Stochastic Structural Dynamics

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Module No. # 10

Lecture No. # 37

Problem Solving Session-1

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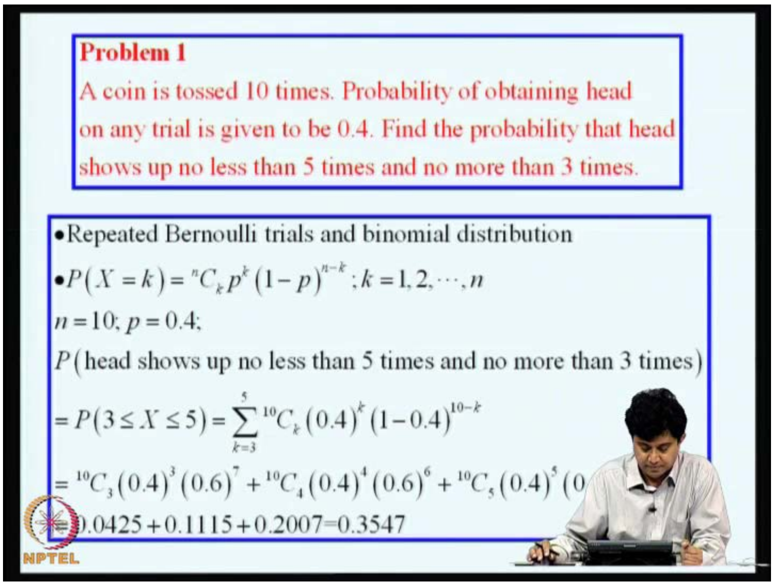

In this and remaining few lectures, we will have problem solving sessions. So, in todays session, we will consider problems that are based on theory the, of probability and random variables which I cover during first few lecture – 4, 5 lectures. So, what I will do is - I will consider a series of problems; explain what the problem is and suggest how it can be solved, and the presentation contains reasonably complete solutions. In some places, there are hints; the solutions are not complete. That, those places you have to complete the steps that still remain at the end of that.

So, we will start with some problem. These problems are not graded in any specific order of complexity or difficulty. So, let us see how it goes. So, first problem - a coin is tossed the 10 times; probability of obtaining head on any given trial is 0.4. Now, find the probability that head shows up no less than 5 times and no more than 3 times.

So, we need to model now. When we toss the coin, there are two possible outcomes. This is the Bernoulli trial and probability of getting head which we make call as event of success remains constant, and these trials are independent. So, we can model this tossing of coin as repeated Bernoulli trials and we know that number of successes in k trials follows the binomial distribution probability of X equal to k is n C k P to the power of k 1 minus P to the power of n minus k - where k runs from 1 to n.

Now, we are given that in this example, n is 10 because there are n trials, and probability of successes is 0.4, and even that we are asking is, we are considering is, head shows up no less than 5 times and no more than 3 times. That would mean we are looking at the probability that X is greater than or equal to 5 or less than or equal to, greater than or equal to 3 or less than or equal to 5. So, we have to sum this probability from 3 to 5. So, if we do that, there will be terms correspond to k equal to 3, 4 and 5. There are three terms, and upon evaluating this, we get the number 0.3547. So, that is the answer to this problem.

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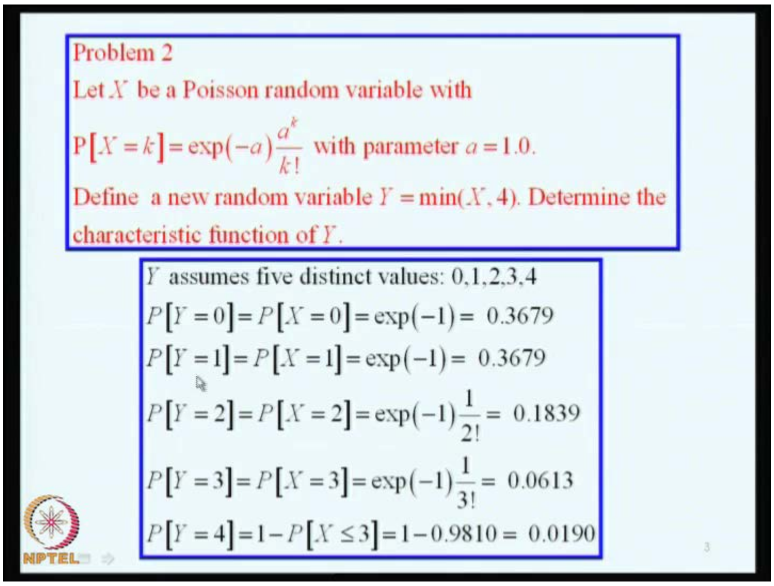

Next, we will consider a random variable x which is a Poisson random variable such that, probability of x equal to k is e raise to minus a a to the power of k divide by k factorial with parameter a given to be 1. Now, I define a new function - function of this random variable - that is defined a y is equal to minimum of x, 4.

Now, the question is how do we characterize y? Now, y is a discrete random variable. Poisson random variable is the discrete random variable with countably infinite sample space; whereas y will be a discrete random variable with finite sample space because it assumes only 5 possible values – 0, 1, 2, 3, 4 - these are the possible values. If we get for example X equal to 5, then according to this transformation, y will assume value of 4.

So, now, therefore, what are the probabilities associated with these out comes? Y equal to 0 is probability of X equal to 0 exponential of minus 1, probability of y equal to 1, probability X equal to 1. Similarly, y equal to 2 is Poisson; x equal to 3 is Poisson. Now, y equal to 4 can happen when outcome is 4, 5, 6, 7, 8 for X. So, that is 1 minus probability of X less than or equal to 3. So, that probability will be 0.0190. So, if you add all these, they will add to 1. So, this is the solution to the given problem.

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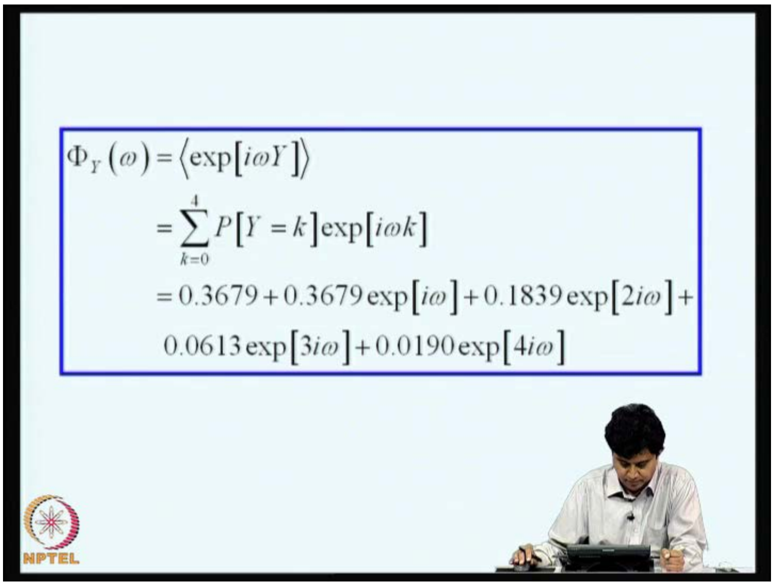

So, we have now obtain the probability mass function. We could evaluate the characteristic function. From which, we can get moments etcetera. So, we, characteristic function is expected value of exponential i omega y and this a discrete random variable assuming five values. So, if you substitute that, we get the characteristic function to be given by this. If it differentiate this with respect to omega and put omega equal to 0, you will be able to find out the mean and so on and so forth.

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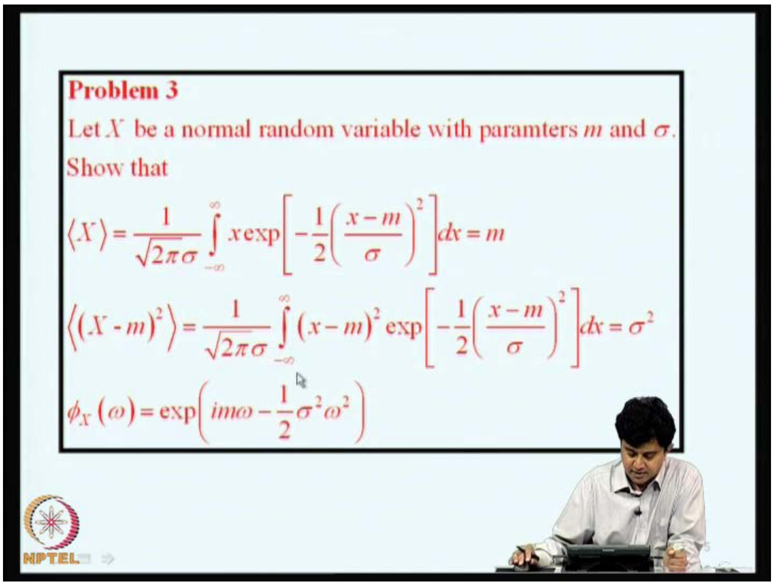

So, once a characteristic function is known, we know the, we can easily evaluated the moments. Now, some simple exercise is related to properties of a normal random variable. When I discuss this aspect during the lecture, I left them as exercise this. Now is a time to look at them. Let X be a normal random variable with parameters m and sigma. Now, how do we show that expected value of X is m and variance of X is sigma square? And characteristic function is given by this.

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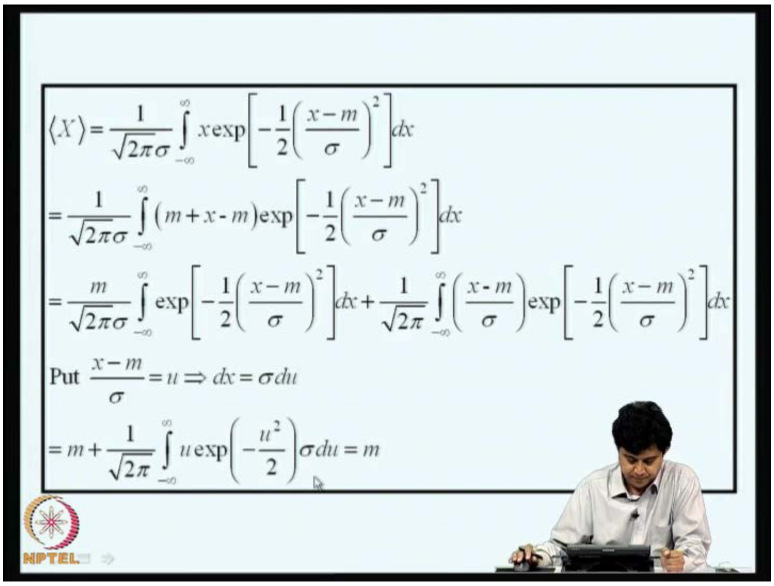

So, this exercise is simply an exercise in evaluating these integrals so that you become familiar with the methods of evaluating these integrals. So, let us start with expected value of X. So, this is 1 by square root of 2 pi sigma minus infinity to plus infinity x exponential of this into dx. So, what I do here x I write it as m plus x minus m. So, the first term m into this integral is taken outside and this integral excepting this term m is 1 because area under probability density function is 1. That is one of the axioms of probability and the remaining term is x minus m by sigma into this. Now, I substitute x minus m by sigma is u and I will get dx is sigma du and we substitute thus into second integral. We can see that this is an odd function and are symmetric limits. Therefore, this value is 0. So, I get the required that expected value of x is m.

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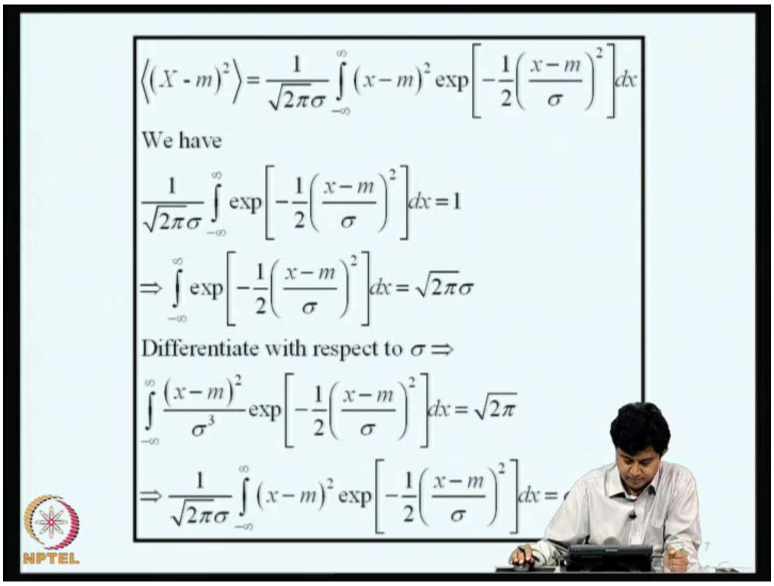

Now, how do we show that variance is, how do you evaluate the variance? So, variance is expected value of x minus m whole square. So, this is the expression that we need to evaluate. Now, what we could we could do is - we can begin by the result that area under the probability density function is 1. This I have shown during the lecture. Now, we can rewrite this as integral minus infinity to infinity exponential of this function dx is square root 2 pi sigma. Now, if I differentiate this with respect to sigma, I will get - Sigma is residing here - I will get this term into exponential is equal to square root 2 pi.

Now, if I rearrange the sigma, rearrange these terms, I will get 1 by square root of 2 phi sigma x minus m whole square exponential this dx is equal to sigma square. So, this is a simple way of showing that variance of a Gaussian random variable is sigma square.

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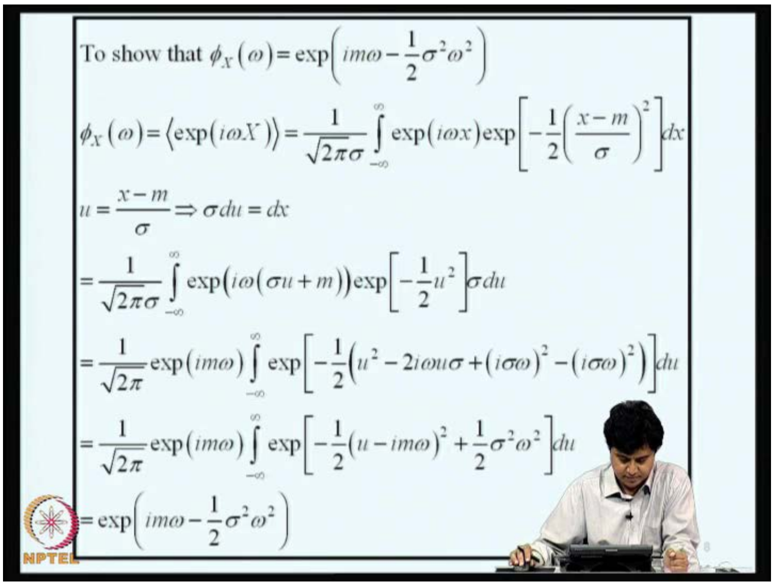

Now, characteristic function is again in terms of m n sigma as we already know, but how do we show that this is correct. So, definition of a characteristic function is expected value of e i exponent of i omega x, exponential of i omega x and this is the integral. So, now, we begin by making the substitution x minus m by sigma is equal to u, and upon this substitution, the exponent becomes u square by 2 and there will be slight rearrangement of these terms and terms not containing u can be pulled outside, and this i omega sigma u can be taken inside the exponent and this can be rearranged, so that we get a proper square of an integral and remaining term. The remaining term is free from the variable of integration u. Therefore, it can be pulled outside, and what remains would be the area under the Gaussian probability density function which is 1. So, we get characteristic function to be equal to 1.

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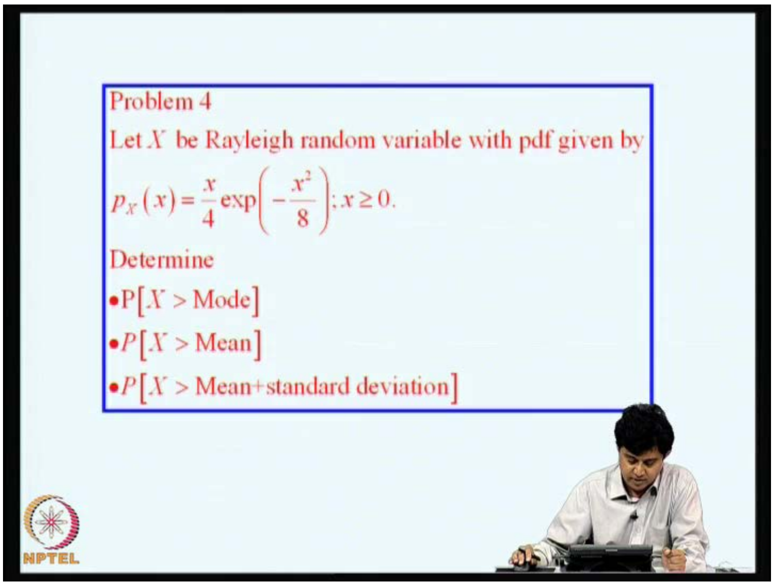

Some more exercises of a similar kind. Let x be a Rayleigh random variable with probability density function as given here. So, we are ask to find probability that X takes value greater than the mode of the random variable. X takes value greater than the mean and X takes value greater than mean plus 1 standard deviation. This again is an exercise that would familiarize with integration process is involved in characterizing random variables.

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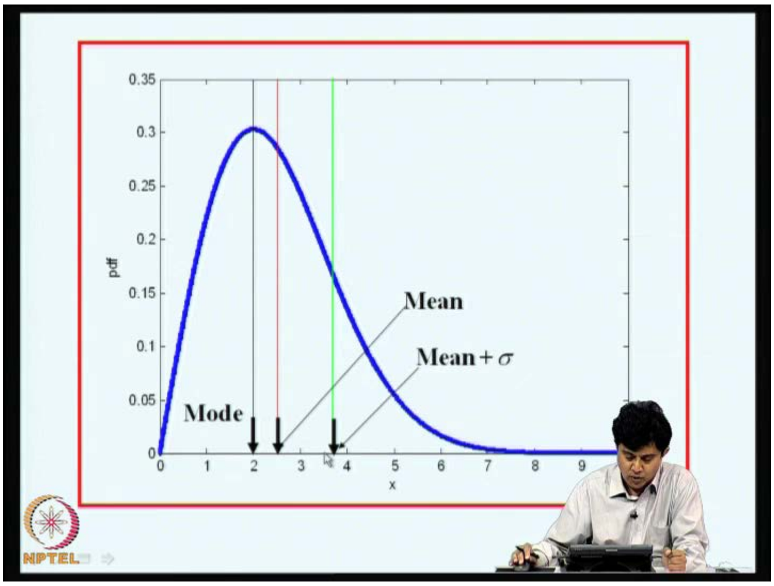

So, we can sketch this random variable for the given parameters, and blue line is the Rayleigh property density function and mode is the point where the probability density function peaks, that is, d p x by dx is equal to 0 at this point, and mean and mean plus sigma are to be evaluated. We still do not know where they are. So, we will do that.

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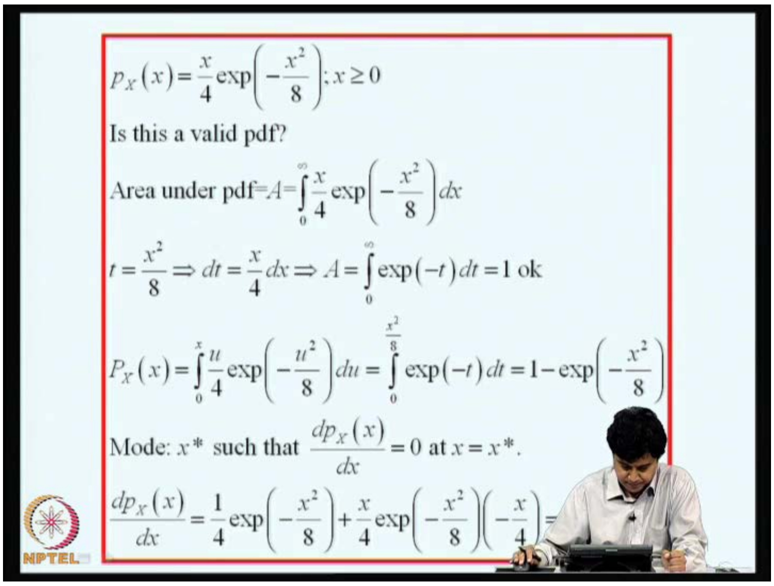

Now, first and for most we can verify whether this is a valid probability density function. So, for that, the area under the probability density function should be equal to 1. So, we can check that. So, to do that, I substitute x square by 8 is equal to t and make this substitution and we can verify that area under the density function is indeed equal to 1. Now, what is the probability distribution function? It is 0 integral 0 to x px of x p p of u du. So, if you do this again, I make the substitution u square by 8 is equal to t and carry out this simple integration. I get the probability distribution function.

Now, we are ask to evaluate probability that x takes value greater than the mode. First, we have to therefore determine what is the mode. The mode by definition is the point at which d p x by dx is equal to 0 at that value of x equal to x star. So, if I do this exercise, if i differentiate the probability density function and equate it to 0, I get x equal to 2. So, the mode of this density function is 2.

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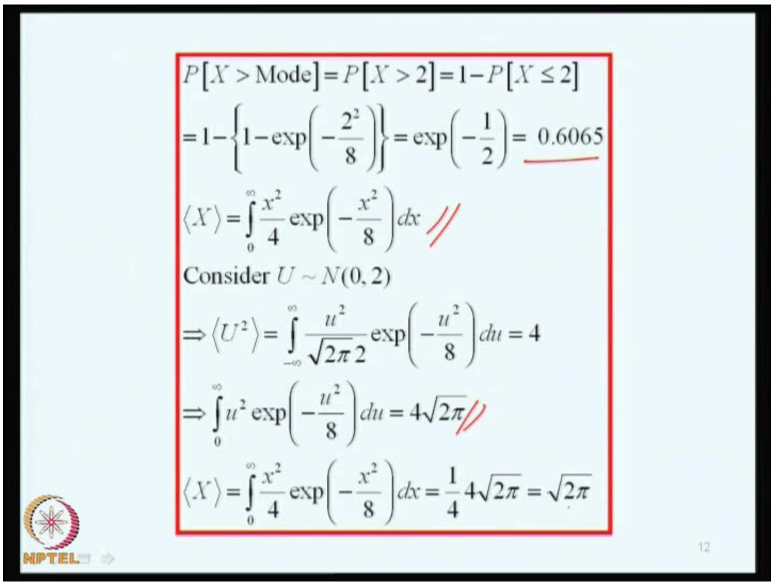

Now, what is probability X greater than mode? The probability of X greater than 2 or 1 minus p of X less than or equal to 2 and we already determine the probability density function. So, we get probability distribution function. So, we get the number 0.6065.

Now, the next is what is probability that X is greater than the mean. So, to answer that, we have to first find the mean of X is x square by 4 integral 0 to infinity x square by 4 exponential minus x square by 8 dx. Now, how do we evaluate this? We can consider random variable U which has 0 mean and standard deviation is 2, and we know that the integral minus infinity to infinity u square divide by square root of 2 pi 2 exponential minus u square by 8 du is 4. Therefore, by this, it will follow that we are interested in this integral and this is actually 0 to infinity where are this from minus infinity to plus infinity, but the variable of integration is the, this integrant is the even function. Therefore, I can write it as 2 into this integral. Using that, I get the required integral to 4 into square root of 2 phi. Therefore, expected value of x is square root of 2 phi.

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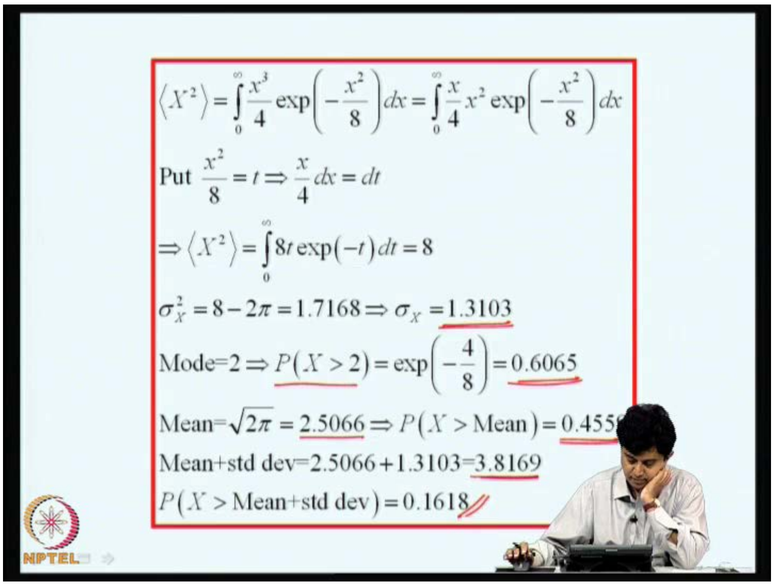

Now, we also need to find out the standard deviation. So, expected value of X square is x 2 by 4 exponential minus x square by 8 dx. So, I will write it as x into x square and make the substitution x square by 8. So, we follow through this steps and we get variance to be this.

So, probability that x is greater than or equal to 2 which is probability that x is greater than mode. I get 0.6065 and mean is found to be this, and therefore, probability of x greater than mean is this number mean plus standard deviation is found out to be this. Therefore, probability of x greater than mean plus standard deviation is this. So, these are the required numbers that we are looking for in this problem.

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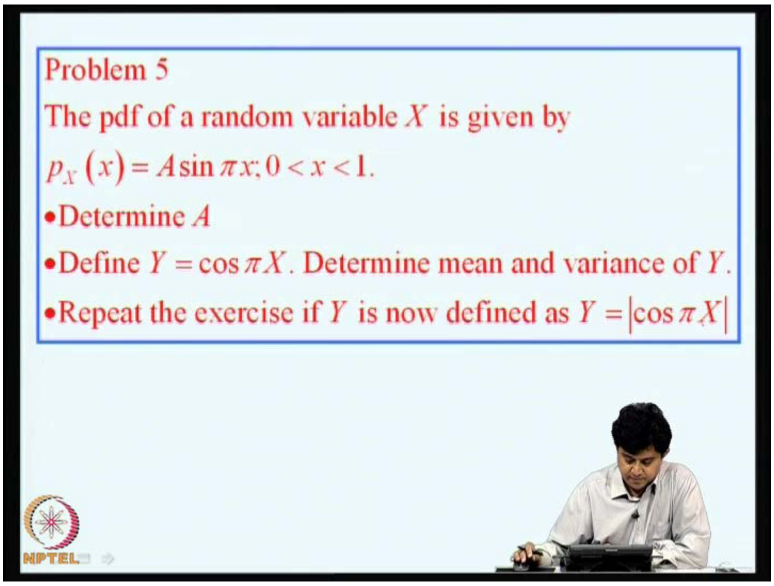

We consider in the next problem. The probability density function of a random variable x is given to be p x of x is a sin pi x, where x varies from 0 to 1. The questions that we need to answer is how to find a, and then, if i define y is equal to cos pi x, we are asked to find the mean and variance of y and we have to repeat this exercise if y is now defined as y is equal to mod of cos phi x.

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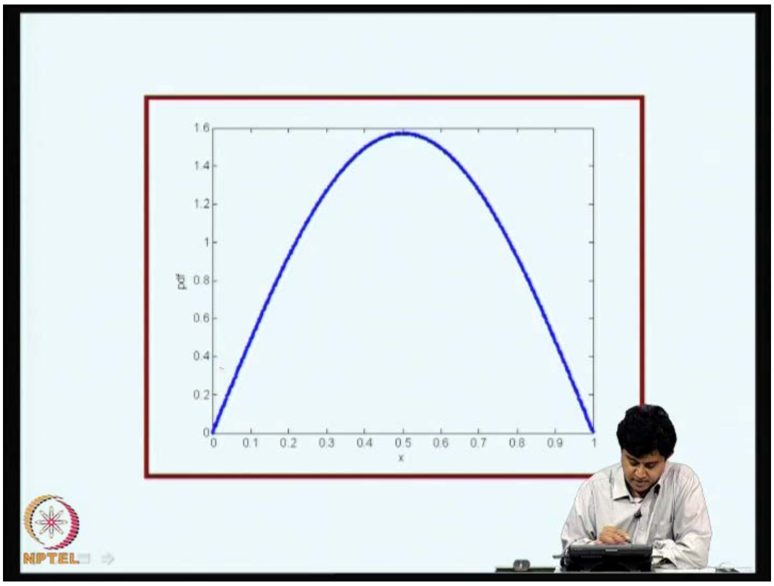

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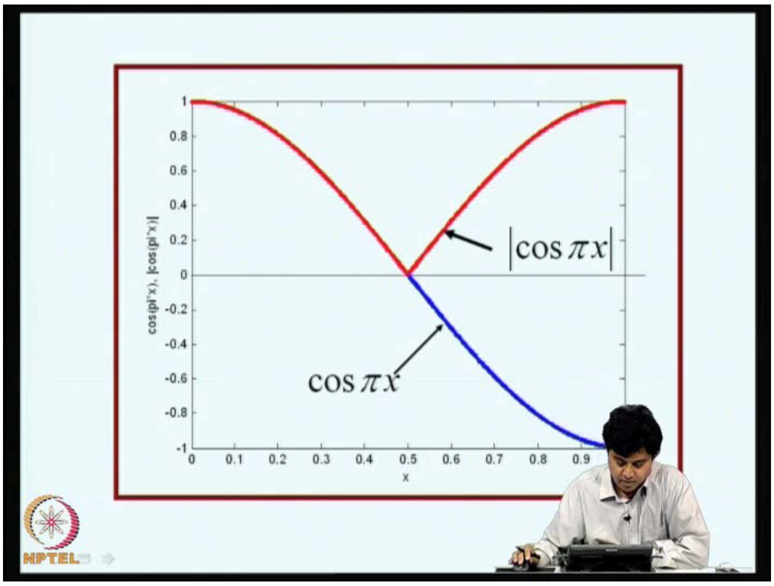

So, how does a probability density function look like? p x of x is a sin pi x x taking value from 0 to 1. The shape of the probability density function looks like this, shown here and x takes value from 0 to 1. We are not yet found the value of a that we need to find out, so that area under the probability density function is 1, and how does cos pi x look like, where the range of interest the cos phi x function is this and mod of cos phi x is this. So, in this portion, cos pi x and mod of cos phi x overlap. Therefore, we see only the red line here which is mod of cos pi x and blue line is cos pi x which runs from, which runs along this line.

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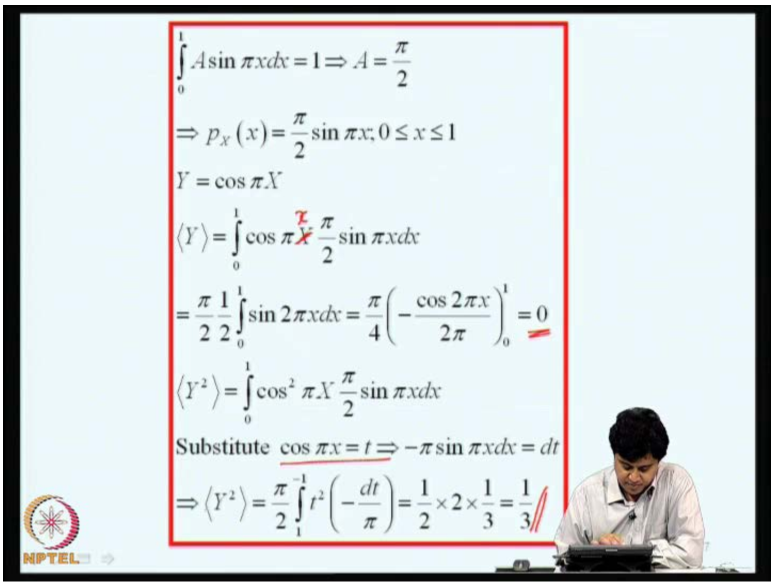

How do you find a? The area under the probability density function should be equal to 1, and if you do quickly this integration, we get a equal to phi by 2. So, the probability density function is phi by 2 sin phi x - where x runs from 0 to 1. Now, consider y is equal to cos pi x. What is expected value of y? It is cos pi x pi by 2 sin pi x dx. So, this we can write it as half of sin 2 pi x, and once we integrate, we show that the mean is 0. What is mean square value? Integral of cos square pi x pi by 2 sin pi x dx, and if you make the substitution cos pi x is equal to t and carry out this integration, we get the mean square value to be 1 by 3. Since mean is 0, the mean square value itself is the variance.

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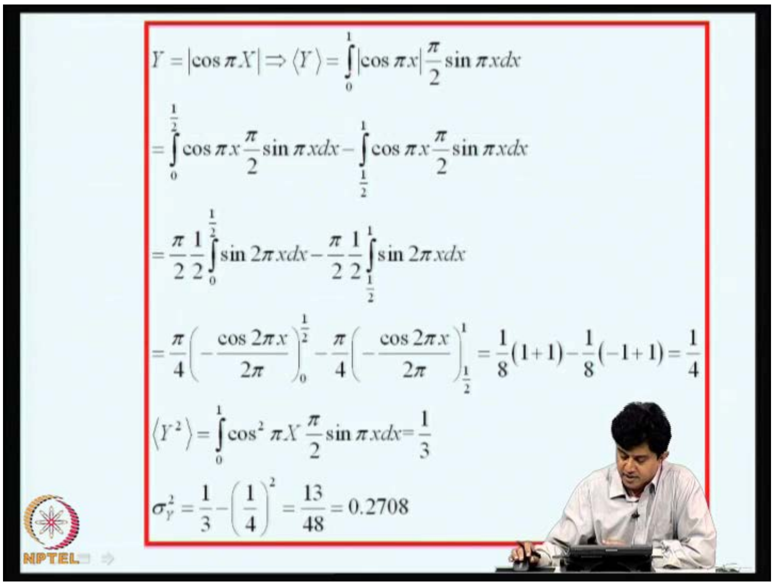

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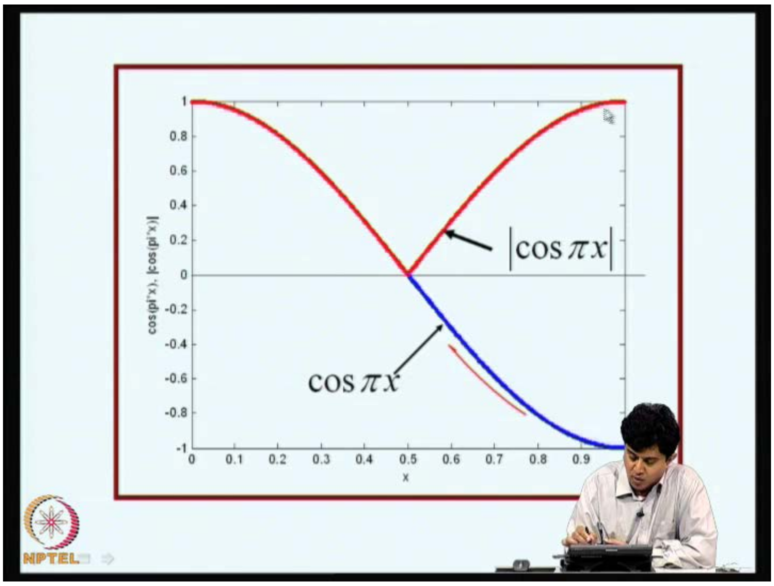

Now, how about modulus of cos pi x? So, Y is modulus of cos phi x. So, expected value of y is 0 to 1 modulus cos pi x pi by 2 sin x dx. Now, we can speed this integral. If you go back to this function, we see in 0 to point 5. Cos pi x is positive, and from 0.5 to 1, the function is negative. So, once you take modulus, we can carry out the integration from 0 to, 0 to half and half to 1 along this. So, accordingly, we can rewrite this as 0 to half cos phi x pi by 2 sin pi x dx and half to 1 cos pi x pi by 2 sin pi x dx. So, this integration can be carried out and we can show that the expected value of y is 1 by 4.

Now, how about mean square value, is 0 to 1 cos square pi x p x of x dx, and this is same as expected value of y square that we obtained in the previous example, that is, 1 by 3. So, from this, I can get the variance to be the means square value minus square of the mean and that answered to be this number 13 by 48.

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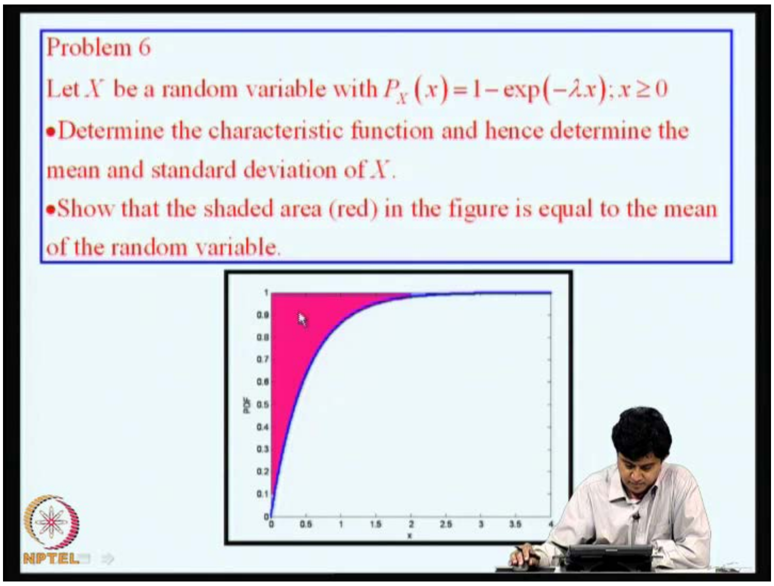

Now, let us consider x to be a random variable which is exponentially distributed with parameter lambda x takes values from 0 to infinity. So, the question that we need to answer is determine the characteristic function and hence determine the mean and standard deviation of X, and we need to show that the shaded area in the figure that is here this shaded area is equal to the mean of the random variable. Mind you this is the probability distribution function, and what we are asking is area under 1 minus p x of x dx as shown here through the shaded region is actually the mean of this random variable.

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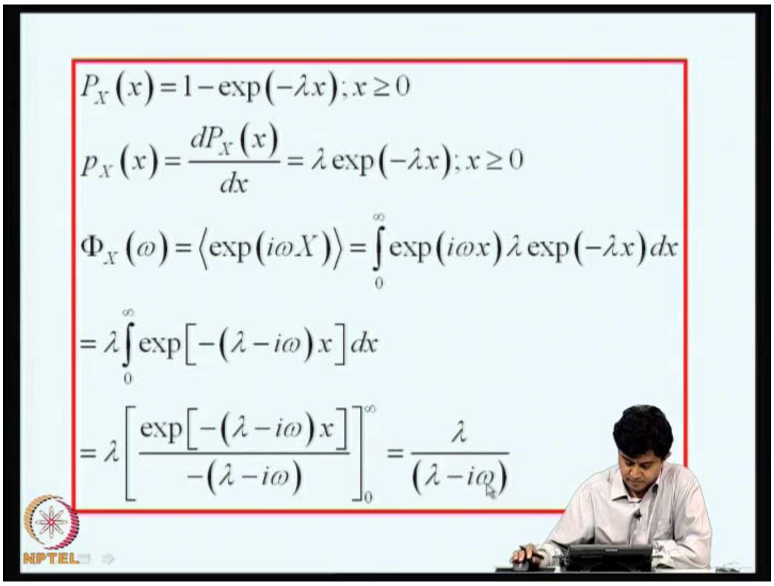

So, the probability distribution function is given to be 1 minus e raise to lambda x. Probability density function is lambda exponential minus lambda x. The characteristic function is defined as phi x of omega is expected value of exponential i omega x. So, this is 0 to infinity exponential i omega x lambda exponential minus lambda x dx and this as this form and this can be integrated, and once we put the limits infinity e raise to minus lambda x, this can be, numerator can be viewed as e raise to minus lambda x into e raise to i minus i omega x; e raise to minus i omega x amplitude is bounded between plus and minus 1 because basically that is cos omega x plus i sin omega x. So, the, that harmonic function is multiplied by e raise to minus lambda x, and as x tends to infinity, the real part e raise to minus lambda x goes to 0. Therefore, the numerator goes to 0. So, we get the characteristic function to be lambda divided by lambda minus i omega.

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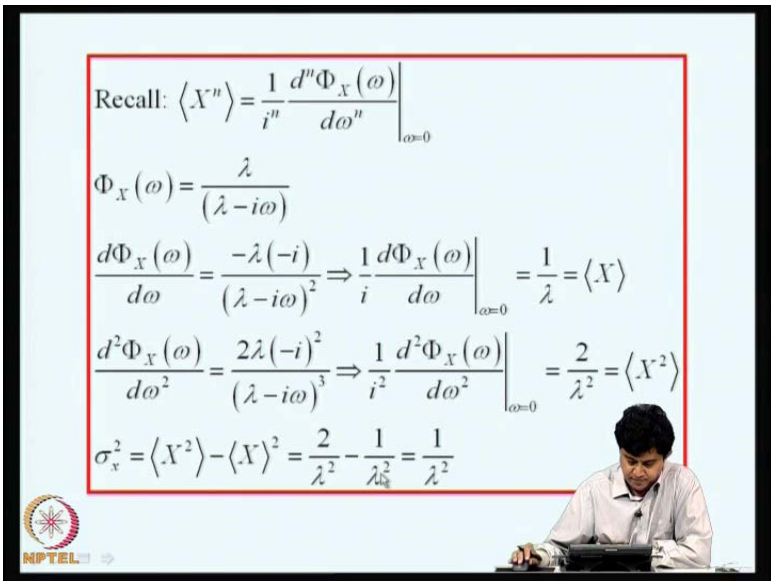

Now, how are the, how is the characteristic function related to the moments? We, we already shown expected value of x to the power of n is 1 by i i to the power of n and n th derivate of characteristic function evaluated at omega equal to 0. So, now, we can use this and evaluate the mean and variance mean square value and variance as is required in this problem.

So, phi x of omega is this and differentiation of that a gives to this and that omega equal to 0 and 1 by i of that is 1 by lambda which is the mean of exponential random variable. Similarly, mean square value can be shown to be 2 by lambda square and consequently the variance becomes 1 by lambda square.

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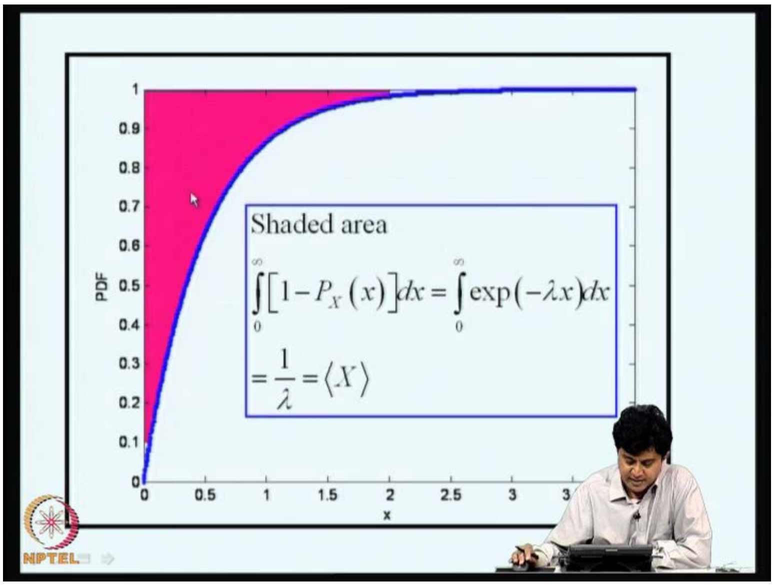

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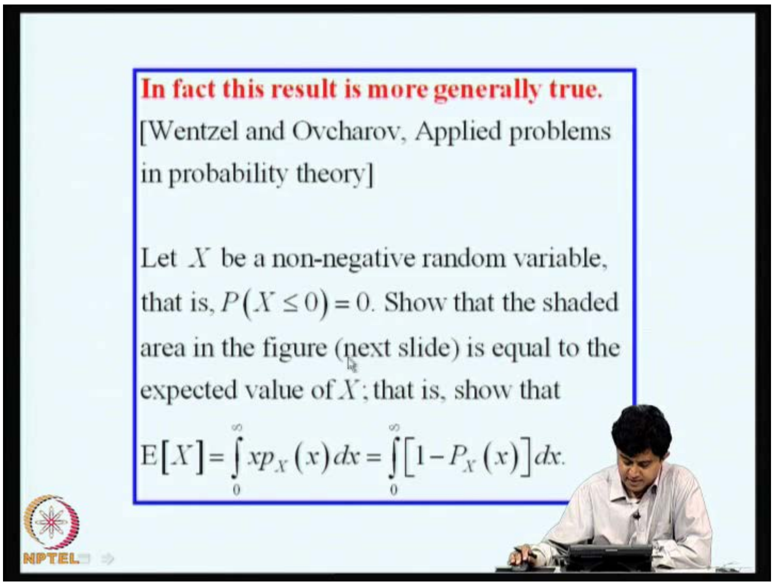

Now, how about this shaded area? This shaded area is nothing but 0 to infinity 1 minus P x of x dx is exponential minus lambda x dx which is 1 by lambda, which is actually the mean. So, it can quickly be verified, but this, this result has more general validity in the sense if you consider x to be a non-negative random variable, that is, probability of x less than or equal to 0 is 0, we can show that the shaded area in the figure that is in the next slide.

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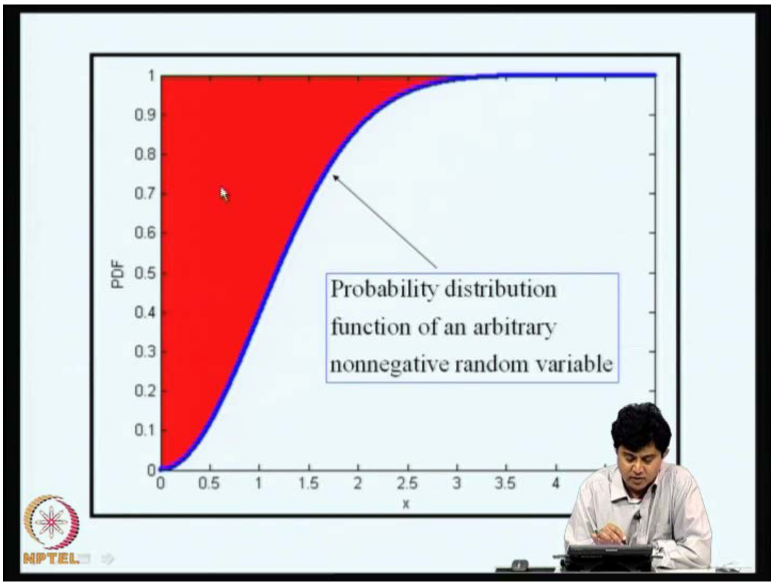

This is now the arbitrary probability distribution function of a non-negative random variable is not necessarily exponential. So, general result is this shaded region, that is, the red area here. This area is equal to the mean of the random variable. So, how do we show that? What we need to show is expected value of x is 0 to infinity x of the probability density function dx. This is the definition of expected value but it turns out that this integral is also equal to 0 to infinity 1 minus p x of x dx.

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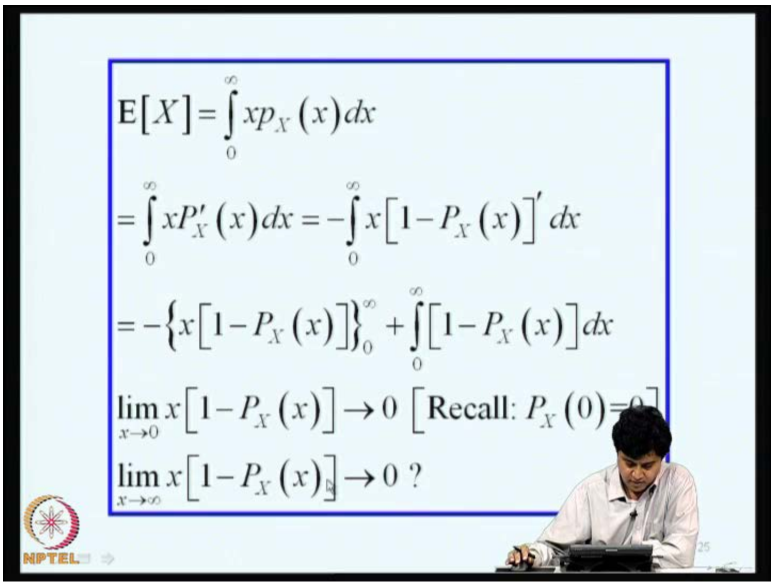

So, how do you show that? You consider this expected value of x is x p x of x dx and p x of x is derivative of the probability distribution function. Therefore, I can write it as x p s prime of x dx and I can write this as 1 minus p x of x prime. So, the prime of derivate of 1 is 0. Therefore, I can write this without making any error and there is a minus sign to allow for this negative sign that we have introduced.

Now, we can integrated by parts. So, this is x 1 minus p x of x is 0, from 0 to infinity plus 0 to infinity 1 minus p x of x dx. Now, this, the claim is the, this quantity. This is second integral is actually equal to the mean of the random variable. So, if that is true, then the first term should be equal to 0. Now, if you consider the lower limit x going to 0, we see that since x of, x is a non-negative random variable p x of 0 is 0. Therefore, the lower limit goes to 0. The upper limit x tends to infinity what happens to this is a question.

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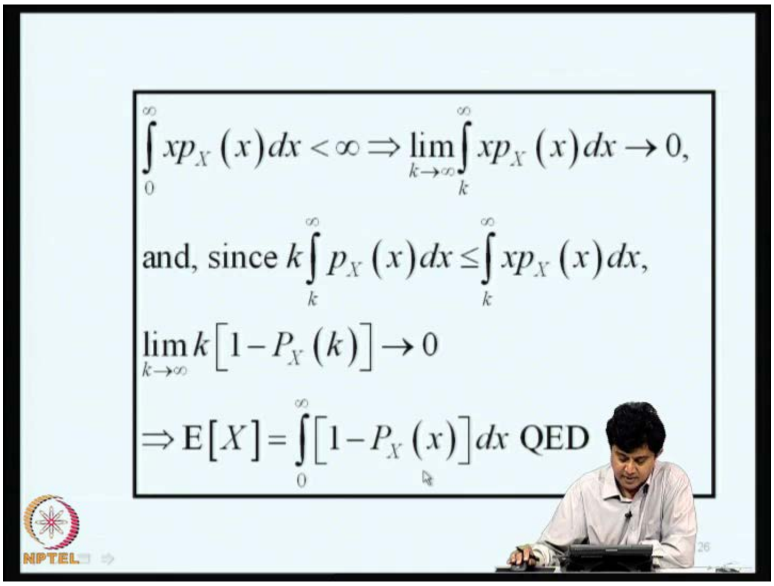

Now, since the, we are assuming that mean is finite, it means that limit k tends to infinity of k to infinity x p x of dx is 0, and since k into k to infinity p x of dx is less than equal to k to infinity x p x of dx. It follows that limit of k tends to infinity k into this function is 0, and consequently, we show that expected value of the random variable is indeed the shaded area that we shown in the figure.

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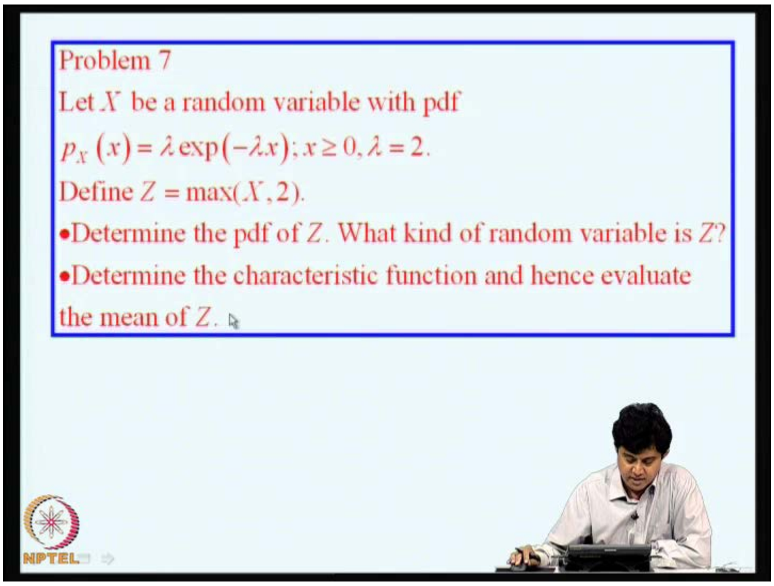

The next example is again with an exponential random variable. So, x is the exponential random variable with parameter lambda and lambda is given to be equal to 2, and we define Z as maximum of x, 2. The question is determine the probability density function of Z and common term what kind of random variable is Z. Subsequently, we are asked to determine the characteristic function and hence evaluate the mean of Z.

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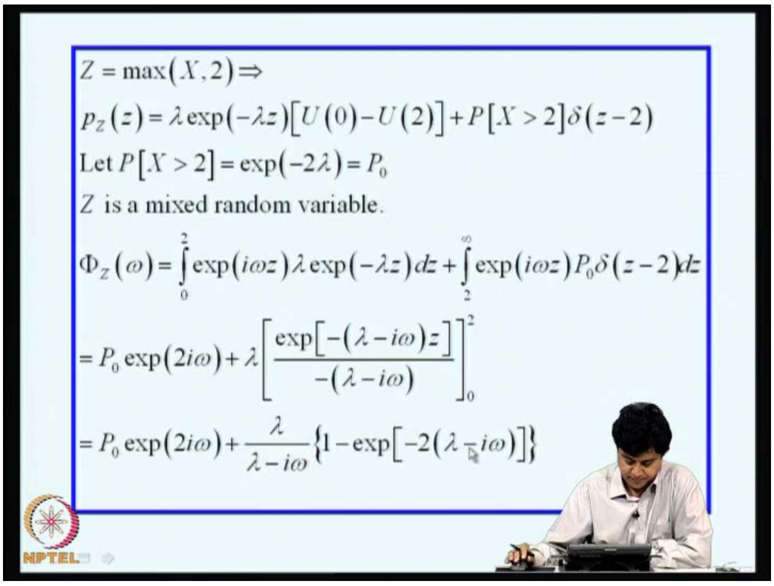

So, how do we do this? Z is maximum of x, 2 implies that p z of z is, from x equal to 0 to 2 it follows the exponential distribution. So, lambda exponential minus lambda z; u is the step function u of 0 minus u of 2 plus it is then a direct delta function centered at Z equal to 2 with probability, the probability of x greater than or equal to 2.

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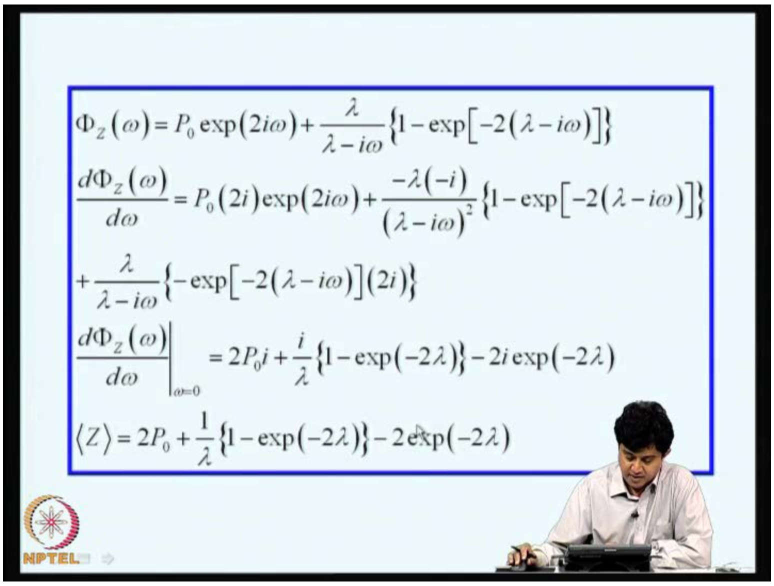

Now, so Z becomes a mixed random variable and we can show that the characteristic function is given by this. We can evaluate these integrals and show that the characteristic function is this. So, this is simplification follows and we can verify that these steps are acceptable.

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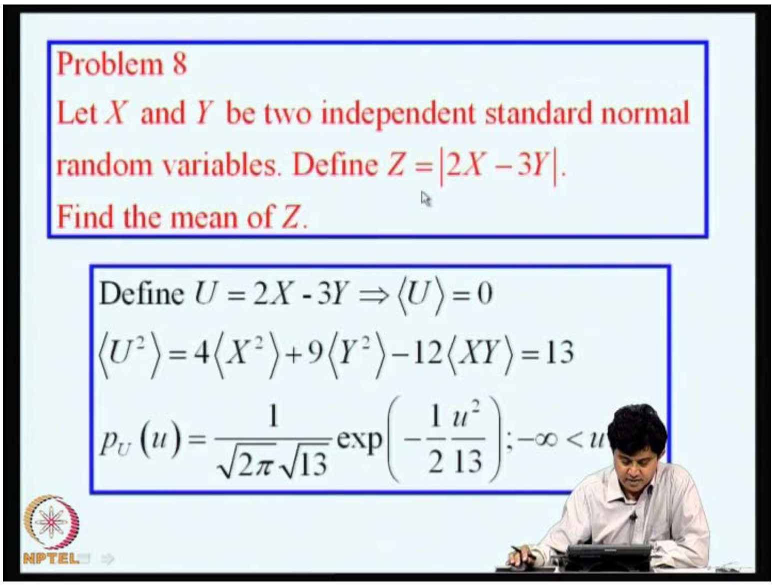

The next problem - we consider two random variables X and Y, and we assume that the independent standard normal random variables and I define z as modulus of 2 X minus 3 Y. The question is find mean of Z. Now, to solve this problem, what we do is, we introduce a random variable u which is 2 X minus 3 Y. Expected value of u is 0 because mean of X and mean of Y are given to be 0. So, how do mean square value? It is 4 into x square plus 9 into y square minus 12 into expected X Y and this is 0. Therefore, I get 9 plus 4 is thirteen because mean square value of X and Y are unit. So, probability density function of u is indeed another normal random variable with this density function, where standard deviation is square root of 13. So, thus the given problem can be viewed as finding expected value of a modulus of a Gaussian random variable.

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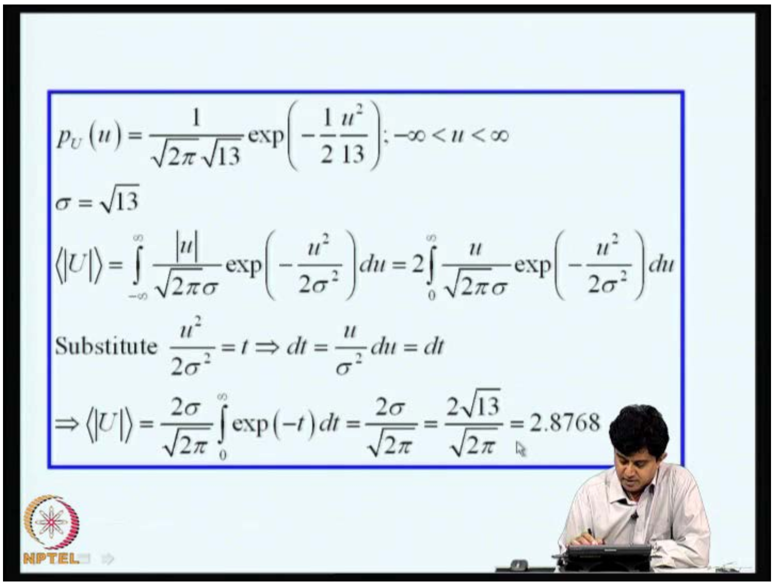

And so, this is there probability density function and modulus of u is expected value of modulus of u is minus infinity to plus infinity mod u into the density function. This can be written as 2 into this area 0 to infinity u by square root 2 pi sigma. Now, if you substitute now u square by 2 sigma square is equal to t and carry out this integration, we can show that the expected value of u turns out to be this number 2.8768.

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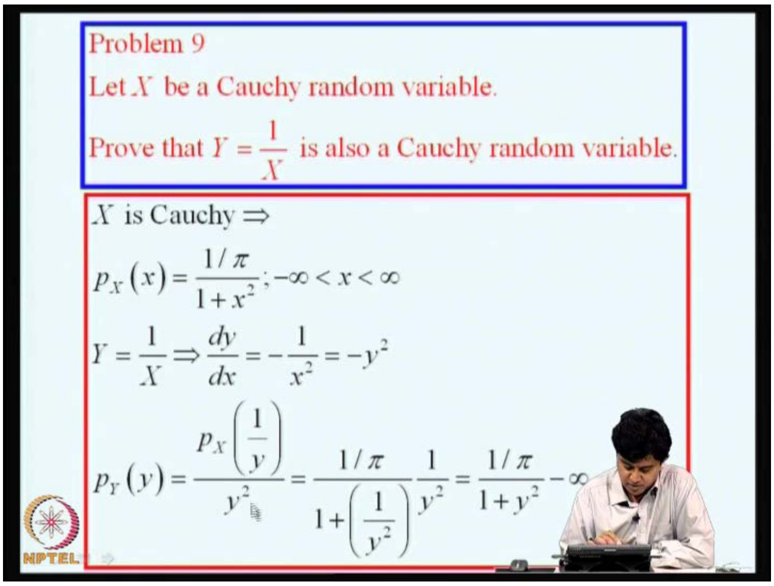

Now, discussion we came across Cauchy random variables. One of the property that Cauchy random variable satisfies is that if x is a Cauchy random variable, 1 by X is also a Cauchy random variable. So, how do you prove this? So, x is Cauchy means the probability density function is 1 by pi divided 1 plus x square where x takes value from minus infinity to plus infinity. Now, we define Y is equal to 1 by X, and therefore, the gradient is minus 1 by x square that is minus y square.

So, using rules of transformation of random variables, p y of y is p x evaluated at x equal to 1 by y divided by the absolute value of the gradient which is y square, and p x of 1 by y is 1 divided by pi divided by 1 plus 1 by y square into 1 by y square is simplify, you get that y is also a Cauchy random variable.

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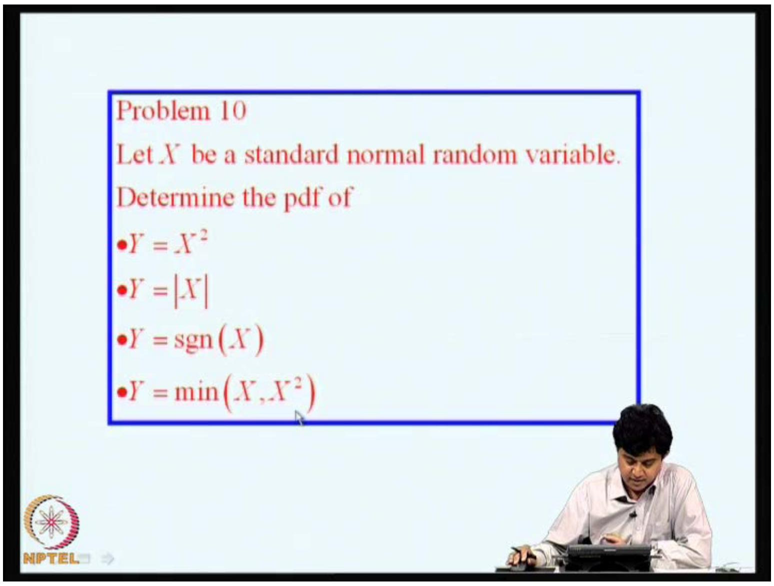

Now, a few examples involving transformations on a Gaussian random variable. So, let X be a standard normal random variable by standard normal random variable. We mean of X is 0; standard deviation is 1, and we are asked to find the probability density functions say y equal to X square, y is equal to mod X, Y is equal to signum of X, Y is equal to minimum of X, X square. So, how do we tackle this problems?

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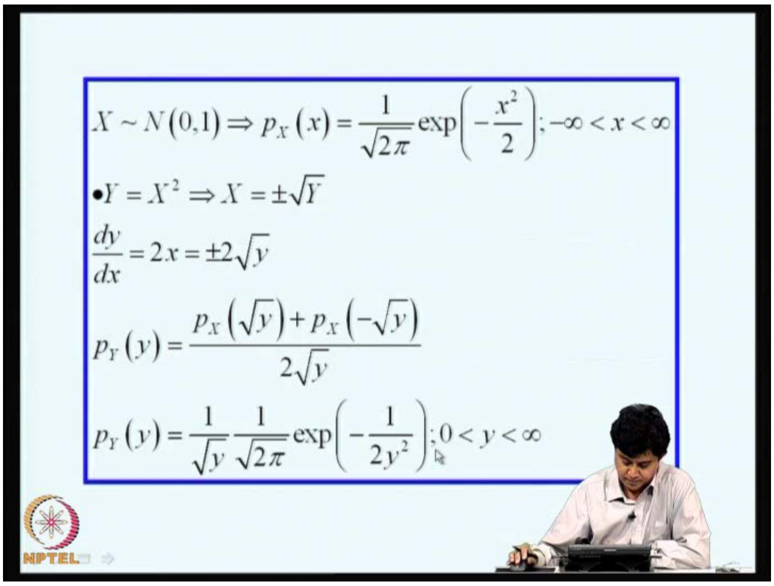

So, we will begin with the definition of p x of x. If x is normally, if x is normally distributed with mean 0 and standard deviation 1, the density function is 1 by square root of 2 pi exponential minus x square by 2 - where x is from minus infinity to plus infinity. Y equal to x square means x can take 2 values plus minus square root of y. Now, dy by dx is 2 x. Therefore, that is equal to plus minus 2 into square root of y.

Now, therefore, p y of y is p x of x evaluated at the 2 possible roots namely plus square root of y and minus square root y divided by the gradient which is 2 into square root of y. Now, we know p x of x is Gaussian, and if we substitute, there is a, since we are having x square here square root of, plus minus square root y we will translate to same number. So, we get p y of y to be this function.

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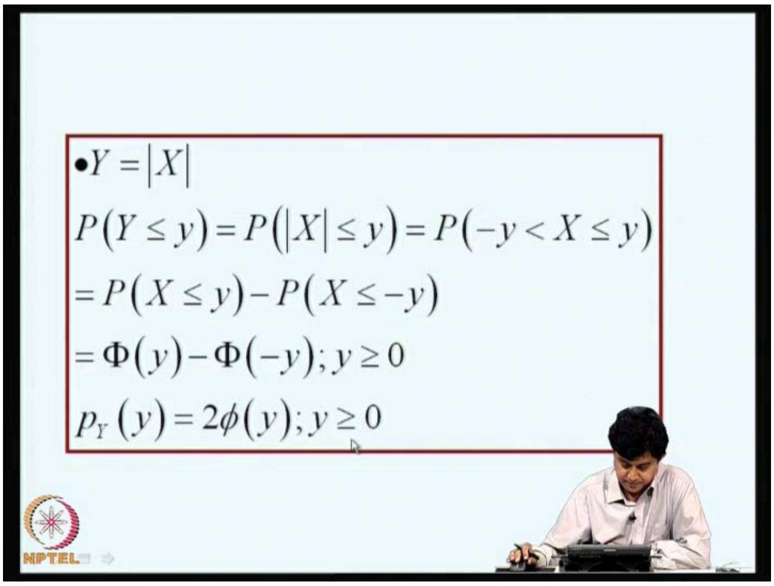

Now, how about y equal to mod x? Probability of Y less than or equal to y is probability of mod X less than or equal to Y. That would mean X takes values, X the probability of Y less than or equal to y is same as probability of x greater than minus Y less than or equal to y. So, this is nothing but probability of X less than or equal to minus y probability of X less than or equal to minus y. So, by denoting the probability distribution using capital phi, we write this as a phi of y minus phi of minus y and upon differentiation and I get 2 into phi of y where y is greater than or equal to 0.

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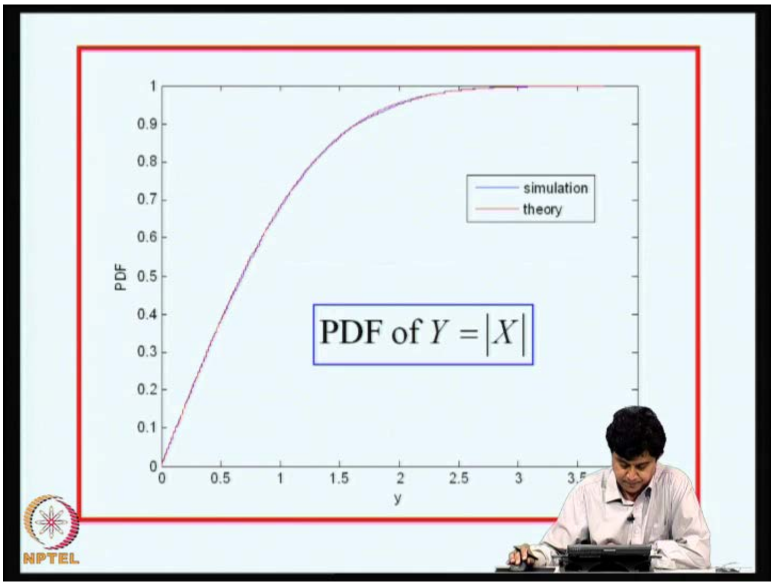

So, the probability distribution function of mod x is shown here. Just for verification a simulation was also performed with 10,000 samples. The blue line is a simulation and red line is the theoretical result.

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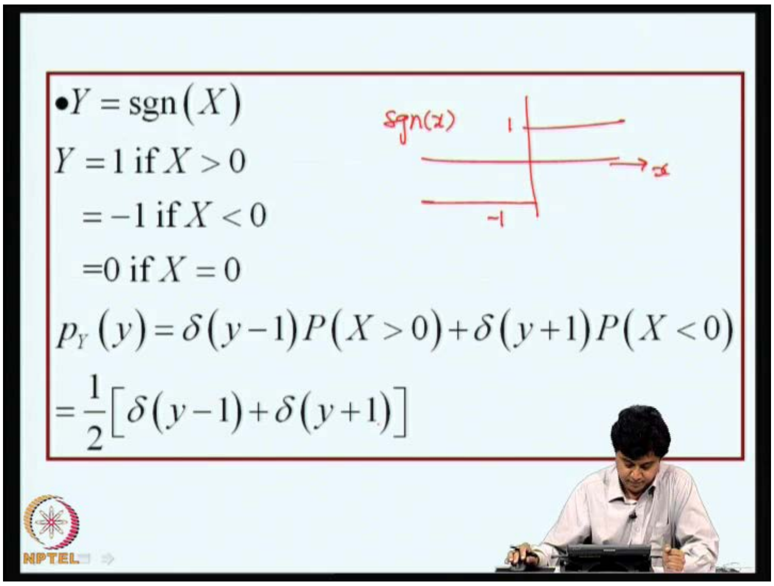

Now, how about y is equal to signum of x? Signum of x - the meaning of that is Y is equal to 1 if x is greater than 0; Y will be minus 1 if x is less than or equal to 0, and y will be 0 if x is equal to 0; that means signum of X is the function. This is plus 1; this is minus 1. Therefore, y is a discrete random variable that takes values 1 minus 1 and 0, but since X is a continuous random variable, probability of X equal to 0 is 0. Therefore, I get the probability density function of y to be direct delta y equal to 1 into probability of x greater than or 0 plus direct delta y equal to y plus 1 probability of X less than or equal to 0.

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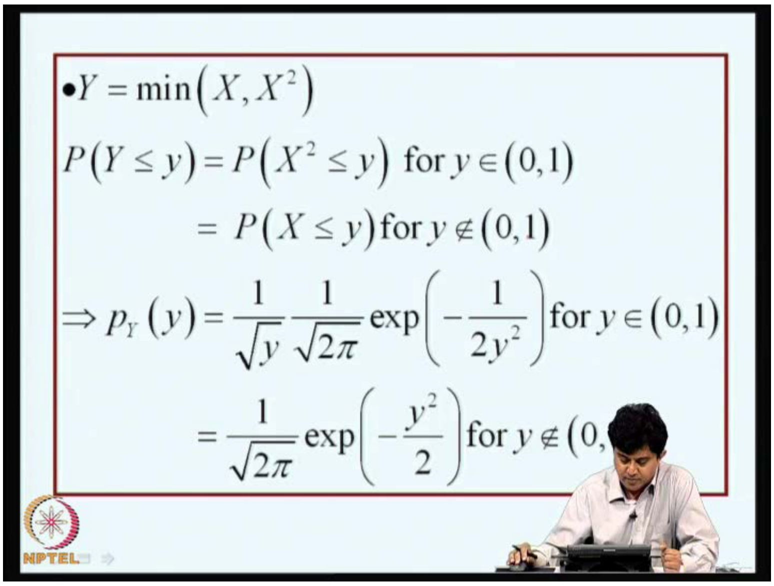

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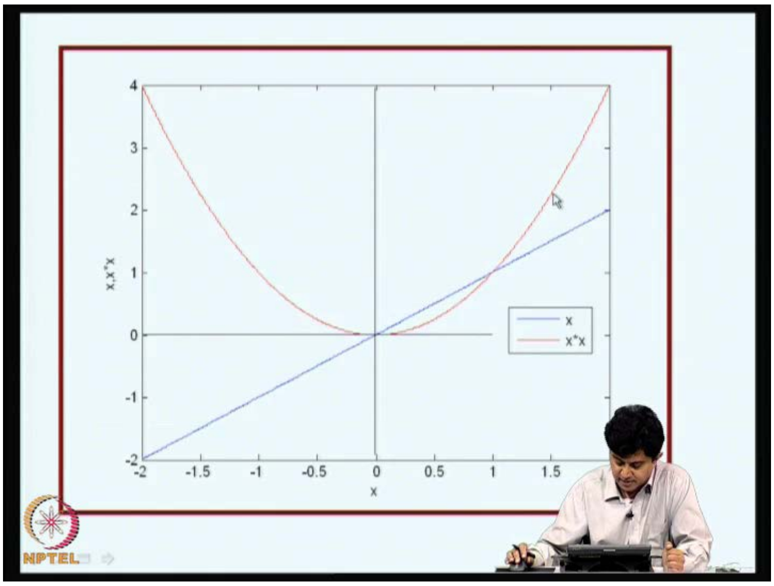

So, with that, I get the density function to a half of direct delta y minus 1 plus direct delta y plus 1. How about the function minimum of X, X square? So, probability, density, distribution of y is probability of y less than or equal to y, that is, probability of x square less than or equal to y if y belongs to 0 to 1, or probability of X less than or equal to y if y does not belongs to 0 to 1; that means if i plot this function here, this blue line is y equal to x and this red line is X square. So, we are taking minimum of these two values, and in this region between 0 to 1, red line is smaller than the blue, and outside, red line is, red line takes a greater values.

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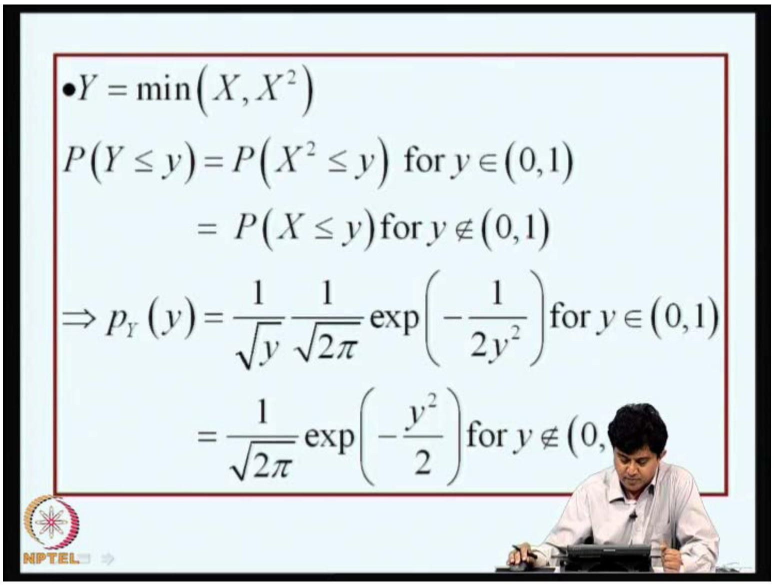

So, therefore, we write probability of X square less than or equal to y for y lying between 0 to 1 and probability of X less than or equal to y for y outside this region. So, using that, I get probability density function to be given by these two functions.

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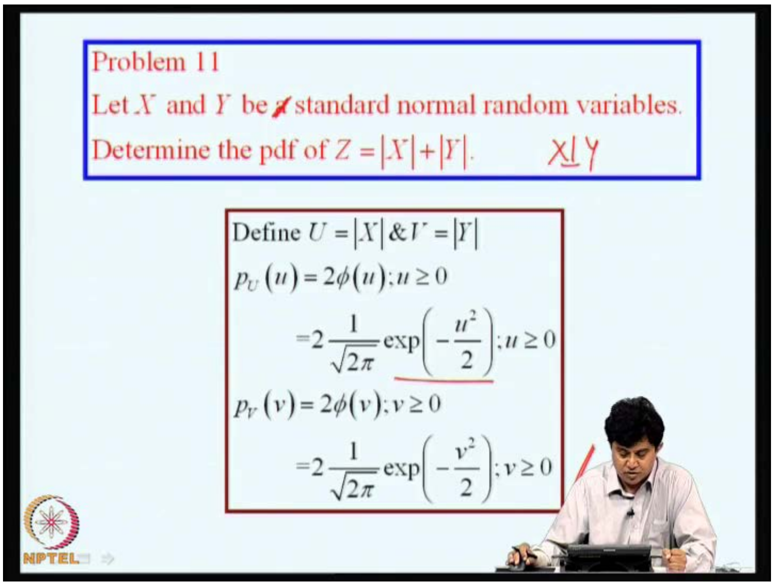

Now, slightly complicated problem. Let X and Y be standard, let X and Y be standard normal random variables, where asked to find the probability density function of z which is some of mod x and mod y. So, what we do is we define U to be mod X and v to be mod y, and we have obtained the probability density function of mod X and mod y just now. So, the probability density function of mod x is given by this and which is u and probability density of v is given by this. Since X and Y are independent, we are assuming that X and Y are independent; u and v would also be independent.

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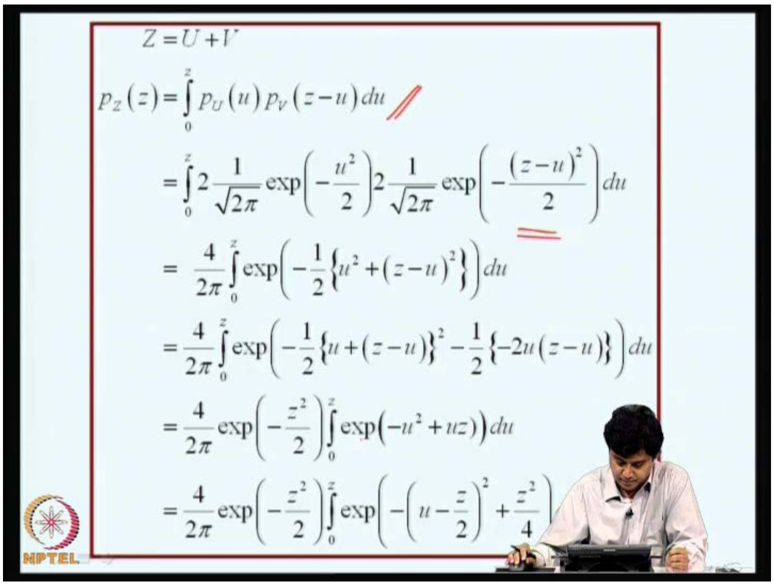

So, we have now Z is equal to u plus v, and we are asked to find probability density function of z, and we as you know, this is given by the convolution of probability density of u and v which is given by this. So, that would mean this integral is in terms of p u of u. This is the in terms of p v of v evaluated Z minus u. I get these terms. So, slight rearrangement, and ensuring that the exponent is expressed as perfect square and rearranging this terms. We need to manipulate some of this expression.

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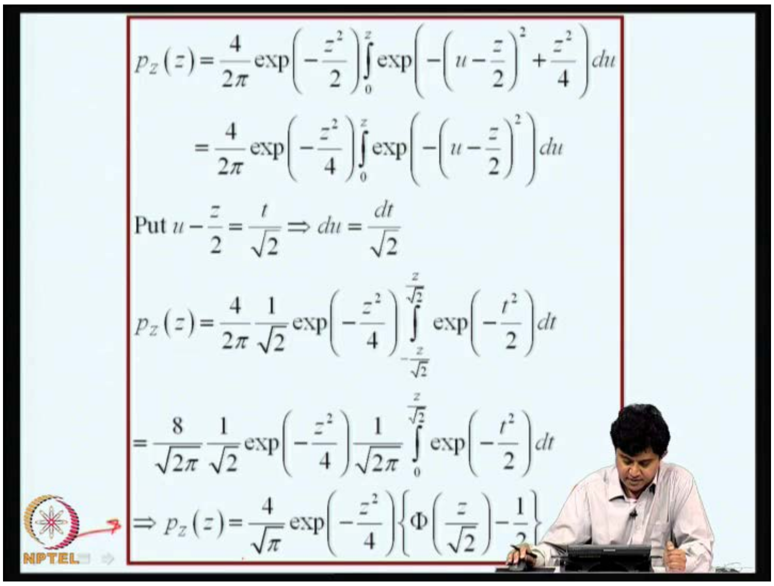

And we can show that the several substitution that we need to make, so that with in the exponent, we get perfect square and then we are able to evaluate this using known results. If you do that, we get the probability density function of Z to be this.

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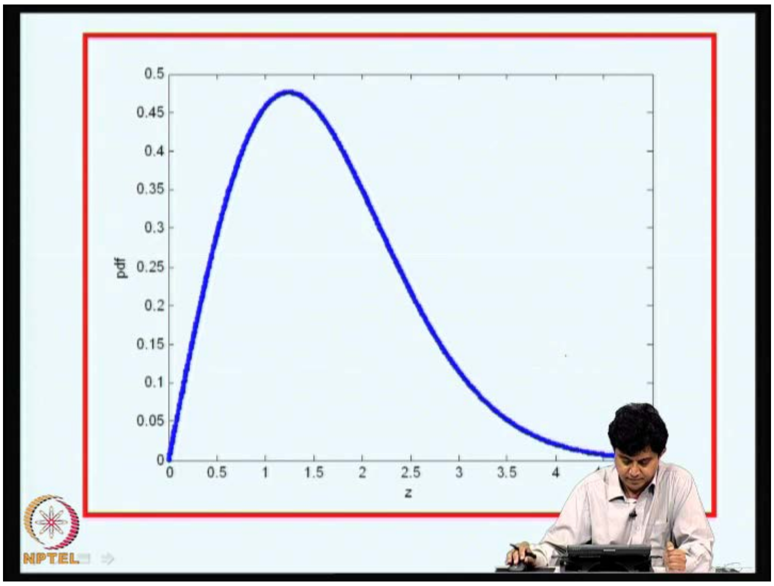

So, this is the fairly involved exercise. You are encouraged to go through this steps and see if they are correct. Now, the plot of this function P Z of Z is shown here and it was numerical verified that area under this function is 1.

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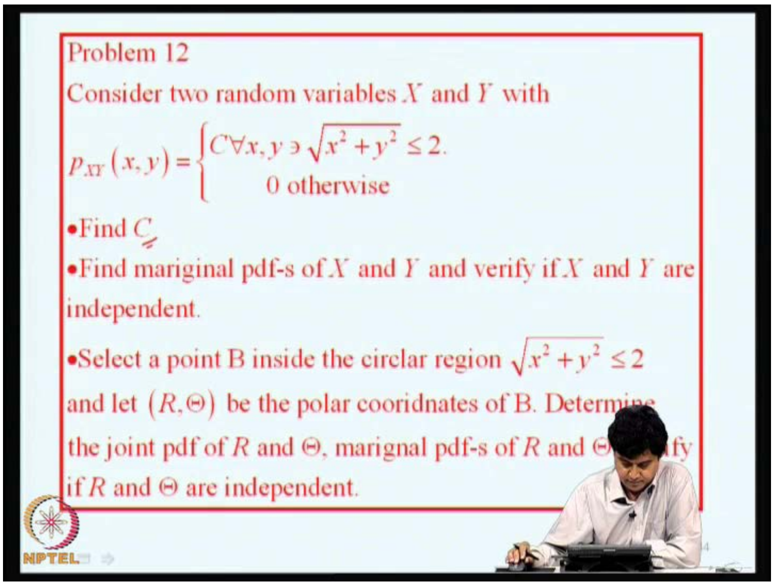

Now, let us consider two random variables x and y. The probability density function is a constant wherever x and y lie within a circle, that is, square root of x square plus y square is less than or equal to 2; otherwise, it is 0. So, the problem on hand consists of finding this constant c. Then finding marginal probability density functions of x and y and verify if x and y are independent. Following this, there is another subsection here. Select a point B inside the circular region square root of x square plus y square is less than or equal to 2, and let R and R, theta be the polar coordinates of B; whereas to determine the joint p d f of R and theta marginal probability density function of R and theta and we need to verify if r and theta are independent.

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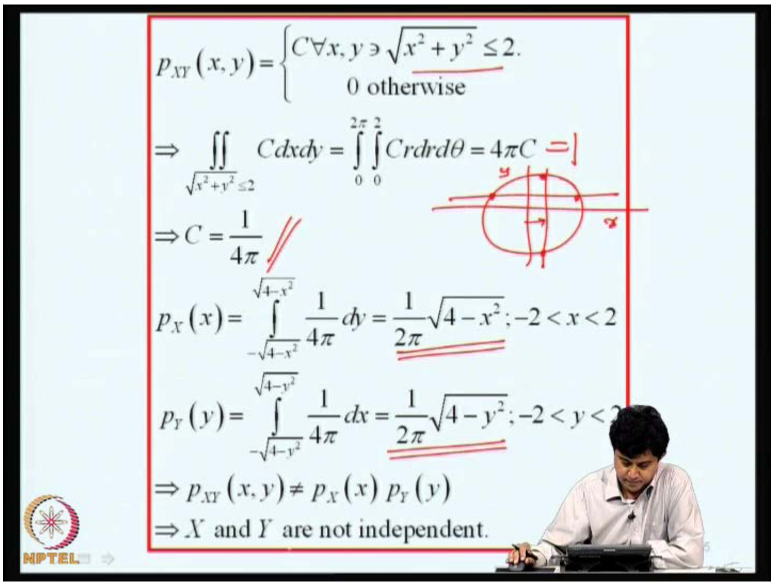

Now, we have the joint density function p x y of x, y is a constant for x and y belonging to this region square root x square plus y square is less than or equal to 2. So, the area under this density function should be equal to 1. That would mean the double integral where the regions square root x square plus y square less than or equal to 2 c dx dy must be equal to 1, and from that condition, we get this should be equal to 1 and I get c to be equal to 1 by 4 pi.

Now, how do you evaluate the marginal density p x of x? So, we need to integrate. We need to select. This is x; this is y. We need to integrate from this point to this point. So, this, if this is x, this will be square root of minus 4 minus x square and this will be square root of 4 minus of square root 4 minus x square plus of square root 4 minus x square 1 by 2 pi d y and we get this density function. py of y by the same token we need to select line like this and integrate from these two points, and accordingly, we get this function.

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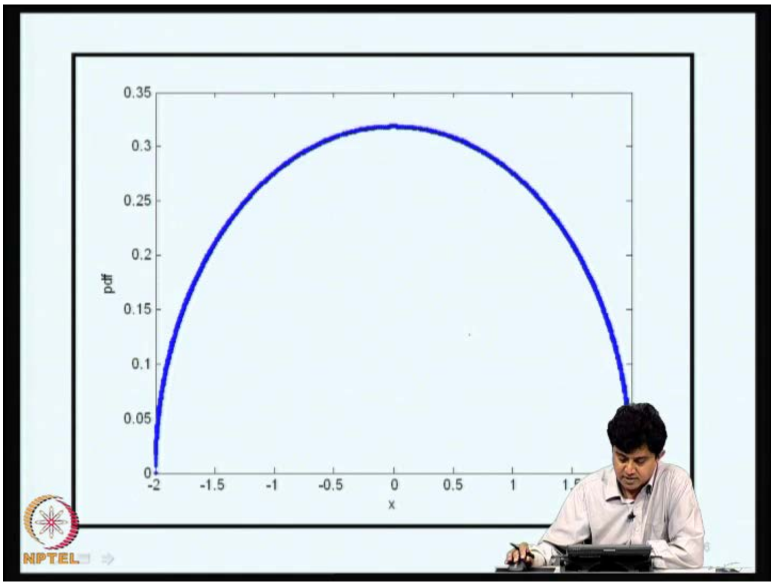

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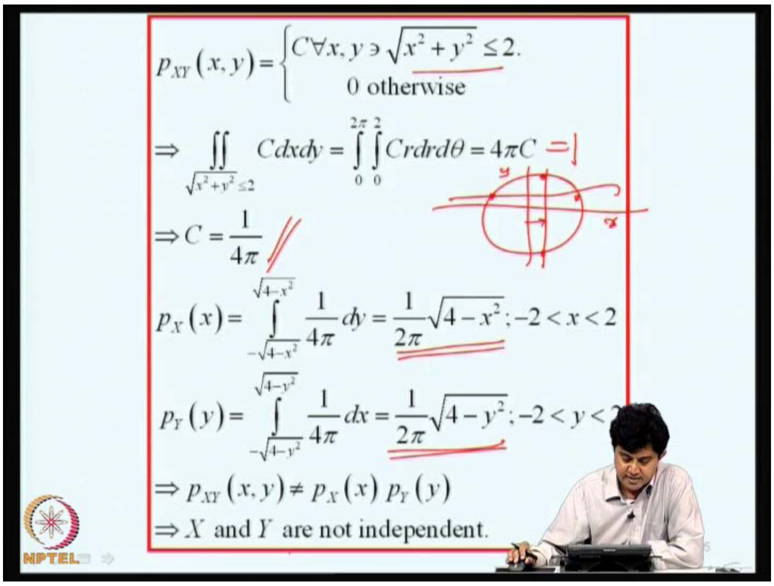

Now, if i multiply p x of x and p y of y, you can verify that their product does not lead to the function which is a constant, that is, 1 by 4 pi. From this, we can conclude that x and y are not independent. So, this is plot of probability density function of theta, that is, this function p x of x 1 by square root of 2 phi 4 minus x square this look like this.

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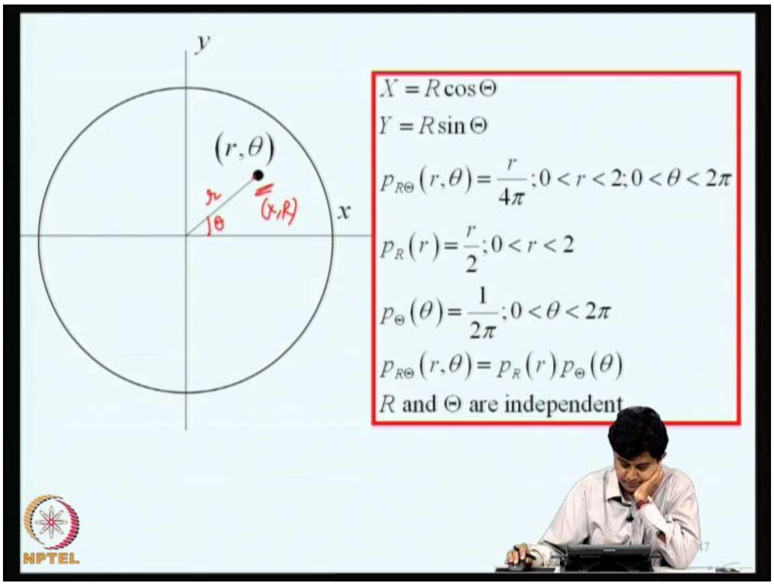

The next part of the question is within the circularly region, we select this point r. This, select this point and this is the coordinates r and theta. So, the coordinates of this is x, r which is X is r cos theta y is the r sin theta. Now, we are asked to find the joint density function between r and phi. P x y joint density of X and Y is given to be 1 by 4 pi whenever X and Y live within this circle, with this circle, within this circle.

So, therefore, r takes value from 0 to 2 and theta takes values from 0 to 2 pi, and if I integrate this over 0 to 2 pi, I get p r of r to be r by 2 for 0 to r varying from 0 to 2 and p theta of theta to be 1 by 2 pi where theta varies from 0 to 2 pi. Now, if I multiply these 2 p r of r and p theta of theta, I get p r theta of r comma theta which is r by 4 pi, which is nothing but r by 2 in to 1 by 2 pi. Consequently, we reach the conclusion that r and theta are independent.

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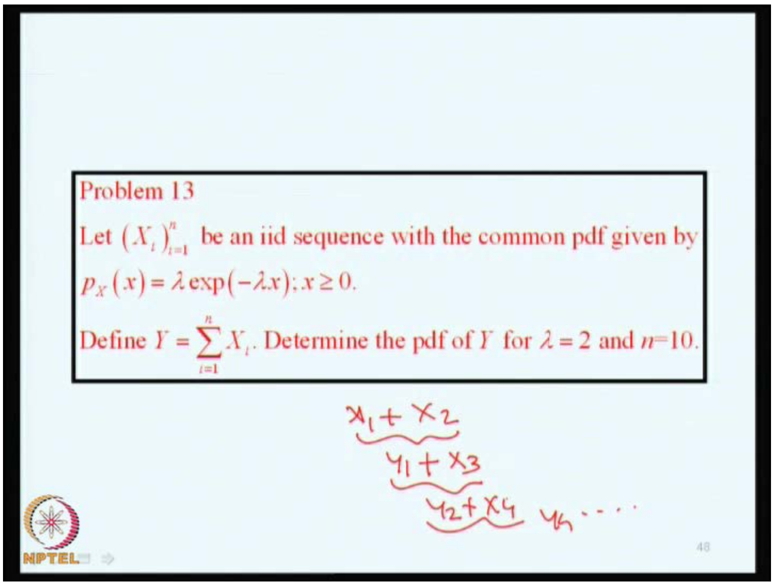

The next problem is we consider a sequence of random variables X i with a common density function, which is an exponential density function. We define Y to be i equal to 1 to Y i. The problem on hand consists of determining the probability density function of y for lambda equal to 2 and n equal to 10. Now, the two possible ways to evaluate this. I will give hints on how to solve this. One is we start by finding x 1 plus x 2 by convolving the probability density function of x 1 and x 2, and suppose if this is say y 1, then we convolve this resultant density function into with x 3 that leads to y 2; convolve with x 4, that leads to y 4 and so on and so forth. We can show that at every step, there will be sums of exponential that will emerge in probability density functions and this process can easily be carried out.

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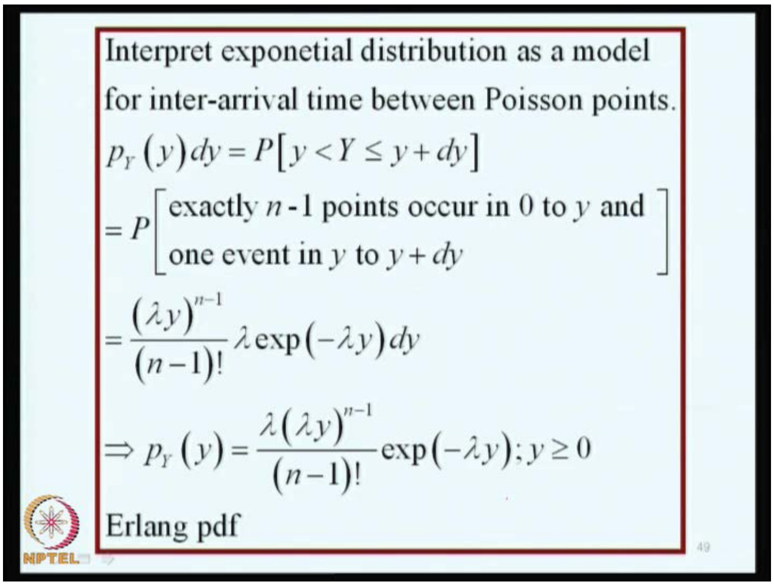

Other approach is to interpret the exponential distribution as a model for inter arrival time between Poisson points, and consider the event p y of y dy which is nothing but probability that exactly n minus one points occur in 0 to y and 1 event in y to y plus dy, and if you use this, we can get this density function. This density function is known as Erlang probability density function, and this is the special case of the gamma density function - where n is an integer.

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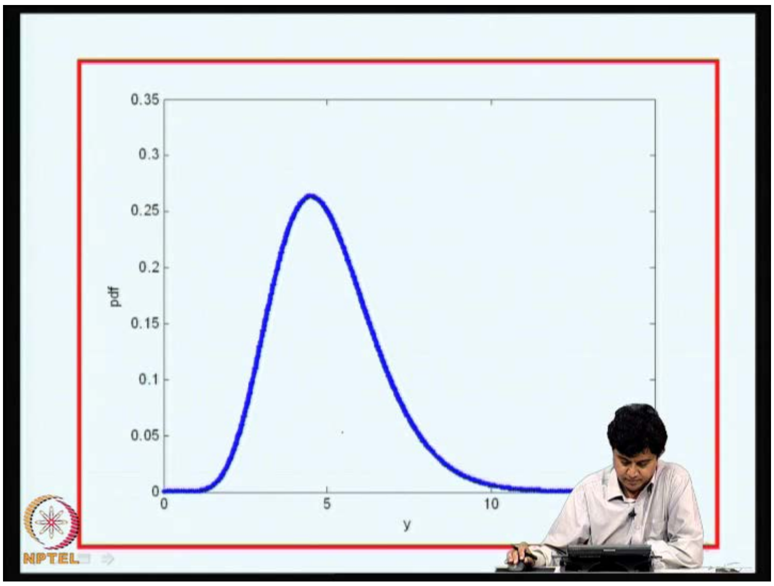

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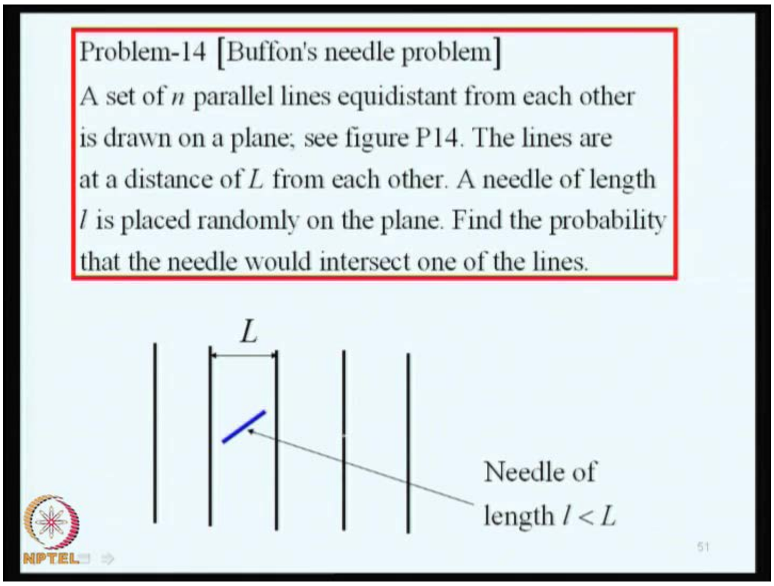

So, this is how this function looks like. (No audio from 40:00 to 40:14) Next, we consider a well-known problem known as Buffons needle problem. This problem is of historical interest. I will clarify what that means. So, here, a set of n parallel lines equidistant from each other is drawn on a plane. That is here in this figure. The lines are at a distance l from each other. So, this this vertical lines here are equally space and distance is l. Now, select a needle of length small l that this needle is of length small l which this length is smaller than capital l, and what we do is we place this needle randomly on this plane and we consider the event whether this needle intersect any of these lines. So, we want to find out the probability that the needle would intersect 1 of the lines.

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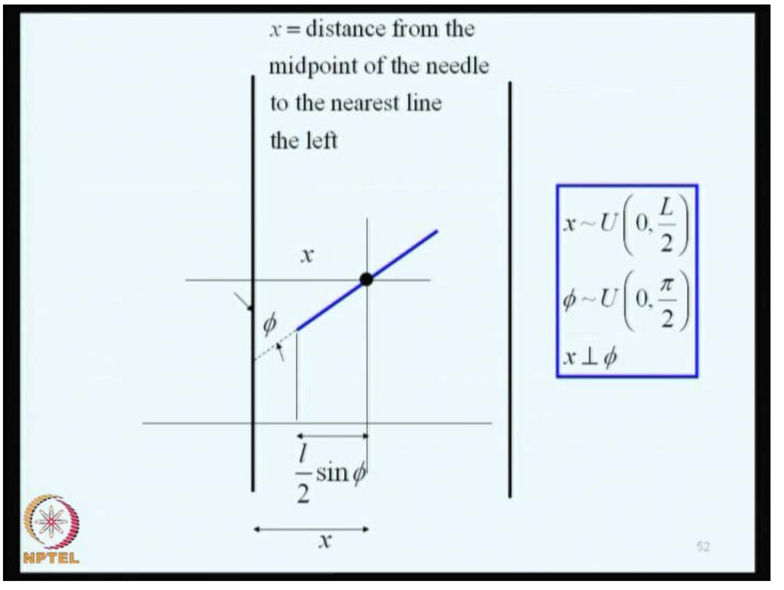

So, what we do is we consider a position, a situation where needle has landed between two lines, and consider the midpoint of the needle and measure the midpoint of the needle the distance of this midpoint to nearest vertical line to the left call it as x and this angle as phi. Now, since the needle is being placed on this plane randomly, you can infer that x is uniformly distributed between 0 to l by 2, and similarly, this angle phi is uniformly distributed between 0 to pi by 2, and x and phi are independent.

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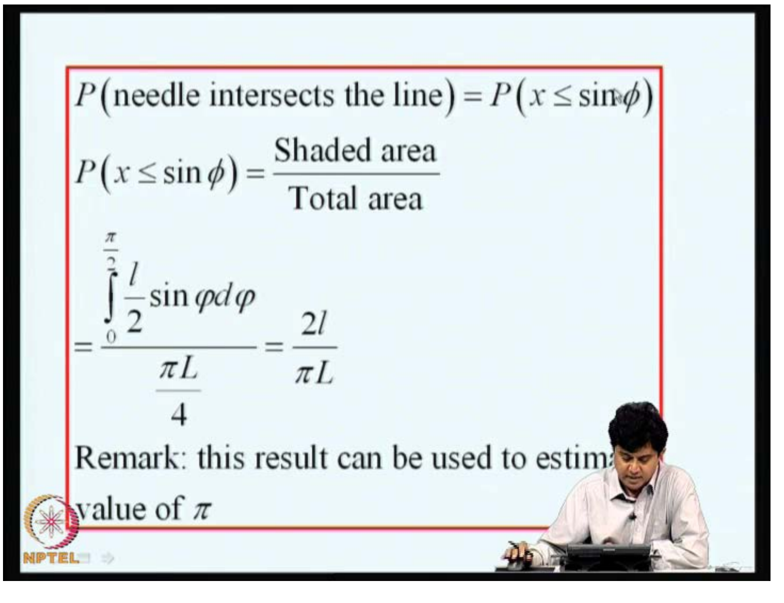

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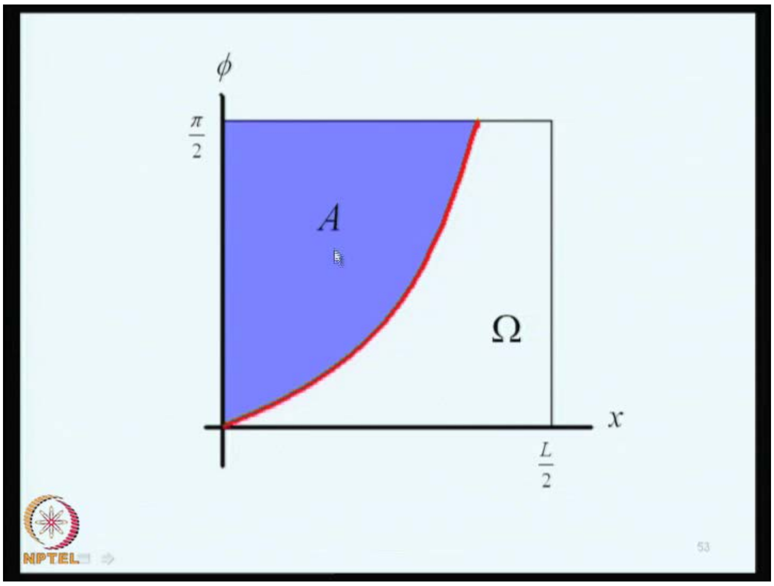

Now, so what is the event that needle intersects the line? Needle will intersect the line when x is less than or equal to sin phi; that means this shaded region a is the, in the phi x plain. Capital omega is the rectangle which is the sample space, and this blue shaded region is the event the subset of sample space which is favorable to the event that we are looking for, and if you do that calculations, we find that probability of x is less than or equal to sin phi is 2 l by pi l, and consequently, you know this, this probability is a function of pi. So, historically, this experiment was used to estimate value of pi.

So, you can task this needle and actually measure the probability that using relative frequency approach, how many times this needle intersects the vertical line, and based on that, once we get that estimate of probability, you can go back to this formula and get an estimate of pi. So, it is in this context that this problem has some historical significant.

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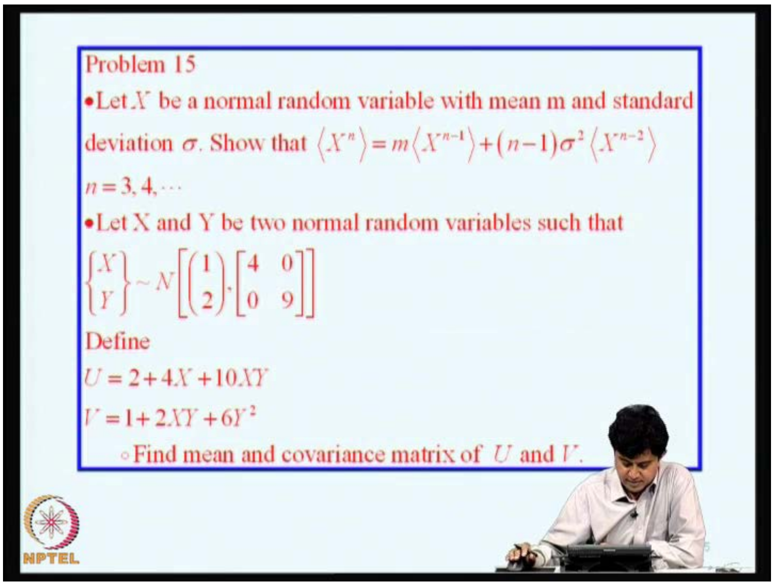

Let us consider now in the next problem X to be a normal random variable with mean m and standard deviation sigma. The first part of this exercise is to show that expected value of X to the power of n is m into expected value of X to the power of n minus 1 plus n minus 1 sigma square x n minus 2.

Now, we define X and y to be two normal random variables, is the next part of the problem - where the mean of X and y is 1 2 and covariance is 4009; that means x and y are independent. Now, I introduce to new random variables u and v through this relation U is 2 plus 4 X plus 10 X Y and v is 1 plus 2 X Y plus 6 y square. The problem is to find the mean and covariance of U and V.

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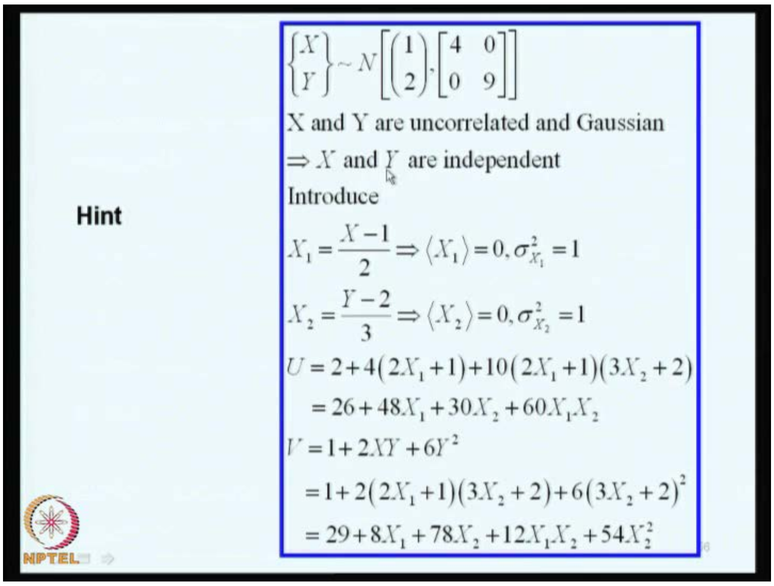

So, this is the fairly straight forward exercise which will tell you how to manipulate moments of Gaussian random variables. So, what we could do is we could remove the mean from X and y and recast the expression for u and v. That is what we do is, we begin by noticing that X and Y are uncorrelated, and therefore, they are also independent because are uncorrelated. Now, introduce new random variables X 1 is X minus 1 by 2, which is the standard deviation of X 1, that is, X and X 2 is y minus 2 which is mean divided by 3 which is standard deviation of y. So, consequently, we get expected value of x 1 is 0 and variance of X 1 will be unity. Then we recast U and V in terms of X 1 and X 2. For X, I will write it as 2 X 1 plus 1, and for Y, I will write 3 X 2 plus 2 and simplify these expressions. I will get now U and V in terms of X 1 and X 2 which are easy to manipulate mean is 0; standard deviation is 1. They are independent.

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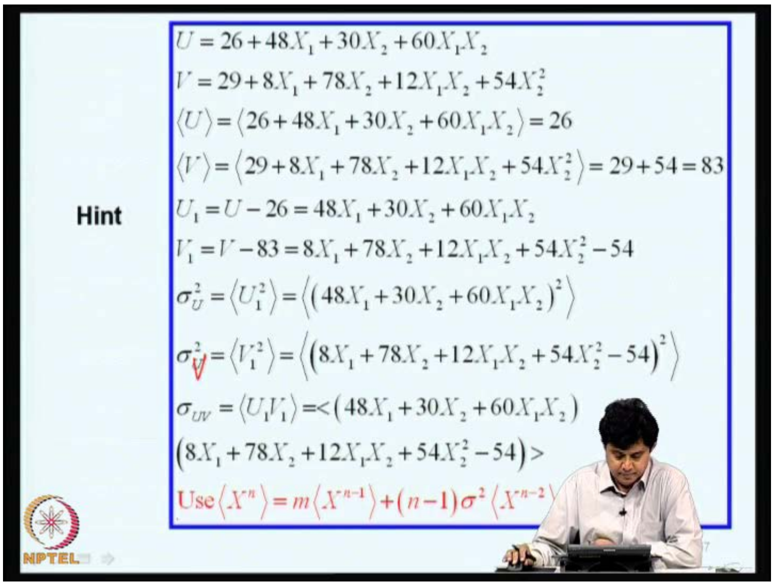

Now, consequently, we can find. For example, we get this these two as the expression for u and v. Now, expected value of u is we can simply operate on this. This will be 26 because expected value of X 1 X 2 are 0; X 1 and X 2 are uncorrelated. Therefore, this is 26. Expected value of V is again we operate on this expression. Expected value of x 1 is 0; X 2 is 0. Expected value of X 1 X 2 is 0, but we have this expected value of X 2 square which is unity. Therefore, the answer will be 29 plus 54 which is 83.

Next, I define U 1 to be U minus 26, which is U 1 is a random variable with 0 mean and V 1 is V minus 83. I get these expressions and this will be helpful in finding variance and covariance of U and V. So, sigma U square will be expected value of U 1 square which is square of this.

So, similarly, sigma V square will be expected value of V 1 square which is this and covariance of u and V will be in this form. So, when we do it for instance, you will get here say terms like expected value of X 1 square X 2 square or X 1 cube and X 2 square etcetera. So, to evaluate that, you need to use the result that you need to show as a first part of this exercise. That this I leave it as a exercise. You can use the characteristic function and show this.

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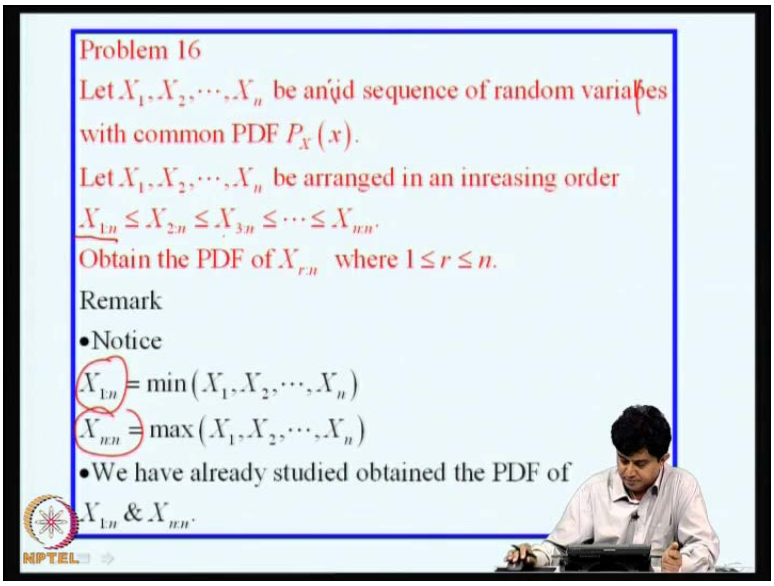

Now, another problem on sequence of random variables. We consider X 1 X 2 X n to be an i d sequence of random variables with common probability distribution function p X of X. Now, what we will do is, we will arrange x 1 a 2 X n in an increasing order and we denote the lowest value as X subscript 1 colon n. Next, 1 is X 2 colon n and so on and so forth X n colon n.

Now, the problem on hand is to find the probability distribution function of the r th member in this ordered list - so, where r runs from 1 to n. Now, the first 1 here X 1 of n is nothing but minimum of X 1, X 2, X n. Similarly, the last one is maximum. We already determined the probability distribution of the minimum and maximum. Now, what is being asked is in this order list, can you pick any number r th candidate in this ordered list and what is the probability distribution function of that?

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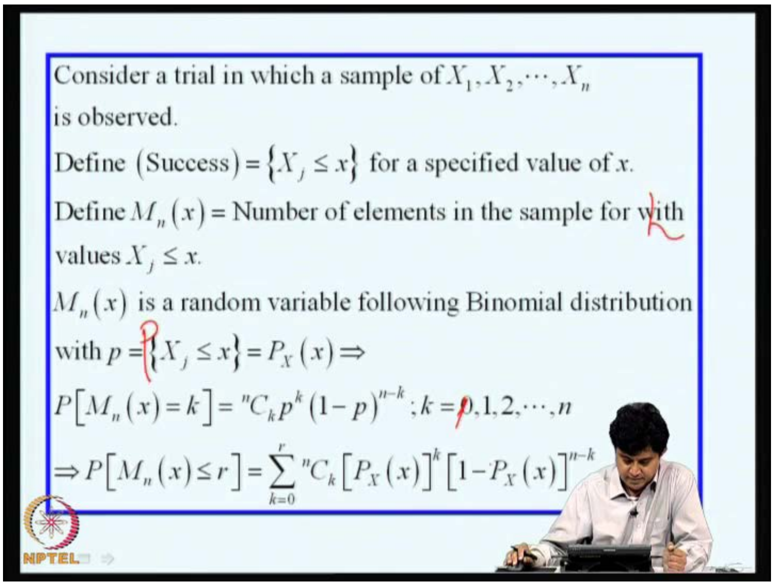

Now, to answer this question, we consider a trial in which a sample of X 1, X 2, X n is observed. We define success as a event x j is less than or equal to x for a specified value of x. Now, then define m n of x is number of elements in the sample for which values x j less than or equal to x. Now, m n of x is a random variable following binominal distribution with p is equal to probability of x j less than or equal to x which is p x of x, and following that, I get probability of m n of x is equal to k is a binominal distribution n c k p to the power of k 1 minus p n minus k k running from 1 to n.

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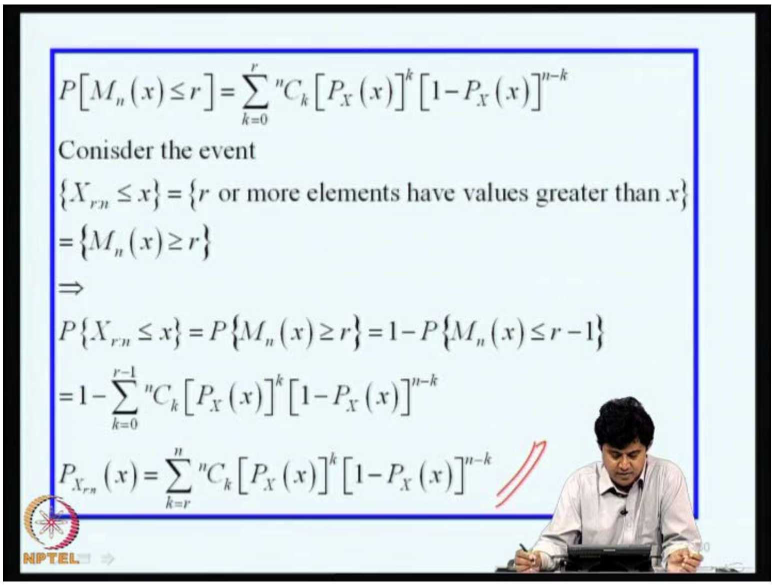

Probability of M n of x less than equal to r is given by this summation and I get M n of x less than equal to r is this. Now, from this if you consider the event x r n less than equal to x, that means the r th candidate in the ordered list has a value less than or equal to x. It would mean r or more elements have values greater than x. Therefore, this is nothing but M n of x is greater than or equal to r.

Now, therefore, now, we have characterized M n of x as a random variable. Therefore, I would be able to derive this probability density function for the r th member in the ordered list. Now, P x of x in this of course you can use the specified density function. It could be Gaussian normal, Rayleigh whatever and you will able to proceed further in simplifying this expression moment you know the details of P x of x.

So, with this, we will conclude this section of problem solving. So, in this section, we have basically considered problems involving theory of probability and random variables. So, in the next part, we will consider problems involving random processes. So, we will cosnclude this lecture at this point.