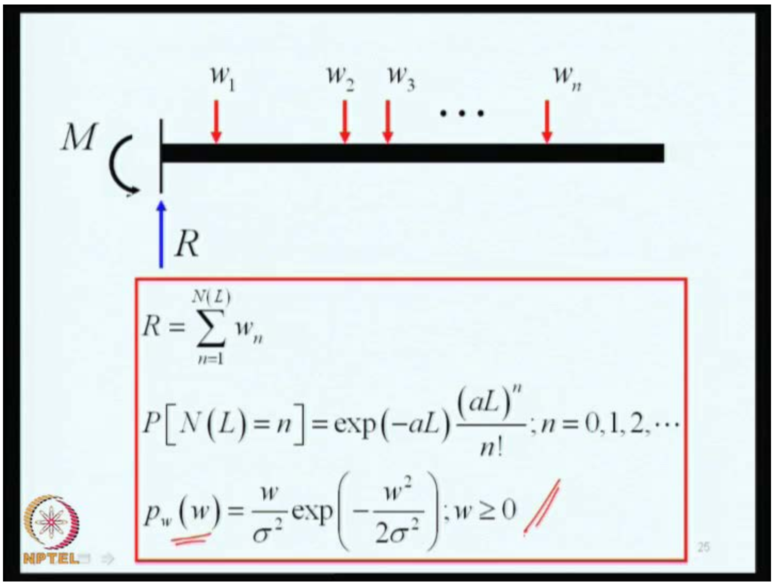

So, what is happening is w 1, w 2, w 3, w n are Poisson points and reaction R and reaction M are to be determined. We are focusing on reaction R; reaction R is nothing but summation of N equal to 1 to N N of L w n; this N of L is a Poisson random variable and w n is a ii- d sequence of Rayleigh random variables.

So, we are required to find the probability distribution of N. Now, what we are given is N of L is Poisson therefore, probability of N of L equal to n is E raise to minus a L a L to the power of n divided by N factorial and w that is w 1, w 2 3 w 2, w 3, etcetera form a ii- d sequence with a common density function p w of w and that is Rayleigh. We are asked to find characteristic function of R.

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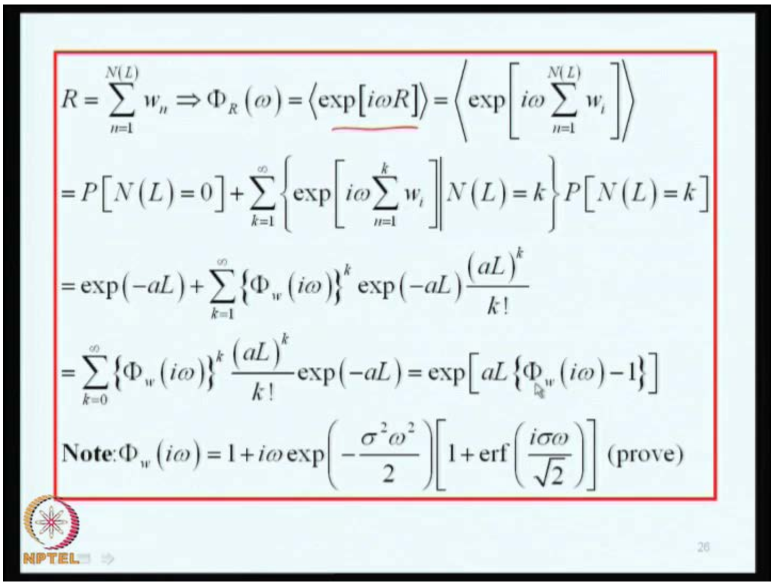

So, what is characteristic function of R? It is expected value of i omega R that is exponential of i omega N to 1 to capital N of L w i. So, what we do is, we condition on N of L, find a conditional expectation and then, take expectation of N of L with respect to distribution of N of L. So, we begin by considering the situation when there are no loads on the structure. So that is probability of N of L equal to 0 plus summation from k to 1 to infinity exponential i omega n to n from 1 to k w i condition on N of L equal to k and probability of N of L equal to k.

So, probability of N of L equal to k is given by this E raise to minus a L a L to the power of k by k factorial and this is nothing but the characteristic function of the ii-d sequence w, we need to multiply them, because they are all independent; so this become phi w of i omega to the power of k.

So, we can rearrange these terms; there is a L to the power of k here and phi w i omega to the power of k; if we arrange these terms together we can show that this characteristic function is nothing but exponential of a L phi w of i omega minus 1. This phi w of i omega is the characteristic function of Rayleigh random variable, that we can show I leave it as an exercise; it involves slightly tedious integration. You can show that this is given by the characteristic function of Rayleigh random variable phi is given by this; so you substitute that here and you got the solution to the problem that is post.

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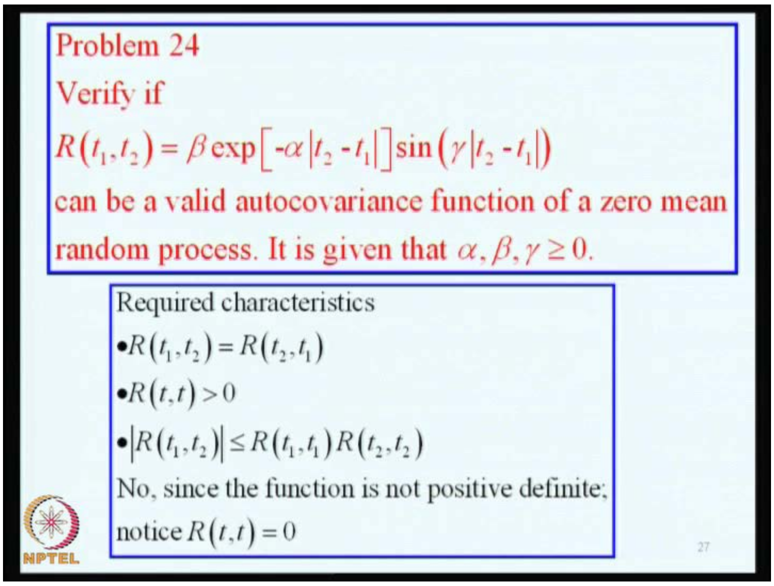

So, another simple example, the problem here is, we are given auto correlation function beta exponential minus alpha modulus t 2 minus t 1 sin gamma t 2 minus t 1. Question is, can this be a valid auto covariance function of a 0 mean random process. It is given that alpha, beta and gamma are positive. If you recall, the required characteristics of an auto covariance functions is should be symmetric. So, this function is symmetric, because if you exchange t 1 and t 2, the function remains unaltered therefore, function is symmetric so it passes this test.

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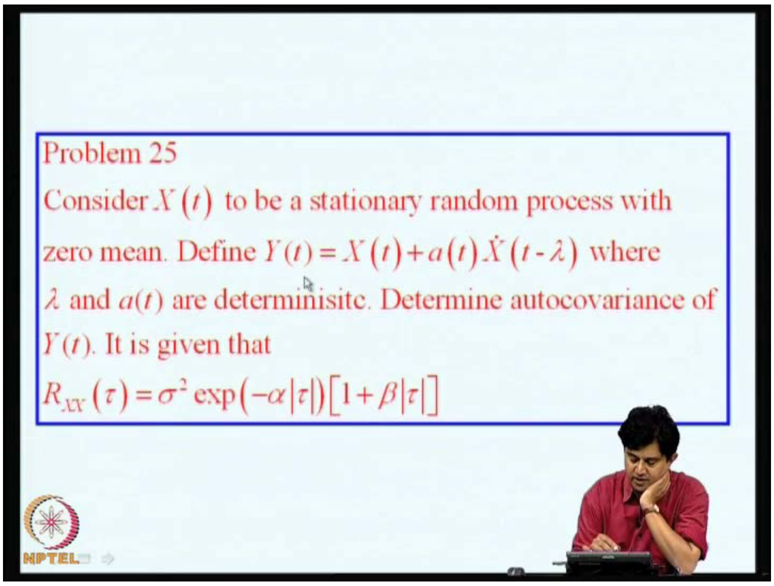

Now, R of (t, t) should be greater than 0 so that if you see here, there will be a problem, because sin gamma of 0 is 0; so the variance becomes 0. So, since the function is not positive definite, we can we can answer to this question is, this cannot be an auto covariance function of a random process. Another example, where you need to manipulate a random process and its derivative; so consider X of t to be a stationary random process with 0 mean, we define Y of t as X of t plus a of t into X dot t minus lambda, where lambda and a of t are deterministic. Determine the auto covariance of Y of t. The auto covariance of X of t is given to be sigma square exponential minus alpha mod tau into 1 plus beta mod tau.

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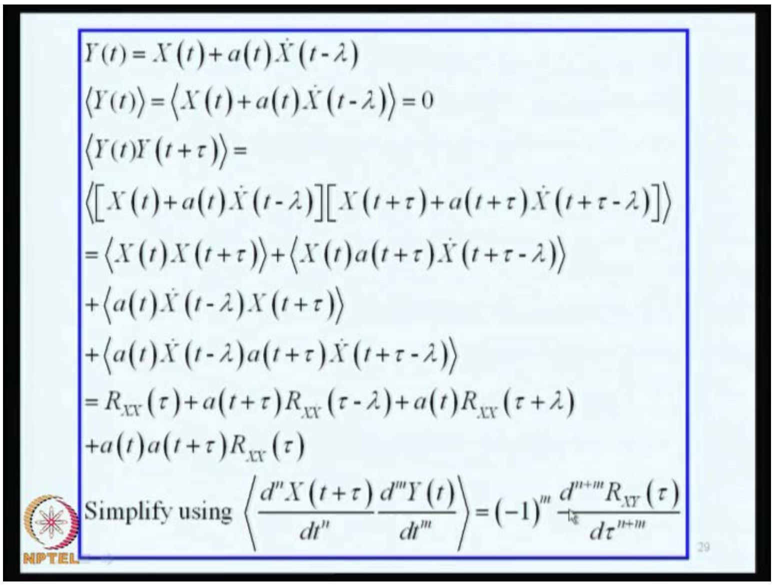

So, this a reasonably straight forward exercise, which we need to do carefully. So, Y of t is given as X of t plus a of t into X dot t minus lambda. The expected value of Y of t is 0, because X of t has 0 mean therefore, derivative process will also have 0 mean; so this is 0. So, the auto covariance is given by expected value of Y of t into Y of t plus tau. So, you need to write expression for Y of t and for Y of t plus tau and multiply all those terms and if you carefully do that, the answer that we are looking for is R x x of tau plus a into t plus tau R x x dot tau minus lambda and so on and so forth.

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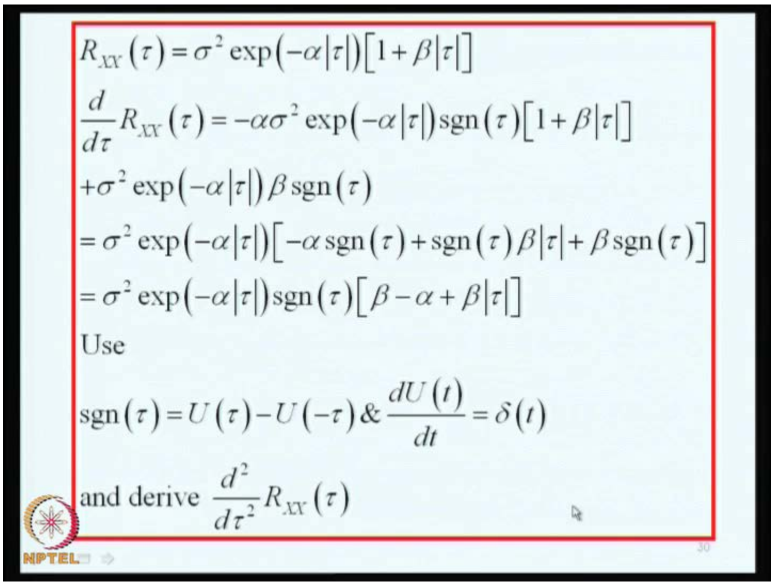

So, here the question is, we need to evaluate R x x dot, R x dot X and R x dot X dot. We are given R x of tau so we need to use this formula to evaluate that. So, R x x of tau is sigma square exponential of this this into this multiplication factor; if you differentiate this, differential of modulus of tau leads to term called signum of tau, that needs to be handle and then, if you simplify, we will get this as a first derivative.

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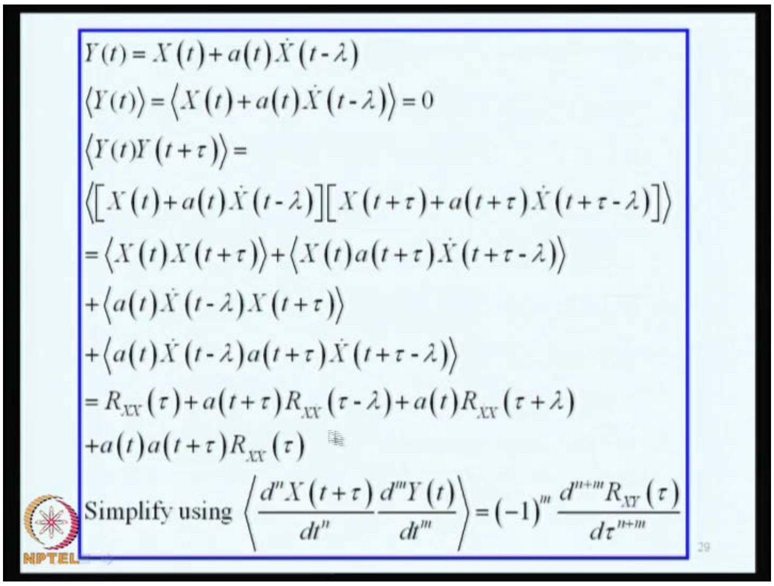

Now, this is nothing but R x x dot, we can get that quickly. Now, we can write signum of tau was u of tau minus u of minus tau and d u by d t of as direct delta function and thus get the higher derivatives also and go back and substitute into this expression, we will get the required auto covariance of Y of t.

Now, you please notice that, R y y of (t, t) plus tau, Y of t is not stationary, because this R y y (t, t) plus tau is not a function of t alone, simply because there is a deterministic function a of t; because of that, this Y of t is non-stationary. But auto covariance is obtained as some kind of super position of R x x of tau, each one of is a function of tau only, because X of t is stationary.

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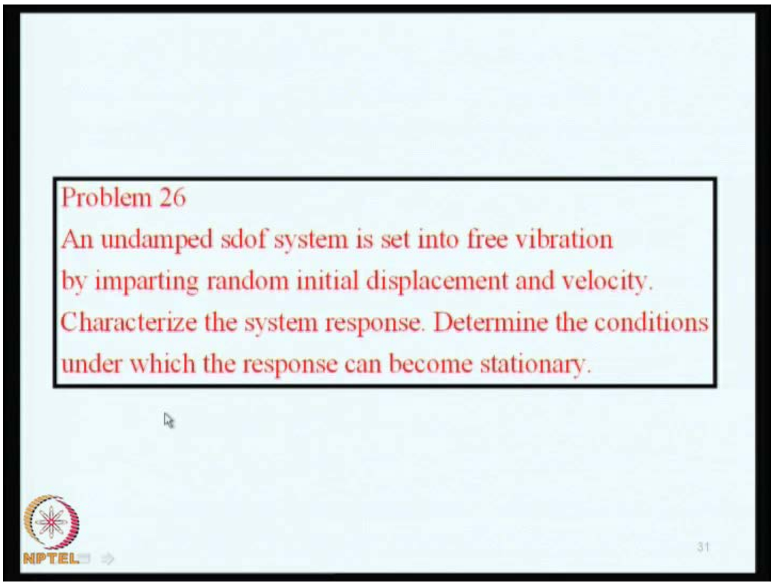

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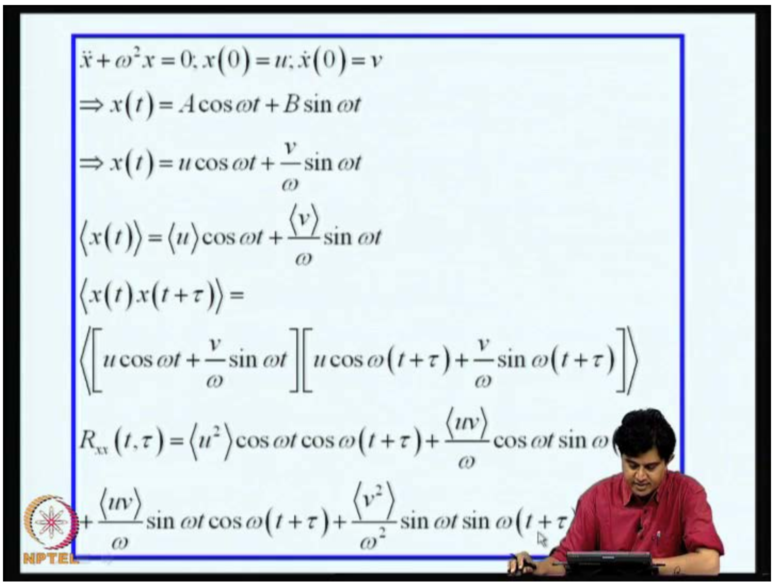

Another simple problem, an undamped single degree of freedom system is set into free vibration by imparting random initial displacement and velocity. Characterize the system response. Determine the conditions under which the response can become stationary. So, these are reasonably simple exercise. So, the problem is x double dot plus omega square x is equal to 0; x of 0 is u; x dot of 0 is v, where u and v are random variables.

So, x of t is a cos omega t plus B sin omega t. If you use the required initial specified initial conditions, I get x of t as u cos omega t plus v by omega sin omega t. Expected value of x of t is expected value of u into cos omega t plus expected value of v divided by omega into sin omega t. Since, we are not yet given the information on expected value of u and v, this is the answer at this stage.

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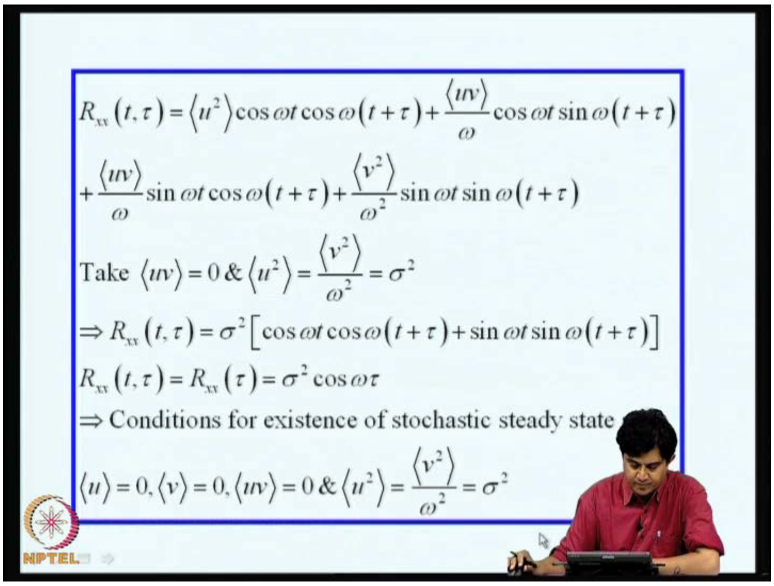

Now, you find auto correlation; you have to multiply u cos omega t plus v by omega sin omega t with x of t plus tau and carry out the calculations and this is the answer that we get. But the question that we are asked is under what conditions on u and v, the process can become stationary? Now, if you simplify this expression, if we take if you look at auto correlation of x of t, this is the expression here and for x of t to be stationary, this R x x of (t, tau) should be function of tau alone.

So, for that to happen, in this expression if expected value of u v is 0 and expected value of u square and v square by omega square are equal, we get auto correlation to be function of time difference alone, but mean is still a function of time. So, we need to make mean equal to 0; see mean is expected value of u into cos omega t plus this into sin omega t. So, this the way that this can become in time invariant is that, the expected value of u and the expected value of v should be equal to 0. So, under this condition, the process becomes wide sense stationary.

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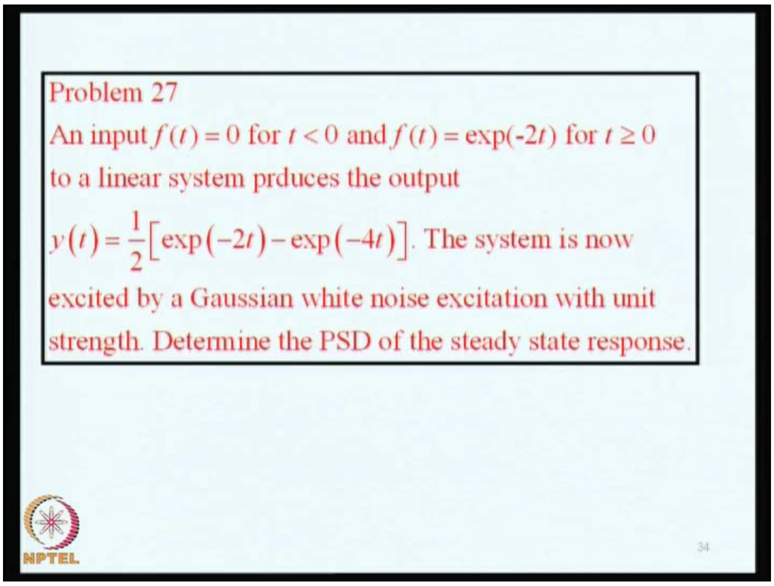

Here, we consider an input F of t which is 0 for t less than 0 and equal to e raise to minus 2 t for t greater than or equal to 0 to a linear system, we observe this output. y of t is the output half e raise to minus 2 t minus e raise to minus 4 t, given this information. The system is now excited by a Gaussian white noise excitation with unit strength. Determine the power spectral density function of the steady state response.

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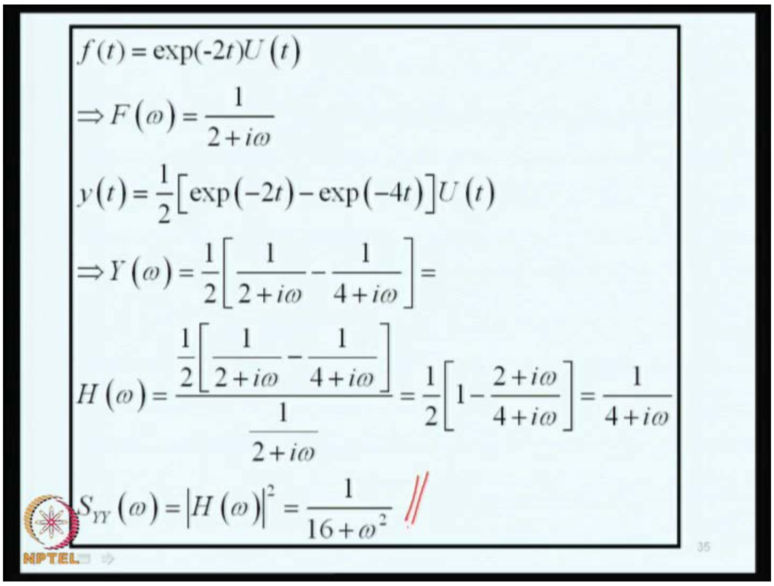

So, notice that we are not given the governing differential equation here, but input output relation in time domain is given. Now, f of t is exponential minus 2 t U of t. So, the Fourier transform of this is, you can quickly verify, it is 1 one by 2 plus i omega. Y of t is given as half of E raise to minus 2 t minus E raise to 4 t into the step function. The Fourier transform Y of t can again be derived and that is shown here and based on this, we can determine the complex frequency response function, which is ratio of Y of omega to f of omega.

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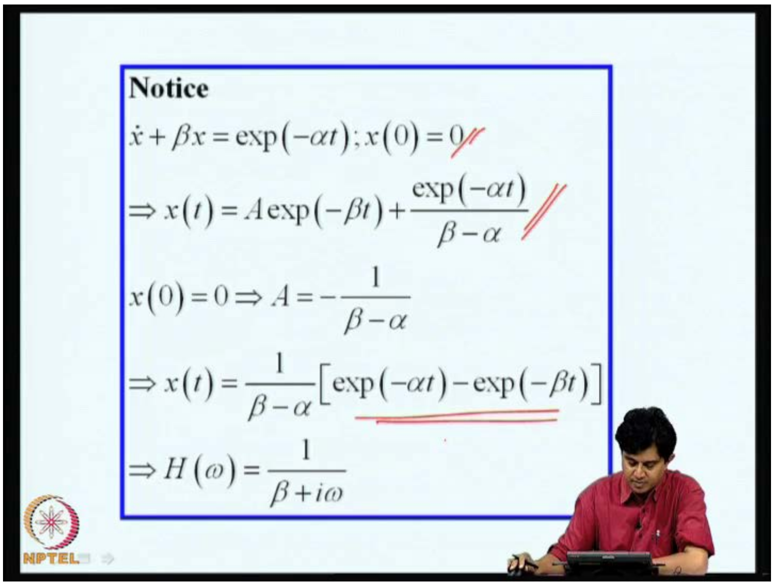

So, if we do that, I get the H of omega is 1 plus 4 1 by 4 plus i omega. Now, if the system is now driven by white noise, the output power spectral density function will be square of this transfer function into unit. So, the output power spectral density function is therefore, this. Now, you can to gain a bit of inside in to this, if you consider the dynamical system x dot plus beta x equal to E raise to minus alpha t, the starting from rest, we can see that the solution to this problem will be in this form and if we take now the 0 initial condition, we can get x of t is 1 minus 1 by beta minus alpha this that would mean that the given problem the underlined dynamical system is simply x dot plus beta x equal to some F of t.

So, now, on this system, if you apply white noise, you can show that that you will get the same transfer function that you got for this system is 1 plus beta plus i omega and you can show that power spectral density function would be similar to what we got by using other argument.

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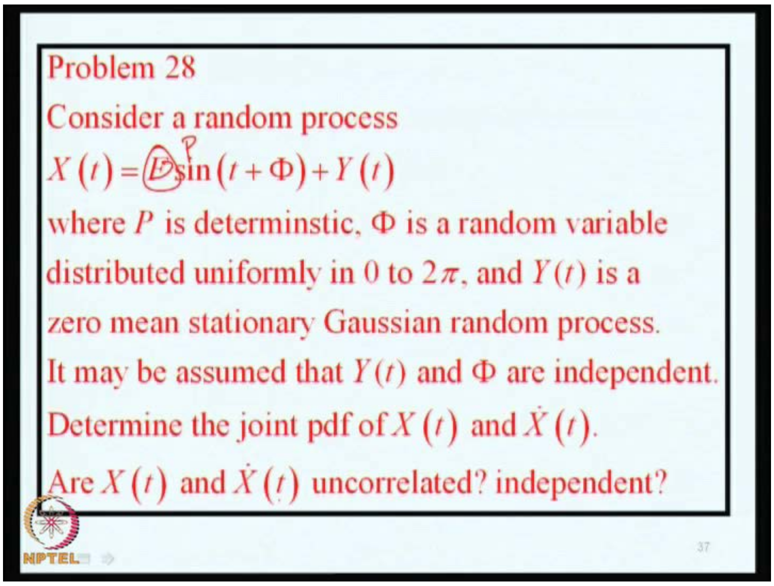

In this example, we consider a non-Gaussian random process. So, here we are considering X of t to be F sin t plus phi plus Y of t; Y is P is deterministic; this is P, is deterministic, capital phi is a random variable distributed uniformly in 0 to 2 pi and Y of t is a 0 mean stationary Gaussian random process. You can also assume that Y of t and phi are independent.

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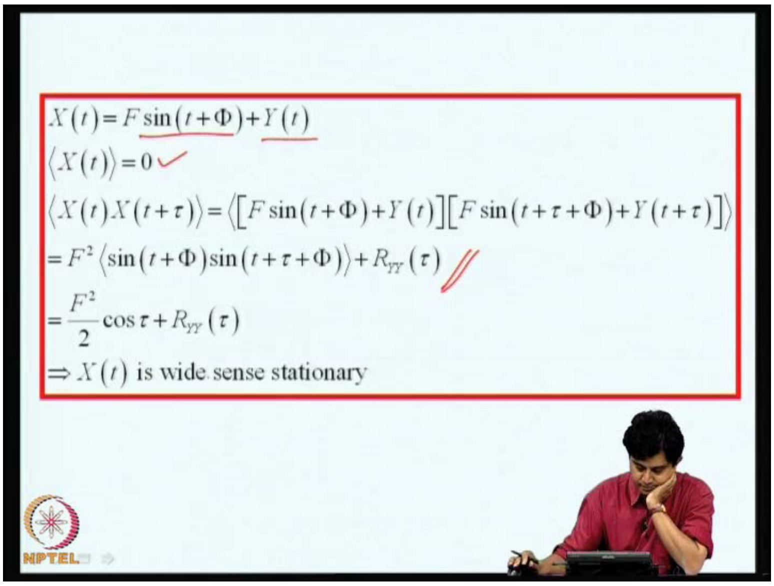

Now, we are asked to determine the joint p d F of X of t and X dot of t. The question is are X of t and X dot of t uncorrelated or independent? So, X of t is F sin t plus phi plus Y of t. You take expected value phi is uniformly distributed between 0 to 2 pi; so expected value of this would be 0; the expected value of this also would be 0; therefore, expected value of X of t would be 0. Now, if you form the product X of t into X of t plus tau into expectation, we get this expression, where we use the fact that phi and Y are independent and if you simplify this, we get thus F square by 2 cos tau plus R y y of tau.

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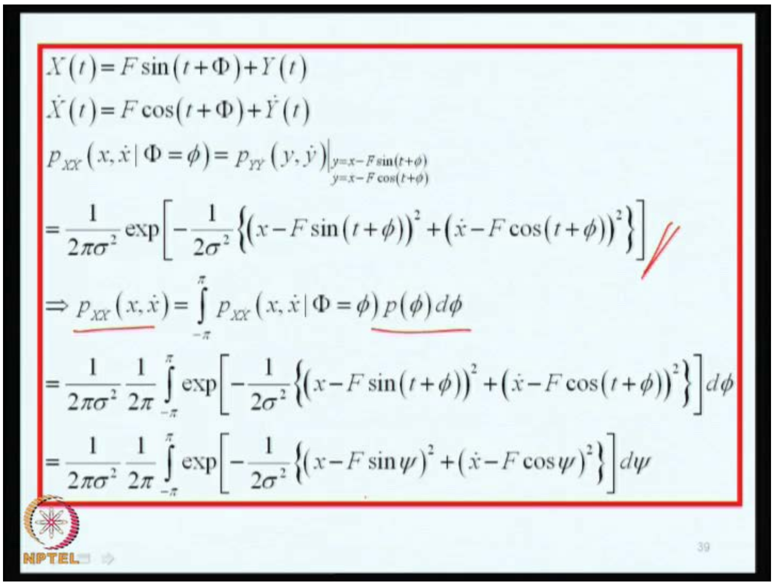

So, the process is the auto covariance function is the function of time difference; so we can conclude that X of t is wide sense stationary. Now, how about the joint probability density function? So, we consider X of t to be F sin t plus phi plus Y of t; X dot is F cos t plus phi plus Y dot of t. First what we will do is, we will find out the joint density function condition on phi. So, this will be P y y dot Y comma Y dot, where Y is X minus F sin t plus phi Y dot is X dot minus F cos t plus phi; we are finding conditional probability density function.

Now, conditioned on phi, X of t is a Gaussian random process, because Y of t is Gaussian and it will have now the mean which is X minus F sin t plus phi; so if you mean will be sin t plus phi. So, if you write this Gaussian density function, this will be the conditional density function. Now, the unconditional density function, we have to do carry out integration with respect to the probability distribution of phi that is as shown here.

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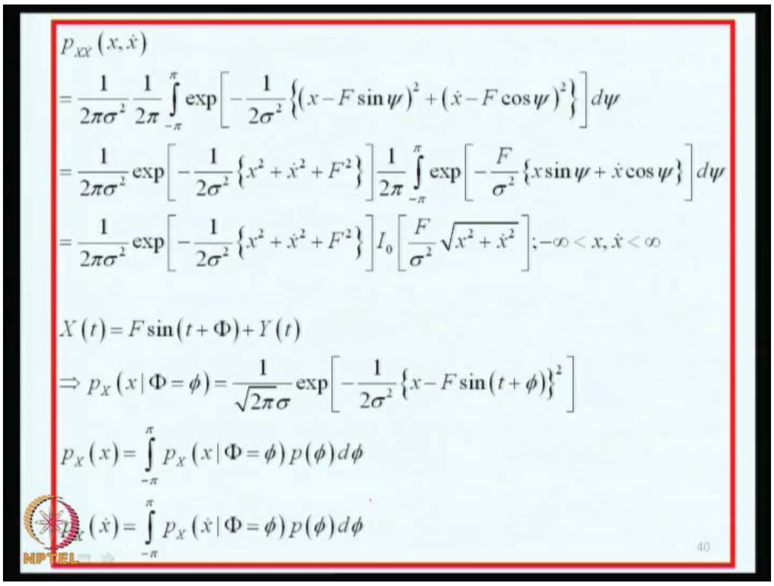

Now, if you make this substitutions, you can carry out this; we can show that P x of x dot, we can square this and take out terms, which are free from psi and look at the terms which contain only this sin sin cos psi; you can see that this integral is nothing but Bessel's function i naught and this has this form, this is first term and this is second term. So, it is clear that x, x of t is a non-Gaussian random process and if you look at the marginal density again, the same logic can be used; probability density function of x condition on phi is this and P x of x is the unconditional density function will be will involve another quadrature with respect to phi and if you carry out this again, you will get answers in terms of Bessel's function. If you multiply the marginal density, you can verify that you would not get the joint density function.

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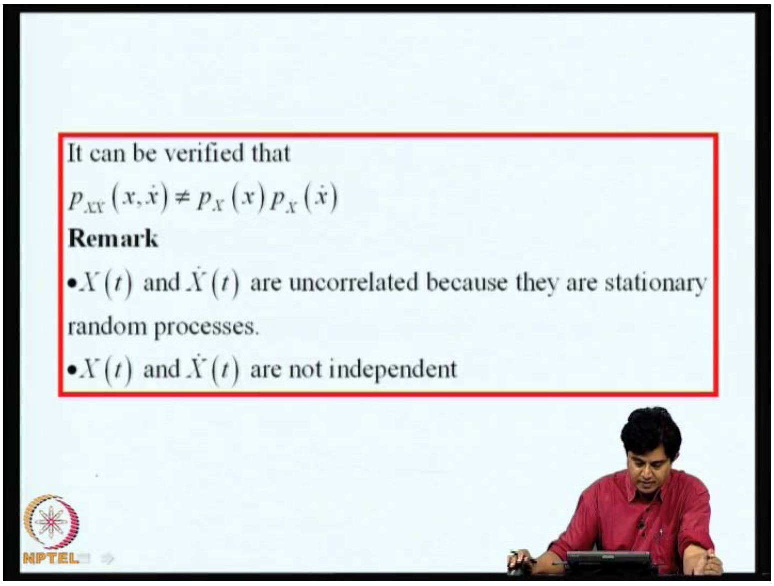

So, from this, what is the conclusion that we can draw? X of t and X dot of t are uncorrelated, because they are stationary random process processes and X of t and X dot of t are not independent. So, it is an example of non-Gaussian random processes, where process and it is derivative are uncorrelated, but they are not independent.

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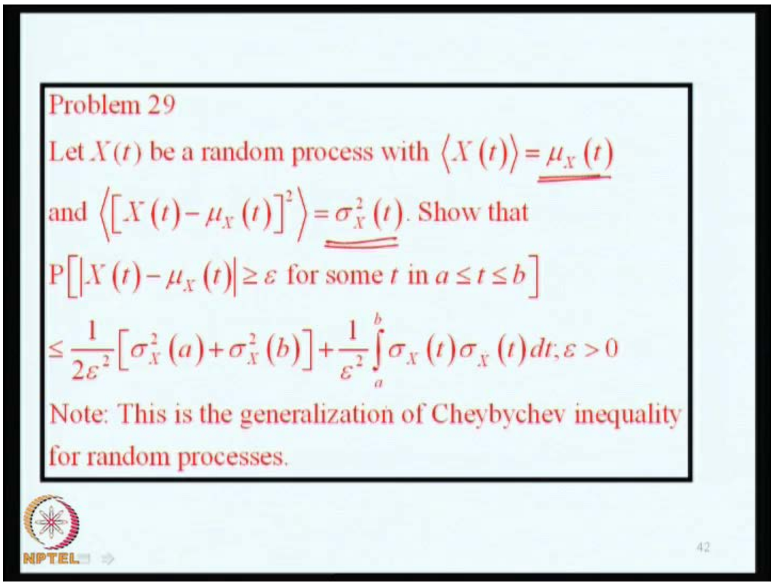

Now, while discussing random variable, we talked about what is known as Cheybychev inequality. Now, what happens to that inequality when we extend the logic to random processes? So, let X of t be a random process with mean mu X of t and variance sigma X square of t; now the inequality that we are asked to show is probability of modulus of X of t minus mu X greater than or equal to epsilon for some t in a to b is less than or equal to a quantity which involves the variance of X evaluated at a and b and an integral over a to b.

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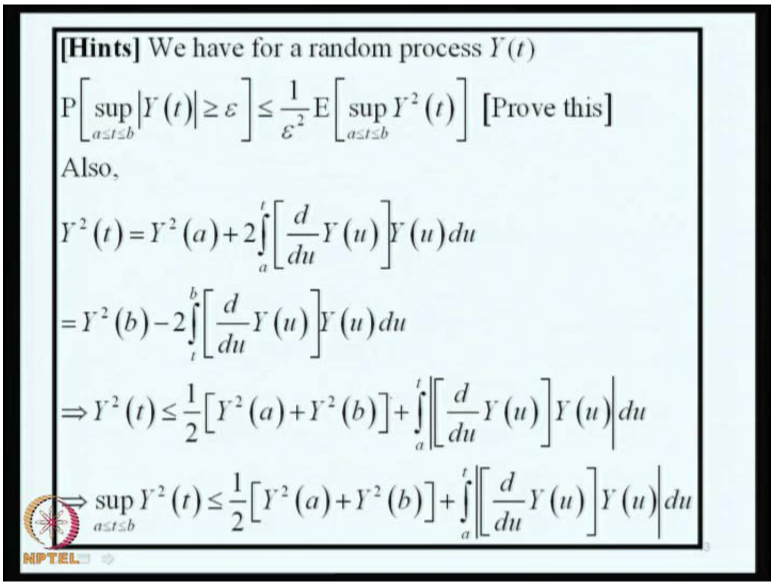

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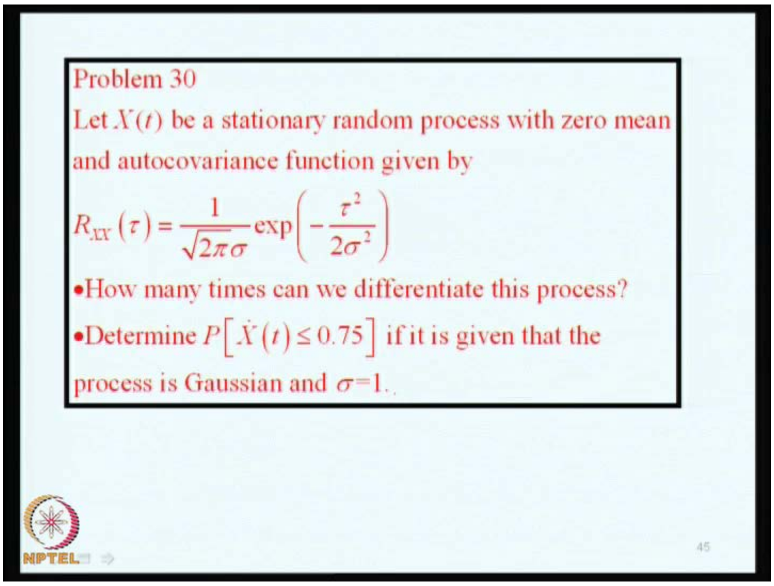

So, this is the generalization of cheybychev inequality and this can be proven. I have indicated the steps; I would not like to go through this; you can take a look reasonably straight forward. Now, let X of t be a stationary random process with 0 mean and auto covariance, which has the form of a Gaussian probability density function. The question is how many times can we differentiate this process? And we are asked to determine probability of X dot of t is less than or equal to 0.75, if it is given that the process is Gaussian and sigma is 1.

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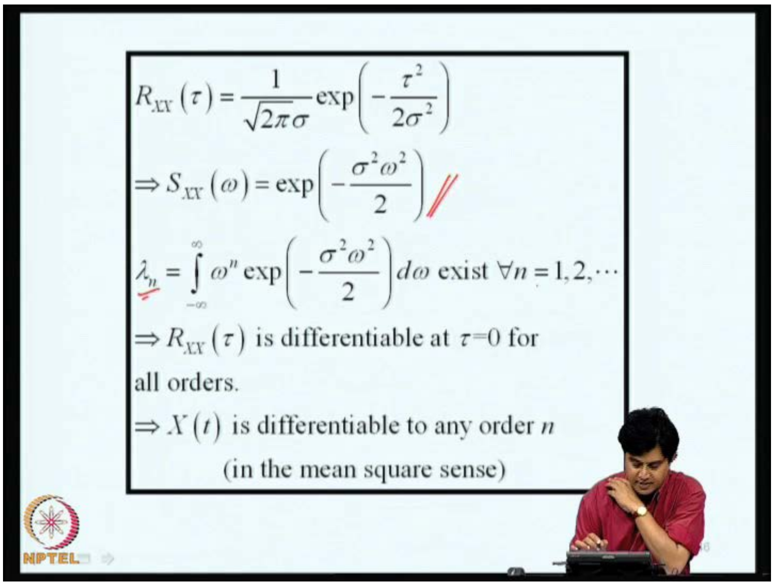

Now, R x x of tau is given to be this; from this, if I take the Fourier transform, I get the power spectral density function to be this. Now, if you consider the spectral moments, we can show that the since this exponent is decaying as minus omega square, all these integrals will exist for n equal to 1, 2, n, etcetera. In spectral moments, that 0 to 4, 6, etcetera are nothing but auto covariance of the parent process and there derivatives evaluated at tau equal to 0 and since all these are finite, it follows that X of t is differentiable in the mean square sense to any order n; so it is a fairly smooth process.

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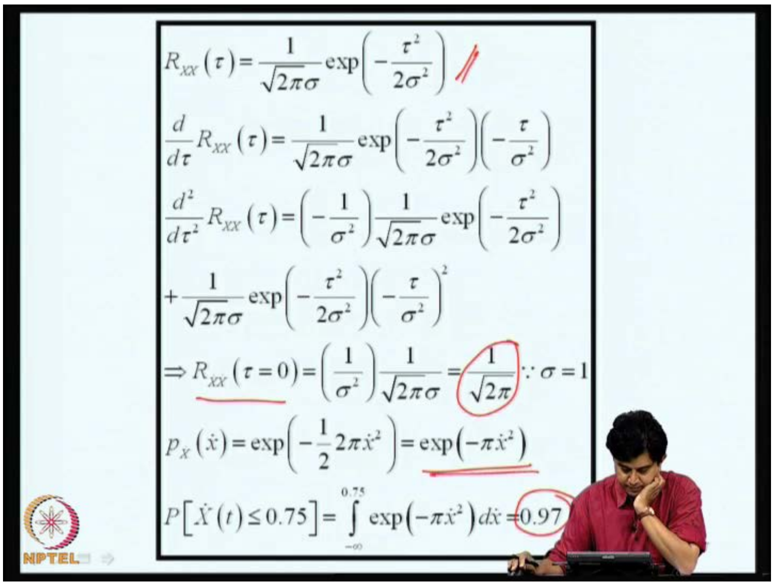

Now, we can differentiate the auto covariance function and we can evaluate the various quantities at X equal to 0. So, R x x of 0 is 1; from this, it follows. So, if you differentiate these twice, you get R x x dot and if you put tau equal to 0, we get 1 by square root of 2 pi. And we can now write therefore, p X dot of X dot is given by this and from this, we are asked to find out probability density probability of X dot of t is less than or equal to 0.75, which is an integral minus infinity to 0.75, this quantity this is given; so the answer that we are looking for is this number 0.97.

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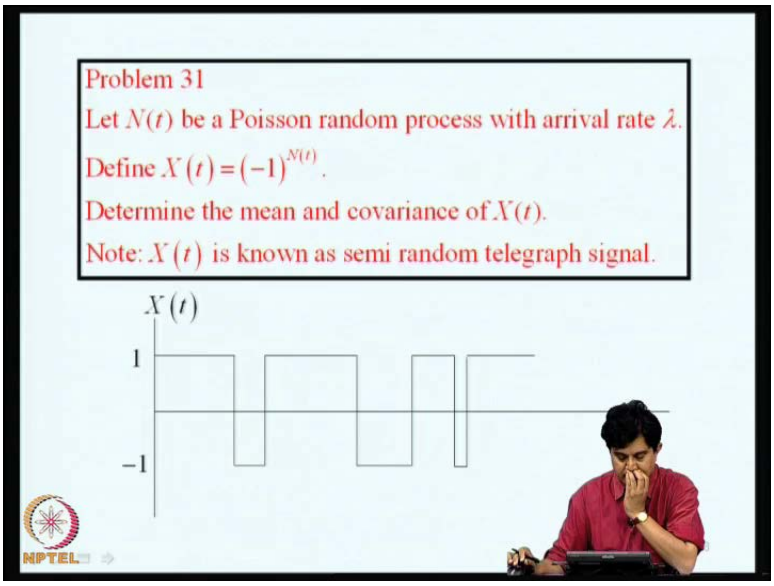

This is an interesting problem; here, we consider a random process N of t to be a Poisson random process with arrival rate lambda. Now, I define a new random process X of t which is minus 1 to the power of N of t. Now, we are asked to find a mean and covariance of X of t; X of t in the literature is known as semi random telegraph signal. So, X of t obviously, N of t is an integer value random process. So, whenever N of t is even, X of t will be 1 and N of t is odd, X of t will be minus 1. So, sample will look like this, depending on even odd combinations of n of t.

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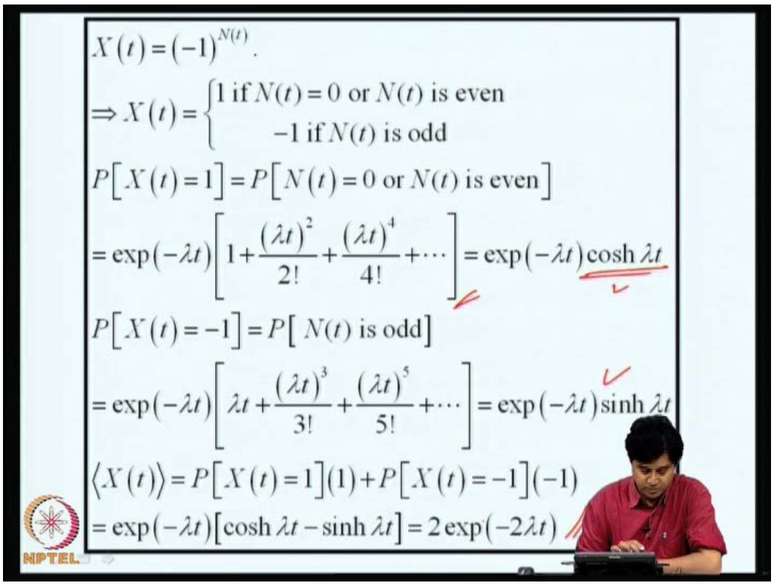

So, X of t is minus 1 to the power of N of t. So, X of t is 1, if N of t equal to 0 or N of t is even; it is minus 1, if N of t is odd. So, what is probability that X of t equal to 1? It is nothing but probability of N of t is 0 or N of t is even and that probability since it is Poisson distributed, I can evaluate and this I get to be that expression and if you carefully look at this series inside the bracket, we recognize that this is cosine hyperbolic term.

Now, how about the case, where X of t is minus 1? We have to sum over all integers- odd values of n- and if you do that, I get E raise to minus lambda t sin h lambda t. Therefore, the expected value of X of t would be X of t takes only two values, with one probability is this; other probability is this. So, probability of X of t equal to 1 into 1 plus probability of X of t equal to minus 1 into minus 1. So, if you do that, we get the answer as 2 into E raise to minus 2 lambda t; so this is the expected value of X of t.

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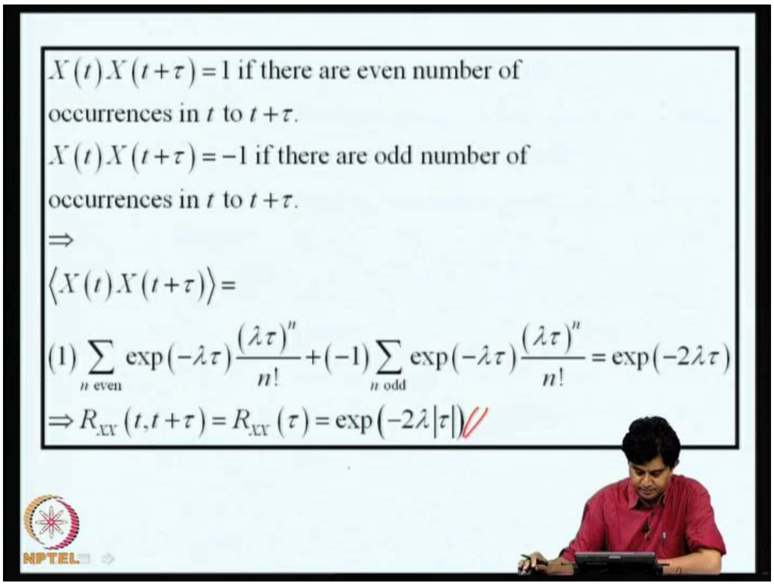

Now, what is correlation function auto correlation? Again X of t takes two values- plus 1, minus 1; therefore, the product X of t into X of t plus tau is 1, if there are even no of occurrences in t to t plus tau; otherwise, it is minus 1. So, based on this argument, we can write the expectation of X of t into X of t plus tau and we can show that the auto correlation function is the function of modulus of tau.

So, this process is wide sense stationary, that is, if I remove the mean, that is, if I define another process, where X of t minus 2 E raise to minus lambda t, if I consider that process, that will have 0 mean and it is wide sense stationary. I will return to some comments on this example shortly.

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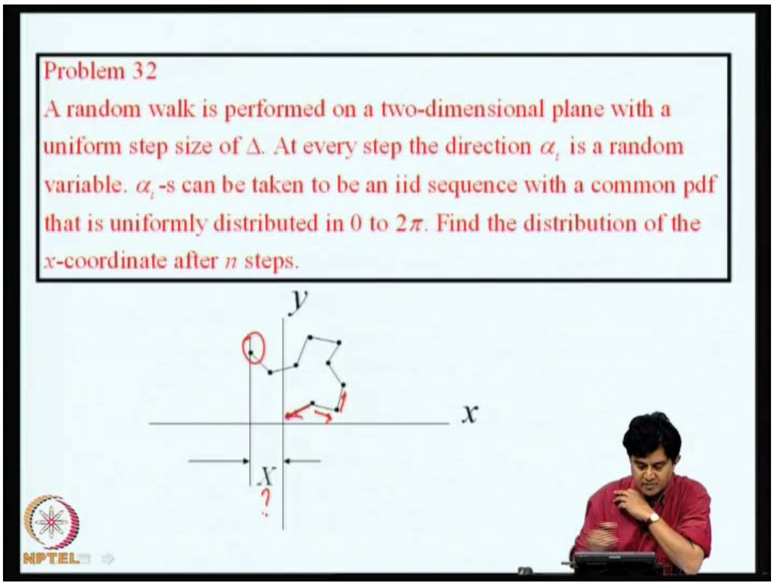

We will consider another problem; here, we consider a random walk that is performed on a two-dimensional plane with a uniform step size of delta. At every step, the direction alpha i is a random variable and alpha i alpha i's taken to be iid sequence with a common p d F that is uniformly distributed in 0 to 2 pi. Find the distribution of the x- coordinate after n steps. That means, we start here, we draw a random variable to be distributed uniformly between 0 to 2 pi and this is our realization. So, we take one step, then next, this step, this step, etcetera so after n steps, I am here and I am asking what is this X?

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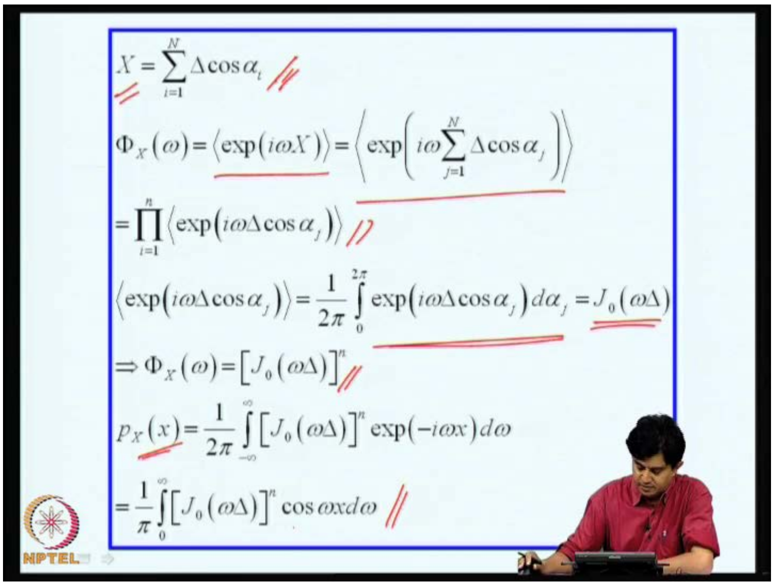

So, X is nothing but i equal to 1 to n delta cos alpha i, where alpha i are the sequence of random variable that we have generated. So, what is the characteristic function of this? It is exponential expected value of exponential i omega X and it is this and since alpha i's are all independent, I can get this. And we can evaluate this expected value of i omega delta cos alpha i and given that alpha i alpha i is uniformly distributed between 0 to 2 pi, I get this and this integral is nothing but Bessel's function j naught.

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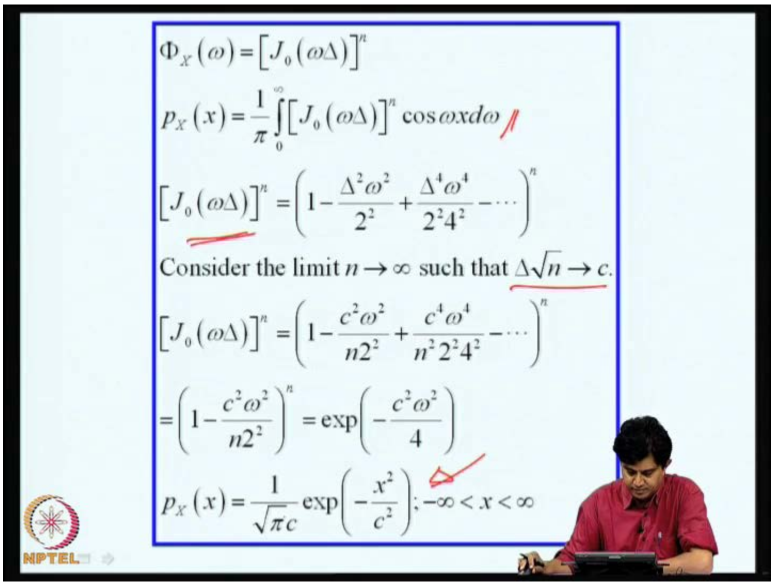

So, phi X of omega which is a characteristic function of this random variable X is actually the j naught of omega delta to the power of n. Now, we have to invert it to get the probability density function and if you do this, we get this integral; this is the integral. And now if you do a series expansion for j naught to the power of n and consider the limit of n tend into infinity, number of steps going to infinity such that delta of square root of n goes to C that is a kind of limit that we used for taking a one-dimensional random walk to a Brownian motion process in the same sense, this is done; if you do that, we can show that this process is this resulting random variable is Gaussian.

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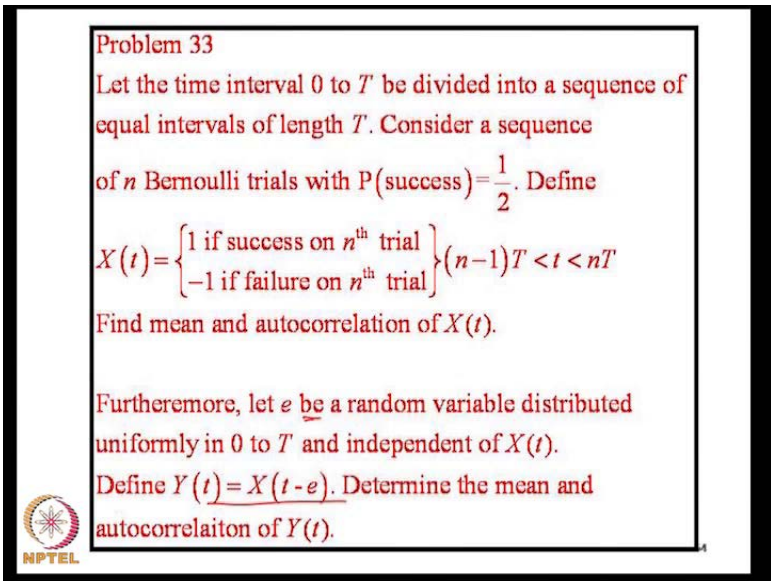

So, this is an illustration of application of central limit theorem. So, you get the limiting density function to be Gaussian. If that n tend to infinity is not reached, the answer that we are looking for is given by this integral.

So, another example which will help you to manipulate simple random processes.; let the time interval 0 to t be divided into a sequence of equal intervals of length T, that means, I am considering some n T. Consider a sequence of n Bernoulli trails, that means, at every step probability of success is half. Now, I define X of t equal to 1, if success if we observe success on nth trails; if it is minus 1 it is minus 1, if we observe a failure so, where t is between n minus 1 capital T into N t.

So, the problem is find mean and auto correlation of X of t. This is the second bit to this problem, where we define another random variable e, which is distributed uniformly in 0 to capital T and independent of X of t. We define Y of t as X of t minus e. So, the problem is again determine the mean and auto correlation of Y of t.

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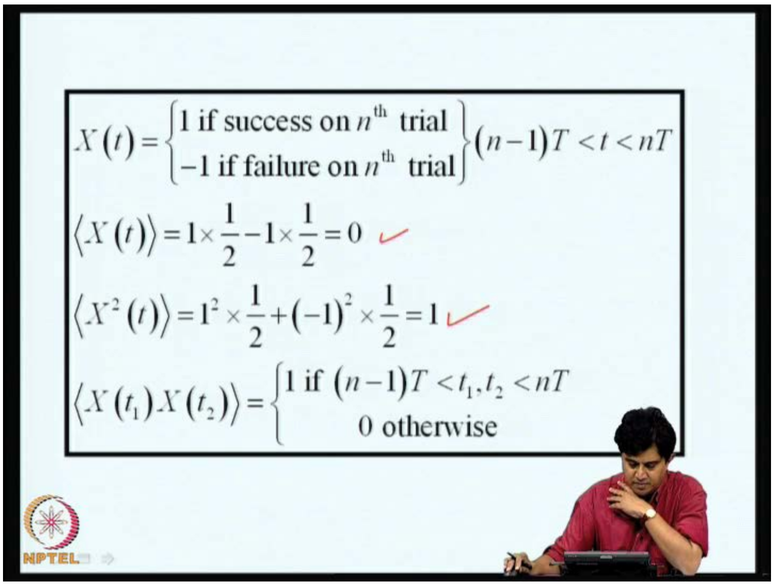

This is the parallely simple exercise that needs to be carefully done. So, X of t is 1 if success on n th; trail X of t is minus 1 if failure on n th trail. So, X of expected value of X of t is therefore, 1 into half minus 1 into half which is 0; the mean square value is again we can easily find out, it is this.

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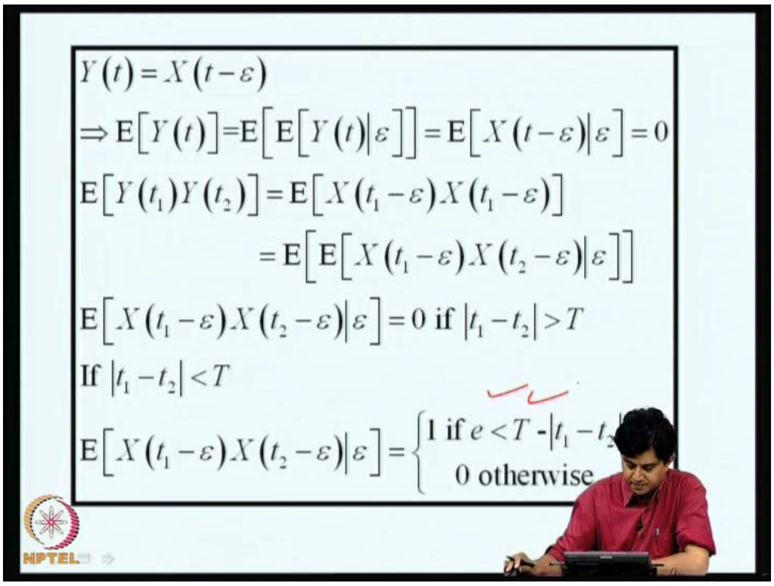

Now, the product X of t 1 into X of t 2 is 1, if both t 1 and t 2 are contained in this interval n minus 1 capital T into 2 n t otherwise, it is 0. So, that completes the solution to this problem. Now, we introduce the this epsilon that is the random variable epsilon; now again we can find expected value on auto correlation of Y of t. We can first find the expected value condition on epsilon and then, integrate with respect to distribution of epsilon. If we do that, we can show I will leave this as an exercise that the auto covariance of correlation of X of t in this case that is Y of t in this case will be given by this function.

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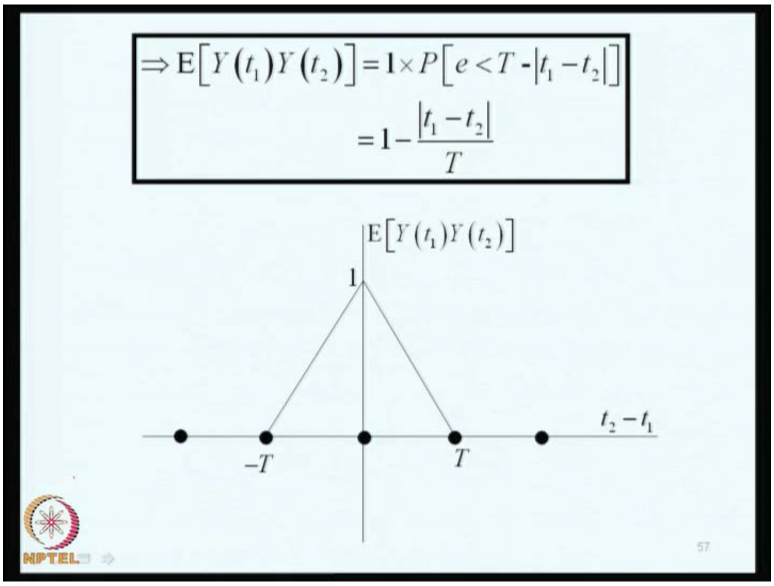

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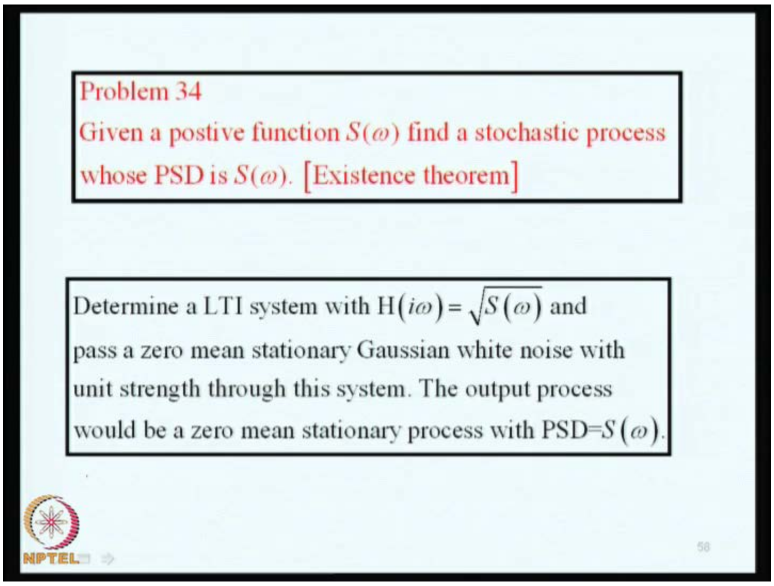

So that is how it looks like here; it is a triangle over the step minus t 2 two steps minus t 0 to t and it is 0 elsewhere. Now, this is a slightly question related to existence of random processes. So, we are given a positive function S of omega and we are asked to find a stochastic process whose power spectral density S of omega.

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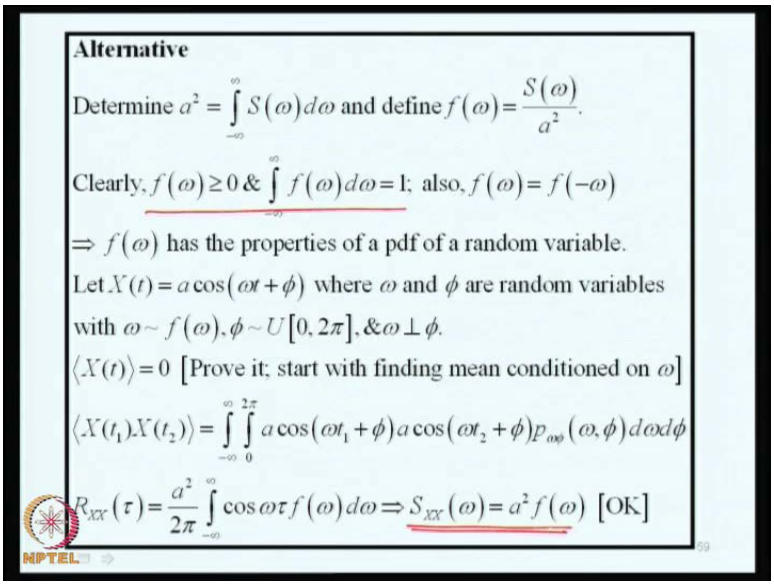

Now, there are two solutions; I will start with the second solution what I will do is I will determine a square, which is area under S of omega and define a function F of omega, which is S of omega by a square. So, clearly area under F of omega is 1 and it is positive.

We assume that S of omega is equal to S of minus omega; so consequently what happen F of omega is F of minus omega. So that would mean based on these two properties, we can conclude that F of omega has the properties of a probability density function of a random variable.

Now, I define a random process X of t as a cos omega t plus 5, where omega and phi are random variables with omega having probability density function F of omega and phi is uniformly distributed between 0 to 2 pi and omega is independent of phi. So, you consider now expected value of X of t, we can show that it is 0 you can start by you can find the mean condition on omega and then, integrate with respect to omega condition on omega this mean is 0 therefore the answer is 0.

The auto correlation function can be evaluated; this is reasonably straight forward and you can show that the auto correlation function is a square into F of omega, which is what we are expecting. that is Auto correlation is this the power spectral density function is this which is what we are expecting.

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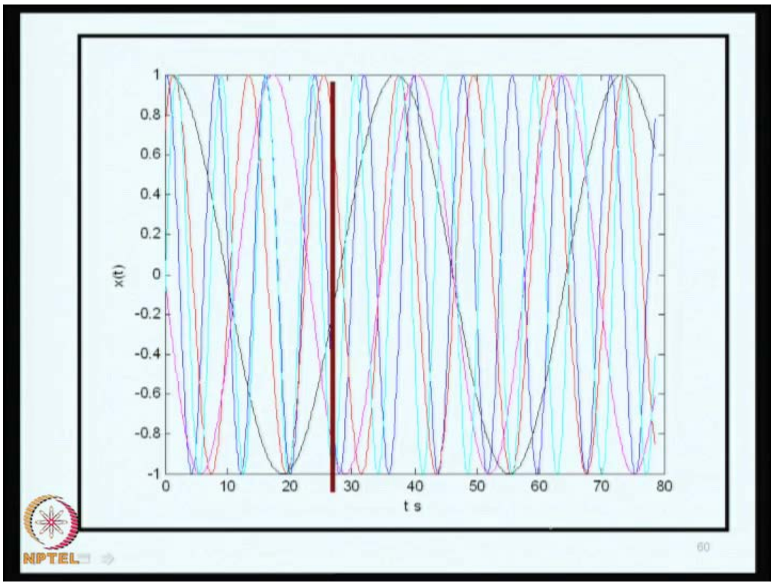

Now, interestingly how does a sample of this function look like? They are If we assume say this as 1 they all are harmonics, where frequency and phase are randomly distributed, but the power spectral density function can be specified by the is a part of the specification of the problem.

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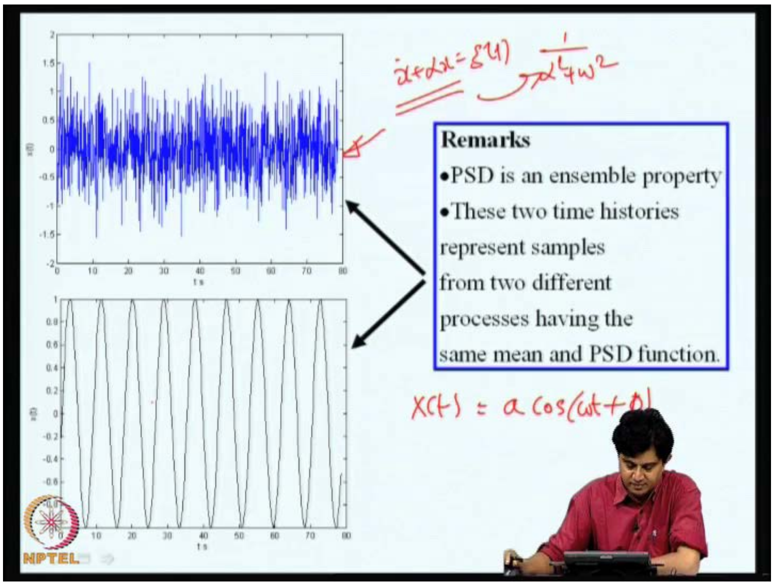

By that what I mean is if power spectral density function is of the form say alpha square plus omega square. If i now assume that this is the output of alpha X equal to psi of t, the power spectral density can be shown to be similar to this and samples of this will look like this. But following the logic that we outlined here, X of t is a cos omega t plus phi, this also has the same power spectral density. So, these two samples come from two different random processes, whose power spectral density functions are the same.

So, this would be question us to the fact that power spectral density function is an ensemble property. The two time histories that we see here represent samples from two different processes having the same mean and power spectral density function.

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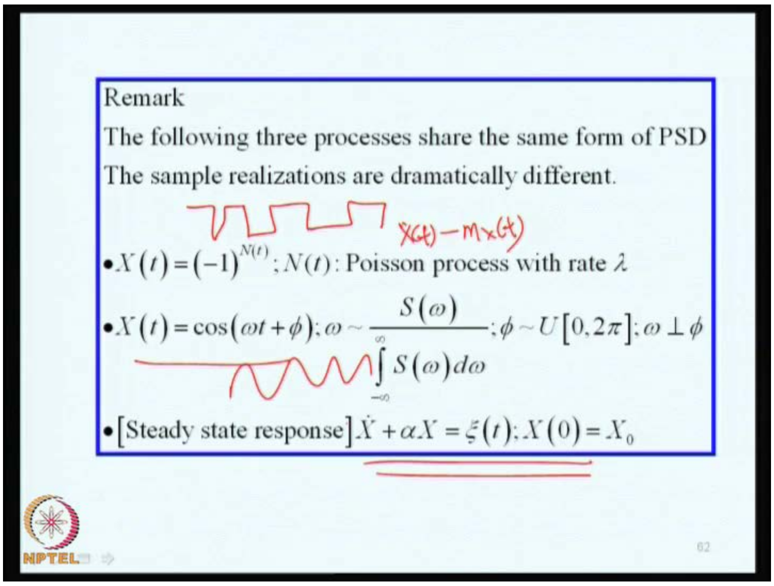

Just to summarize, we saw that one example earlier X of t is minus 1 to the power of N of t, where samples were like square waves. So, in this if we detect that mean, this quantity had power spectral density function or auto covariance function, which is same as the power spectral density of this; the power spectral density of response of this system. But in this case, the sample looks something like this; in the first case, it is a square wave pulse; in this case, it is harmonic.

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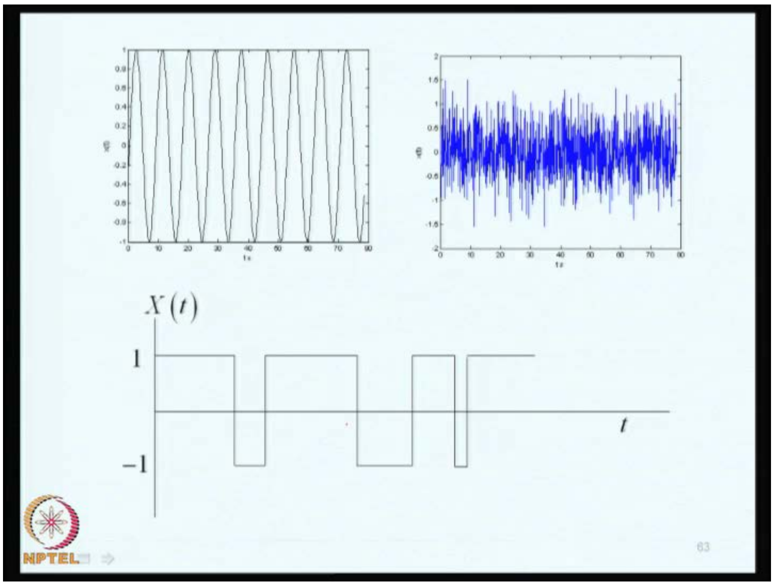

So, while interpreting power spectral density function in modeling, you have to take cognisense of this fact. At the end of the day, it is a not a sample property; it is an ensemble property. So, this is what I am showing here; this is harmonic function; this is an erratic function and this is a square wave function. It is intuitively not clear; if i say that these three samples come from three different processes having the same power spectral density function.

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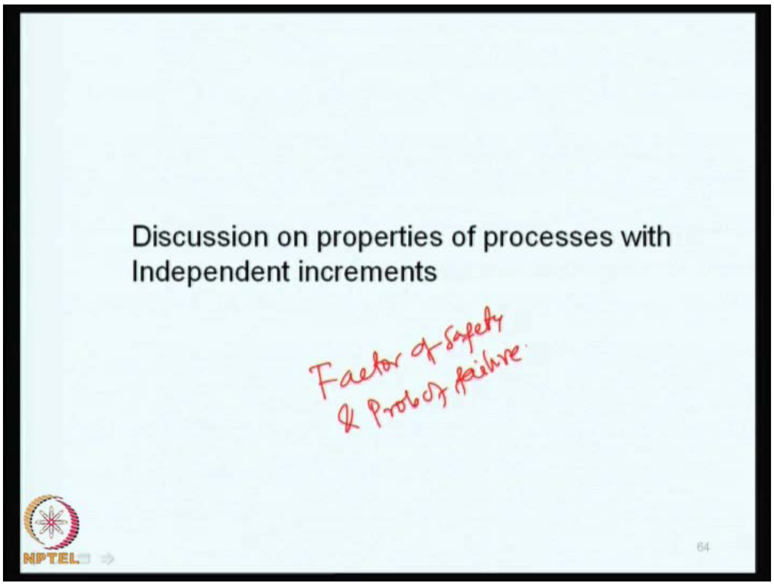

Now, there are few properties of random processes with independent increments, that I will take up in the following lecture and that discussion will be followed by some which problems on first persist time and extreme value distribution and so on and so forth and then, we will consider some discussions on the notion of say factor of safety and probability of failure.

The notion of factor of safety essentially originates from deterministic outlook, whereas probability of failure when we talk about how we are modeling uncertainty using theory of probability random variables and random processes so how they are related. So that is one of the questions, will briefly consider the philosophical issue there and then, we will consider some more problems in response of dynamical systems to random excitations; specifically, we will consider excitations which are non-Gaussian in nature and systems which are non-linear and we will apply in some problems the theory of Markova processes and derive the response moments and all these will consider in the following lecture.

So, we will close the present lecture at this junction.