Stochastic Structural Dynamics

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Module No. # 09

Lecture No. # 36

Fatigue Failure & Vibration Energy Flow Models

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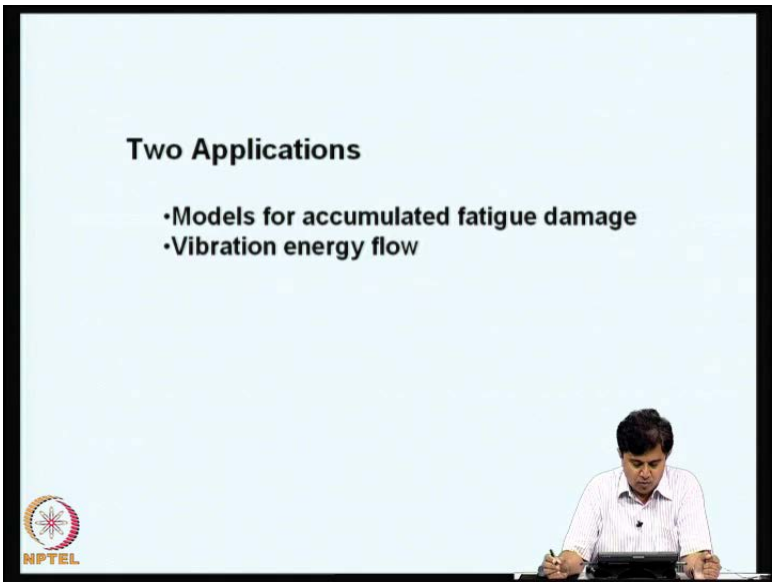

We have been discussing an application of principles of random vibration to problems in earth quake engineering. We will continue with discussion on applications. So, in this lecture, I will consider two more applications - one is in the area of models for accumulated fatigue damage in randomly vibrating structures, and the second one is the area of vibration energy flow that is important in the area of high frequency vibrations of engineering structures.

So, what I intend to do is to be favor of these two applications areas and illustrate how principles of random vibration analysis are employed here. The objective is not to describe the problems of fatigue damage nor the problems of energy flow, but to simply state the basic problem and illustrate how random vibration principles could be used.

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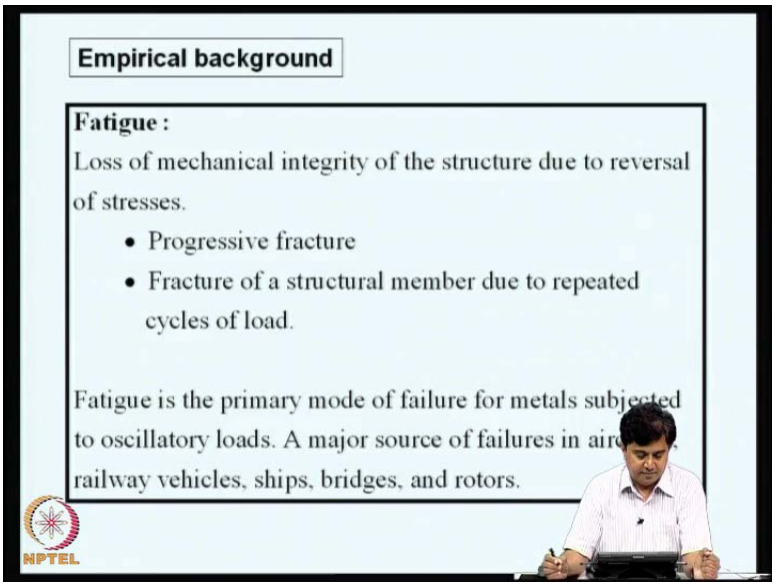

So, we will start with problem of fatigue damage accumulation. So, some, we start with some empirical background. What is fatigue? So, we can say the fatigue is loss of mechanical integrity of the structure due to reversal of stresses. So, any structural material on some scale, on any scale, on, even on very fine scales will have some imperfections defects. Consequently, the strength of the structural material in a given volume depends on amount of defect that are present.

Now, when the material is subjected to vibratory loads, there will be reversal of stresses, and as a consequence of this, the defects and imperfections grow within a given volume of structural material, and consequently, the strength of the material reduces and this phenomena is what is known as fatigue. So, fatigue is a essentially a of phenomena involving of progressive fracture. So, it is a fracture of a structural member due to repeated cycles of load. Fatigue is the primary mode of failure for metals subjected to oscillatory loads. So, this is a major source of failures in aircrafts, railway vehicles, ships, bridges and rotors.

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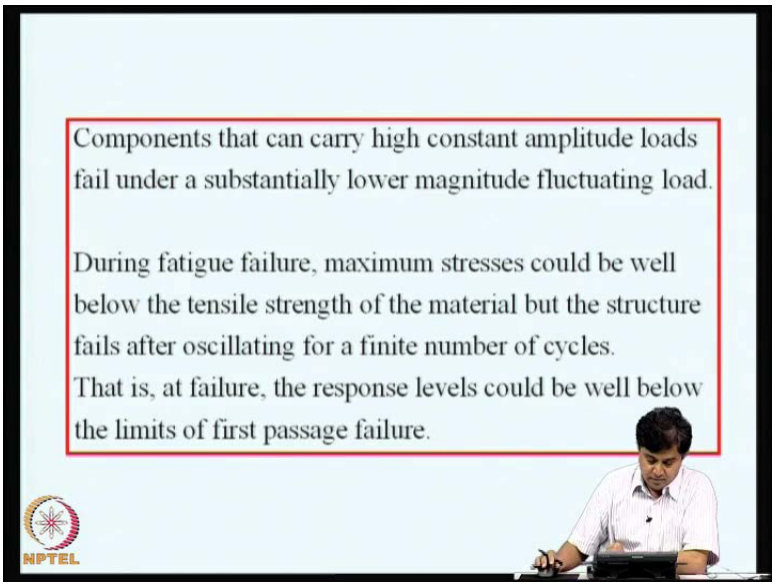

Actually structural components that can carry high constant amplitude loads fail under a substantially lower magnitude fluctuating load. This is due to, basically due to fatigue, and during fatigue failure, maximum stress could be well below the tensile strength of the material but the structure fails after oscillating for a finite number of cycles, that is, at failure, the response level could be well below the limits of first passage failure.

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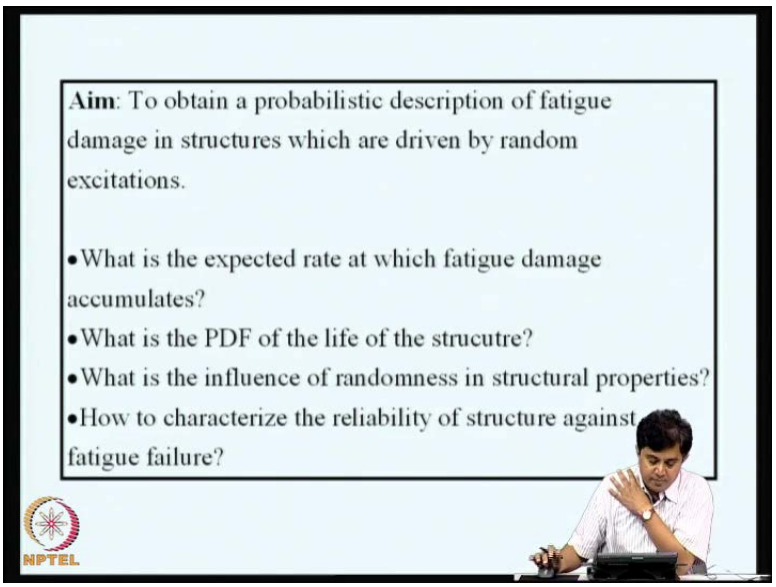

So, it is a failure could be quiet catastrophic. So, it is important to obtain a rational description of how to model these failures. So, the objective of this discussion is to obtain a probabilistic description of fatigue damage in structures which are driven by random excitations. For example, the types of questions that would like to ask are - what is the expected rate at which fatigue damage accumulates? What is the probability distribution of the life of the structure? What is the influence of randomness in structural properties? How to characterize the reliability of structure against fatigue failure? So, we will consider some of these questions and see how we can progress.

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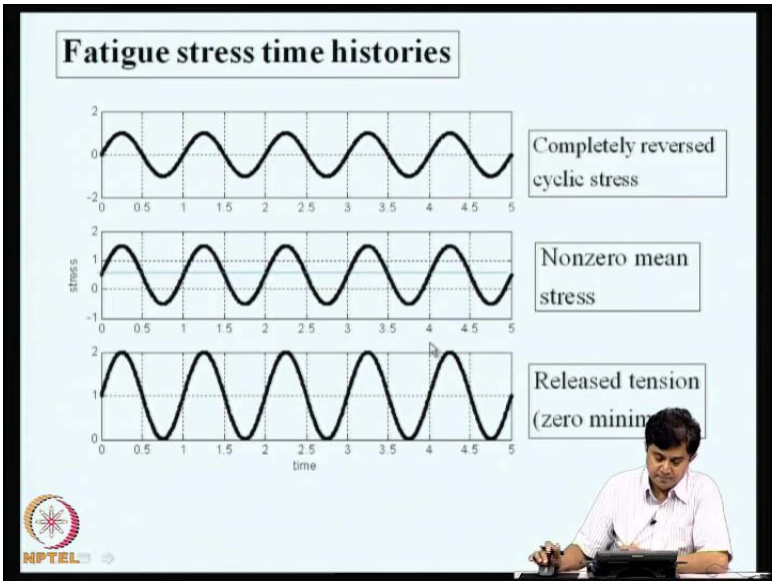

Now, if you look at time histories of a stresses, the various possibilities exist. For example, in this first one, there is an oscillatory load about a mean level. That would mean the stress cycles are completely reversed and these are called completely reversed cyclic stresses. This oscillation could occur about a non-zero mean. So, the cycling occurs about a non-zero mean which is not 0 or it could be such that there is no stress reversal, but actually there is a for instance released tension. Here, the stress never enters the compression zone. It remains tensile but the tensile stress magnitude keeps oscillating.

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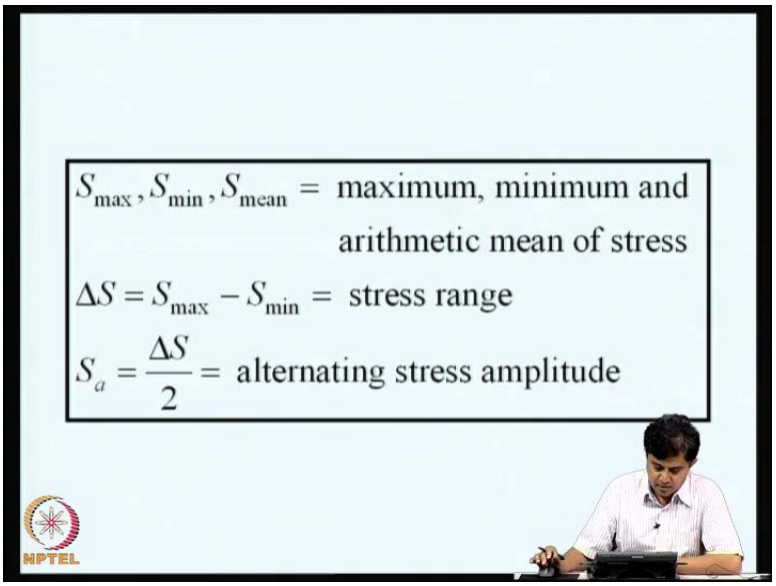

Now, here, we introduce certain terms S max S min and S mean are the maximum minimum and arithmetic mean of this stresses. Delta S is a stress range S max minus S min and the alternating stress amplitude is given by delta S divided by 2.

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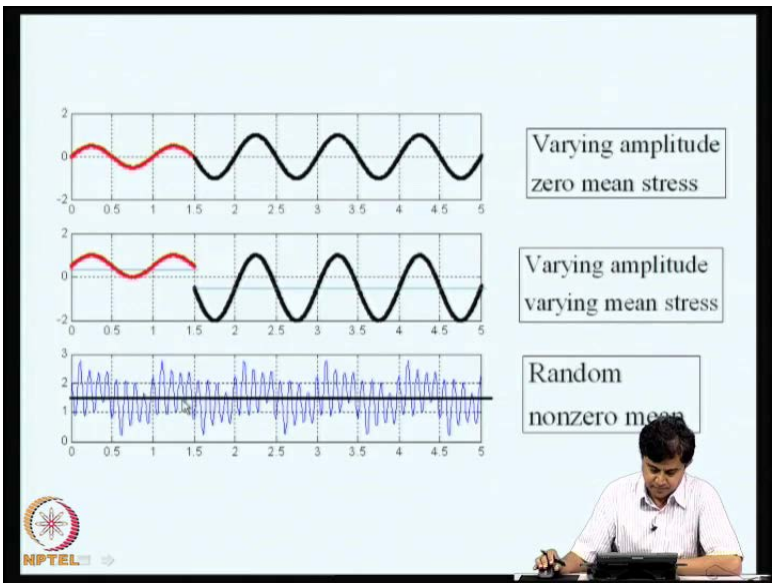

The stress amplitude need not be harmonic. The amplitude could vary. For example, here, there is a stress time history where up to say 1.5 second there is one amplitude. Beyond that, the amplitude increases.

Now, the change in amplitude could also be accompanied by change in the mean levels about which these oscillations occur. These are varying amplitude, varying mean stresses, or all the stress could be a randomly varying in time. It could be multi frequency. The, here, what we are shown is a narrow band random process. So, here, there is no, the definition of a cycle here is not immediately evident unless we clarify what exactly that is.

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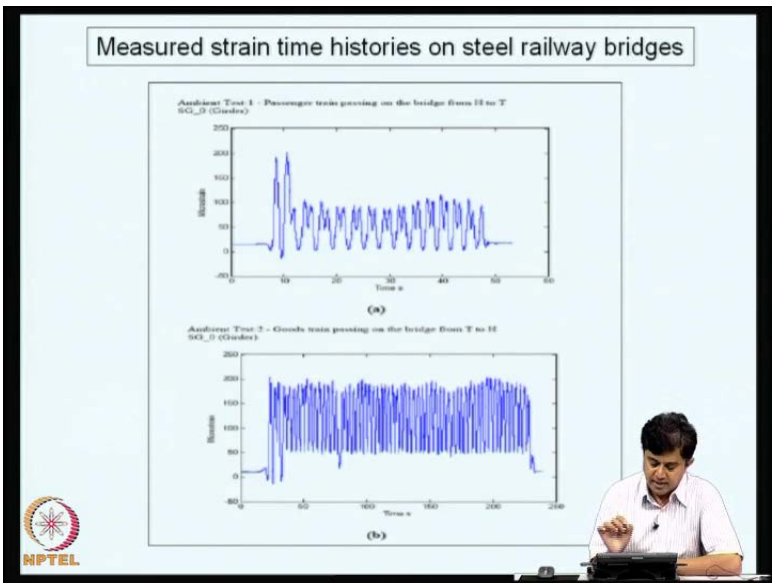

Just we will illustrate here I am showing certain measure time histories. For example, in this graph, we are showing the time history of the strain on a steel girder bridge as a train formation crosses the bridge at a particular velocity. So, this is second graph is similar result when a longer formation may be free traffic crosses the bridge. So, the cycle that we see here typically corresponds to the passage of one locomotive and carriage wagon and this wagon themselves we will have multiple access and these small oscillations about a higher level represent, you know, oscillation due to passage of axils of the same wagon.

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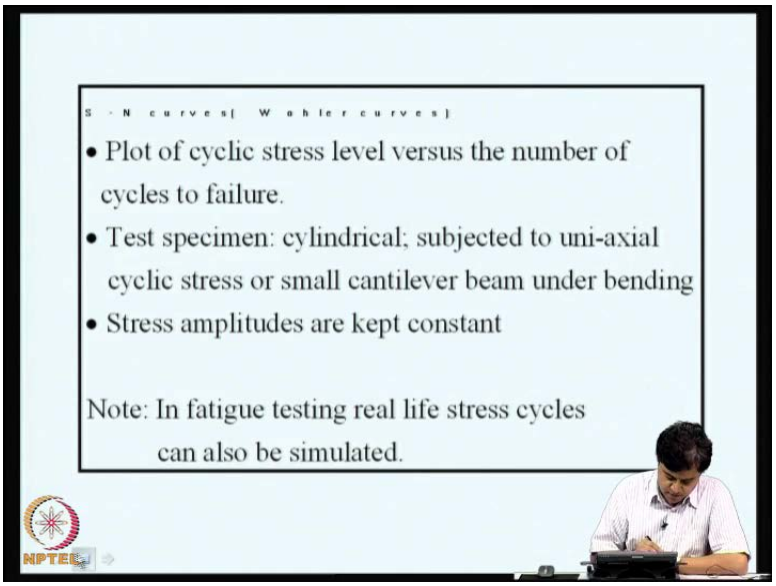

Now, we use what is known as S N curves to characterize fatigue failure. This S N curve essentially is a plot of cyclic stress level versus number of cycles to failure; that means if you take a test coupon and subject it to cyclic loaded into a given stress value after certain number of cycles of oscillation, this coupon will fail and the number of oscillation as a function of stress level is known as a S N curve. So, this is something that is experimentally obtained. This also is known as Olar curve.

The test specimen is typically a cylindrical specimen subject to uni-axial cyclic stress or a small cantilever beam under bending oscillation. The stress amplitudes are kept constant. We could also similar actual real life stress cycles in a fatigue test but we can, this is not what we are discussing in this lecture.

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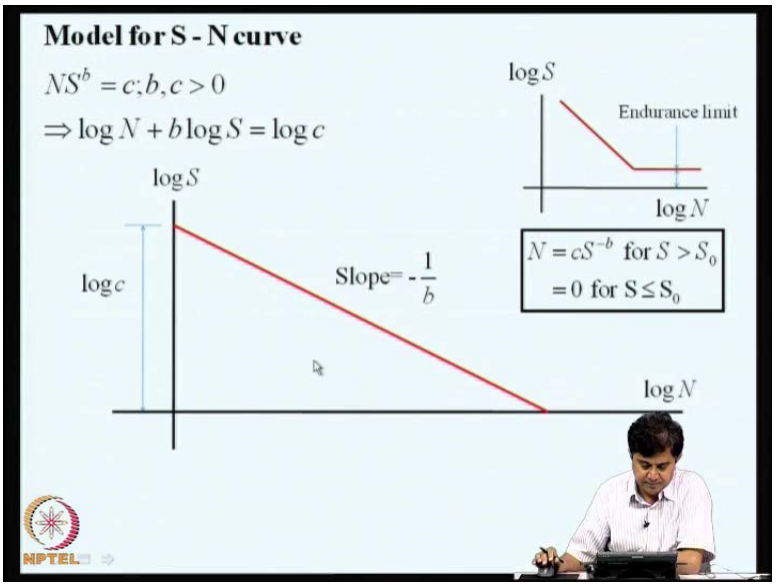

A simple model for this so called S N curves is of this form N S to the power of b is equal to c - where b and c are constants, and if you take every them of this, we get log N plus b log S is equal to log c. That would mean on log scale. S N curve is a straight line with slope minus 1 by b and intersect log c.

Now, in reality, what happens is - below a particular stress level, the, there would not be failure due to cyclic oscillation is not possible and this limit of stress is known as endurance limit, and we are considering stress levels above the endurance limit in our studies. So, number of cycles from this formula N S to the power b equal to c. We can, in, for this case, it is N S to c to c into S to the power minus b for S greater than S not is 0 for S less than or equal to S naught if you taking to account the effect of endurance limit.

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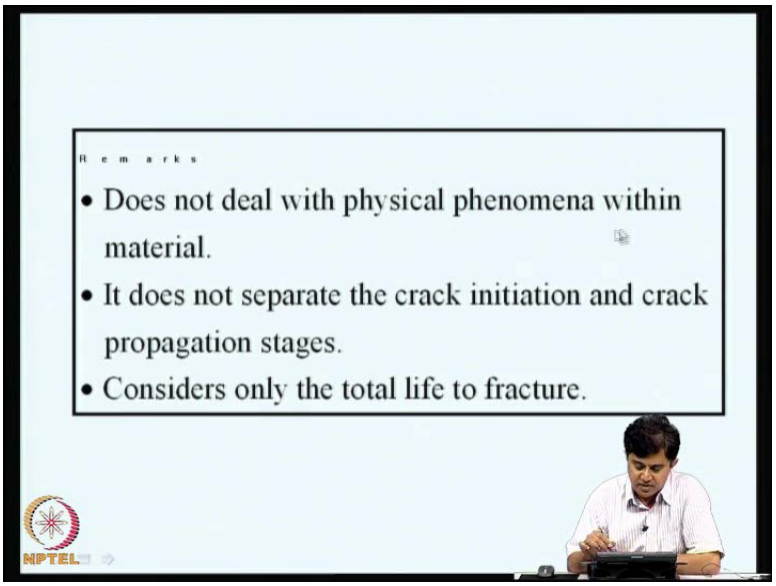

Now, in the S N curve approach, it is an integral approach which actually does not deal with physical phenomena that would be taking place within the material which results in loss of mechanical integrity. Actually does not separate the crack initiation and crack propagation stages, considers only the total life to fracture.

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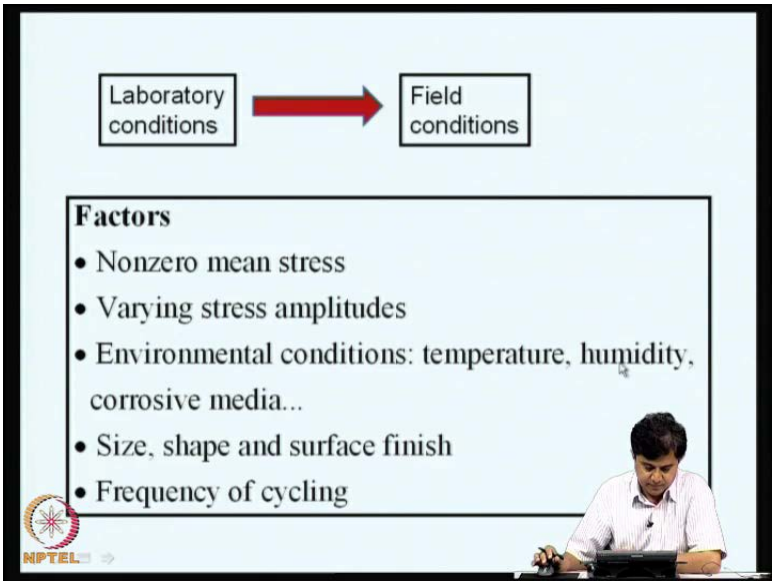

Now, we obtain S N curves in the laboratory conditions and then we want to extrapolate this information to the field conditions. So, laboratory testing is done on coupons just like with the way we find Young's Modulus of a, say material using coupons. In the

same sense, S N curves are also obtained in a laboratory conditions on test coupons, but these results need to be extrapolated to free condition.

When we do that, there are factors like nonzero mean stress, then varying stress amplitudes, an environmental conditions like temperature, humidity, corrosive media, etcetera will start influencing the results, and also the size shape and surface finish of the member under consideration also have significant influence on the fatigue performance. The frequency of cycling could also important but many times this is not a very crucial factor.

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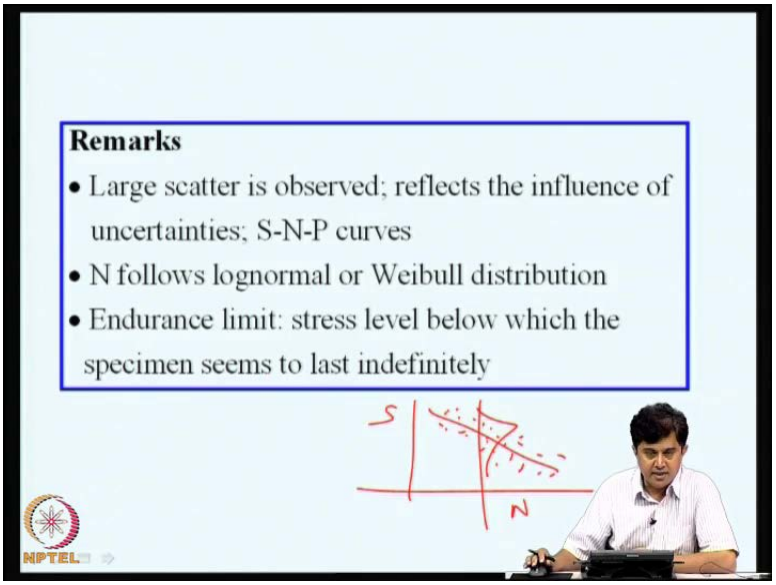

In the experimental data, actually large scatter is observed is reflects the influence of uncertainties, and one actually plots, for example, the S N curve, one gets an, if, in experiment there where green straight line would not emerge. So, at any given point, you can draw a probability distribution curve, density function, and one can specify the S N curves in terms of S N and probability curves. This density function could be modeled as lognormal or Weibull. Endurance limit have already explained; it is a stress level below which the specimen seems to last in definitely.

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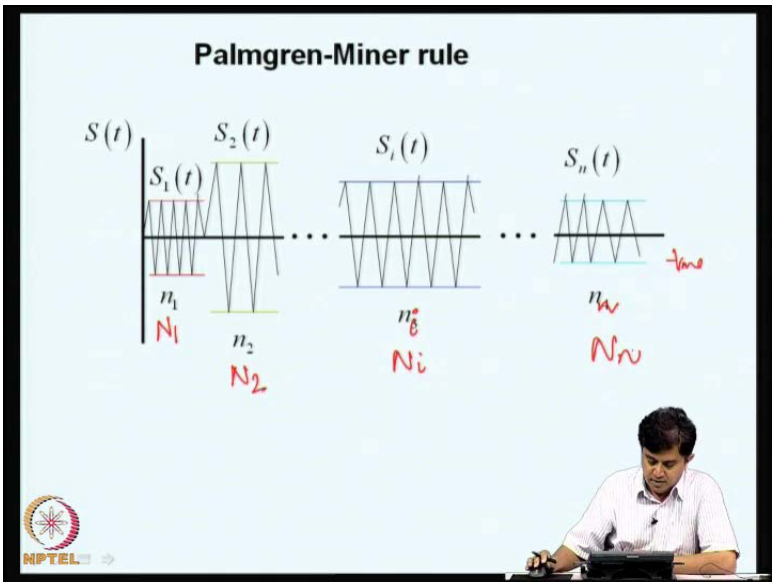

Now, we use what is known as Palmgren Miner rule to find out accumulation of a fatigue damage. Suppose during the life of the structure on x axis, I am plotting time, and on y axis we are plotting stress amplitudes. Suppose the structure undergoes N 1 oscillations at stress level S 1 (Refer Slide Time: 11:00) and N 2 oscillations that stress level S 2 and N 3 at N i; N i at stress level S i and N N at stress level S N. Now, if the specimen is left to oscillate stress level S 1, it will required as a capital N 1 number of cycles for failure. Similarly, if the specimen is left to oscillate only at S 2, it may required N 2 number of cycles to failure and these capital N 1 capital N 2 etcetera are obtain from the S N curve.

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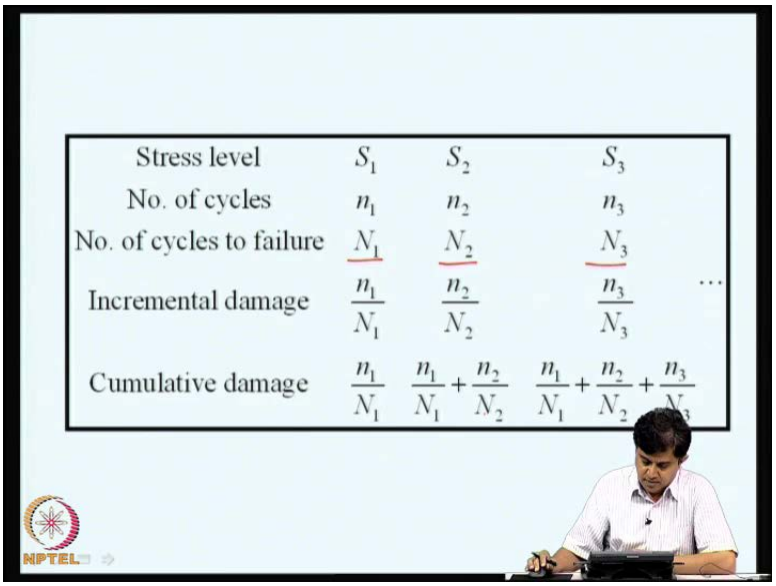

Now, according to Palmgren miner hypothesis, what we do is we find out the number of cycles at different stress levels - S 1 is n 1, S 2 is n 2, S 3 is N 3 and so on and so forth, and for this stress level from S N curve, we get the number of cycles to failure. We define in incremental damage as the ratio of n 1 by capital N 1. Similarly, incremental damage at stress level n 2 by capital level N 2 and so on and so forth. We define what is known as cumulative damage as the sum of this incremental damage at any given stage. For example, at the end of application of S 2, stress level S 2 for N 2 number of cycles, the accumulated damage would be n 1 by capital N 1 plus n 2 by capital N 2.

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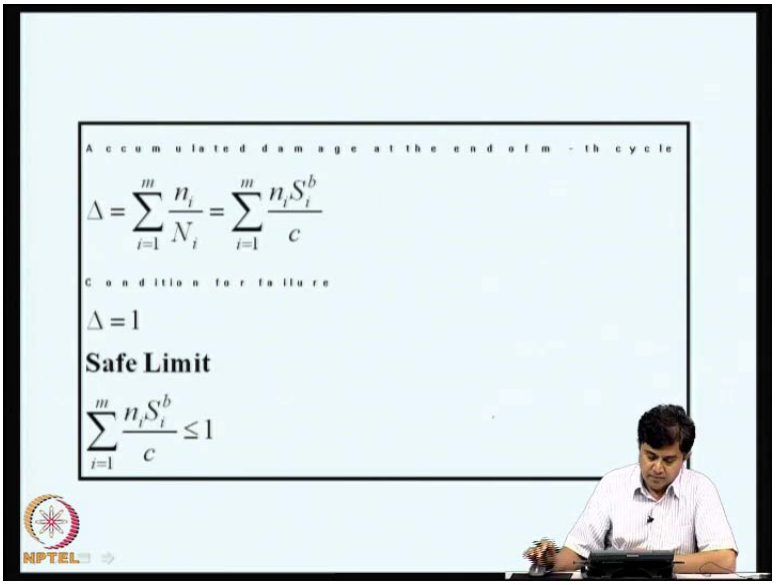

Now, according to the palmgren miner hypothesis at the end of m th cycle of oscillation, we get the accumulated damage as capital delta is summation i equal to 1 N i by capital n i. Now, if I use the formula for the S N curve, I can write this as N i S i to the power of b by c because capital n i is related to S i and the constant b and c. According to palmgren miner hypothesis, the condition for failure is that this cumulative damage should reach a value of 1. So, consequently the safe limit is i equal to 1 n i S i to the power of b divided by c must be less than or equal to 1. If this miner sum crosses 1, the specimen deemed to have failed according to this theory.

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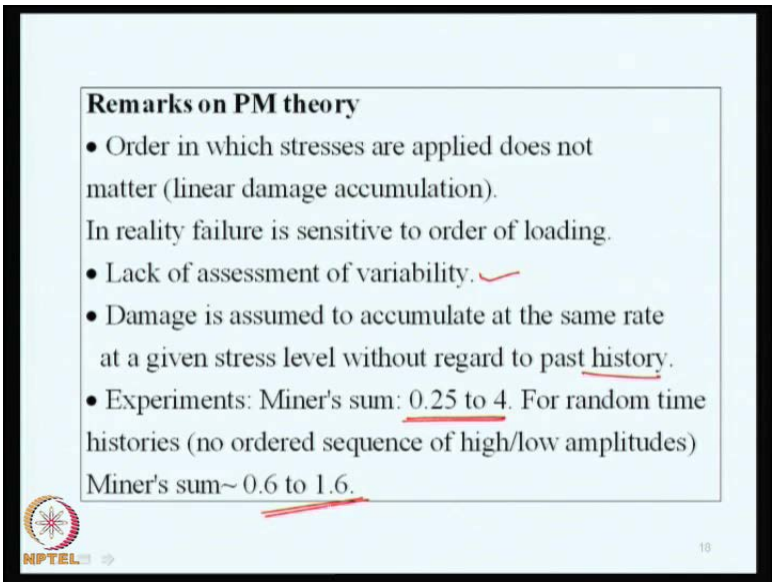

A few remarks on the palmgren miner theory could be made. In this theory, the order in which stresses are applied does not matter; that means the damage is assume to take place in a linear manner. In reality, actually failure is sensitive to order of loading. A high stress level followed by low stress level would produce accumulated damage of a different kind than the other way round where a low stress cycle is followed by high stress cycle.

The method as such does not provide any means of assessing the effect of variability in constant b and c etcetera, and damage is assume to accumulate at the same rate at a given stress level without regard without regard to the past history; that means it does not take in to account the memory. In actual experiments, the miner sum at which the failure as occur actually varies over very large range, that is, 0.25 to 4 under harmonic loading, but for random time histories where no ordered sequence of high and low amplitudes exist, the miner sum is reasonably good 0.6 to 1.6.

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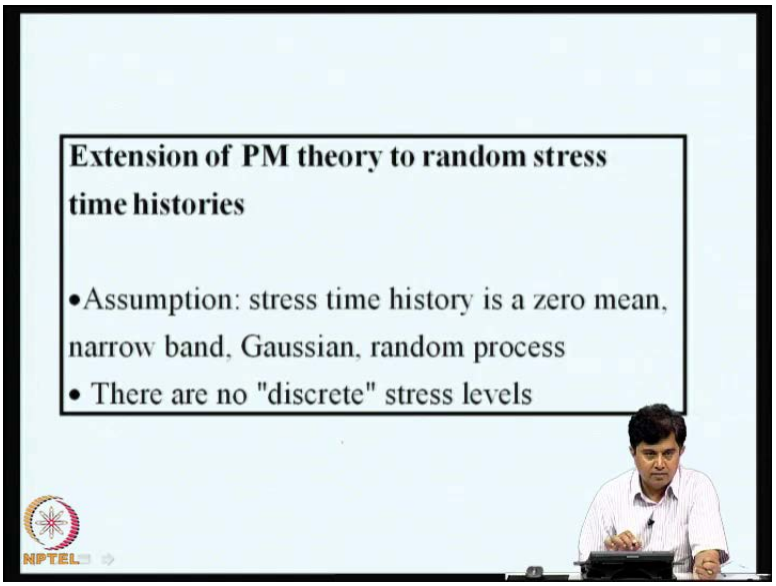

Now, the question is we are not so much interested in the limitations and scope of palmgren miner hypothesis etcetera. What we would like to consider is how can we extend the palmgren miner hypothesis to random stress time histories. Now, let us assume that the stress time history is a 0 mean narrow band Gaussian random process, and consequently if you consider stress to be a time history, there are no discrete stress levels here, is there not harmonics of different amplitudes following each other. So how do we proceed?

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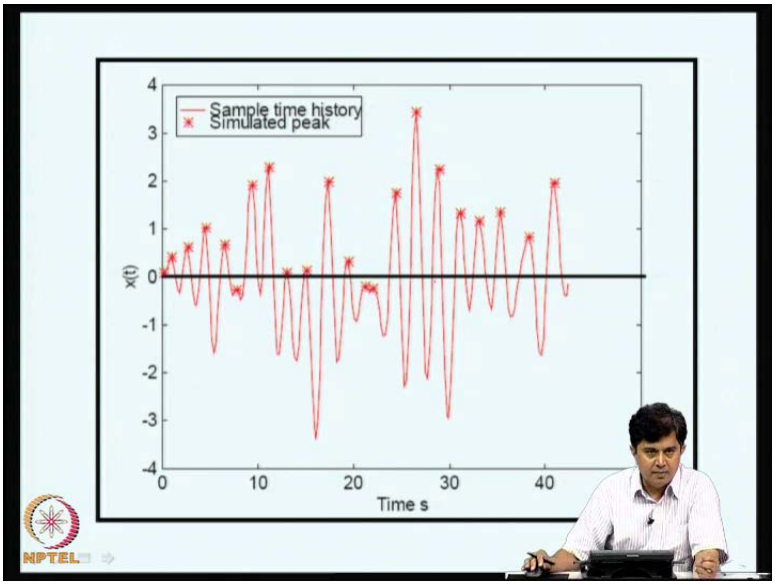

So, for example, this is a sample of a narrow band time history, time history of a narrow band random process. The notion of cycle is not very evident here. This is a narrow band process, but in a broad band process, as we already seen the notion of a cycle becomes much less apparent.

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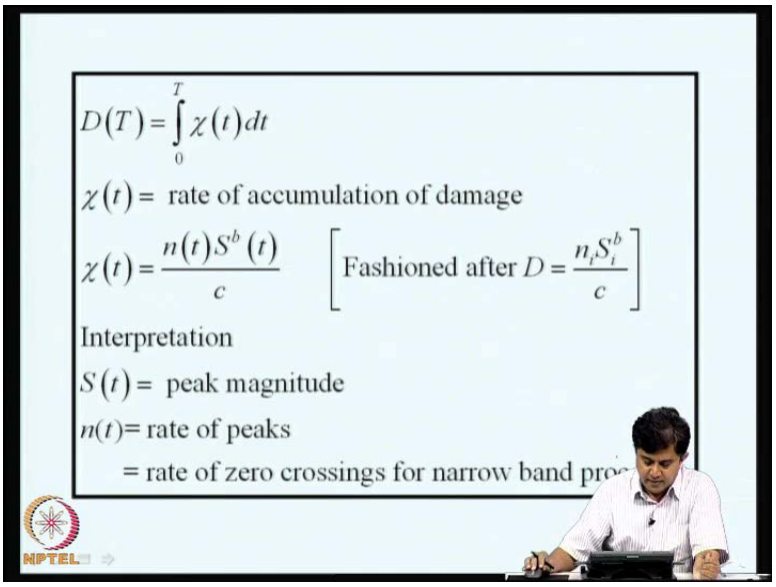

Here, what we define is function D of T. We define as integral 0 to t chi of T dt - where chi of t is a rate of accumulation of damage. This is, this we write as N of t into S to the power of b of t divided by c and this is fashioned after the formula for accumulation of damage given by n i S i to the power of b divided by c.

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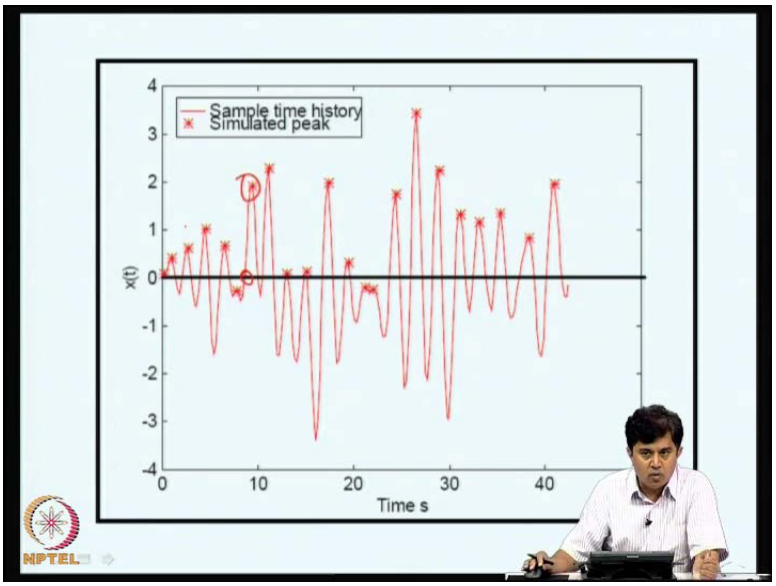

Now, the question is - if S of t is a as I said 0 mean narrow band stationary Gaussian random process, how do interpret these quantities? If the stress time history is a random process, how do interpret S of t n of t in this expression? So, what we do is, we take S of t to the peak magnitude and n of t to be the rate of peaks, and these we have studied already. For a narrow band process, rate of peaks is same as rate of 0 crossings for narrow band processes; that means every time a 0 is crossed, there will be a peak as you see here. For example, there is 0 crossing; there is a peak. So, that is a property of a narrow band process; that is not true for wideband process but I will to start with will limit or discussion to narrow band process.

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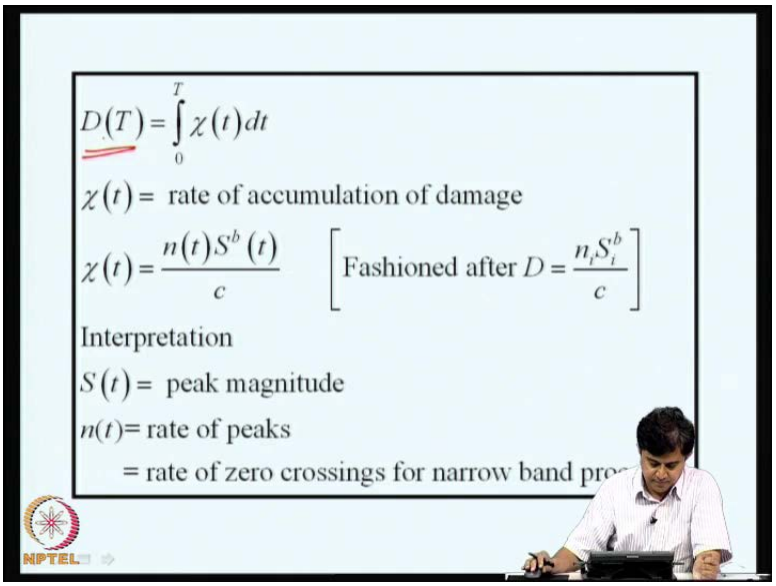

Now, our objective is to characterize this function D of capital T, is we are interested in knowing for example, what is probability that d of capital t is less than or equal to 1. That is a question that we would like to answer.

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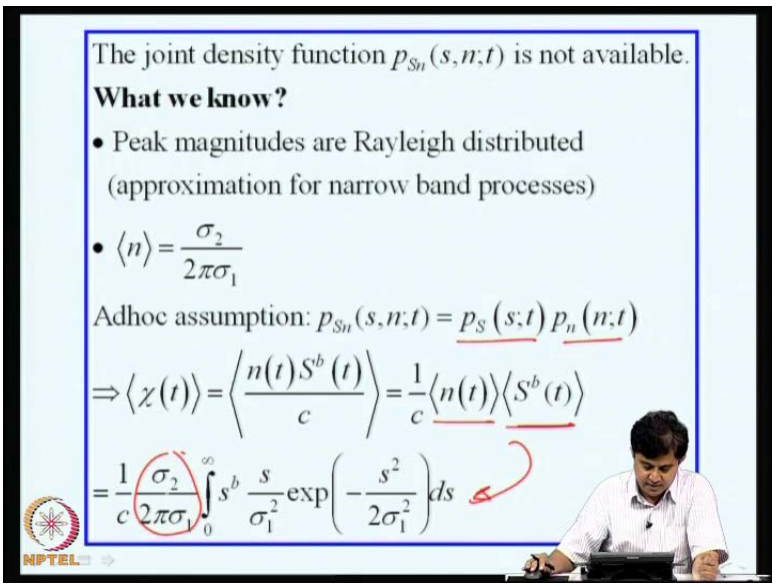

Now, the join density a function of S and N is actually not available. So, what we actually know? We know the peak magnitudes are Rayleigh distributed for narrow band processes and expected value of n is given by sigma 2 divided by 2 pi by 2 pi into sigma

1 - where sigma 2 is the standard deviation of the derivative of the process and sigma 1 is the standard deviation of the palmgren process.

Now, we make in Adhoc assumption that S and N are independent. So, we assume that join density function PSN s, n is product of the marginal density functions. So, based on that, we find the expected value of chi of t which is the rate at which the damage is accumulating, and since we know this expression and we know this probability density function, we can write for this expected value this form. This is Rayleigh density function. So, consequently this will be the expected value of S to the power of b, and this is, this quantity is the actual expected value of n of t, which is again known from study of low level crossings and 0 crossings of random process.

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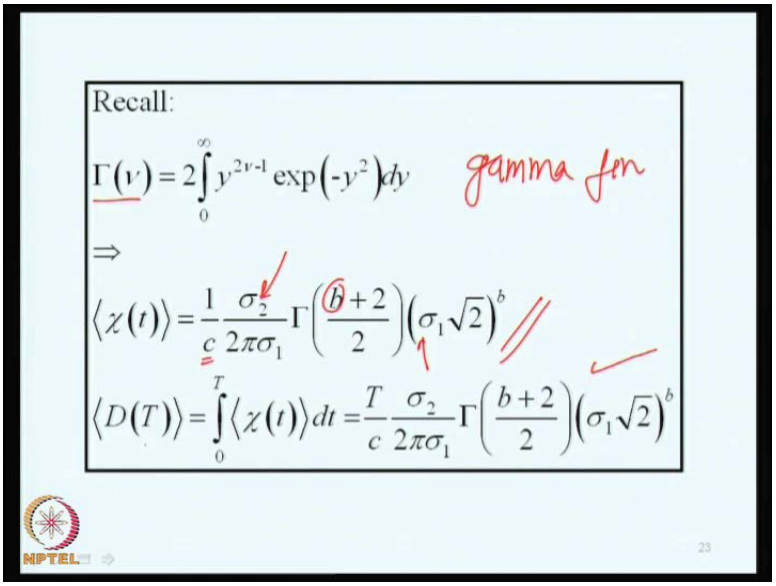

Now, we recall that the integral of this type 0 to infinity y to the power of 2 nu u minus 1 exponential minus y square dy is actually given by the gamma function. Now, consequently we can express the expected value of chi of t in terms of the gamma function and this is what we get. So, at least we are able to proceed and get the certain Adhoc assumptions the average rate at which fatigue damage is accumulating in terms of the standard deviation of the parent process, standard deviation of the derivative process and the material concuss c and b.

Now, we can return to the problem of finding expected value of d of t. Now, if process is stationary, we see that sigma 2 and sigma 1 are independent of time. So, they can be pulled out and this 0 to t d t will become this. So, we have this expression for d of t. That is average of the accumulated damage at time instance capital T.

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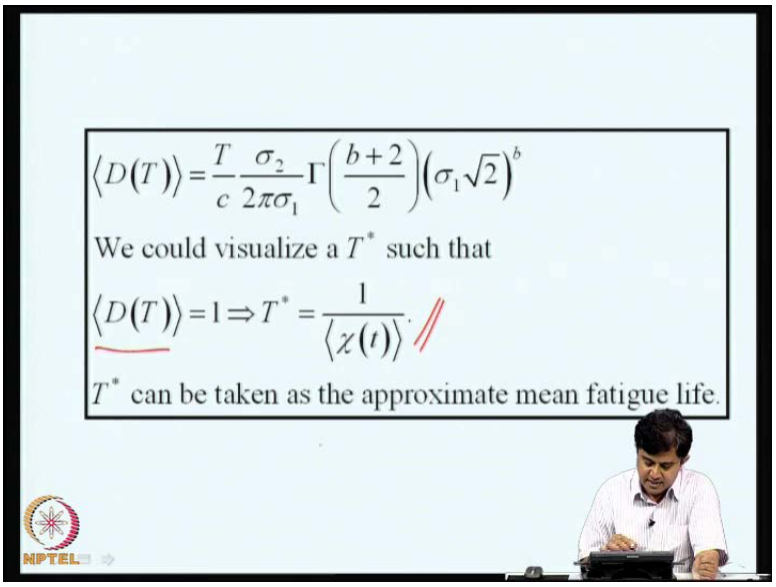

Now, we could visualize a T star such that these expected value of D of T becomes 1. Now, if you do that, that T star is will be 1 by expected value of chi of t. So, to a first approximation, we can take T star to be the mean fatigue life. So, if you know the random stress time history and you know its prospect density, you can derive the variance of the process and its derivative process, and if use this formulary, you have a handle now that, you can at least, you are at least in a position to evaluate the mean fatigue life. This is for a stationary narrow band Gaussian random process.

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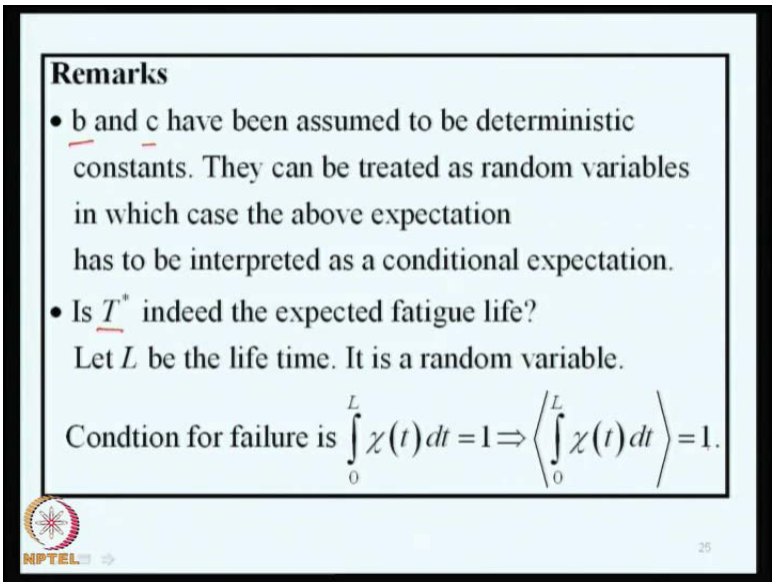

Now, a few remarks are in order. In this particular derivation, we have taken the constant b and c to be deterministic, but if you happen to know the join probability density function of b and c, we could incorporate that into the analysis. So, what the expectation that we have got can be interpreted as a conditional expectation and the unconditional mean can be evaluated by an integration with respect to the conditional probability density function to the joint probability density function of b and c. Is worth considering the equation is T star indeed the expected fatigue life. Now, we will this needs to be consider carefully. So, for that purpose, we will consider capital l to be the life time. Life time is a random variable. So, condition for failure that is a definition of capital l is 0 to l chi of t d t is 1; that means the expected value of this integral 0 to l chi of t dt is 1.

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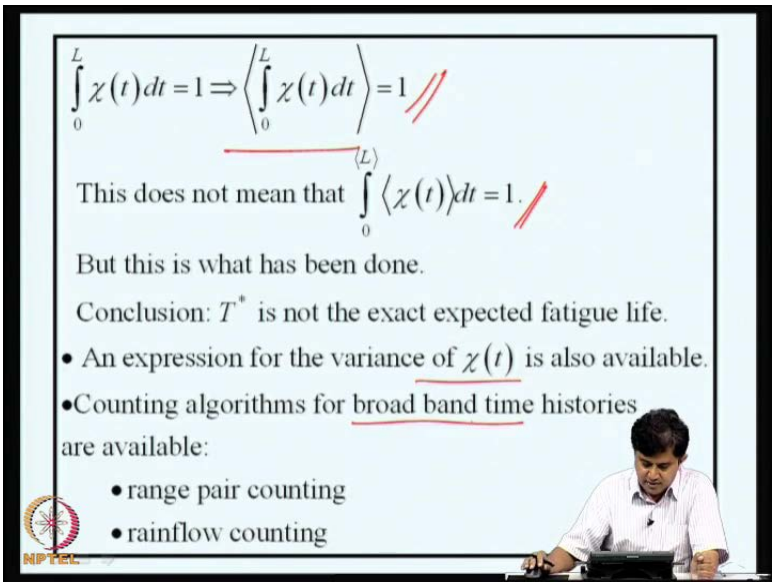

Now, this is a condition that we actually need to analyze, but what we have done is we have place this integration, the random quantities in this expression is L as well as chi of t and there mutually dependent. This integral obviously not equal to this. This is 0 to mean of L. Then mean of the integrant dt is not the, actually the condition that we are looking for all those. This is what we seen to have utilized. So, the conclusion is T star, is not the exact expected fatigue life, but if first approximation, we could take it as an approximate, an acceptable estimate, and if you want to really evaluate this integral in exactly, this is the, we do not have adequate information on the uncertainties here. Therefore, we will not able to proceed anyway.

Now, we have derived now the expression for expected value of chi of t. With little effect, we can also derive the variance of chi of t and we can also derive variance of capital d of t, and we have based our assumption on the fact that where assuming that the stress time is to the narrow band process. If the process is broad banded, then the notion of a cycle and amplitude becomes less apparent as I have already said, but there are algorithms for counting cycles and some of the algorithms are range per counting range flow counting etcetera. They are available one could also deal with broad band time histories.

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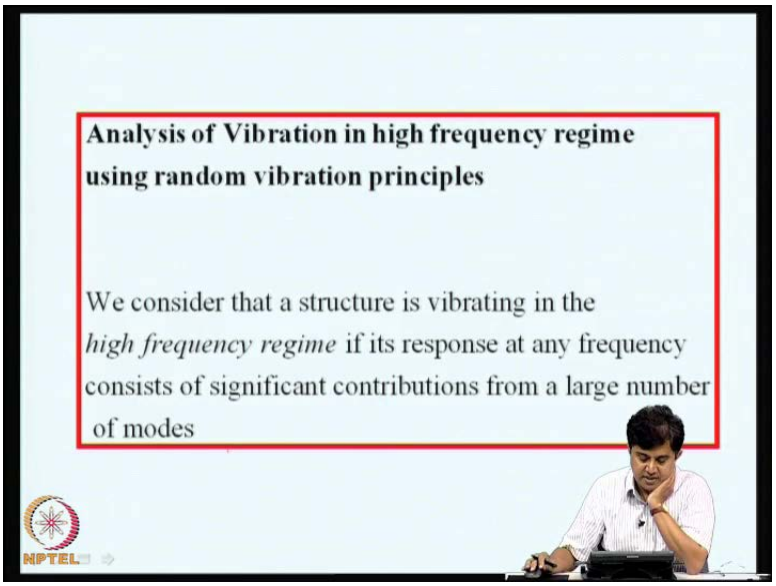

So, what have done in this brief presentation is to illustrate how properties of level crossing and peak distribution etcetera could be utilized to get a theoretical estimate of average, the expected life due to when failure is defined with respective fatigue. Now, we will move on to another application. This is analysis of vibration in high frequency regime and how we can use random vibration principles for this. Now, what is high frequency vibration? We consider that a structure is vibrating in the high frequency regime. If it is response at any frequency, consists of significant contributions from a large number of modes.

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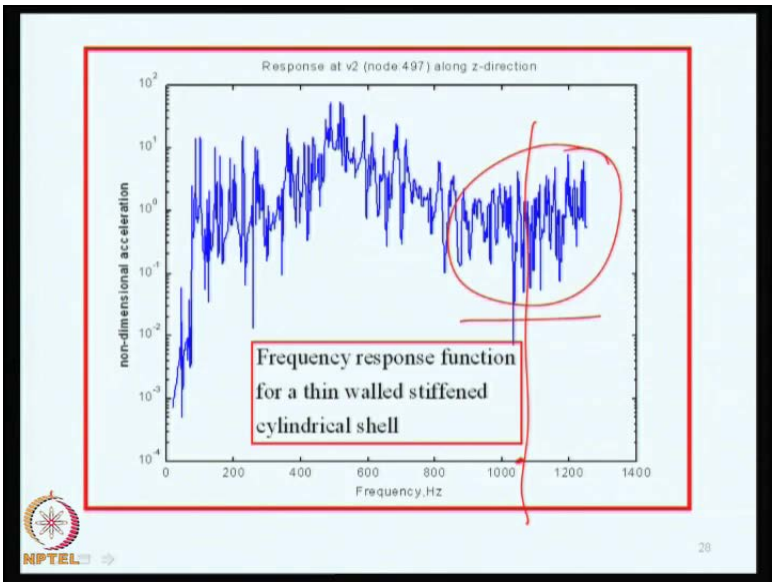

Now, what are the basic problems in such, such, basic difficulties in such problems? For purpose of illustration, I have shown here the transfer function of a, frequency response function of a thin walled stiffened cylindrical shell, and as you can see here, there are several peaks which are closely space. We know that in frequency response function, peaks occur wherever there is a (( )) frequency and their variation is also governed by value of mode steps at point of driving at point of measurement.

So, if we consider segments like this, it is very clear that there are large number of natural frequencies that are crowded in this frequency range. So, if you were to drive the system harmonically at a single frequency, it is expected that to compute the response, we need to consider contributions from several neighboring modes. In that sense, we call this type of oscillations is high frequency oscillations. This is not true for example in the case of say earthquake response of tall buildings, but whereas the in acoustic response of missile shells or aircraft wings etcetera, there will be large number of natural frequencies or in the frequency range that we are interested. I will clarify the nature of the problem as we go long.

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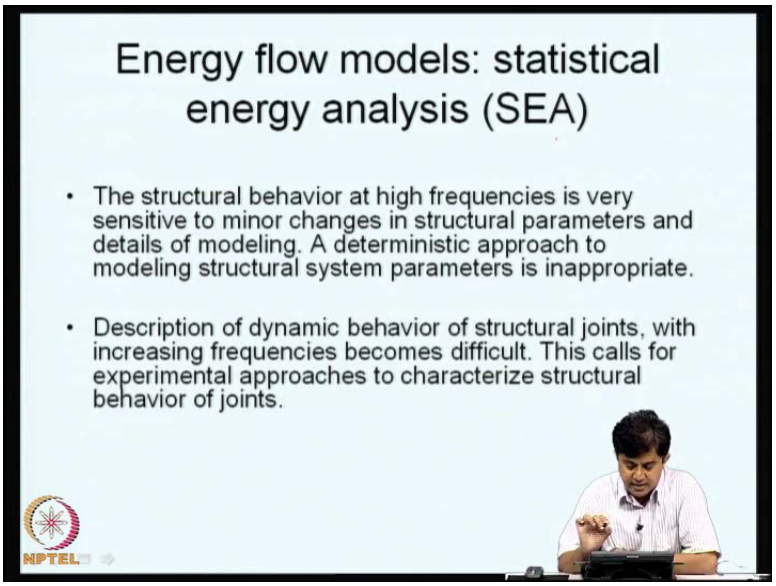

Now, there is one approach to studies such high frequency vibration and that is known as energy flow models, and one of the techniques are one of the methodology there is what is known as statistical energy analysis, is a frame work for analyzing high frequency vibration analysis. Now, the structural behavioral at high frequencies is very sensitive to

minor changes in structural parameters and details of modeling. A deterministic approach to modeling structural system parameters is inappropriate; that means if a structure is undergoing high frequency oscillations, a slight change in say boundary conditions or some of the parameters of the problem will dramatically alter the behavior of the structure.

So, the essence here is that we need to consider this extreme sensitivity of the response with respective changes in system parameters. Description of dynamic behavior of structural joints with increasing frequencies become difficult. This calls for experimental approaches to characterize structural behavior of joints. So, in typical structure like automobile vehicle or aircraft structure, there will be several joints, client joints, point joints, etcetera and their behavior at high frequencies becomes quiet complicated, and if use methods like finite element method modeling, the flexibility of the joints becomes a very complicated issue as you go very higher up in the frequencies.

So, the, this statistical energy analysis procedures essentially take in to account these difficult issues. First thing, it does is it treats the vibrating system as being randomly parameter; that means a natural frequency is more shapes of this structure is treated as being random in nature. The word statistical in statistical energy analysis refers to uncertainties in specifying the system parameters. The excitations could be random or could be deterministic. So, the stochastic nature in these problems come originate due to randomness in system properties.

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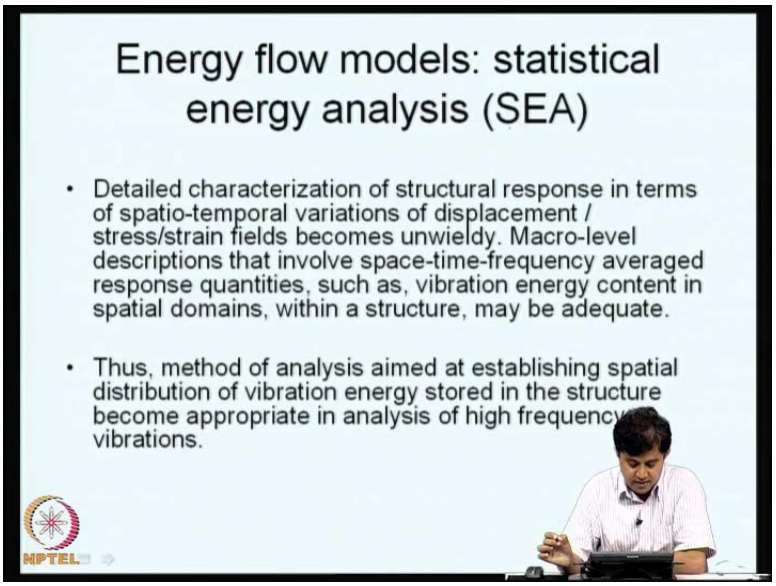

In such problems, detailed characterization of structural response in terms of spatiotemporal variations of displacement stress and strain fields become unwieldy. Macro level description that involves space time frequency averaged response quantities such as vibration energy content in spatial domains within a structure may be adequate. So, the method of analysis is aimed at establishing spatial distribution of vibration energy stored in the structure as a function of frequency.

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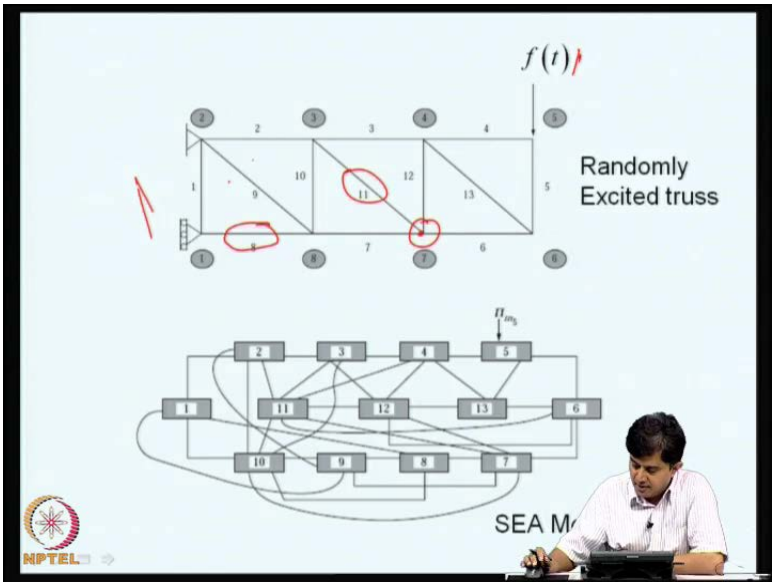

So, I will just illustrate what, what the, what that means, what I said. Consider a simple truss here which is carrying a force f of t. Let us assume that the frequency range of f of t spans several decays of frequencies. It may go up to a few tense of hertz to couple of thousands of hertz. We could use finite element method to analyze this problem, but then, if we need to compute the natural frequencies of this structure over a frequency range of say 2 kilo hertz, there could be several may be hundreds or couple of hundreds of natural frequencies which could lie in that frequency range.

Now, if we indeed perform, that would mean that we need to model this structure in greater detail. We need to select element which are quiet small in size. We should have large number of such elements that intend requires, that we should be able to know the property of this structure at smaller levels, smaller spatial levels and that may not be always possible. Similarly, the question on how does the join behave as a frequency increases becomes crucial.

So, any slight change in system parameters here can dramatically alter the behavior of the system. So, therefore, if you are interested in for example displacement stress or strain at any given point in the structure, we end up doing huge amount of calculations, but that type of information may not be needed to take decisions in this types of problems. What do we may like to know is what is the total energy that resides in this member or how does a total energy gets spatially distributed in the system.

For that purpose, what we do is we divide the truss into a set of subsystems. Each one is an energy carrying unit, and we would like to now consider the problem of this force into the velocity here will be a power input to the system. That power should be stored at various points in the structure and they, which are they more energetic members, and if that level of energy is high, how do I reduce it? What are the vibration paths that take this energy two different parts in the structure? This type of questions are asked in, in, statistical energy analysis.

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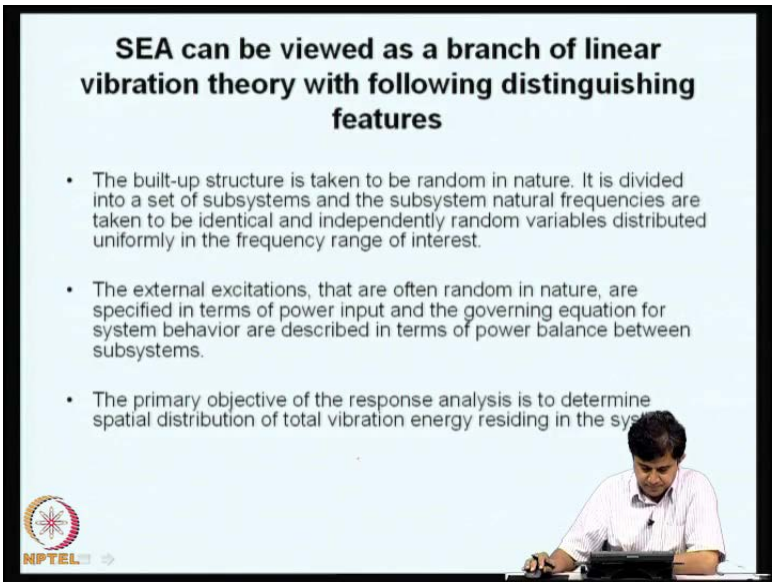

So, statistical energy analysis can be viewed as a branch of linear vibration theory with some of the following characteristic features. Here, the built up structure is taken to be random in nature. It is divided into a set of subsystems and the subsystem natural frequencies are taken to be identical and independent random variables distributed uniformly in the frequency range of interest. That could mean the natural frequencies of the subsystems are taken to constitute Poisson points on the frequency axis; that means natural frequencies are like number of natural frequencies is a Poisson random variables. The external excitation that are often random in nature are specified in terms of power input, and the governing equation for system behavior are described in terms of power balance between subsystems. We do not write the equilibrium equation the way we do in finite element method. We simply keep track of energy balance. I will come to some details.

The primary objective of the response analysis is to determine the spatial distribution of total vibration energy residing in the system. It is not to obtain, for example, time histories of stresses, strain and displacement at every point on the structure. Here, it is a macro level description where certain spatial extends are identified as subsystems, and what happens within that spatial extend in terms of a single number is what is being sort.

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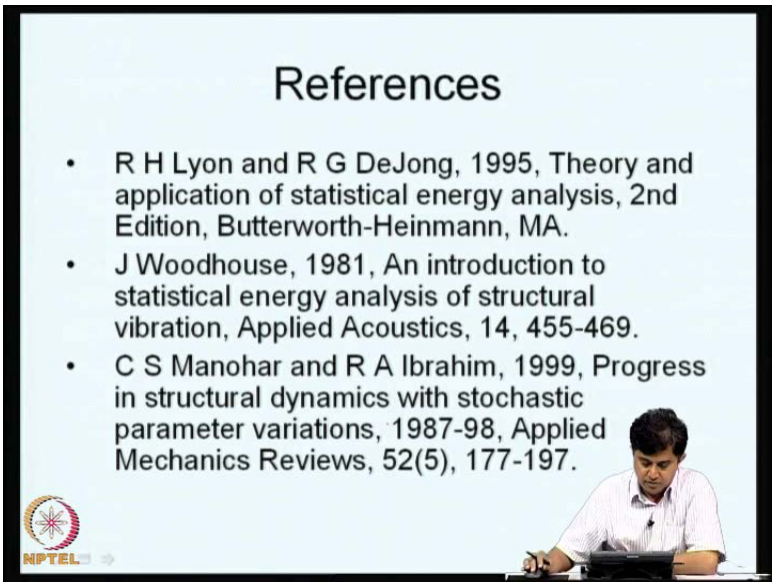

Here are some references - the book by the Lyon and Dejong theory and application of statistical energy analysis. There is an introductive paper, and a review paper which provides some background to this subject.

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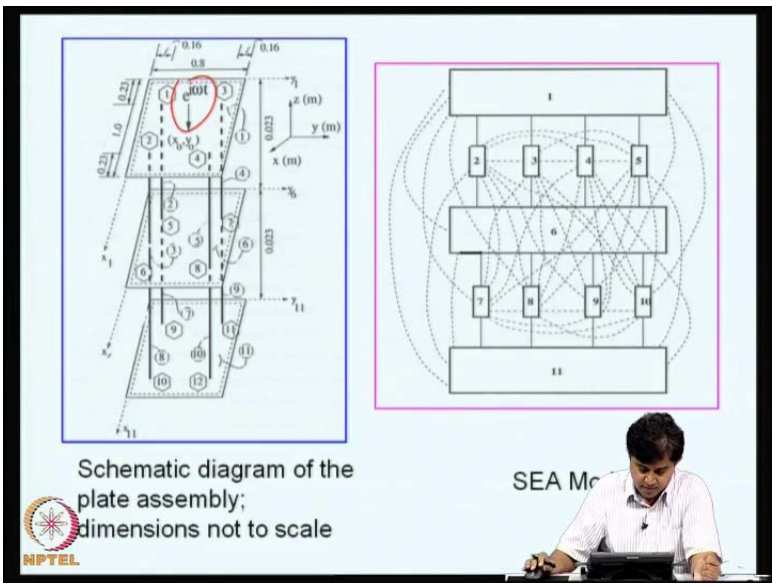

Here, there is another example where there are a stock of three plates. This could be a typical situation in solar panel arrays in satellite. This e raise to i omega t is a forcing function that acts on this and this frequency can covers several, you know, it could run from twenty to say two kilo hertz. What we are interested in knowing is under this kind

of excitations, how does vibration energy reside in different parts? So, we could model this in terms of subsystems as shown here 1, 6 and 11 correspond to these three plates. This, 1, 6 and 7, 1, 6 and 11 correspond to these three plates and 2 3 4 5 etcetera, 7 8 9 10 are this coupling members. We are not interested. It may again emphasis in finding stresses strain and displacement in various points on the structure.

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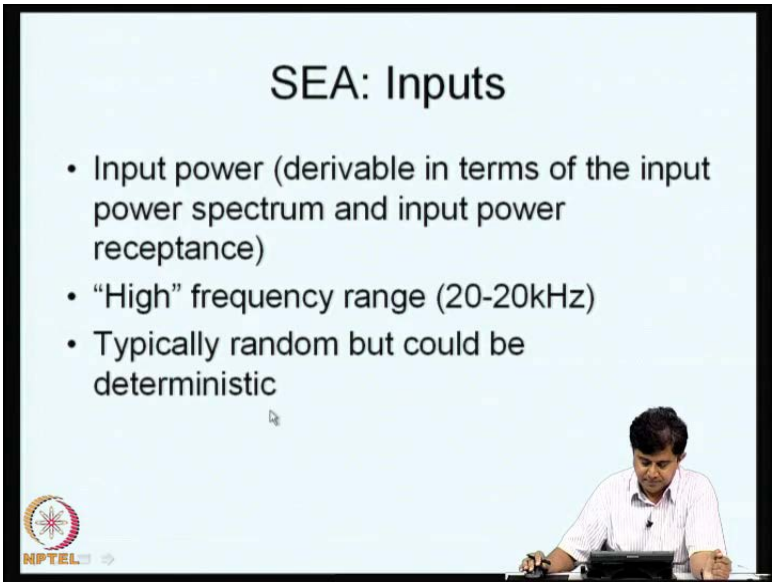

Now, how do we specify inputs in statistical energy analysis? We do not know time history of the force but we would like to provide the input power. This input power is derivable in terms of the input power spectrum and the input power receptance; that means the force into the velocity, applied force into the velocity in the direction of the force is a power. Force into displacement is a work done and for that work done for a unit time is the input power, and in terms of a spectrum if you want, you have to do certain calculation. Now, we consider inputs to be in the high frequency range say 20 to 20 k kilo hertz, these excitations could be random or they could be deterministic also. It is not important that the randomness resides in the excitation. The randomness is in the system parameters.

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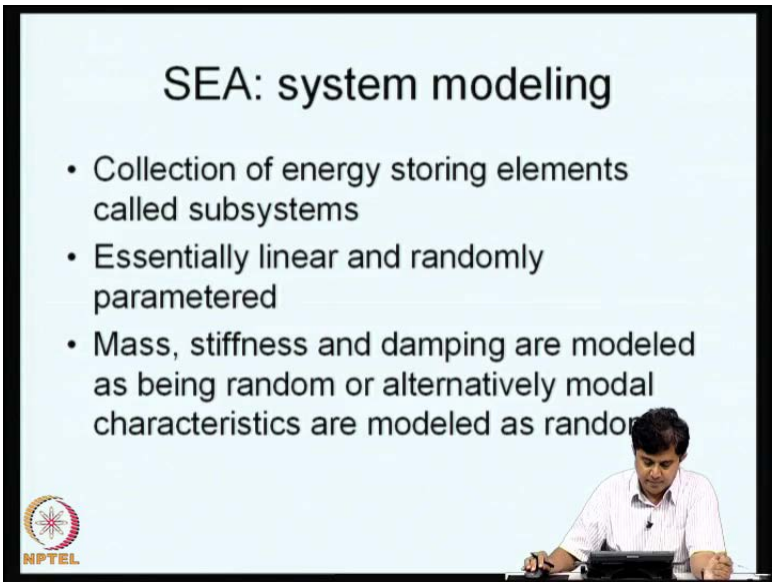

How do we model a system? We model a system as collection of energy storing elements called subsystems. They are not like finite element elements but they are much more macro level description. They involve much, you know, macro level description of the system behavior. We focus only on linear and randomly parametered systems. The behavior is linear but the system is randomly parametered.

This randomness in system parameters could be introduce at the level of definition of mass stiffness and damping as being random or alternatively we can directly characterize the natural frequencies and more shapes of the subsystems as being random. We need not have to specify mass stiffness and damping as random but since solution is being obtained in model domain. The model parameters like natural frequency, more shapes participation factor model damping etcetera could be taken as random.

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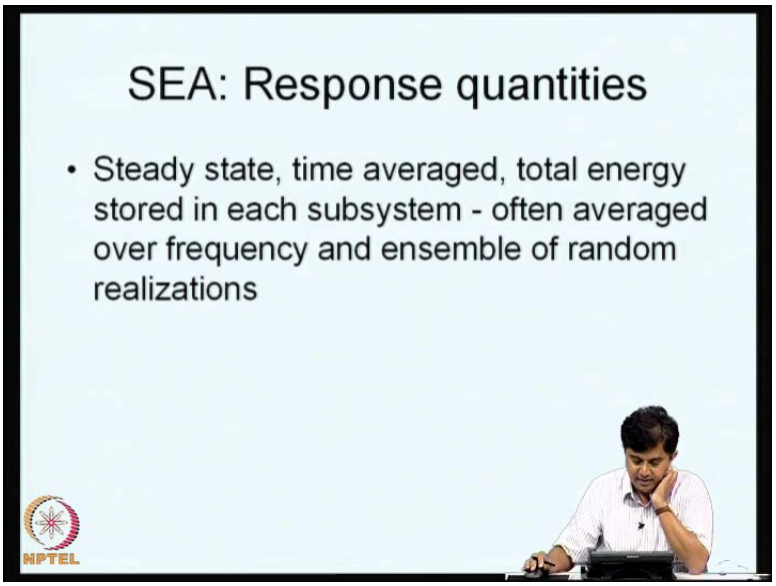

And what are the response quantities we look for? We look for steady state, time averaged, total energy stored in each subsystem often averaged over frequency and ensemble of random realizations; that means we are looking for grossly averaged features. We are not looking into details. It is average over time, average over space, average over frequency band and it gives a broad idea about different levels of energy. They stay at, they decide at different levels.

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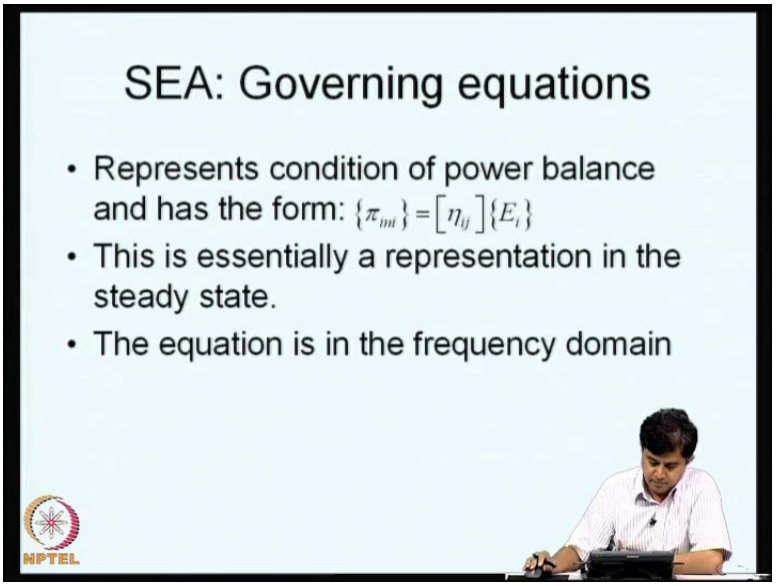

How do we derive the governing equation? What are the governing equations in SEA? The governing equation in SEA represents condition of power balance and it has the form the vector of power input is equal to a matrix multiplied by energy at different energy residing in different subsystems. This matrix is known as coupling loss factor matrix. I will talk about this shortly. This is essentially a representation in the steady state. We do not normally consider transient behavior using SEA, and the whole analysis is done in frequency domain.

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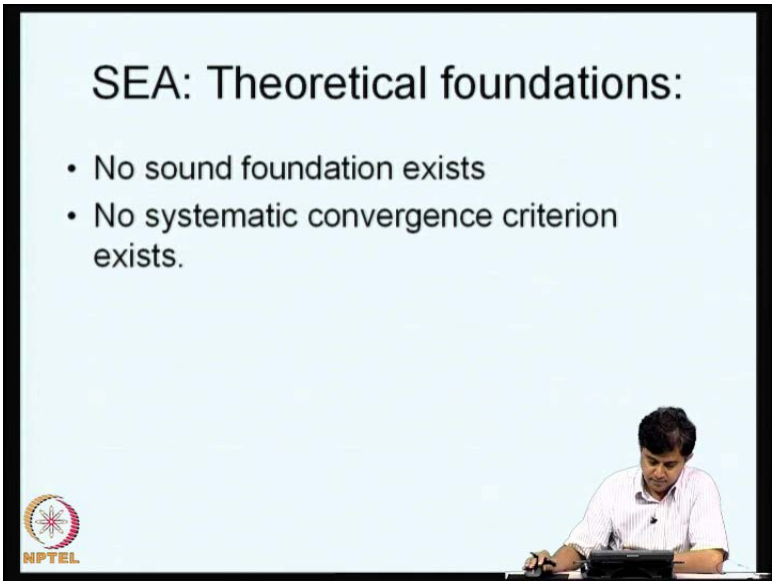

Now, we can ask are there any theoretical foundations for this procedures? Actually there is no such regress theoretical foundation. There is no, for example, a variational principle from which we can derive SEA equations nor there is any systematic proof that as we increase the number of subsystems in SEA formulation. The answer would converge. So, it is not like a method like finite element method but it is a heuristic procedure which is found to work well when other methods do not perform.

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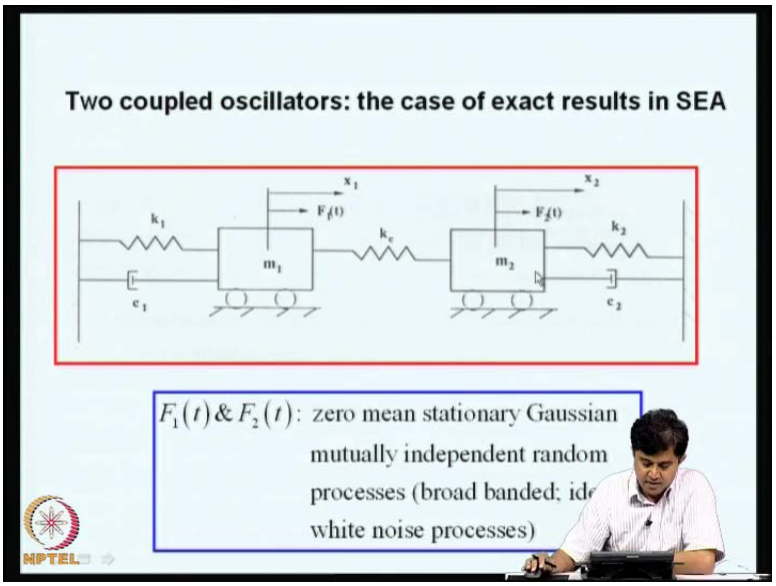

Now, the, having said that there is no theoretical basis but there exist an idealize situation where statistical energy analysis basic results of statistical energy analysis can be shown to hold good. So, what is that system? This is simple system where two oscillator which are couples, a coupling spring. The two oscillators are driven by F 1 of 2, F 1 of t and f 2 of t. These are taken to be independent Gaussian White Noise Processes. Now, the question that would like to answer is this F 1 of t will do some work on this displacement. Therefore, there will be a power input here and there will be a power input here and how does vibration energy gets shared between this subsystem and this subsystem.

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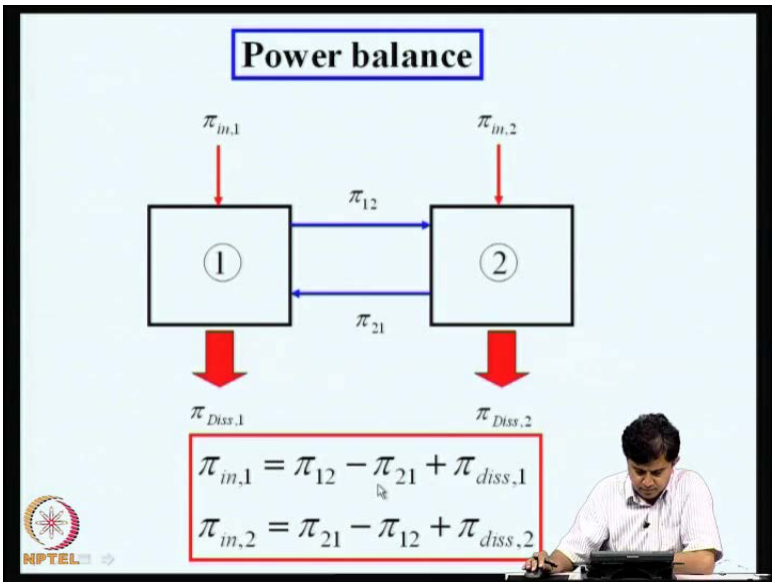

So, what we do is we model the problem using two subsystems - this is one oscillator this is second oscillator there is a coupling k c, and please note that the coupling element do not have, does not have damping. It is a conservative coupling. So, what happens? There is a power input to the first system through F 1 of t and part of the power is dissipated through damping and part of that flows to the, energy flows to the second subsystem.

In the second system because F 2 of t, there is some power input, and part of it is dissipated and part of it flows to a subsystem one. So, now, we set up this simple equation that pi in 1 is equal to pi 1 2 minus pi 2 1 plus pi diss 1. So, pi in 1 must equal to this. Similarly, what comes in to second system? Pi in 2 plus pi 1 2, and what goes out? Pi diss 2 and pi 2 1. So, pi in 2 must equal to pi 2 1 minus pi 1 2 plus pi diss 2. So, this is, there is no approximation here. This is an exact.

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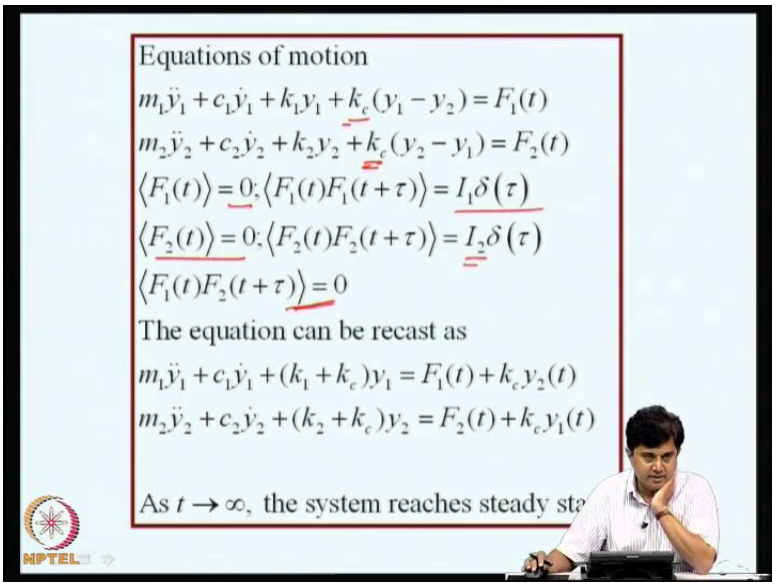

Now, let us try to evaluate the expressions for this input powers and maybe I will take expected values and see what happens. So, to do that, we will write the equations of motion. Now, the equation of motion consist this simple to derive the 2 degree approximation, 2 degree freedom model. It guess couple through the spring k c and this is reasonably straight forward, and we are assuming that F 1 of t is 0 mean and it is a white noise with strength i 1 and F 2 of t also is a white noise with strength i 2 is 0 mean, and we assume that the uncorrelated; that means they are independent because we are assuming Gaussian White Noise models. The equation can be recast as in this form where we are taking the force acting on the first system through the coupler on the right hand side.

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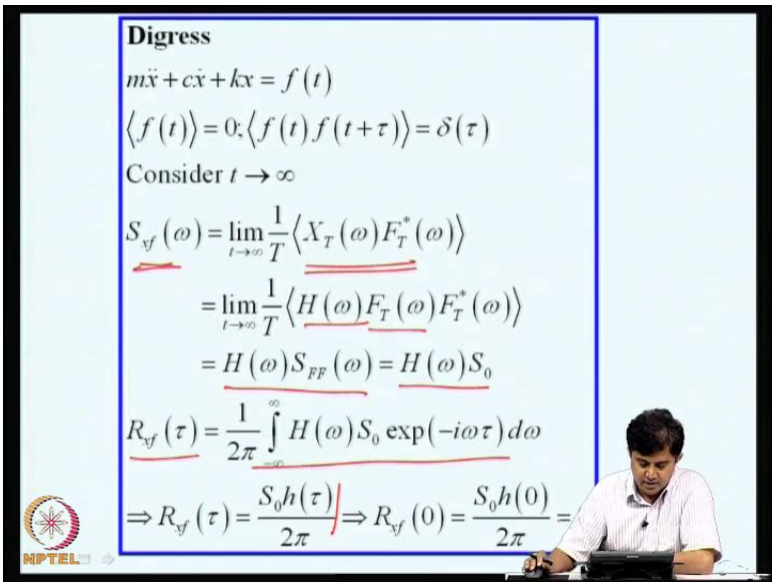

So, we assume the system since the system is damped, we assume we assume that as c tense to infinity the system reaches study state. The input stationary; system is damped. Now, we will digress for a moment. If you consider a single degree freedom system under a white noise excitation in the steady state, if you want to find out the cross power spectral density function between displacement and applied force, this can be derived in terms of this well-known expression, and the Fourier transform of the response is related to the Fourier transform of the input through the system transpose function, and we get this S x f of omega to be H of omega into S F F of omega. Since excitation is white noise, I get this H of omega into S naught.

Now, it is a cross covariants is a Fourier transform of this. So, I get this expression. Now, a tau equal to 0. I will get R x of f 0 R x f of 0 can be obtained by putting tau equal to 0 but you can see that the integrant is nothing but Fourier Transform of the impulse response function. So, R x f of tau is S not H of tau by 2 pi. Now, in this if we put tau equal to 0, I get R x f of f is 0 which is S not h of 0 by 2 pi. Now, we know definition of h of 0 is 0. Therefore, r x of 0 is 0. So, that result I will be using so that this miner digration to make sure that, that is, registers on your mind.

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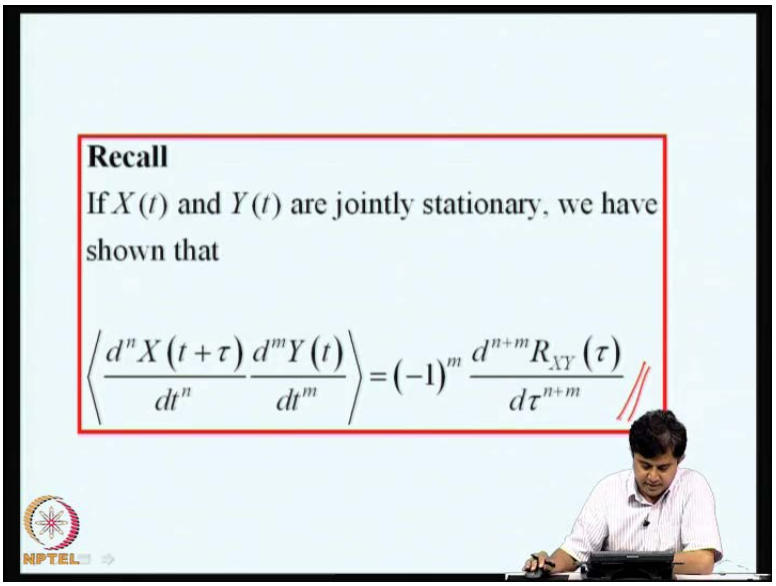

Now, we would also need another result. If X of t and Y of t are jointly stationary, the expected value of the product of n th derivative and m th derivative of X and Yy as shown here is given by this. This we have derive when we introduce the basic motions of mean square derivative for random processes.

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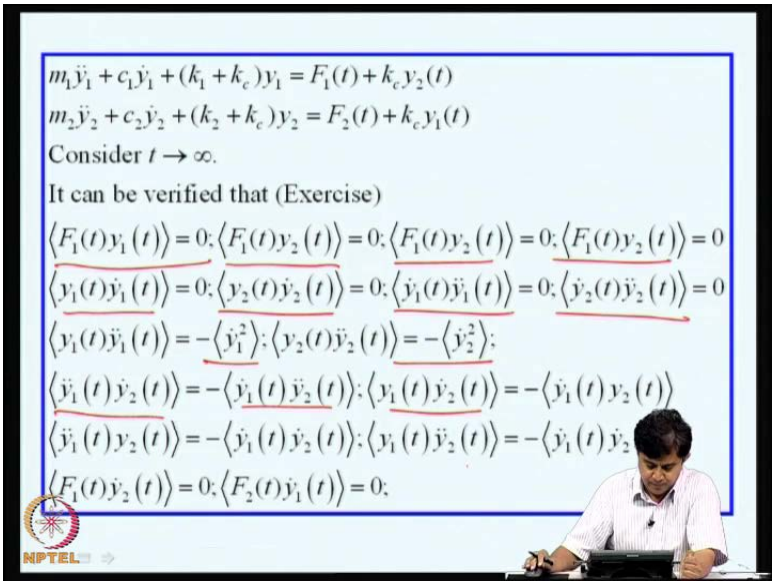

Now, we can consider certain expectations F 1 of t into y 1 of t is 0 using the logic that I just not described. We can also show that sum of these are 0, and since process is stationary, the process and its derivative at same time are uncorrelated. So, again, that

could mean these expectations are 0, and expected value of y 1 into y 1 double dot using this formula can be shown to be given by this, and we have several other expectation that we need when work this equations and I leave it on an exercise to verify each of these statements. These are already covered in the lectures. So, it is a good time to check you are understanding by showing that each one of these statements are true.

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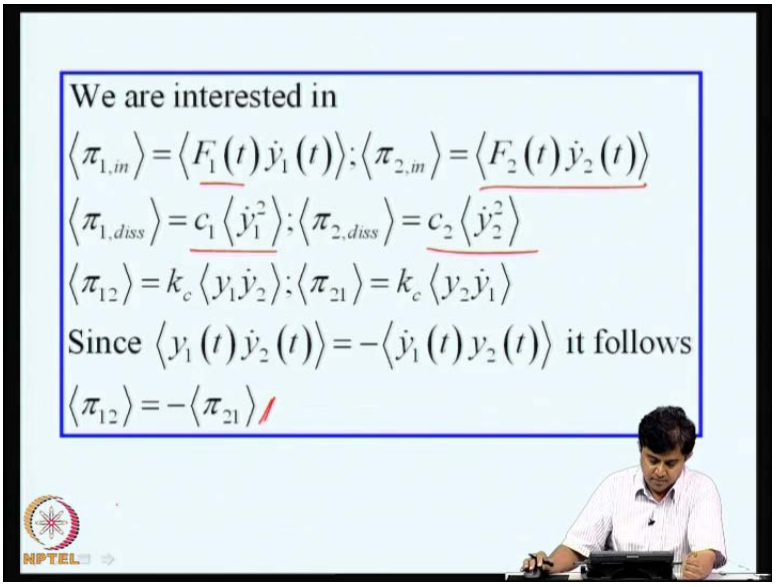

Now, what is that we are interested in? We are interested in the average power input, that is, expected value of pi 1 in. What is power input to the first mass that is applied force F 1 of t into dy 1 would give the work done and into dy 1 by d t would give the work done for unit time. So, pi 1 in expected value nothing but expected value of F 1 into y 1 dot. Similarly, pi 2 in will be f 2 into y 2 dot. What is power dissipated it is c 1 y 1 dot square; pi 2 dissipated is this. What is pi 1 2? Pi 1 2 is, pi 1 2 is the force in the spring acting on velocity of this. So, that would, you can show that, that would be given by k c into y 1 into y 2 dot and pi 2 1 will be k c y 2 into y 1 dot.

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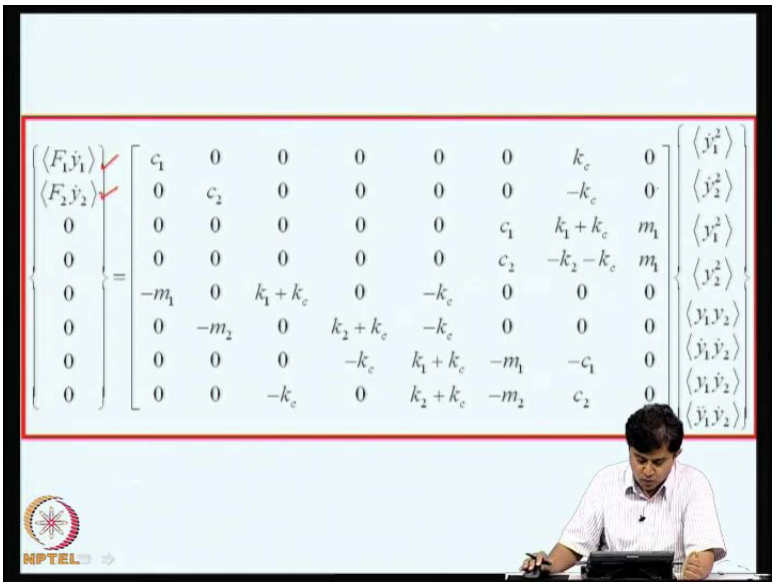

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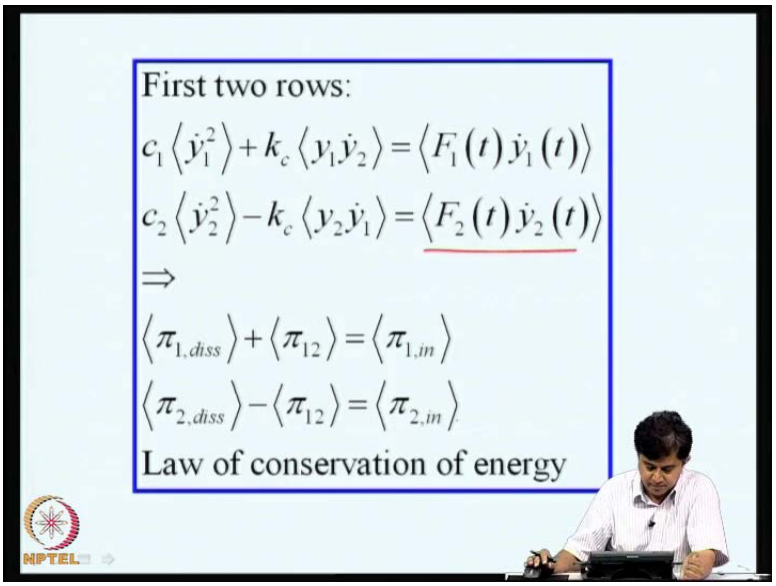

Now, if since y one of t and y 2 dot of t is 0 and it not 0, it is equal to minus y 1 dot y 2 t follows that pi 1 2 is equal to minus pi 2 1. Now, we can recast all these equations in a matrix form. So, on the left side, I have pi one in; here, pi 2 in, and all other expected values I mean I am writing, I mean you have to recast that set of equations in this form we get this equation. This is a straight forward representation of the equation that we consider just now.

Now, in this, we will now consider some of the rows. Suppose if you consider the first two rows, I get c 1 into y one dot square expected value plus k c into y 1 y 2 dot is equal to F 1 of t into y 1 dot. Similarly, c 2 y 2 dot minus k c into y 2 y 1 dot is this. These two equations are nothing but the statement of law of conservation energy, that, is pi one diss plus pi 1 2 is pi 1 in pi 2 diss minus pi 1 2 is pi 2 in. So, things gel-well, there is no problem.

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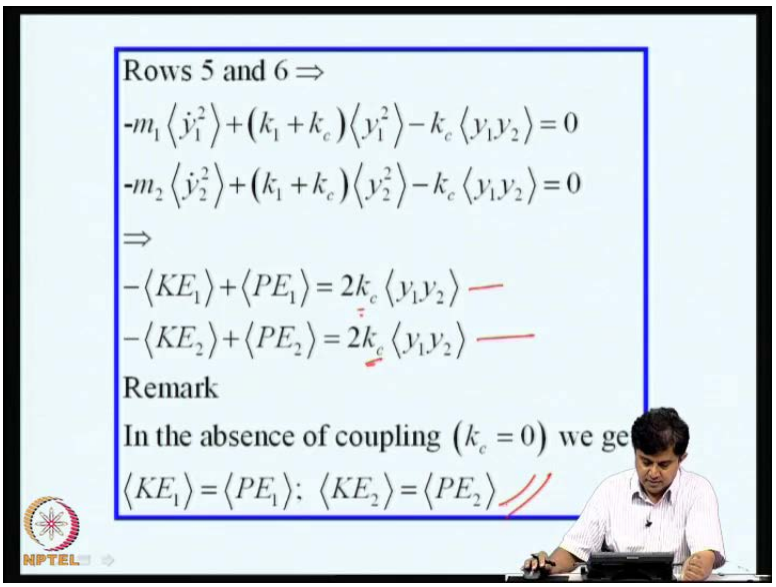

Now, if you consider rows five and six in that equation and take a look in this equation. You can show that what they are telling as is the average kinetic energy in the first system minus plus average potential energy in the first system is given by this. Similarly, this for the second system; this is the equation. Now, if for instance, if coupling spring is absent, then we know that in the steady state average kinetic energy, same as average potential energy and that again this consistent.

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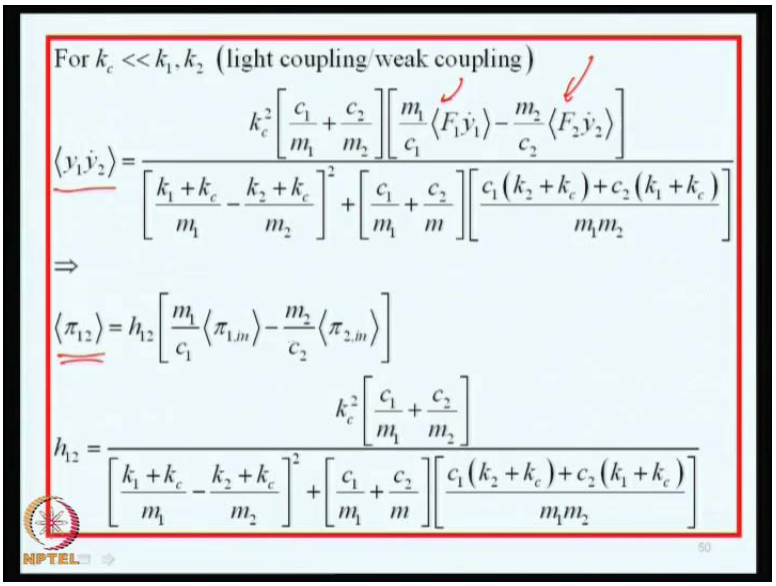

Now, if we assume that this coupling spring is much smaller than k 1 or k 2, that means the coupling is weak. This is one of the assumption that is made in statistical energy analysis, and if I now consider expected value of y 1 into y 2 dot, you can derive this expression and you have to carefully look at this expression. This is expected value of y 1 into y 2 dot is nothing but expected value of pi 1 2 multiply for the k c. We denote this multiplier.

See, the term inside this bracket is of importance. You can see that this is pi 1 divided by c 1 into expected value of pi 1 in and is m 2 by c 2 into pi 2 in, the second term. So, what this is telling is that the energy flow from 1 to 2 is proportional to the, difference in energy at difference in the power input between the first and the second system some scaled factor. Now, we will manipulate this and see what it tells us.

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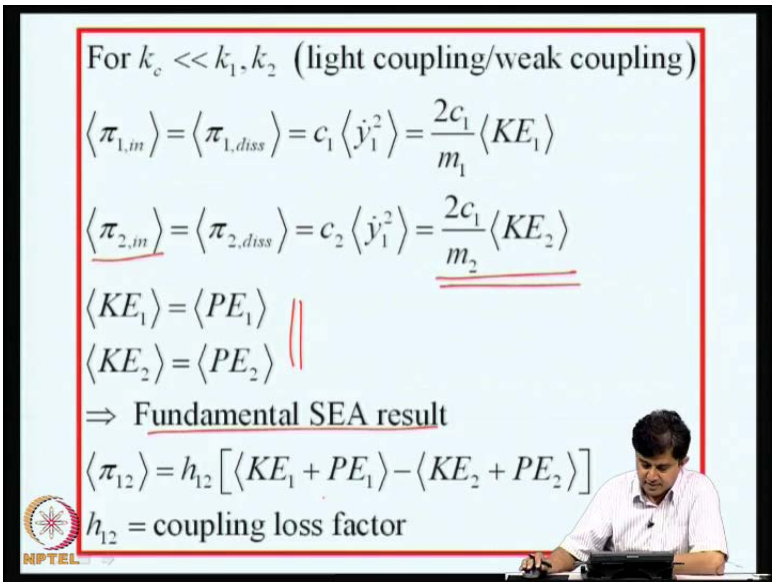

Now, again under the assumption of light coupling, we can show that the power input is in steady state is dissipated by the system. So, that is c 1 y 1 dot square. So, I can write for pi 1 mean 2 c 1 m 1 into k e 1. Similarly, expected value of pi 2 in can be given by this expression, and we use the fact that in steady state, average kinetic energy and average potential energy for the two systems are equal, and if I now substitute, I get what is known as fundamental statistical energy analysis result namely that, average energy flow is proportional to the difference between, the energy between two subsystems, the energy in the two subsystem; that means just like thermal energy flows from hot spot to cold spot, the vibration energy also flows from places of high energy concentration to low energy concentration.

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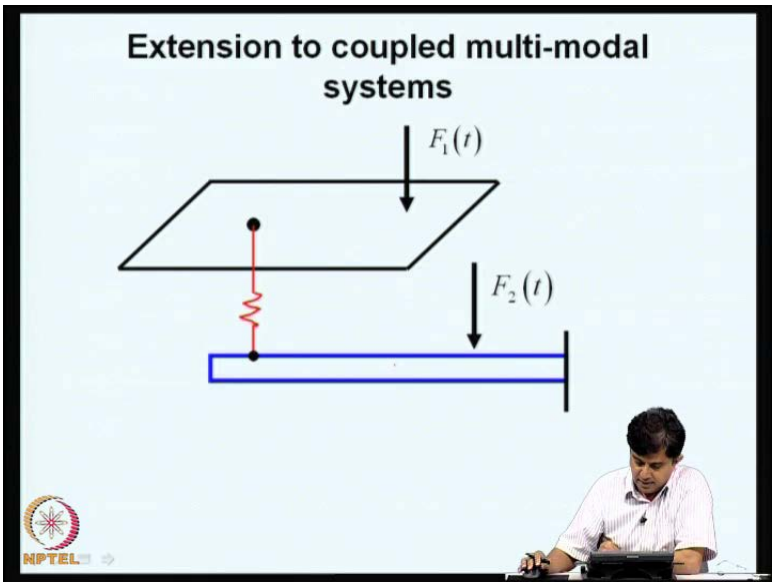

So, this is the essential fundamental result of SEA and this is, this has basis in random vibration principles. That is what I am trying to discuss. Only that aspect of SEA is what I am discussing. We have consider two oscillators. Now, what happens if there are two continuous systems? So, for example, plate is coupled to a cantilever beam. This plate itself is a multi-model subsystem. It has several natural of frequencies in its uncoupled state. If I were to perform a model analysis in the frequency range of interest, there may be several modes; that means this plate will be represented by a set of single degree freedom systems corresponding to the normal modes in the independent, generalized coordinates of this plate. Similarly, the cantilever beam also can be represented as set of oscillators.

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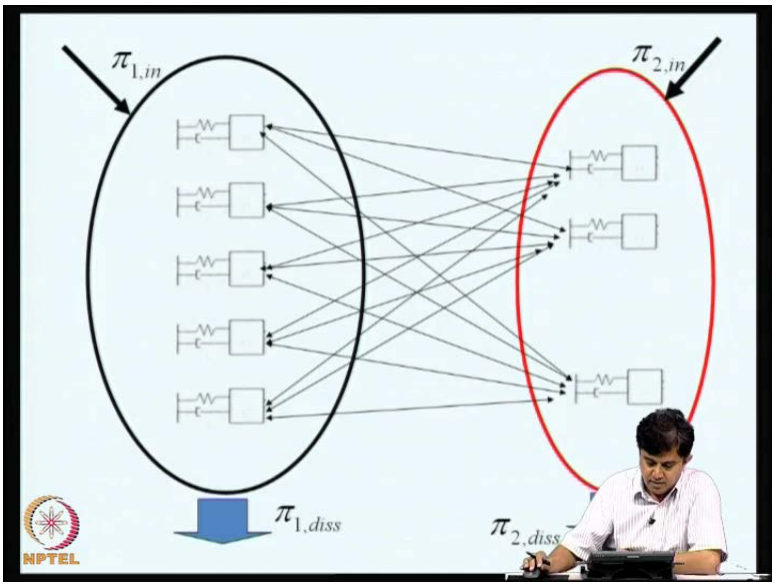

Now, that we notionally represent like this. This first picture here is, this is a plate and these are the oscillators that correspond to generalize coordinates in the frequency range of interest. Similarly, this is a beam where there are few more oscillators here which corresponds to the generalize coordinates of the cantilever beam. This system receives some power pi 1 in and it dissipate some power pi 1 diss. Similarly, this system receives pi 2 in and dissipates pi 2 in. Now, we have studied how energy exchange takes place between two oscillators. Now, what we do is we assume that the energy flow from plate to cantilever consists of energy exchange between the oscillators in this set with oscillators in this set; that means this oscillator exchanges energy with this, this and this. Similarly, this oscillator exchanges energy with all the oscillators here. So, that means all these oscillators exchange energies.

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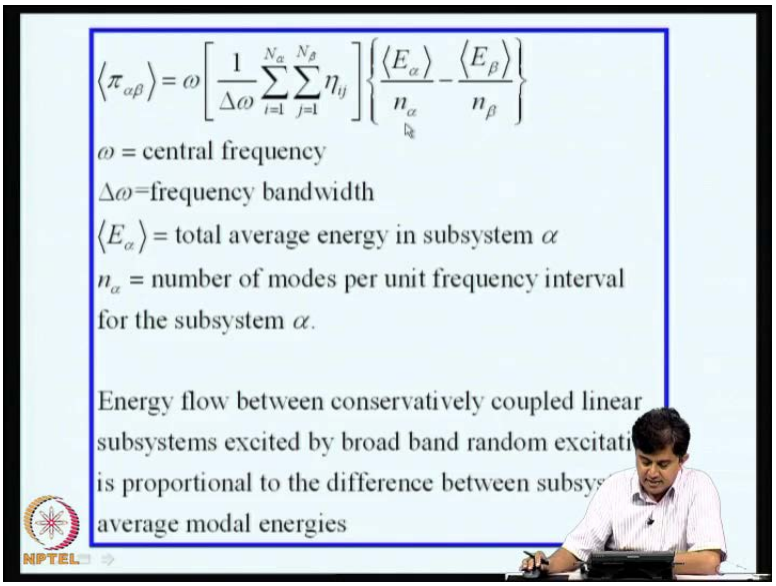

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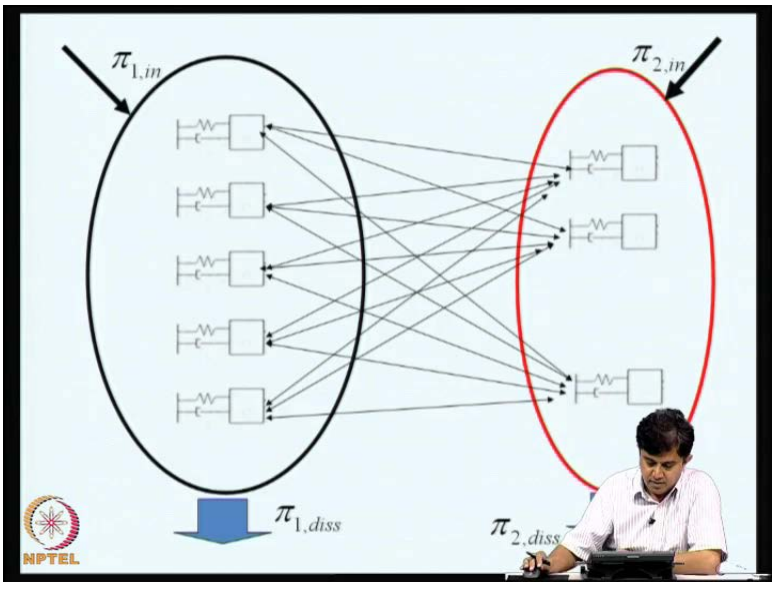

Now, the average power flow from alpha to beta - that means plate to the cantilever - is given by the, a constant multiplier which is again dependent on the plate and cantilever characteristics and in some sense in an energy difference between the two plates, that is, total energy divided by number of oscillator that are participated. What is an energy density or the concentration of energy is total energy divided by number of oscillator? It is like temperature in thermal analogy. So, it is a hot means total energy divided by number of modes is high. So, here, total energy divided by number of modes here. So, the flow takes place from hot spot to the cold spot and that the energy flow is

proportional to the energy difference and its proportionality constant involves the properties of the structure.

So, we have omega is a central frequency; delta omega is a frequency bandwidth and e expected e alpha is total average energy in the subsystem alpha and n n alpha is a number of modes per unit frequency interval for the subsystem alpha; that means omega, that defines this that the way energy is defined this, this, factor is relevant. So, what this means is - energy flow between conservatively coupled linear subsystems excited by a broad band random excitation is proportional to the difference subsystem average model energies. Now, this proportionality constant etcetera - this is known as coupling loss matrix, coupling loss factor matrix - the elements all that have to be determined. By, in this particular case, when we discuss two oscillators, we derived this h 1 2. That is shown here using random vibration principles.

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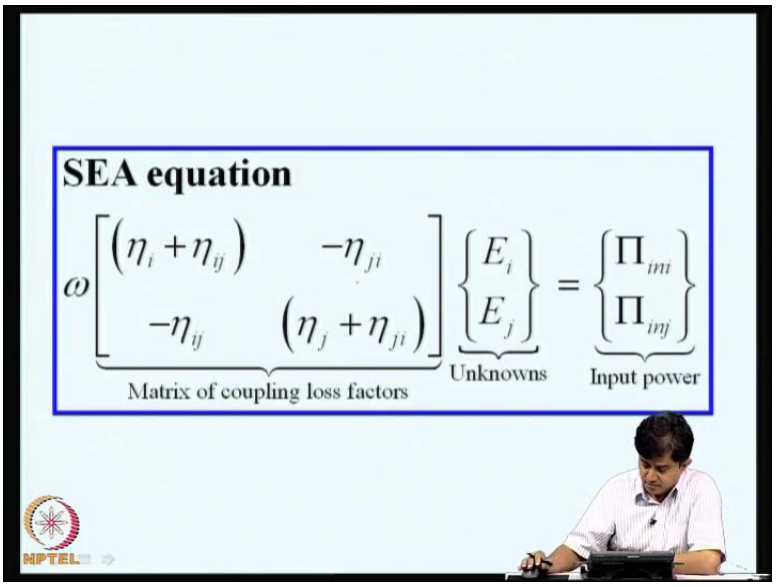

So, the S e a equation, the equation that you will actually solve when you apply statistical energy analysis will be in this form. The unknowns are these energy levels E i and E j. This is E 1 and E 2 in this particular 2 subsystem problem, and the excitations or the inputs in terms of power pi in 1 pi in j. This is known; these are unknowns, and this is analogs to are stiffness matrix but this is known as matrix of coupling loss factors. Typically, this coupling loss factors could be estimated experimentally where you know the difficulty is associated with joint flexibilities, I mean randomness etcetera the system

properties can be taken in to account or analytically using wave propagation concepts. I will not get in to that.

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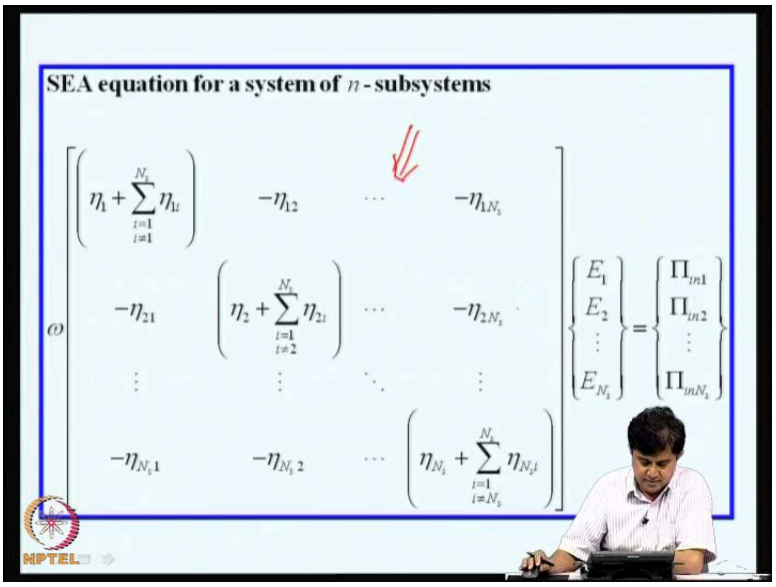

The same logic if you applied to a set of n subsystems, we get the unknowns are E 1 E 2 E 3 E n s - where n s are the number of subsystems. Pi in 1 pi in 2 and pi in n s are the inputs, and this is the matrix of coupling loss factors which has to be determined as a experimentally or by analytically, and the principles of random vibration analysis are relevant in this context.

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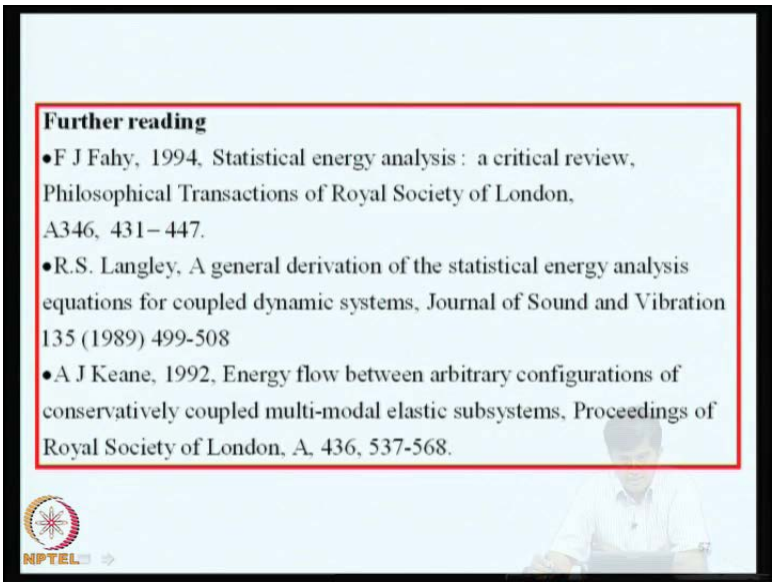

So, this subject of statistical energy analysis quiet worst and I have given few of you suggestions for further reading. There is a review paper by Fahy's statistical energy analysis a critical review, and the two papers - one by Langley and another by Keane which contain quiet useful information. Now, with this, we will conclude this present lecture. In the next set of four lectures, we will consider certain problems which basically would help you to learn the subject. So, the discussion on principles of random vibration analysis and the methods and the basic problems etcetera now closes in the remaining four lectures of this course. We will consider a set of problems and tackle them. We conclude this lecture at this stage.