Stochastic Structural Dynamics

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Module No. # 08

Lecture No. # 35

Probabilistic Methods in Earthquake Engineering-4

In the previous lecture, we considered the modeling of earthquake ground accelerations at any point in terms of three components. And, we saw that there exist three axes along which the three components can be approximated has being uncorrelated and that facilitates the formulation of the problem.

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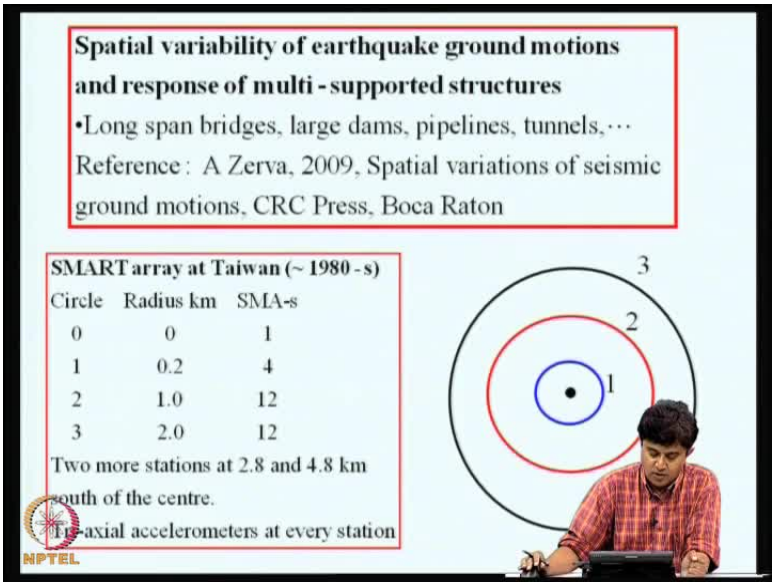

We also briefly saw some issues associated with spatial variability of earthquake ground motions. And, we summarized the major phenomenal-logical features that are associated with spatial variability of earthquake ground motions. So, as I mentioned, the data for making such models have become available since 1980s, where dense strong motion seismic arrays have been established at a few places on the earth's crust. One of that is in Taiwan. And, in a region of radius of about 2 kilometers, there are about 30 strong motion oscillograms distributed along three concentric circles. And, it is this array has

recorded a few events and that data has formed the basis for developing models for strong ground motion, which takes into account spatial variability features.

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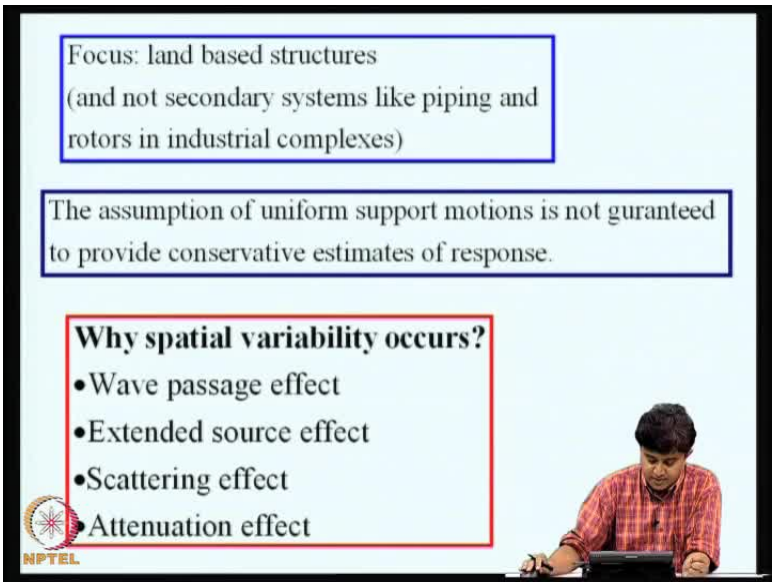

So, we are discussing in this part of the lecture – our focus will be on land-based structures. There are multiply supported structures like piping in industrial complexes, where the piping structures are supported on the primary system at different say points along the height. And, in the event of an earthquake, these piping structures are again subjected to differential support motions. We will briefly touch upon this problem, but our discussion will be mainly focused on land based structures like bridges, dams and so on and so forth. For this class of problems, it has been found through studies in the existing literature that the assumption of uniform support motion is not guaranteed to provide conservative estimates of the response. It could be conservative or it may not be conservative.

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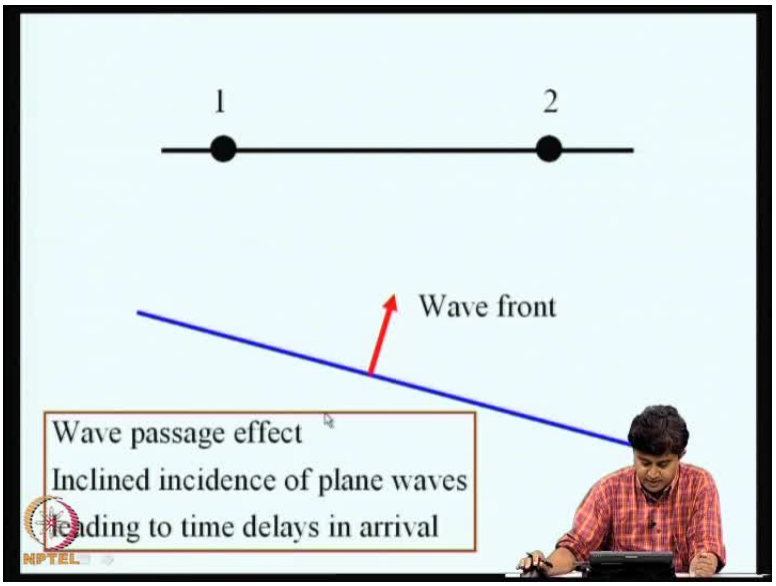

In the last lecture, we also saw why spatial variability occurs and we considered four different effects. The first one was wave passage effect, where a wave front reaches to recording stations at different times, because the wave front is inclined to the plane of the ground. That leads to delays in arrival times.

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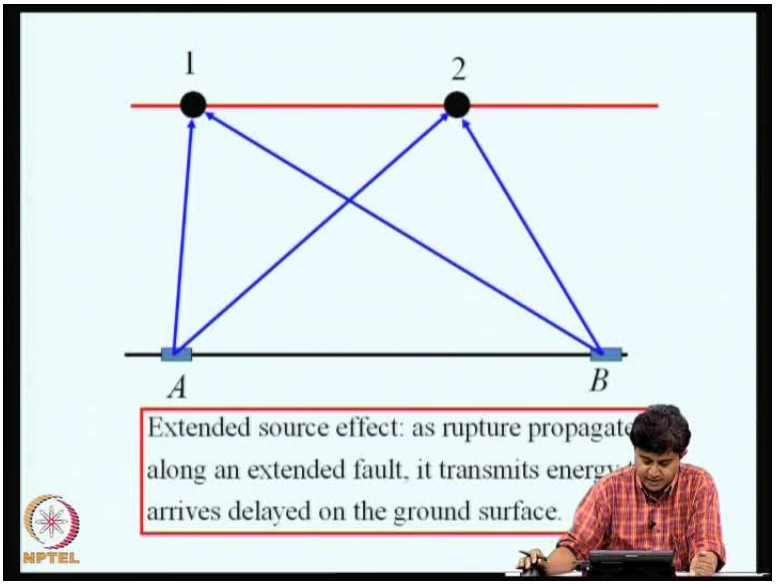

And, the other one was the extended source effect, because energy gets released along a fault line. The source of energy is an extended line and consequently the energy is

released in packets along this line at different time instance. And, the effect of that is felt at stations 1 and 2 at different times. And, this effect is known as extended source effect.

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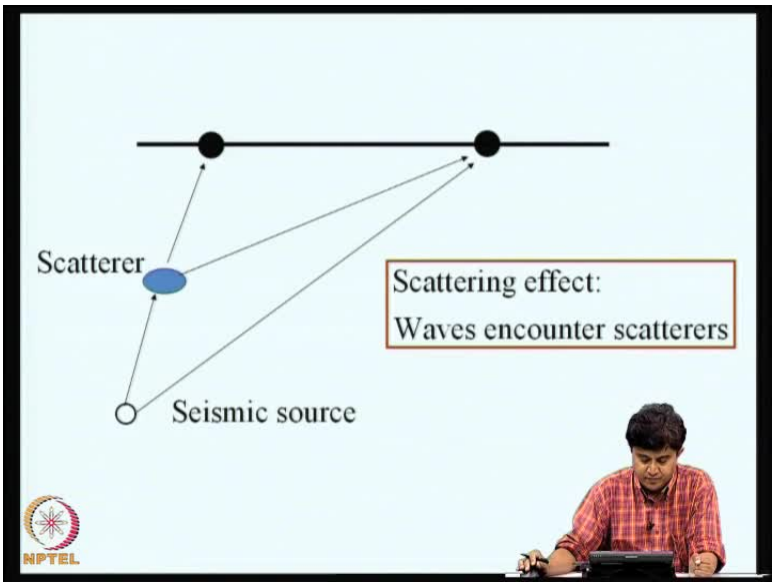

And, the waves propagate through inhomogeneous medium. They will be scattered and that also induces certain variability in the ground accelerations.

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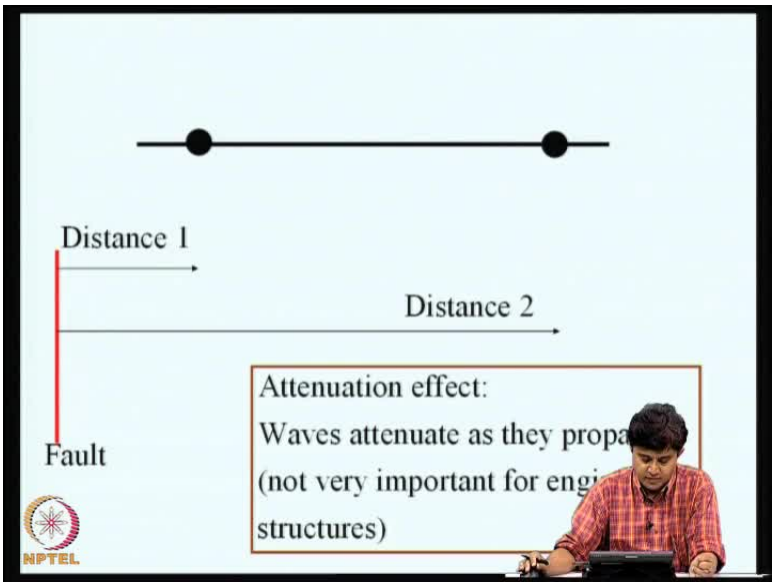

And, attenuation effect refers to the decay of the waves as it travels through the earth medium. And, if the distance through which the waves travel changes the attenuation effect also changes. And, this perhaps is not that very important for land-based

engineering structures. So, this effect is not that crucial compared to the other three effects.

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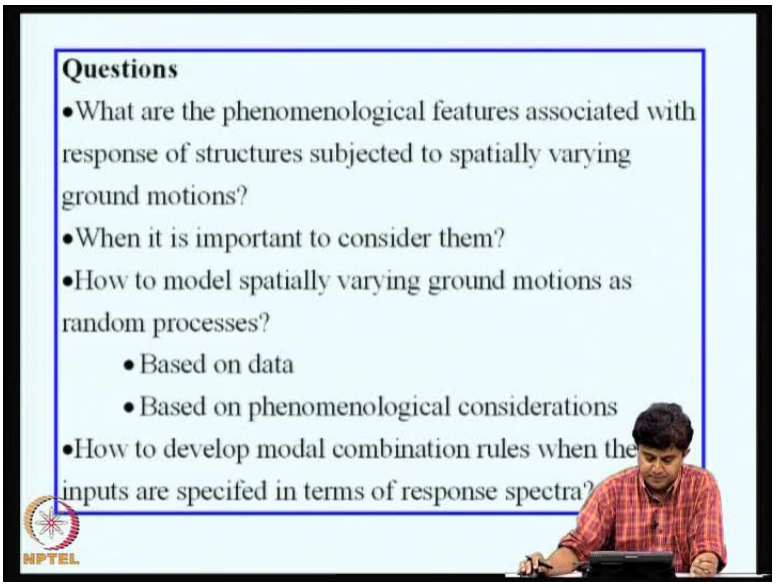

So, the questions that we wish to ask now are – what are the phenomenological features associated with response of structures subjected to spatially varying ground motions? When is it important to consider these effects? And, how to model spatially varying ground motions as random processes? And, how to develop these models based on data and based on phenomenological considerations? And, how to develop model combination rules when the inputs are specified in terms of a set of response spectra?

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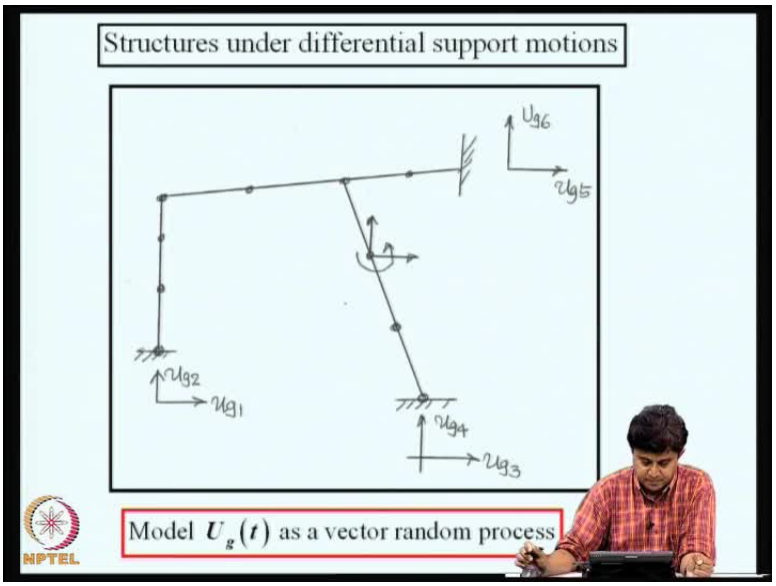

Let us consider some of these questions. Now, if you consider an engineering structure schematically shown here, this structure has three supports and it is subjected to… say it is a planar structure. So, there are six components of ground motion: u g 1, u g 2, u g 3, u g 4, u g 5, u g 6. Now, we can model these six components as a vector random process. And, our objective is to characterize the response of the structure when these six components are mutually correlated random processes.

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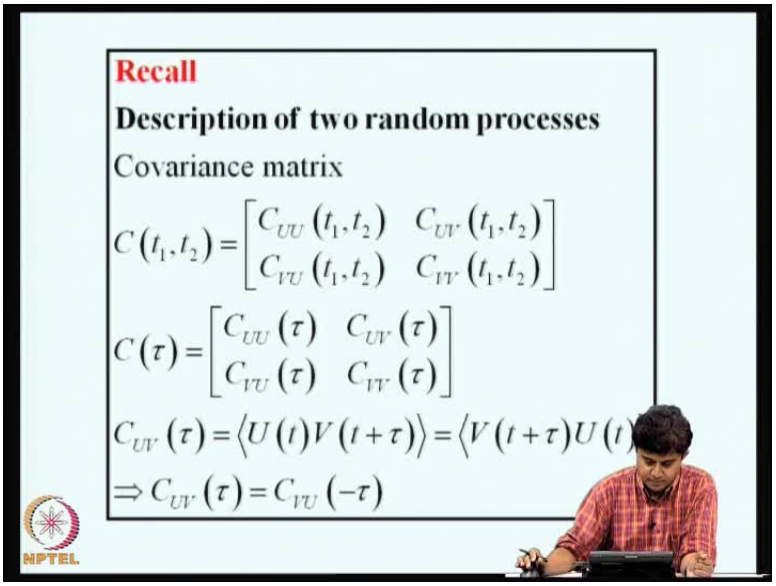

Now, we can quickly recall how do we describe two random processes. Suppose we have random processes u of t and v of t, we can define their covariance matrix. It is given by C UU of t 1 comma t 2, C UV of t 1 comma t 2 and so on and so forth. And, if process is stationary, we get this as C UU tau, C UV tau, C VU tau and C VV tau. And, C UV of tau is given by U of t into V of t plus tau. This is same as V of t plus tau into U of t. Therefore, C UV of tau is C VU of minus tau.

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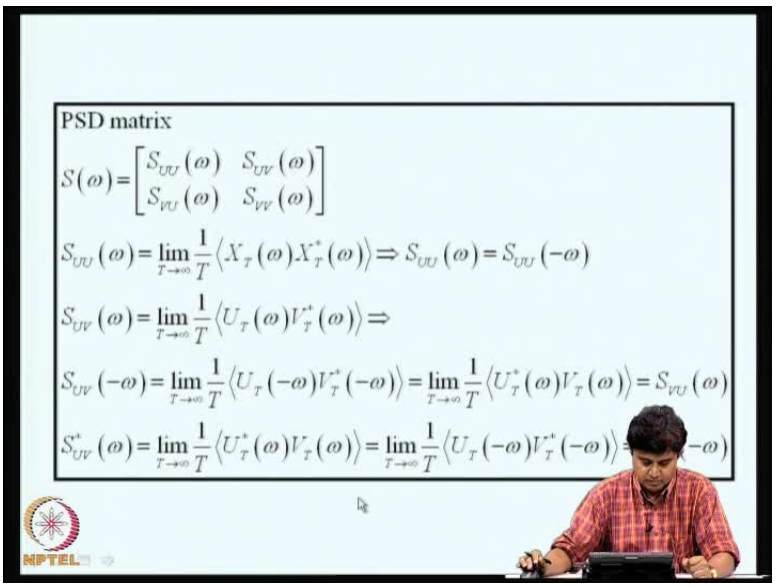

The associated power spectral density function matrix has diagonal terms, which are the auto power spectral density function and the cross terms, which are the cross power spectral density function. The auto power spectral density functions are real valued; whereas, the cross power spectral density functions are complex valued. And, the definitions are shown here. This we have discussed in some of the earlier lectures. And, the power spectral density function has certain properties like S U V of minus omega is same as S V U of omega; and S U V conjugate of omega is S U V of minus omega.

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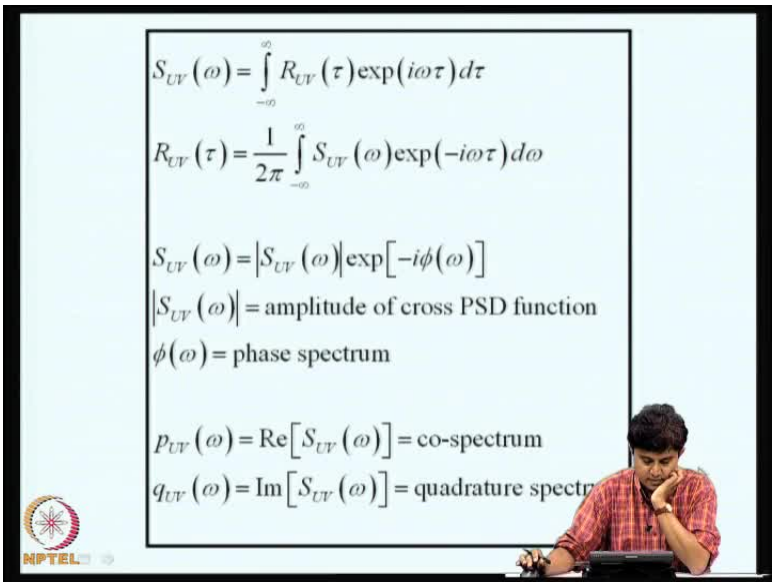

So, this is the definition of the relationship between cross power spectral density function and cross correlation function or the cross covariance function. We write the cross power spectral density function in terms of an amplitude and a phase; and, this quantity is the amplitude of cross PSD function and this phi is the phase spectrum. We call… The real part S U V of omega can also be expressed in terms of real and imaginary parts; the real part is called co spectrum; and, the imaginary part is called quadrature spectrum. So, we have amplitude and phase, co spectrum and quadrature spectrum.

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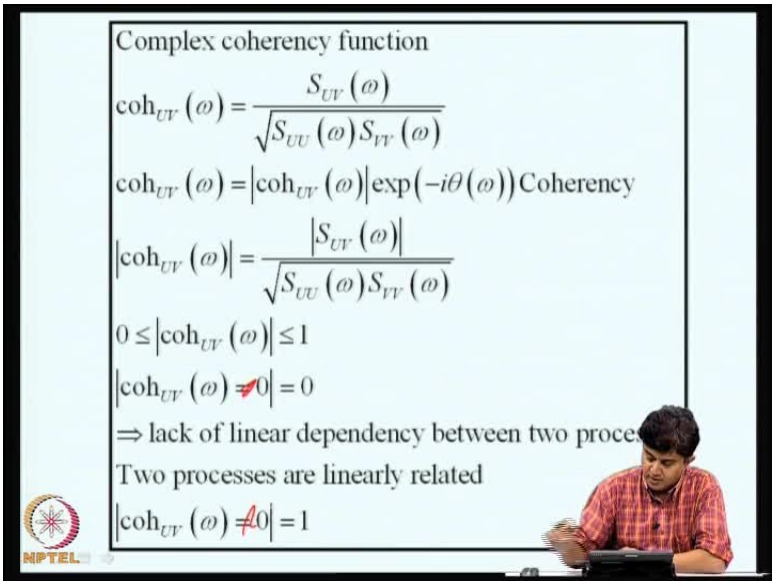

We define a quantity known as complex coherency function, which is the ratio of the cross power spectral density function to the square root of S UU of omega into S VV of omega. Here it is assumed that the denominator is not 0; if denominator is 0, this coherence function is taken to be 0. This itself is a complex valued function and we can write this again in terms of an amplitude and a phase. And, this quantity, the amplitude is known as coherency function. So, that is the ratio of modulus of S UV of omega to square root of S UU of omega into S VV of omega. We can show that the coherency function is bounded between 0 and 1. And, if coherence function is 0, it implies lack of linear dependency between two processes. And, if two processes are linearly related, the coherency function is 1.

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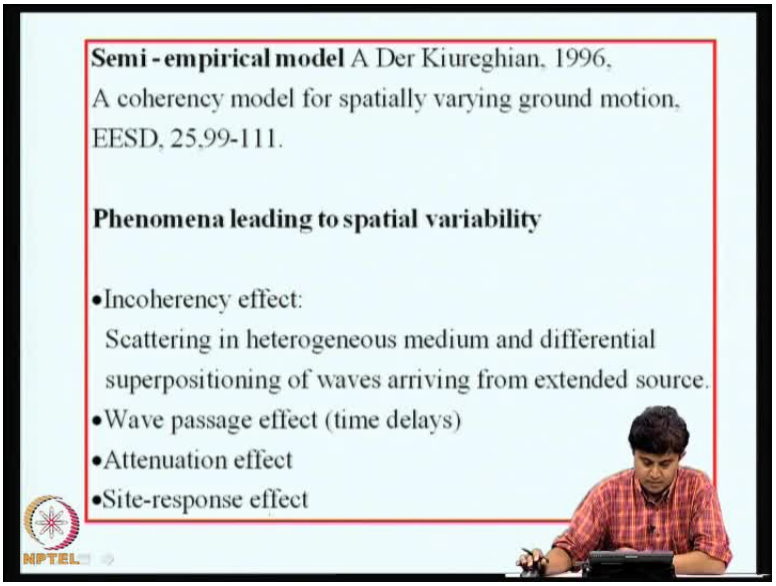

If we consider now for purpose of discussion, two points on the ground surface and we want to model the ground accelerations at these two points as pair of random processes; there are two components. We have already seen how to model the individual components through their auto power spectral density functions. So, our basic objective would be to model the cross power spectral density function. That essentially boils down to modeling of the coherency function. So, we will focus on modeling the coherency function. The coherency function captures the spatial variability characteristics.

Now, I am going to discuss a semi empirical model to start with, proposed by the Kiureghian. He considers four sources of phenomena that lead to spatial variability. And,

according to him, the first effect is known as the incoherency effect. This is caused due to scattering in heterogeneous medium and differential superpositioning of waves arriving from an extended source. The next one is wave passage effect, which causes time delays and the attenuation effect and the site response effect.

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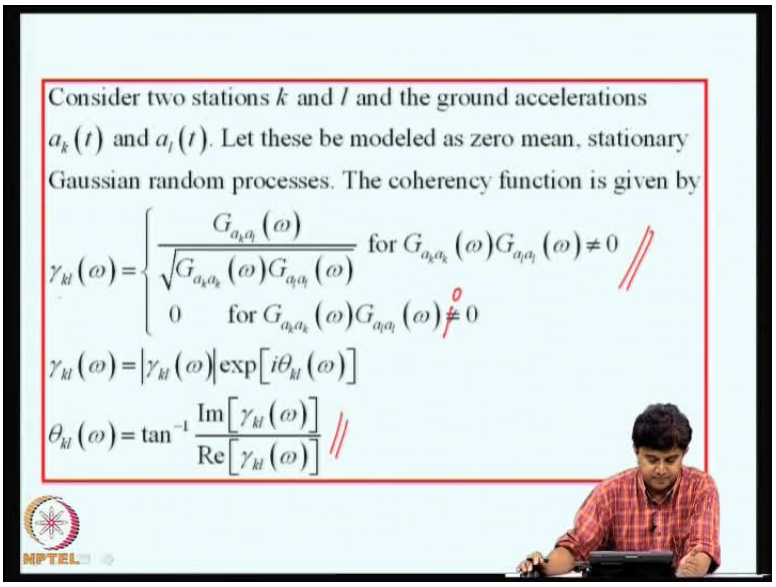

Now, if you consider two stations k and l and the ground accelerations a k of t and a l of t; that is, a k of t is a ground acceleration at station k; a l of t is ground acceleration at station l. We will model these two random processes as having zero mean and stationary Gaussian random processes. The coherency function is given by ratio of the cross power spectral density function. Here I am using capital G to denote the power spectral density function; G a k a l of omega divided by square root of G a k a k of omega G a l a l omega. So, this is when the denominator is not equal to 0; it is 0 when denominator is 0. The amplitude of coherency function – this is again a complex function. So, I can define its amplitude and phase. The amplitude is modulus of gamma k l of omega. And, we introduced a phase, theta k l of omega, which is related to real and imaginary parts of coherency function through this relation. Our objective in modeling would be to arrive at models for coherency function.

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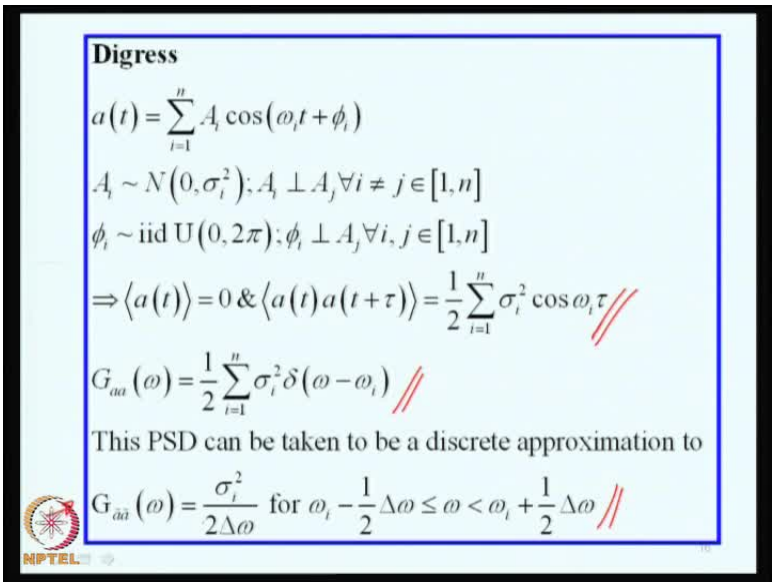

Now we will digress slightly. This is something that we have already seen, but we quickly recall. Suppose I write a of t is a random process and I write it as i equal to 1 to n; A i cos of omega i t plus phi i. These A i's are taken to be normal with 0 mean and variance sigma i square. And, A i and A j are independent for every i not equal to j. And, this phi i are taken to be i i d sequence of random variables distributed uniformly in 0 to 2 pi. And, this phi i and A j are taken to be independent. So, under these assumptions, we can show that the expected value of a of t would be 0 and the covariance of a of t will be given by this function (Refer Slide Time: 11:16). So, this is a mean square periodic random process. And, its Fourier transform is given by this. It is a sequence of direct delta functions stationed at omega i's.

Now, this power spectral density function itself can be taken to be a discrete approximation to a continuous power spectral density, which is given here (Refer Slide Time: 11:36). Therefore, a of t can be taken to be an approximation. This representation can be taken to be an approximation for generating samples from this power spectral density.

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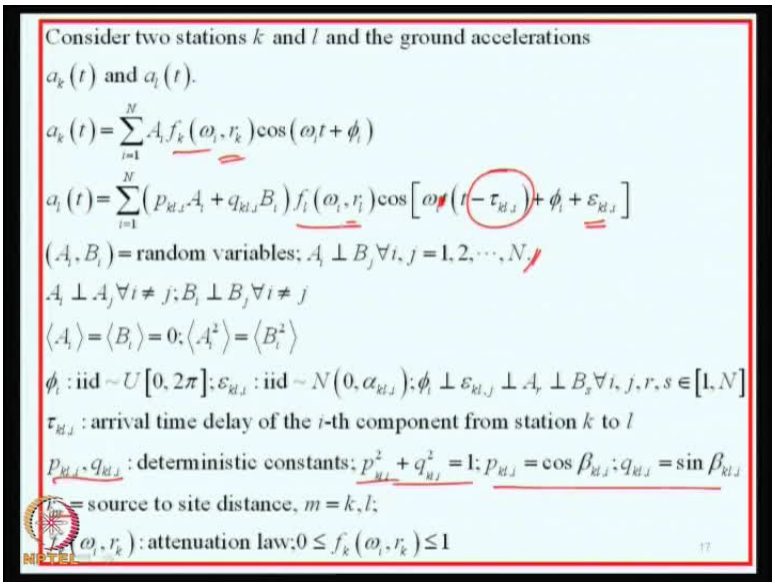

Now, we will return to the discussion on ground accelerations. We will consider two stations: k and l. And, the ground acceleration: a k of t and a l of t. At a k of t, I will write the representation as i equal to 1 to N A i f k omega i r k cos omega i t plus phi i. And, at a l of t, this wave will now get modified due to the four effects that I mentioned. I am not going to get into the physics of those arguments, but I just would like to highlight the nature of the model. This amplitude now gets modified. It is no longer A i, but it is some P k l i A i plus q k l i B i. And, this function – I will explain what are these f k and f l; they are different. And, this cos omega i t – now, there is a time delay, t minus tau k l; k l refers to stations k and l. And, i subscripts indicates that these time delays are functions of frequencies.

Now, this phase also (Refer Slide Time: 12:56) gets modified by an additional random term. So, now, A i, A B are random variables. A i is independent of B j for all i and j; A i is independent of A j for all i not equal to j; B i is uncorrelated with B j for i not equal to j. They have 0 mean; their variances are equal. And, phi i's are i i d's, and epsilon k l i are i i d's; and, phi i is independent of epsilon, independent of a r and b. So, all these random variables are mutually independent. This tau k l, which is appearing here is arrival time delay for the i th component from station k to l. These p k l and q k l are deterministic constants, which need to satisfy this condition. And, they are represented in terms of sine and cosine functions having these angles beta k l comma i. This r m (Refer Slide Time: 13:52) – where does is it appear? That is r k and r l for m equal to k and l;

they appear here. They are sourced to site distances; and, this f k that is sitting here is the attenuation law from source to the site.

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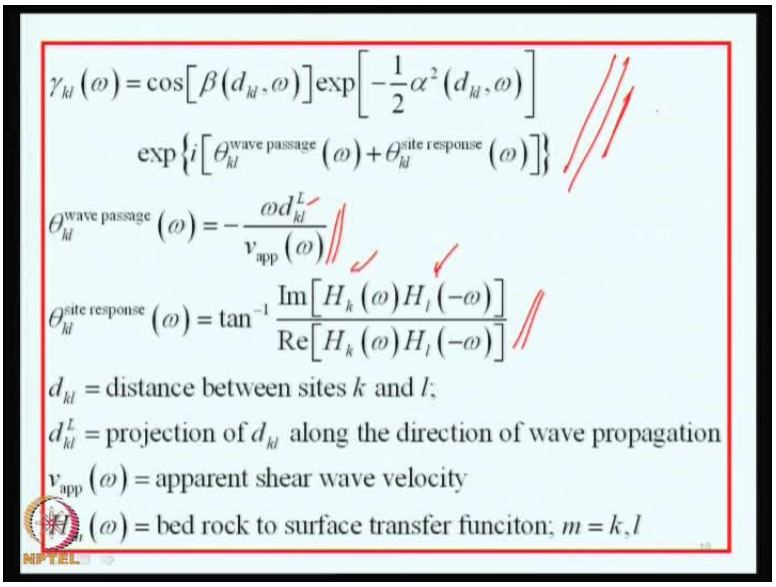

Now, if this representation is taken to be acceptable, it is constructed based on various arguments; I have summarized it here, the salient features. If we accept this as a model, then we can show that the coherency function associated with this model is given by this expression. There are two components in the phase: one is due to phase-wave passage; other one is due to site response. This is the time delay (Refer Slide Time: 14:47) actually due to wave passage effect; and, this is due to the wave propagation in the local (( )) conditions. This d k l is the distance between sites k and l; d k l of l, which appears in the definition of wave passage effect here, is a projection of d k l along the direction of wave propagation. Nu a p p is apparent shear wave velocity. This H there, which are appearing here (Refer Slide Time: 15:13) are the transfer function from bed rock to the ground surface; it is like (( )) transfer functions. So, this is a model for coherency function.

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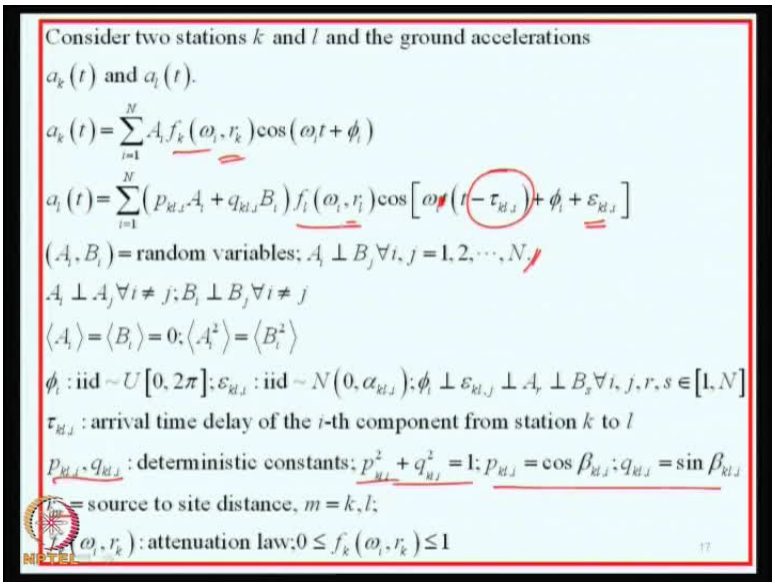

This is based on… As I said, it is somewhat semi empirical different effects that we expect in modeling ground accelerations are explicitly identified. And, a model is constructed deliberately to allow for these effects (Refer Slide Time: 15:45).

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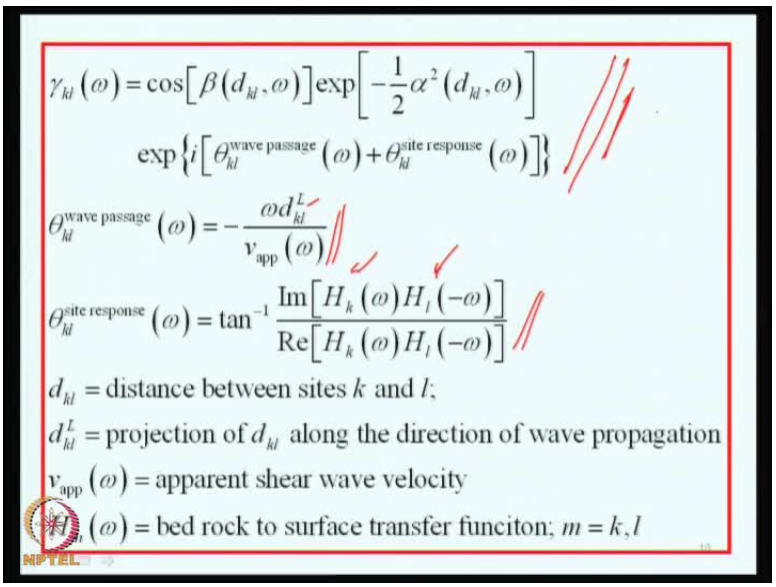

The model parameters here like this (Refer Slide Time: 15:49) beta k l, etcetera are to be now calibrated against instrumented records and they need to be identified if you have to use this model.

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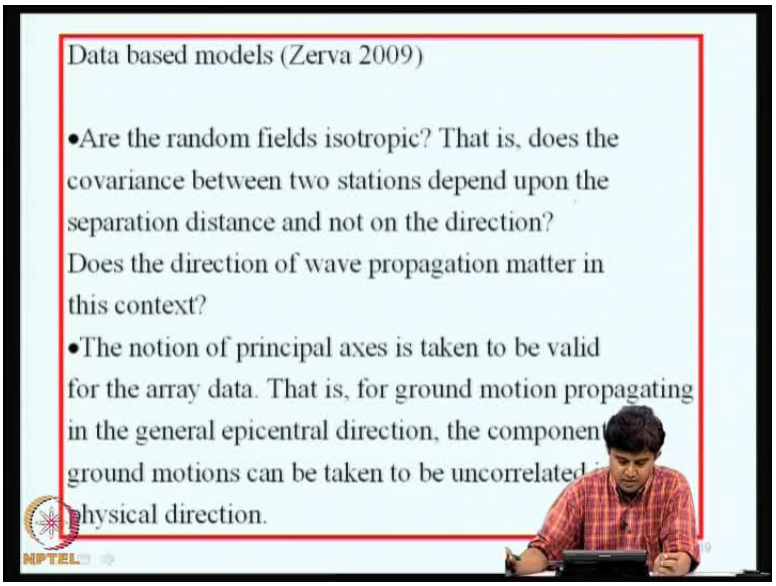

There are actually in the existing literature models, which have been directly made from the observed random fields, that is, purely database models; now, some of the questions that are considered in analyzing these data are following. Are the random fields isotropic? That is, does the covariance between two stations depend upon the separation distance and not on the direction? That means any direction you take in the domain of interest; as long as you are taking two stations, which are at the same distance separated by same distance, their covariance will be the same. Then, the random field is taken to be isotropic. But, given the way the seismic waves propagate, there is a directionality effect and these fields may not be isotropic.

Then, at any point within the domain, the excitations are having three translation components. So, at different points, you can identify principle axes for excitation in different points. But, given that the wave front is propagating roughly in the same direction for the entire region, one could assume that the principle axes for the three translation components at different points have the same principle axes; I mean they have the same principle axes. That is another assumption that we can make.

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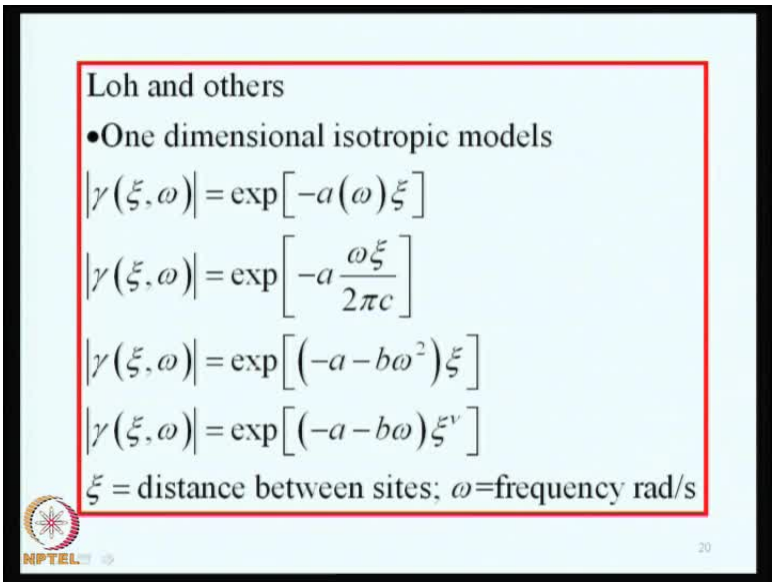

So, based on this, several authors have suggested models. And, I will just run through some of them. One of these models is one dimensional isotropic models, where the amplitude of coherency function is given as exponential minus a omega into x i; where x i is the distance between the sites. There are various forms of this exponent that are suggested.

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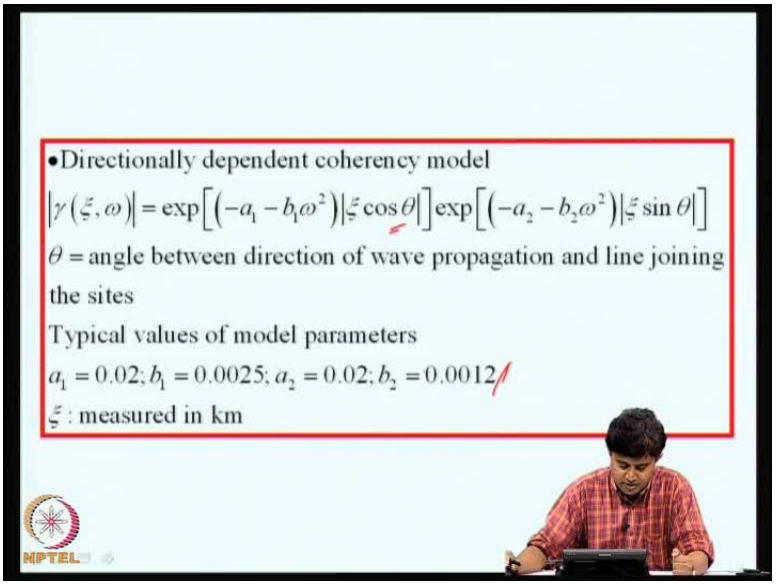

And, for instance, a directionality… This is isotropic (Refer Slide Time: 17:54). And then, subsequently, there is a model where directionality effect is introduced by

considering this angle theta, which is angle between direction of wave propagation and line joining the sites. Now, typical values for these model parameters like a 1, b 1, a 2, b 2, etcetera when psi is measured in kilometers are shown here. This is just for illustration. And, these numbers are arrived based on from actual instrumentally recorded data.

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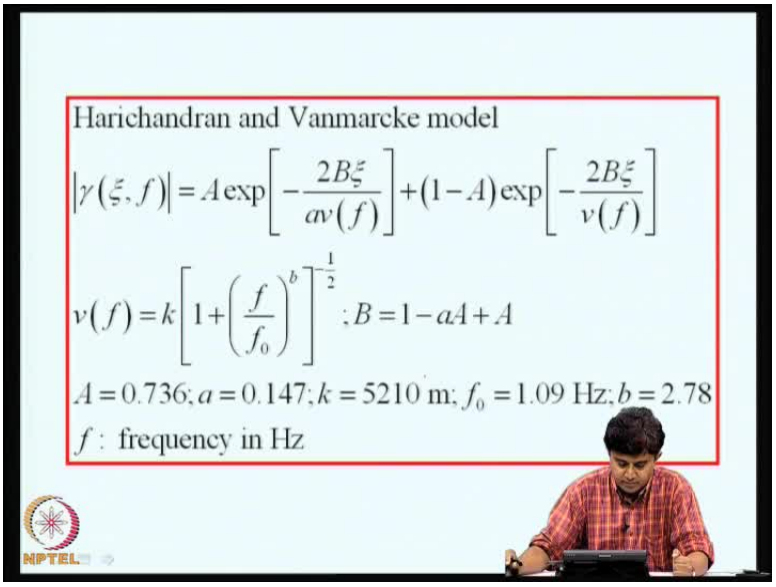

This Harichandran and Vanmarcke model has this expression, A exponential and this exponent; plus 1 minus A exponential another exponent. Now, this nu of f has this functional form; B has this functional form. And, these are some of the constants that are obtained for one event from the Taiwan smart array data.

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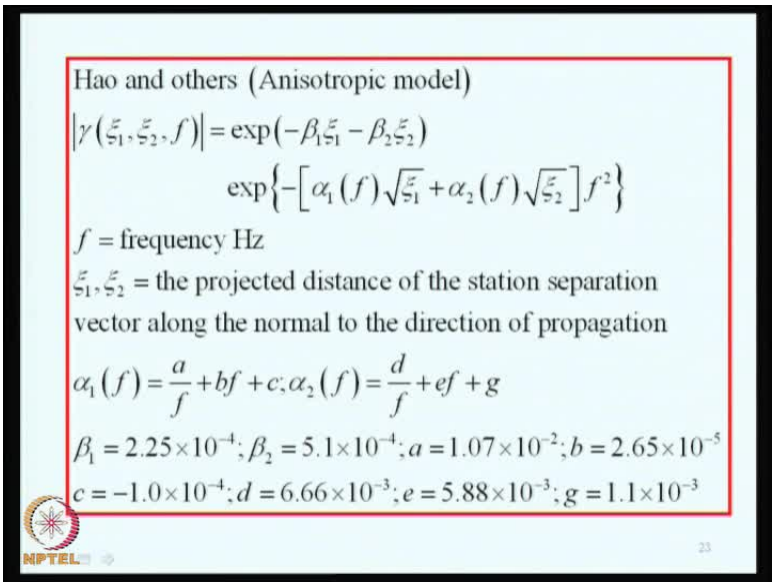

So, there are other models. For example, Hao and others models, where these are anisotropic random field models, where psi 1 and psi 2 are actually the projected distance of the station separation vector along the normal to the direction of wave propagation. So, there are various model parameters here. And, some typical numbers I have included for illustration.

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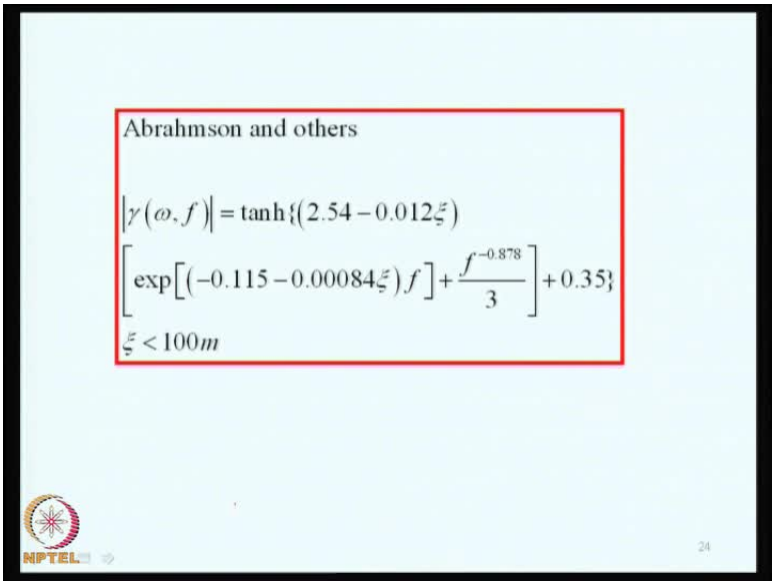

So, this is yet another model for short distances. When separation distance is less than 100 meters, there is another model. So, these are all empirical models, which are fitted to observe data.

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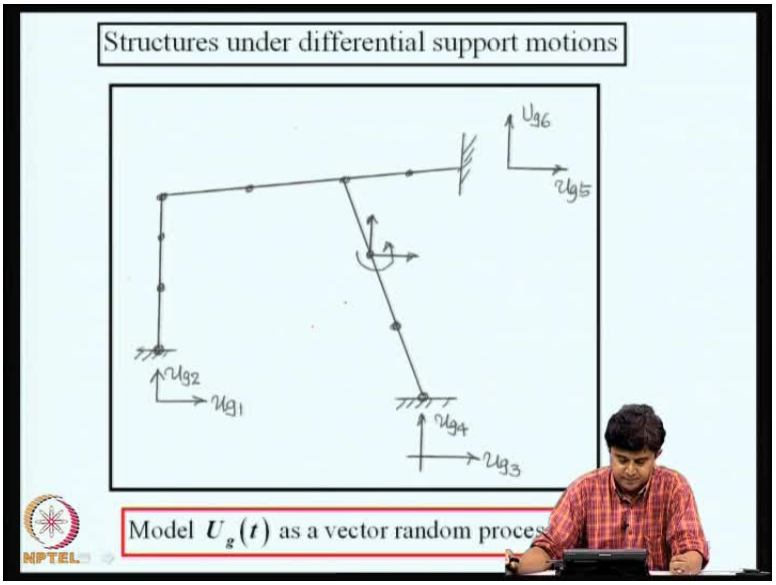

Now, the point that we can conclude at this stage is based on data; or, based on combination of data and certain phenomenological arguments, we can construct the coherency models for the spatial variability characteristics of ground accelerations. So, once we reach that conclusion, the next question that we need to consider is how do we model the structural behavior under differential support motions especially when the support motions are modeled as vector of random processes. So, this is the view graph that we saw a while before.

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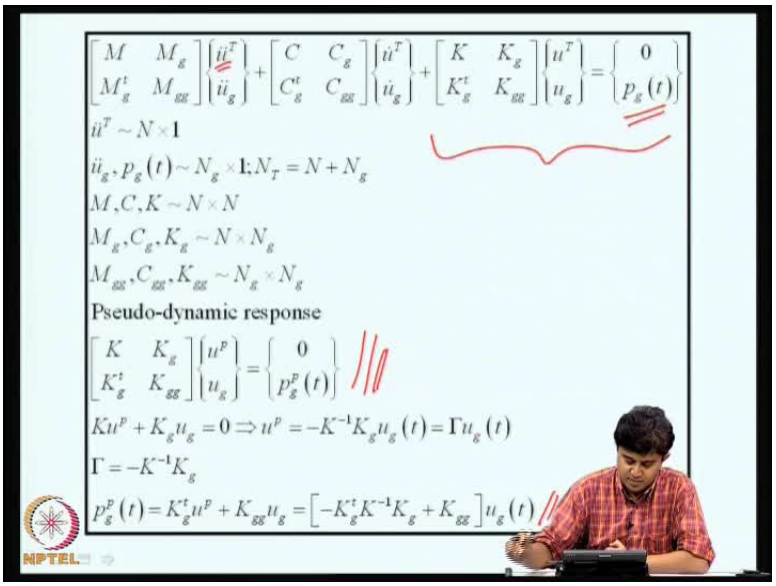

And, the associated governing equations can be written; this we have seen earlier in our discussions. The superscript capital T indicates total displacement, absolute displacement. And, there are N nodes; the size of u T is N and u g is the ground degrees of freedom; that is, applied ground displacements, velocities and accelerations. So, the structural matrices are partitioned as per this partitioning of the nodal displacements. And, what are unknown here are this u T; u g double dot is given. And, P g of t is a reaction transferred to the supports. So, we have, the size of this u g is I think M cross N g cross 1. So, there is N plus N g number of equations with capital N number of unknowns, which are u t; and, N g number of unknowns, which are the reactions. So, we can write two sets of equation: one for the displacements and other for the reactions. So, that is what we should be able to do here.

Now, in analyzing multiply supported structures under differential support motions, we have already seen that there is what is known as pseudo-dynamic response component. That means without the inertial and dissipation effects coming into play, there will be stresses in the structure due to the differential support motions and that is known as pseudo-dynamic response. And, that we obtained by considering the equation by omitting the inertial and dissipation effects; and, we get this equation (Refer Slide Time: 21:58). We call this response as u p. And, we can solve for u p in terms of applied support displacements. And, we call this matrix minus K inverse K g as capital gamma and this is called influence matrix. Now, based on that, the reaction transferred due to

pseudo-dynamic action can also be evaluated. If u g of t is a random process, we can evaluate properties of u p by using… This is a linear transformations on… Say for example, Gaussian random processes, we can characterize u p.

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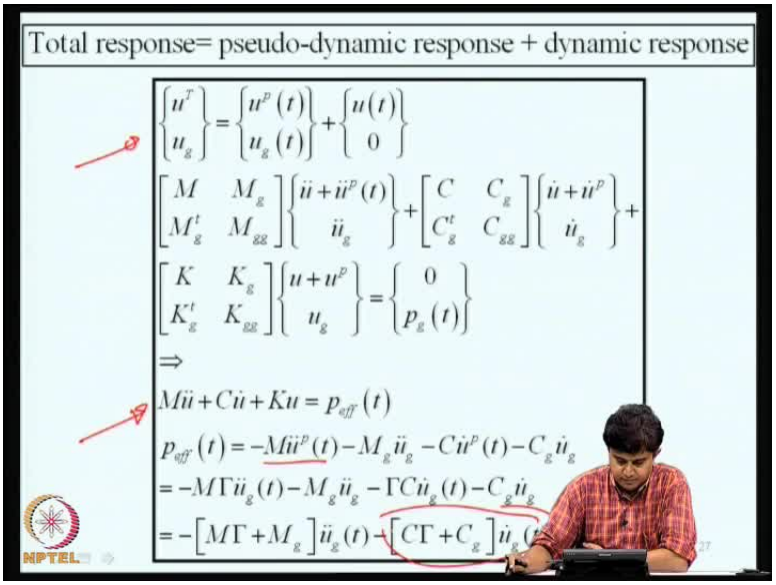

Now, what we do is, we write the total response to be the sum of pseudo-dynamic response plus the dynamic response. That would mean we will write the displacement vector u T u g as u p, u g plus u and 0. And, once we substitute into this governing equation and take into account the fact that u p satisfies certain equilibrium equations, we can show that the equation governing the dynamic component has this form, M u double dot plus C u dot plus K u is equal to an effective force t; where, this effective force is given by this expression (Refer Slide Time: 23:07). And, most often, we assume that the effect due to the inertial effect due to the ground acceleration far exceeds the effects due to the dissipation characteristics; and, this second set of terms is often ignored.

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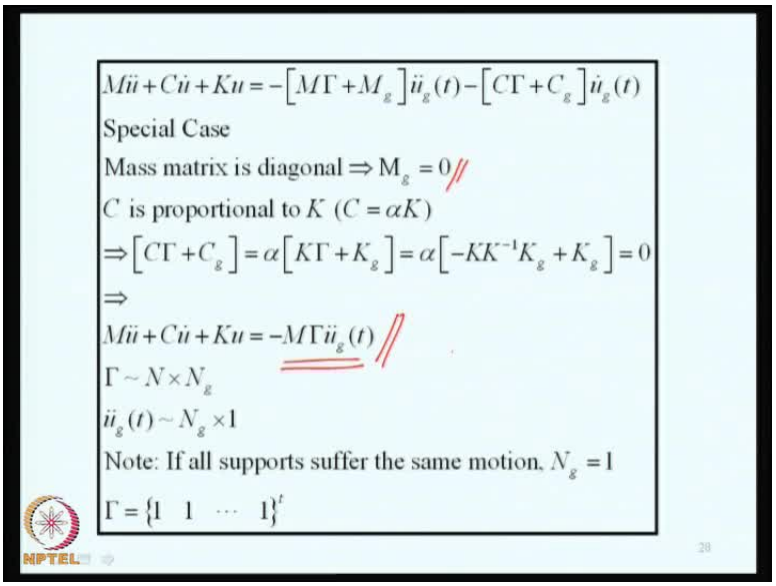

Now, under various conditions, either by… Actually if mass matrix is diagonal, M g would be 0. And, if C is proportional to stiffness matrix, we can show that the effective force is exactly given by this. But, in other situation also, we generally assume that the effect of inertial actions predominates and we ignore the terms that involved damping on the right-hand side.

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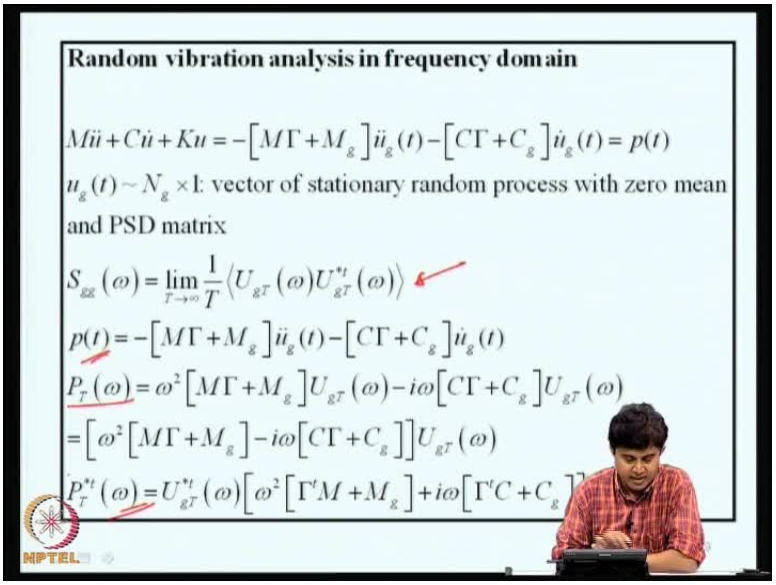

Now, how do we perform random vibration analysis for this problem? Now, this is reasonably straightforward; we have discussed this already. We need the power spectral

density function of the ground displacement and for this effective force. And, p of t is given in terms of u g double dot and u g dot.

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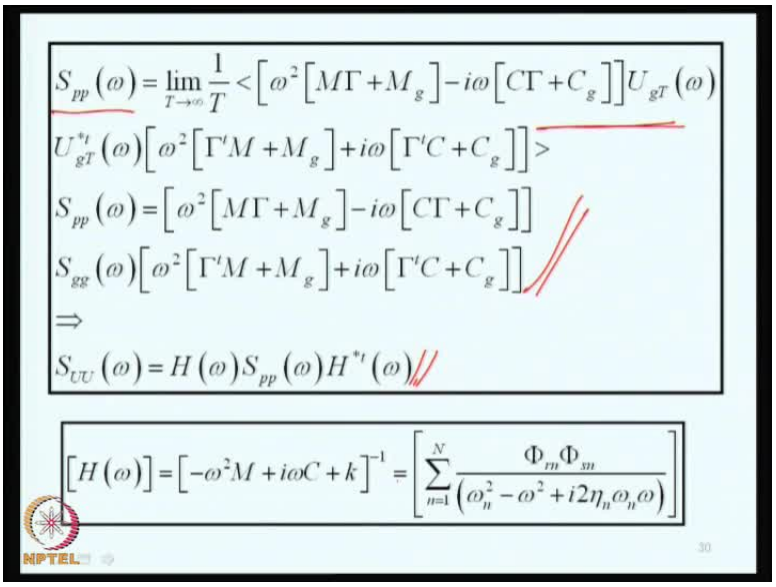

And, we write the Fourier transform of this and the conjugate of this; multiply and take expectation, etcetera. That is, this as shown here. And, we get the power spectral density of the effective force as shown here. And, once we get that, the power spectral density function of the displacement vector can be determined in terms of the matrix of system transfer functions; where, H of omega is minus omega square M plus i omega C plus k inverse. That itself can also be expressed in terms of normal mode. So, all these we have seen in one of the earlier lectures.

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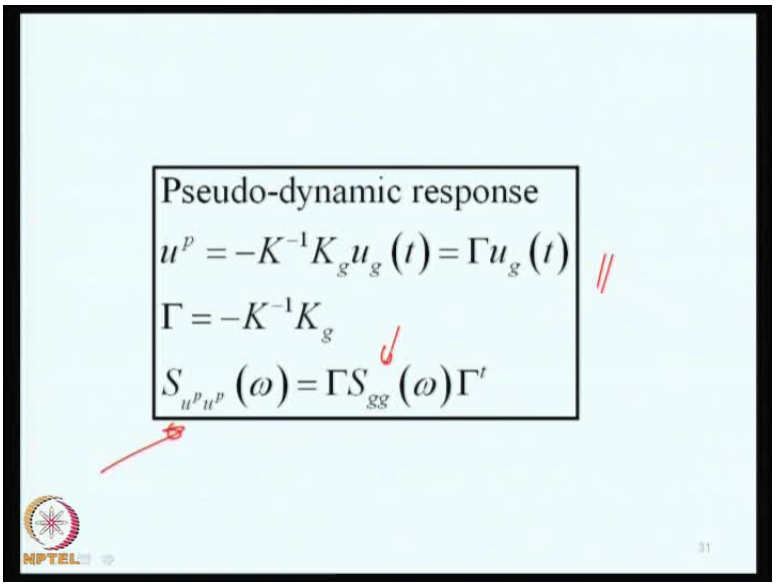

Now, pseudo-dynamic response can be determined using this. And, the power spectral density of that can be found out in terms of the influence matrix. Here of course, we need the power spectral density of the displacement.

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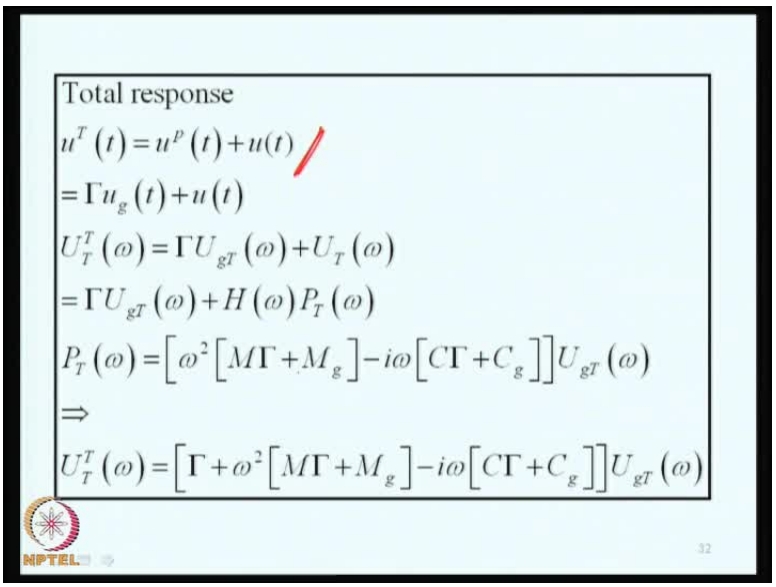

The total response is in terms of pseudo-dynamic component and dynamic component.

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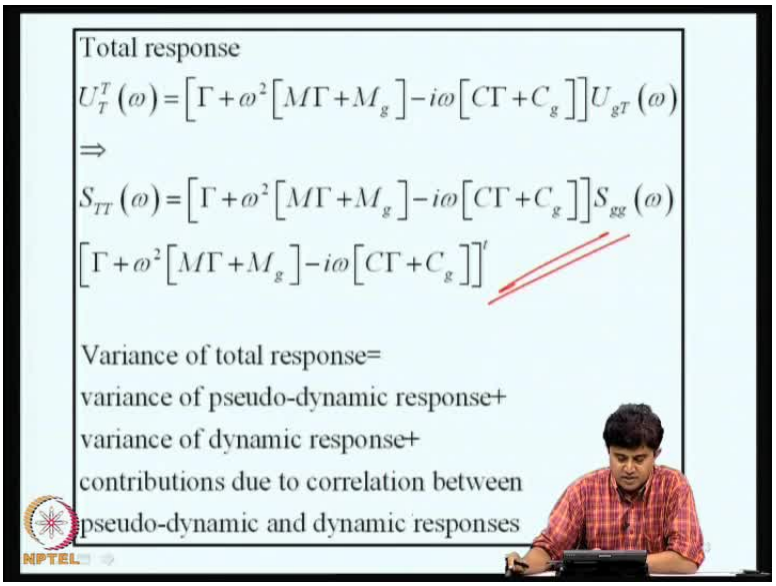

And, again the power spectral density function of the total response can also be derived by manipulating these expressions. And, we get this to be the power spectral density of the total response. The variance of the response can be found out now by integrating of frequency domain. This has a variance due to pseudo-dynamic component, variance due to dynamic component and contributions due to correlation between pseudo-dynamic and dynamic responses. So, these are the characteristic features associated with structures subjected to differential support motions.

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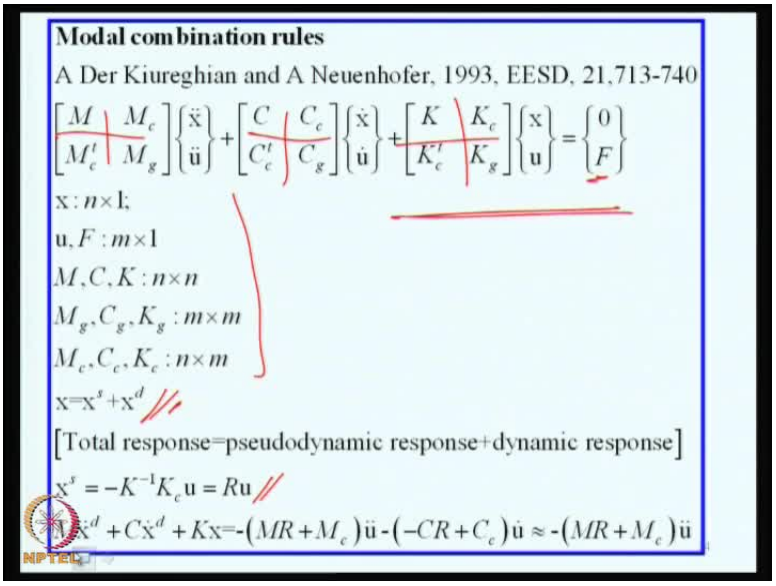

So, at this stage, what we have done is we have outlined how random process models can be made for spatially varying ground motions and how we can formulate the equations of motion. If we assume that inputs are stationary, we can perform analysis in frequency domain and we can get the power spectral density function of the response quantities of interest. The response here consists of a pseudo-dynamic component, a dynamic component; and, sum of that is a total response.

Now we will consider the question how to determine the response if the inputs are specified not in terms of power spectral density function, but in terms of response spectra and coherency functions? If that information is given to us, how to use that and get the highest responses as is implied in the use of response spectrum base methods? So, here we consider the equation of motion in a slightly different notation. So, we call by x the total displacement, u as a support displacement. The form of the equation is quiet similar to what I discussed just now. This form confirms to the one that is used by Der Kiureghian and A Neuenhofer in the paper in Earthquake Engineering and Structural Dynamics.

So, these are the various sizes of these matrices. Again, we partition the nodal displacement vector into x and u and that induces a partition on the structural matrices of stiffness, mass, etcetera. And, here unknowns are x and this reaction F of t. We decompose a total response into pseudo-dynamic response and dynamic response. And, pseudo-dynamic response – we obtain by solving the static part of the equation involving only the stiffness terms and we get x to the power s as R into u; where, u is a support displacement. Then, we substitute that into the governing equation. This equation (Refer Slide Time: 27:59) into that and obtain the equation for the dynamic component.

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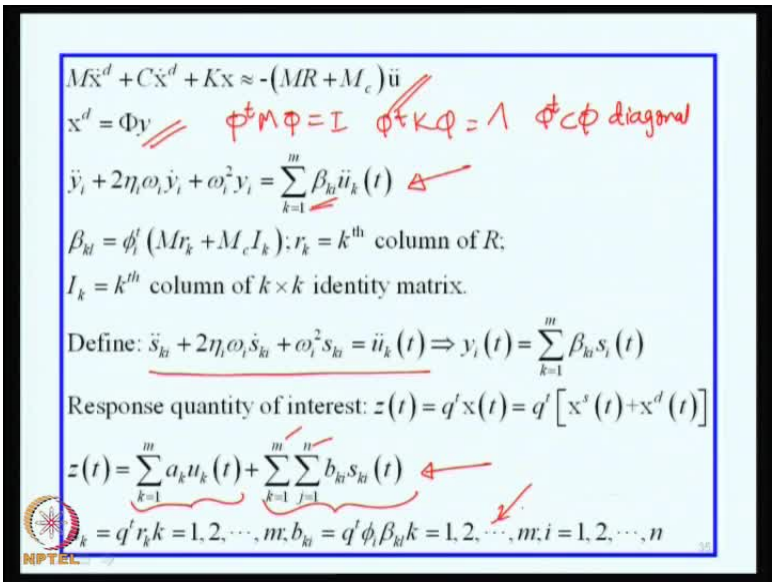

So, this we approximate. We ignore the terms involving damping on the right-hand side. And, this is now in a form that we can apply the model decomposition method to analyze the problem. So, the response is represented as phi into y; where, phi is the matrix of eigenvectors. So, phi transpose M phi is taken to be diagonal and phi transpose K phi is this (Refer Slide Time: 28:42) – diagonal; and, phi transpose C phi is again diagonal. So, based on that, we get now an equation for the generalized coordinates – i th generalized coordinates, y i double dot plus 2 eta i omega i y i dot plus omega i square y i. On the right-hand side, we get contributions due to the M distinct support motions. So, each mode responds to each of the sub components of the support accelerations. And, beta k i is a participation factor for the i th mode corresponding to the k th element in the support displacement vector. So, we get…

To simplify this discussion, what we do is we define a quantity known as s k i, which is defined here; where, it is response of a quantity associated with the i th mode to the k th component of support acceleration without that participation factor. So, if this is accepted, then y i of t can be written as k equal to 1 to m B k i s i of t. Now, if z of t is response quantity of interest, which I write it as q transpose into x of t. That is, q transpose into x s plus x d. This again as I said, it could be interest oriented drift or reaction and so on and so forth.

I can write for z of t an expression that is shown here (Refer Slide Time: 30:19). There is a summation over support displacements. This is pseudo-dynamic response; this is a dynamic response. The pseudo-dynamic response has only static contributions from static behavior, but from m components of the support displacements. The dynamic component has contributions from n generalized coordinates and m support displacement components. So, there is a double summation. This is a generic form. We can relate this a k and b k i to the various quantities that we have introduced. That is a made explicit here.

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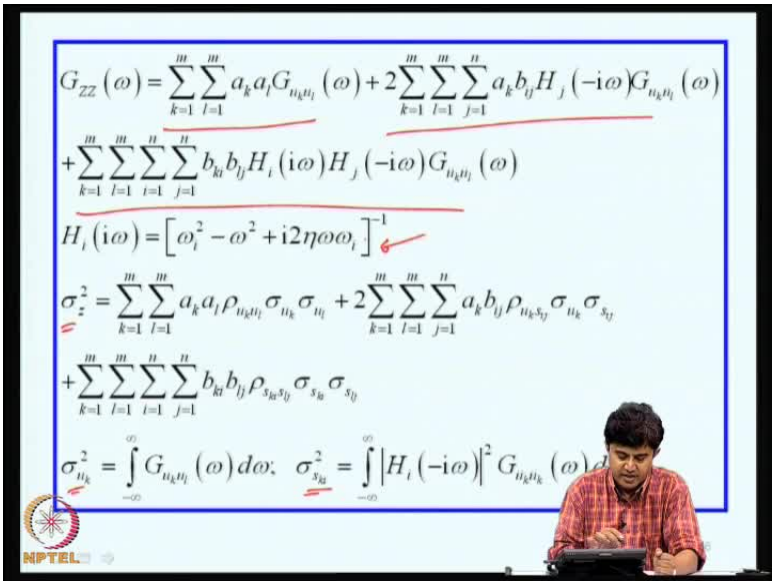

Now, the power spectral density function of z of t can now be derived. And, here there will be terms involving (Refer Slide Time: 31:10) only pseudo-dynamic component, only dynamic component, and cross correlation between dynamic and pseudo-dynamic components. So, I have the first term, (Refer Slide Time: 31:21) which is a double summation corresponding to the pseudo-dynamic component; the third term is the contribution from dynamic component. Single summation becomes double summation, because power spectral density function is a second order property. And, a double summation becomes a quadruple summation. And, the cross correlation is a triple summation.

This is reasonably straightforward. This (Refer Slide Time: 31:46) H of omega is a transfer function for the i th generalized coordinate. Now, area under this function is the

variance. Now, this variance itself is written in terms of variance of the generalized coordinates and this quantity s k i of t that we have introduced. This H i of omega is as shown here. So, this is a straightforward random vibration analysis made explicit for individual components and for a generic response quantity.

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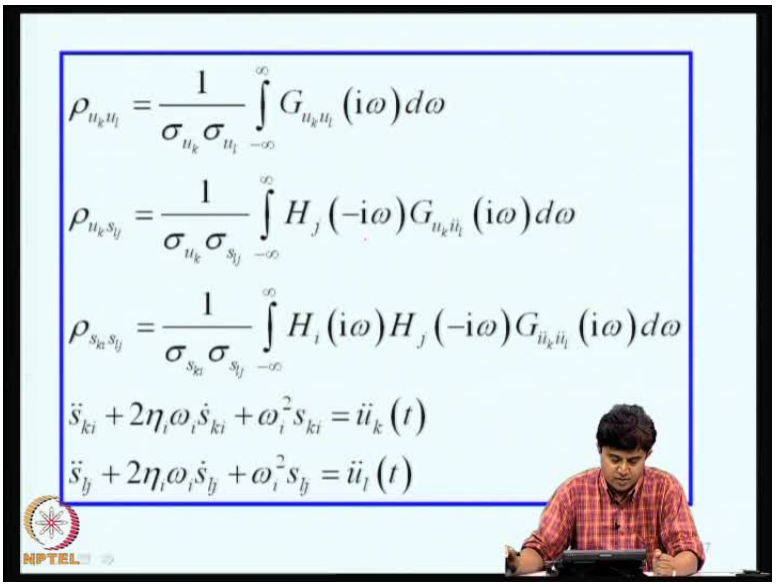

Now, I can normalize the cross correlation or cross coherence terms with respect to standard deviations. I introduce non-dimensional quantities, rho u k u l, rho u k s l j, rho s k i s l j; where, s k i and s l j just to emphasize are response of these two single (( )) systems. So, these are non-dimensional.

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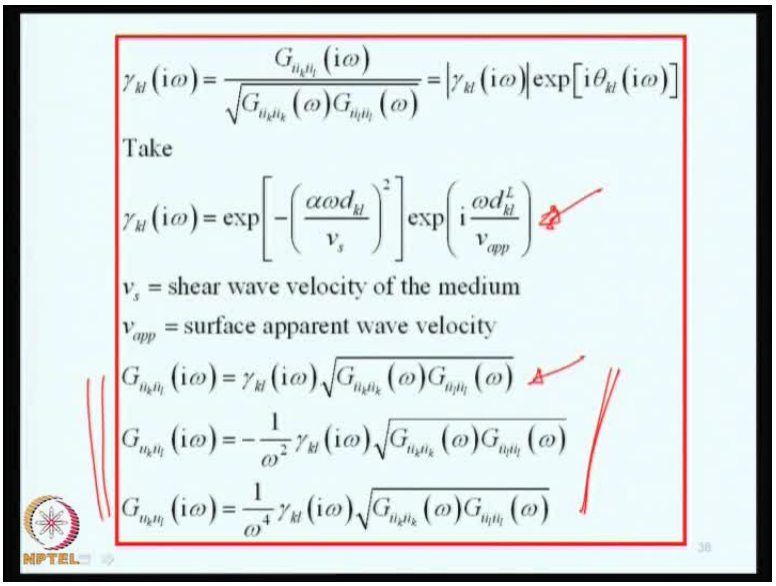

Now, I talk about the coherency function. So, for purpose of discussion, we can take the coherency function to be given by this. To apply the method that we are going to discuss, we need to have a model for coherency function. From this, we can get the model for…

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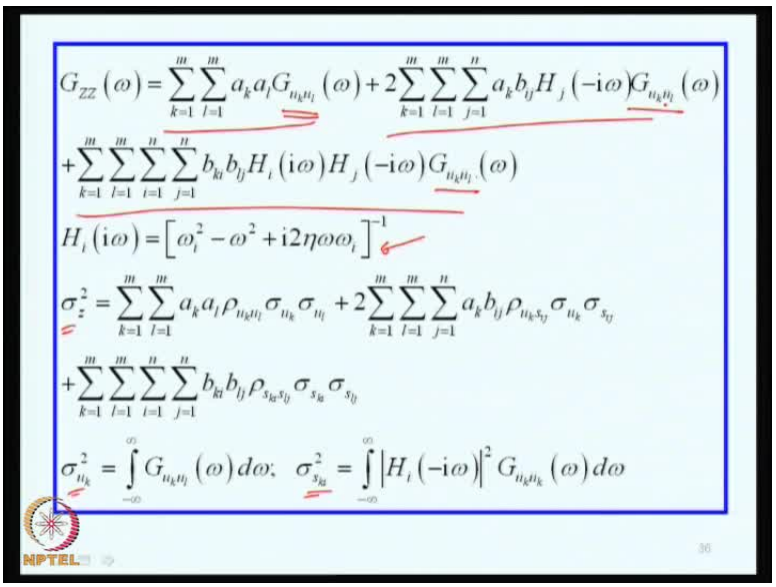

See if we look at the expressions that we are interested in computing, we need the cross power spectral density function between displacement components at k and l (Refer Slide Time: 33:13). The cross PSD between displacement component at k and acceleration at l. And of course, the cross PSD between acceleration at k and acceleration

l; k and l are the stations or the components in excitations. This is (Refer Slide Time: 33:36) coherency model for accelerations. So, from this, I have to derive the models for displacement and acceleration and displacements alone. And, this we use the standard definition (Refer Slide Time: 33:45) of power spectral density function; we divide by minus omega square and omega to the power of 4 to get the required functions as shown here (Refer Slide Time: 33:53).

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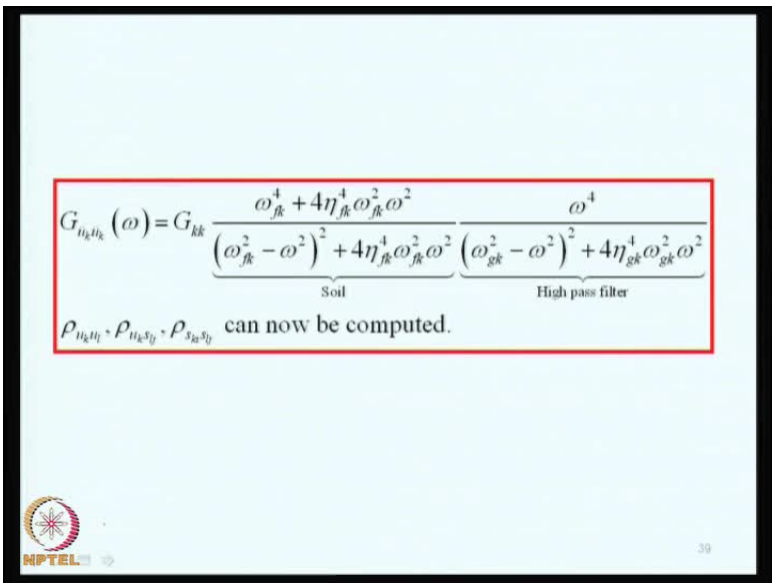

And, the auto power spectral density function itself is taken to be the (( )) type of power spectral density function. It could be anything else that we are ready to use.

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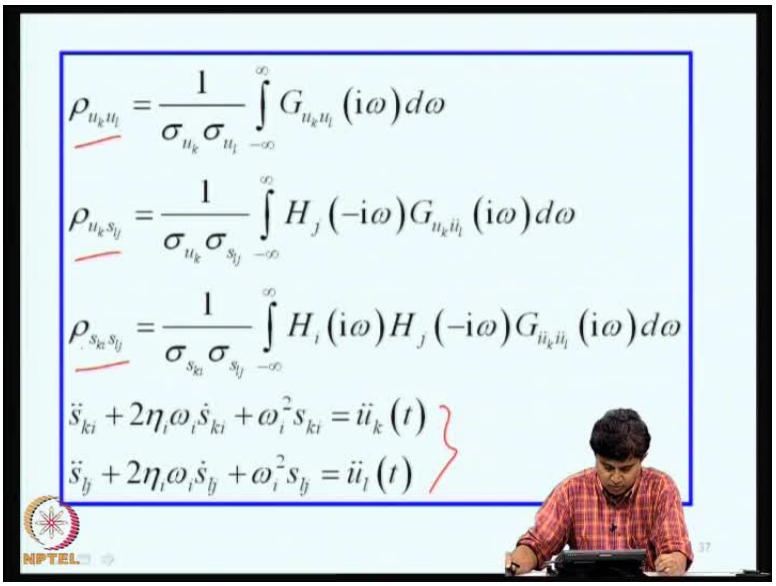

So, based on this, the quantities, these non-dimensional quantities can now be evaluated – this. So, everything that we need to evaluate these three quantities is now known. So, that completes our random evolution analysis.

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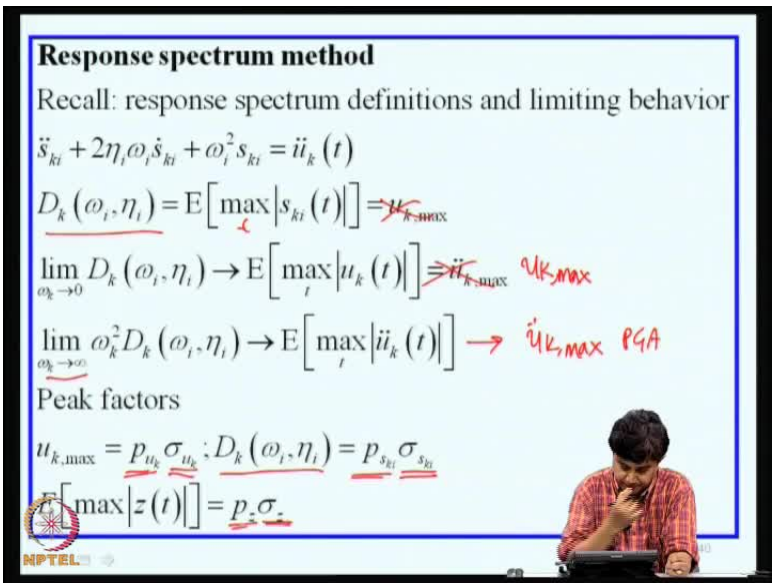

Now, let us consider the question of response spectrum based method – how to analyze the response when inputs are specified in terms of individual response spectra and the coherency function? Now, we will quickly recall the response spectrum definitions and the limiting behavior. If you are considering the acceleration u k double dot of t and we

consider the response of an oscillator with natural frequency eta and omega i, the response spectrum for relative displacement is given here, which we interpret a expected value of maximum of s k i of t over time t. This (Refer Slide Time: 35:24) response spectrum is omega k goes to 0. We have shown that this is u k max. And, the pseudo-acceleration response spectrum as omega k becomes very large; we have shown that this is equivalent to u k double dot max. This is known as the peak ground acceleration or the ZPA, etcetera. These are the limiting behavior. This is a definition of response spectrum.

Now, we introduce what are known as peak factors. For example, the maximum displacement is related to the standard deviation of the displacement through this factor (Refer Slide Time: 35:59) p u k, which is the peak factor for displacement. Similarly, the peak factor for response can be written in terms of standard deviation multiplied by the associated peak factor. So, if z of t is the response quantity of interest as has been the case, the peak factor associated with this is given by the standard deviation into peak factor p z.

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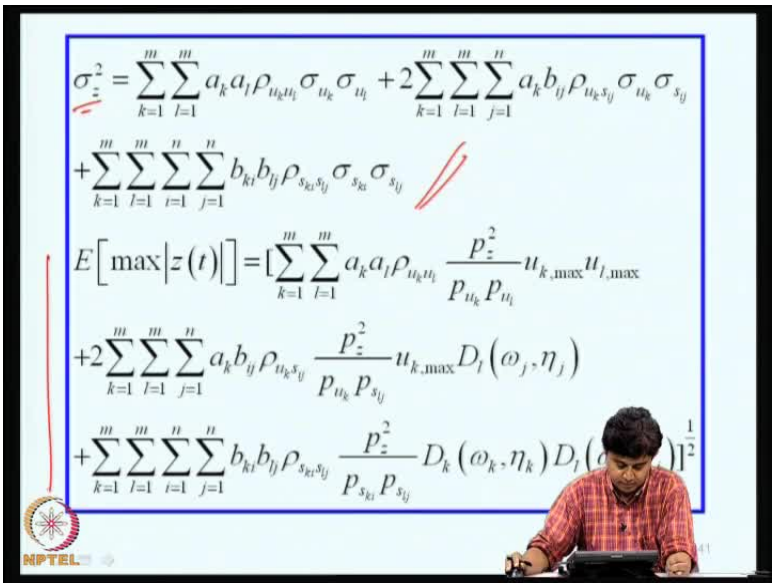

Now, this is the expression sigma z square that we have obtained through standard random evolution analysis. Now, I am writing the variances u k u l. Now, I will write in terms of the peak factors. If I am interested in expected value of maximum of z of t, that will be p z square into sigma z square. And, that sigma z square – that means, on the right-hand side, I should multiply by sigma z square. And, for sigma u k, I will write it as

u k max by p u k. Sigma s i j, for example, will be again expression in terms of its associated peak factor. So, in terms of peak factors, we get this expression.

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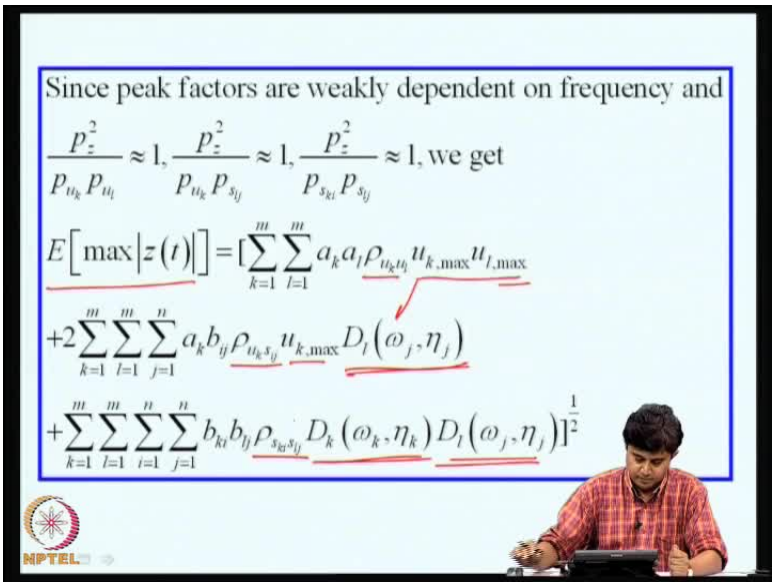

Now, we have seen in the earlier discussion that the ratio of p z square by p u k p u l is approximately unity. This is an assumption anyway we are going to make, because peak factors are weakly dependent on frequency and their ratios are nearly unity. If we do that, then the expected value of the maximum z of t is obtained in terms of u k max and the response spectrum ordinates. And, u k max is in fact obtained as limiting value of these response spectra, omega j goes to 0. So, these are related. So, on the right-hand side, I have now all the quantities that I know off; and, this in fact is the desired combination rule.

This involves the determination of these quantities, (Refer Slide Time: 38:14) rhos. And, from the response spectra, we have to find out the limiting behavior as omega goes to 0 and obtain these displacement values. So, to be able to use this combination rule, it is not enough if only the response spectrum is given for individual components. There should be an acceptable limiting value for this response spectrum as omega goes to 0; care should be taken to ensure, that is a meaningful limit. And also, these quantities, rho – rho u k u l, rho u k s i j and rho s k i and s l j has to be evaluated. That would require the definition of coherency function. Once all these are in place, we can evaluate this function.

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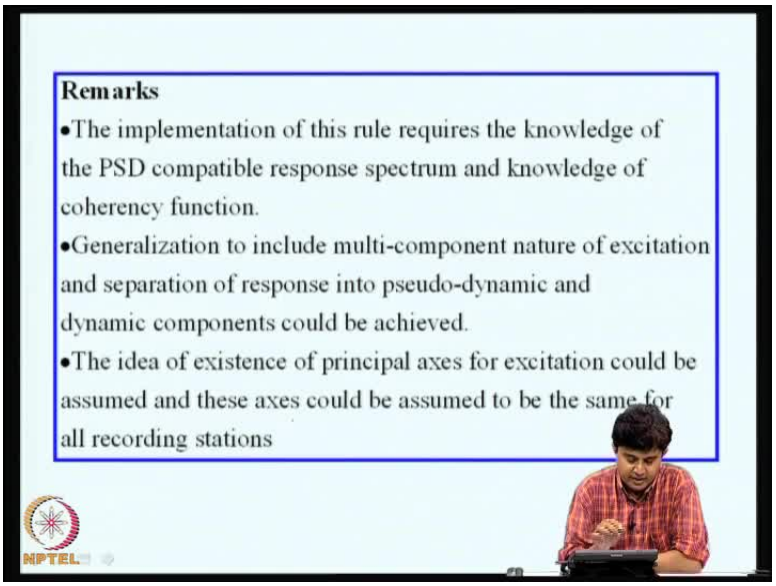

So, in summary, we can say that the implementation of this rule requires the knowledge of the PSD compatible response spectrum and knowledge of coherency function. Generalization to include multi-component nature of excitation and separation of response into pseudo-dynamic and dynamic components could be achieved although this has not been discussed. And, the idea of existence of principle axes for excitation could be assumed and these axes could be assumed to be the same for all recording stations. So, under these assumptions, we can now develop further combination rule, where there are multiple components and spatial variability together. So, that has not been done, but that can be done as a straightforward extension if you assume that the principle axes are the same at all stations.

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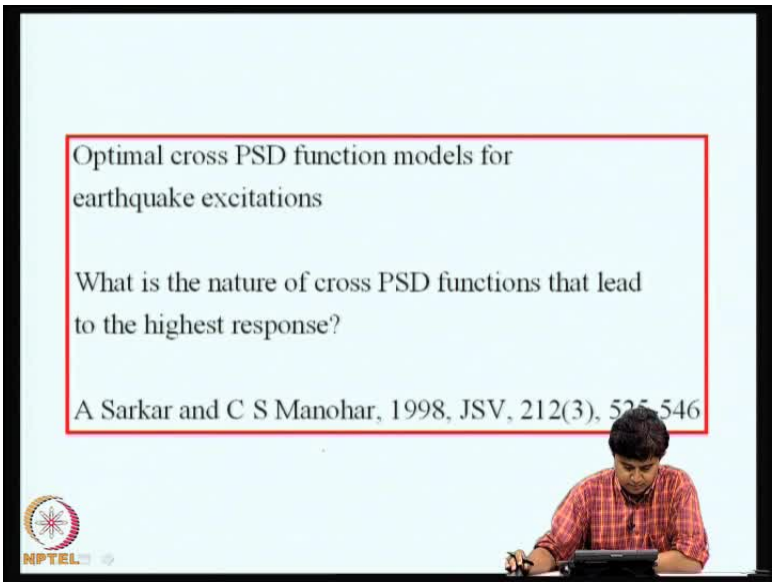

Now, given the lack of adequate knowledge on cross PSD functions of earthquake ground accelerations, it makes sense to ask the questions – what are the optimal values of these cross PSD functions for which the response reaches their highest values? So, this question has been discussed in this paper by Sarkar and Manohar. And, I will just briefly outline the problem and the solution.

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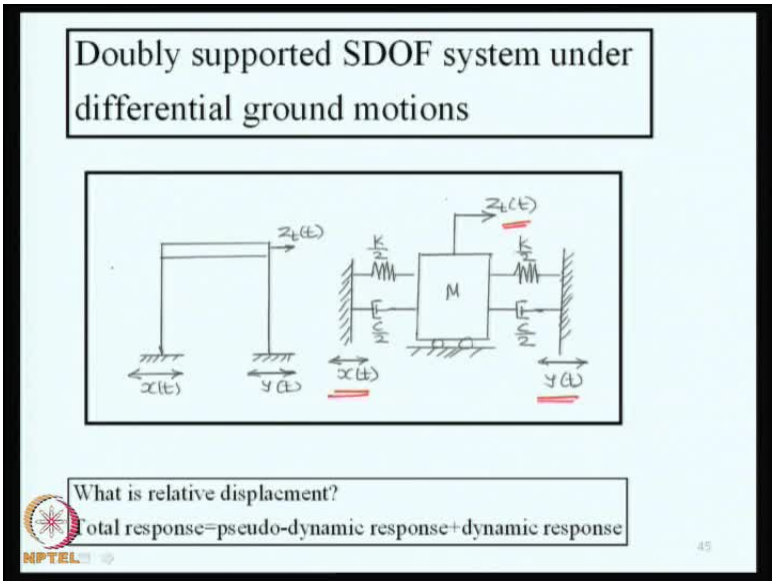

So, for purpose of discussion, we will consider a doubly supported single degree freedom system, which is subjected to differential support motion: x of t and y of t; and, z t of t is

the total displacement. The total response here is pseudo-dynamic response plus dynamic response.

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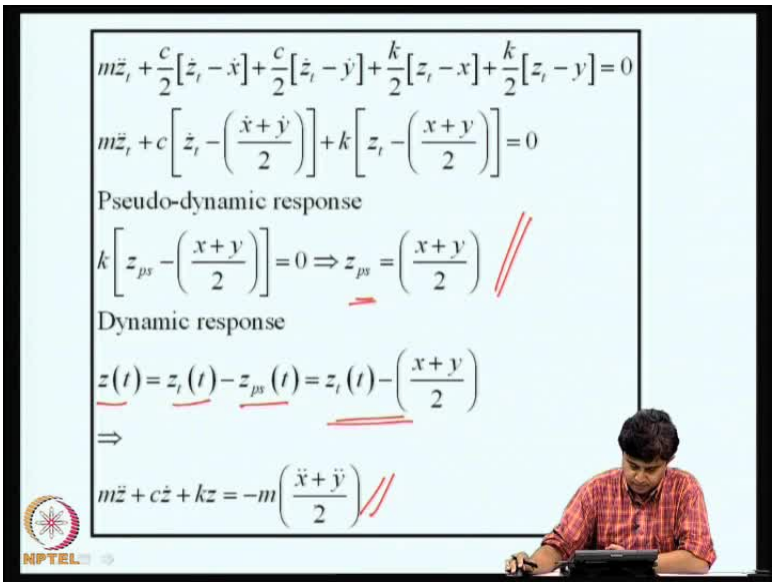

So, we can write the equation of motion quite straightforward. And, the pseudo-dynamic response can be evaluated by considering only the terms involving stiffness and we get this as a pseudo-dynamic response. And, the dynamic response can be obtained as the difference between the total response and the pseudo-dynamic response and we get this as this expression for the problem on hand. And, the governing equations consequently is obtained in this form.

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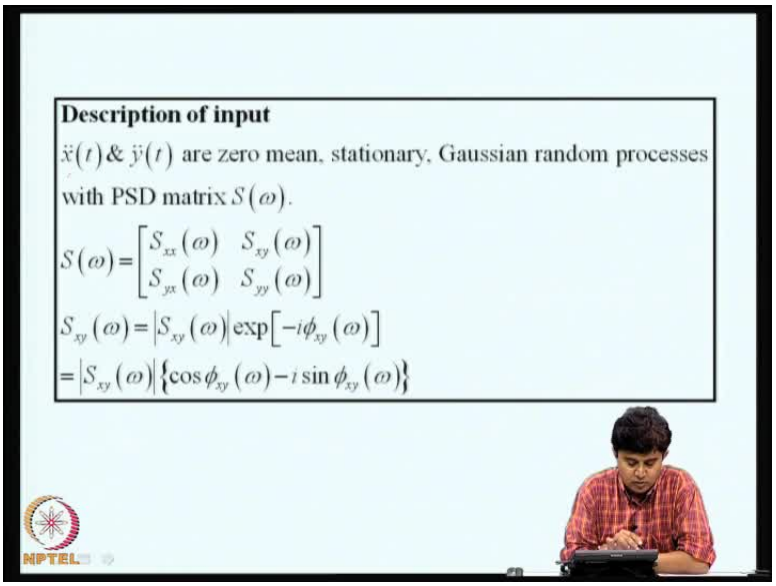

Now, the input – we have two inputs: x double dot of t and y double dot of t. We assume that zero mean, stationary, Gaussian random processes with power spectral density function given by this 2 by 2 matrix. This cross term, that is, the cross power spectral density function we write as modulus and a phase function. And, e raise to i phi x y can be written as cos phi x y minus i sine phi x y.

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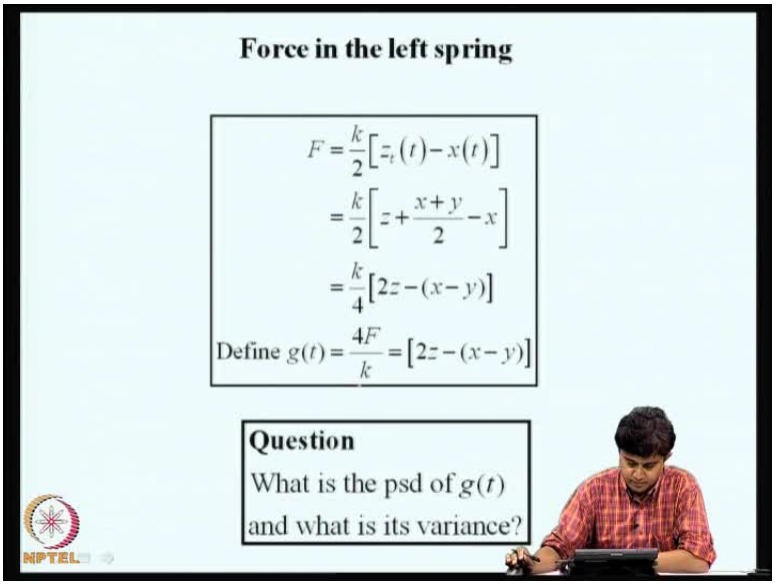

The response quantity of interest for purpose of discussion, we take it to be force in the left spring and that is given by k by 2 into z t of t minus x of t. And, that turns out to be

this quantity – k by 4 into 2 z minus x minus y. And, we define a quantity g of t, which is 4 F by k, which is 2 z minus x minus y. It is something like a displacement quantity of interest. Now, let us focus on analyzing this quantity g of t. So, to start with, we can ask – what is PSD of g of t and what is its variance?

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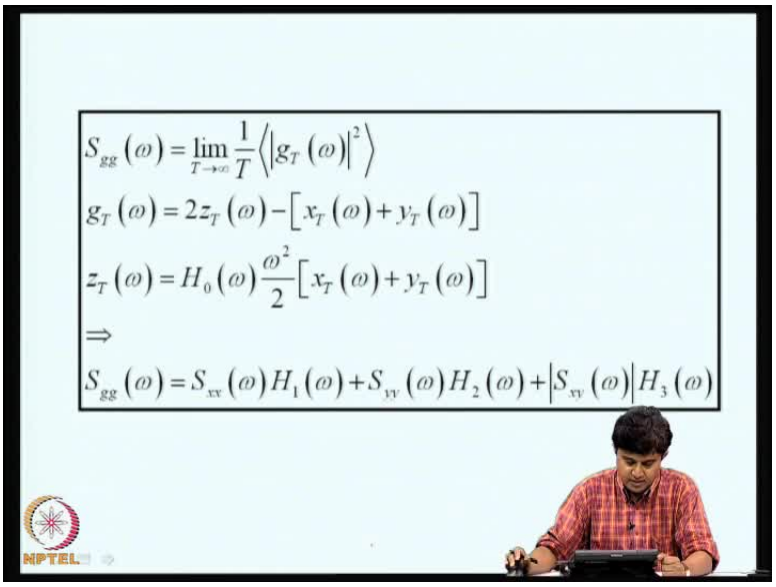

This is reasonably straightforward. We get by using the definition of the power spectral density function and the input-output relations. We get the power spectral density of S gg of omega to be in this form. And, the form that is written here is to be noted carefully. There is one transfer function, which multiplies auto PSD at x, that is, auto power spectral density of x double dot of t. Another one that multiplies y double dot of t. And, there is a third transfer function, which multiplies the amplitude of the cross PSD functions. So, the terms are arranged in this form.

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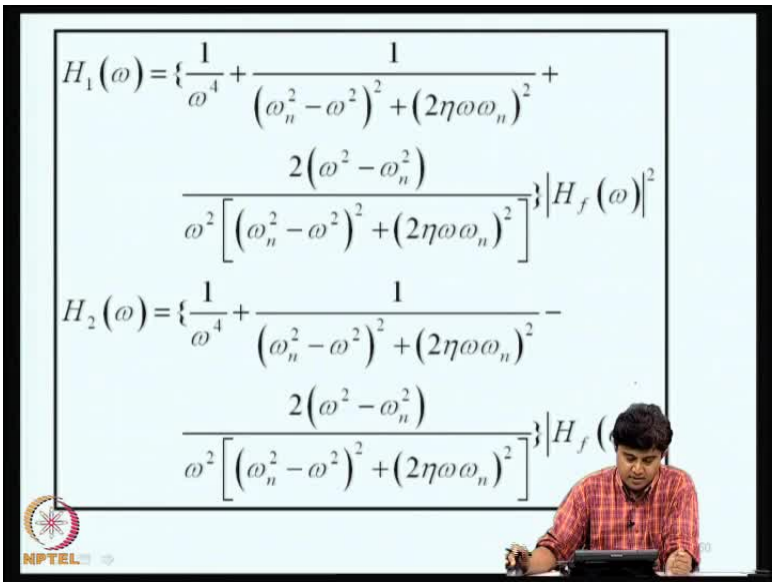

And, it can be shown that this H 1, H 2 in terms of system natural frequency and damping and parameter omega can be written in this form.

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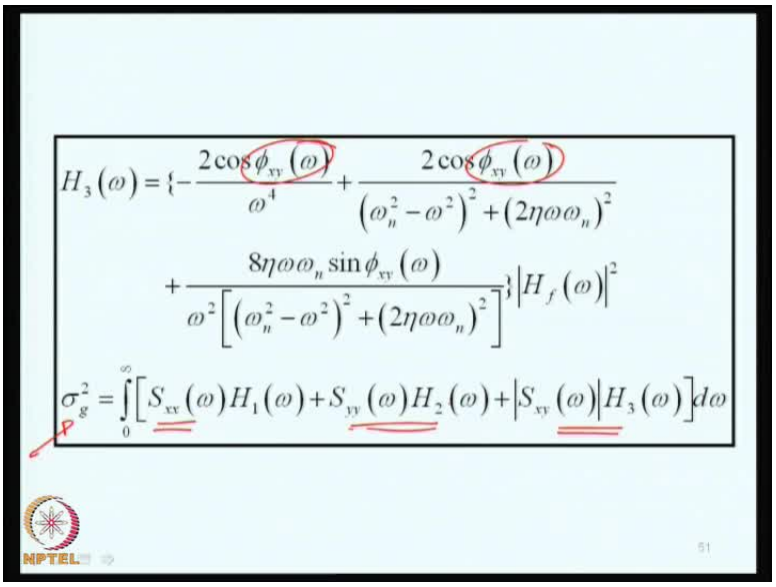

And, this is H 3 of omega, is in this form, where H 3 of omega is also a function of the phase spectrum. So, the variance of the quantity of g of t can be written as area under the power spectral density function. That has again three components. These three components are different from the three components that we discussed earlier. Earlier what we did was we had a component due to pseudo-static response, a component due to

dynamic response and a component due to correlation between pseudo-static and dynamic response. But, the way we are writing here is slightly different. We are writing here as contribution to variance due to x double dot of t, contribution to the variance due to y double dot of t and contribution to variance due to correlation between x double dot and y double dot of t. It is in this form it is done. Therefore, H 1, H 2, H 3 have a different meaning here.

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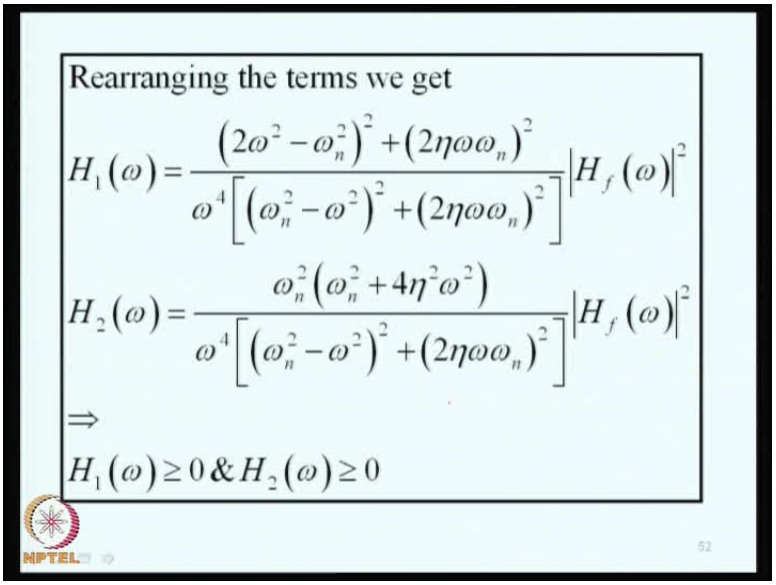

Now, if we rearrange the terms, we can write H 1, H 2 in this form. And, if you carefully observe this, we can show that H 1 and H 2 are positive. That should be expected given the definition of the quantities.

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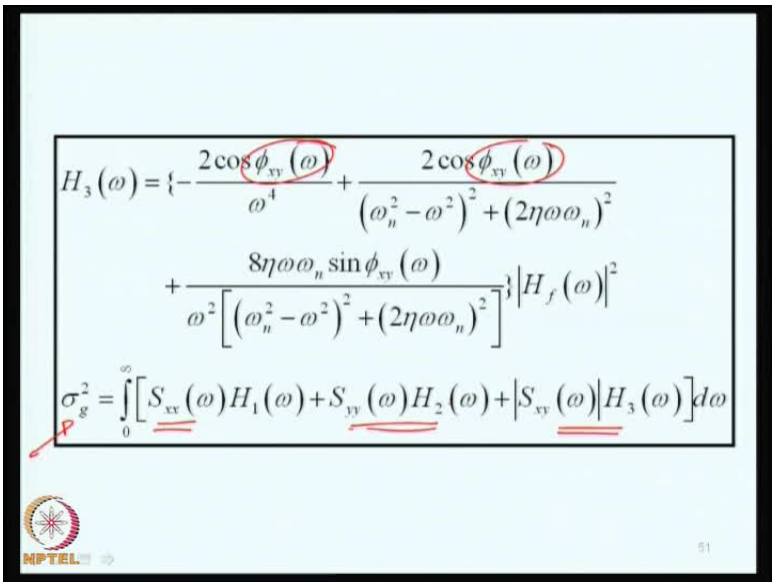

And, H 3 of omega can take both negative or positive values. H 3 of omega is written here (Refer Slide Time: 44:14). This does not lead to any further simplification. So, given the presence of trigonometric terms, sine, cosine, etcetera, there is no guarantee that this is strictly positive or negative. It depends on phi x y of omega whether it is positive or negative.

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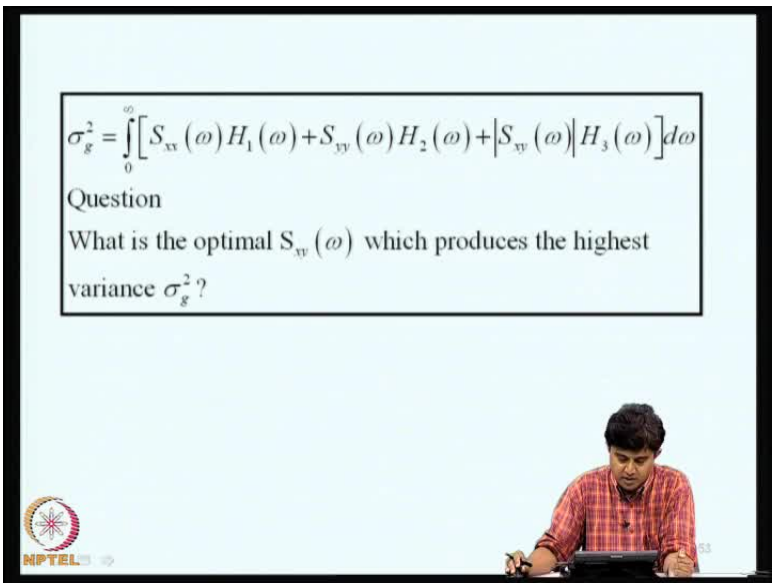

So, the question now is we have the variance of the response consisting of three terms; the contribution from the first two terms are positive; the contribution from this term can

be either positive or negative for a given omega. Now, if S xx and S yy are given – that is a basic assumption we are making; we are assuming that S xy of omega is not available. The knowledge on cross power spectral density function is not available. So, we are trying to find out that cross power spectral density function, which maximizes sigma g square. In absence of any knowledge on spatial variability, what is the worst that might happen is a question. So, what is optimal S xy of omega, which produces the highest variance sigma g square?

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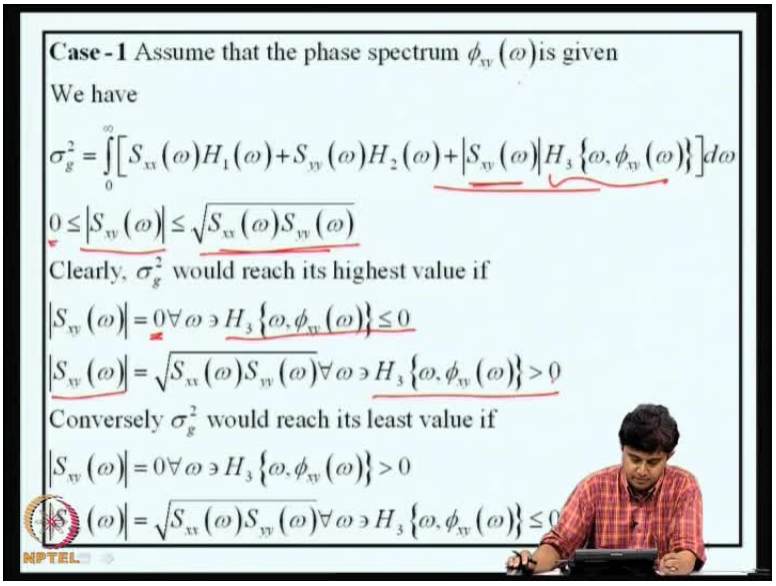

Now, suppose if we assume that the phase spectrum is given. Suppose the distance between two stations is known and the phase is essentially due to time delay; then, we can assume that the phase spectrum is available. In that case, if we look at the expression for the variance, which again is repeated here, we need to notice that the amplitude of cross power spectral density function is bounded between 0 and this quantity, (Refer Slide Time: 45:43) because amplitude of coherency is bounded between minus 0 to 1.

Now, for any given value of omega, the contribution to sigma g square from this term – this is (Refer Slide Time: 46:00) unknown; S xy of omega is unknown. So, what we will do is, we look at the value of H 3. If H 3 is negative, we will put S xy of omega to be 0; that is, this limit. On the other hand, if H 3 of omega is positive, S xy omega will assign it to be its maximum possible value. So, this we have to do for every omega. If we do that, then that corresponding S xy of omega would produce the highest response. If you

want the least response, the most favorable excitation, we have to revise the argument. If H 3 is positive, we will put S xy of omega to be 0; and, if it is negative, we will put it to the highest value, so that the maximum value is deducted from these positive contributions. So, this is a definition of critical – say cross power spectral density function when the phase spectrum is available.

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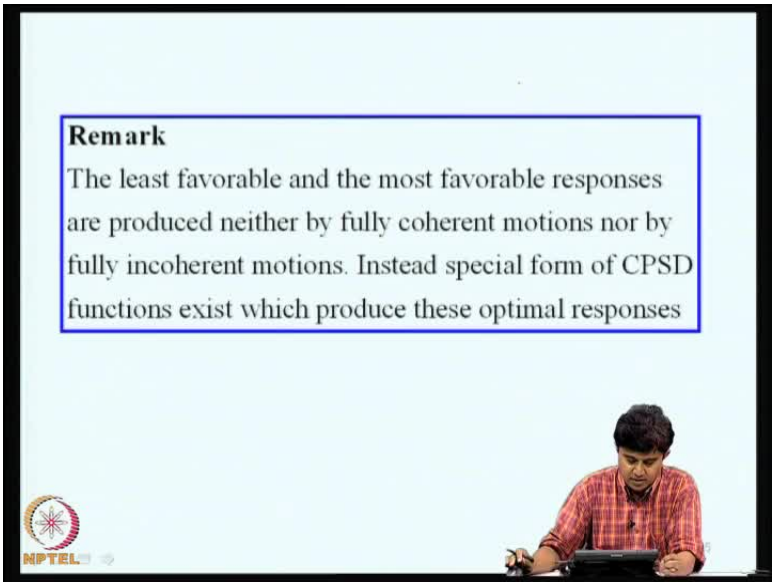

What is to be noted here that the least favorable and the most favorable responses are produced neither by fully coherent motions nor by fully incoherent motions. Instead special form of cross power spectral density function, which depends on the system characteristic exist, which produce these optimal responses. I mean this is of some interest especially when we are talking about highest response as is implied in the philosophy of response spectrum based methods. So, the notion of this optimal cross PSD function needs to be interpreted in that context.

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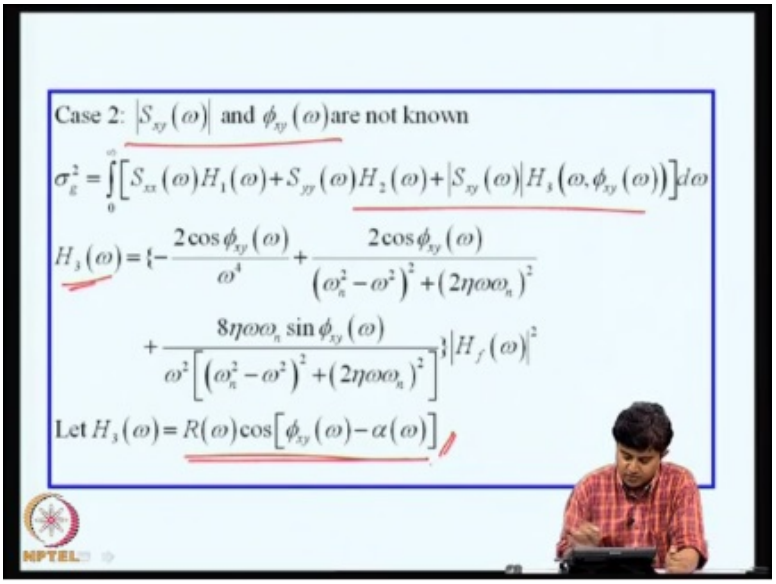

Now, the second case is when the cross power spectral density – nothing about it is known. So, both amplitude and phase are unknowns. Then, we can write again. We return to this expression and look at H 3 of omega and we recast H 3 of omega in this form. We collect terms containing sine and cosine terms separately and define an amplitude and phase function associated with those terms; I can write H 3 of omega in this form.

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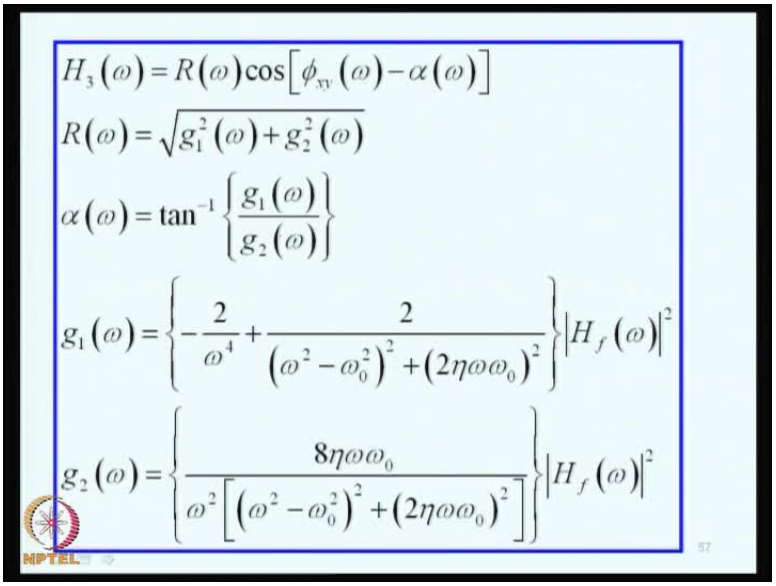

So, the details of these g 1, g 2 functions are shown here. You have to simply collect the terms, which (Refer Slide Time: 48:21) multiply cos and sine separately and define g 1 and g 2. So, these are defined here. So, R of omega is square root g 1 square plus g 2 square; alpha is tan inverse g 1 by g 2.

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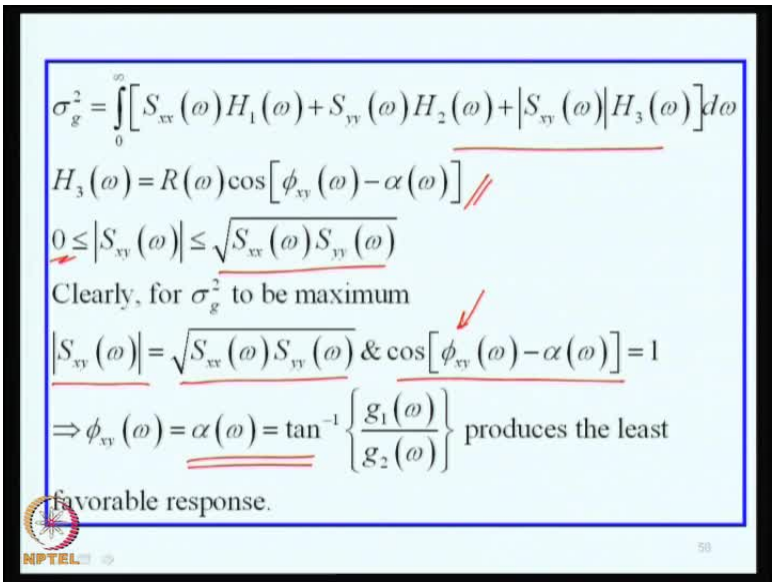

So, equipped with this, now, we have the expression for the variance, where H 3 of omega is now written in this form. Now, S xy of omega takes values between 0 and this quantity and cosine function takes values between minus 1 and plus 1. So, what we can do, what we are interested is in finding sigma g square, which is maximum. So, what we will do is we will set the amplitude of S xy to its highest value if cosine of this function is positive. And, that depends on relative value of phi xy and alpha. And, we will set it as H 3 of omega. S xy – we will put it as 0 if cosine of this function is minus 1. That would happen when phi xy of omega minus alpha of omega is pi. So, this will produce the favorable response; this will produce the least favorable response.

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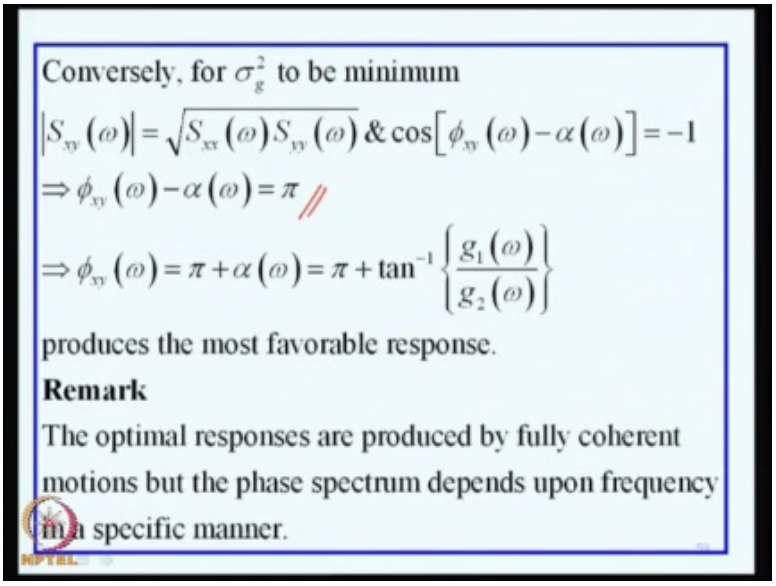

So, this to be minimum, as I was telling, you have to set it to pi and we get this. Again, we notice that here the responses produced by fully coherent motions, but the phase spectrum depends upon frequency in a specific manner.

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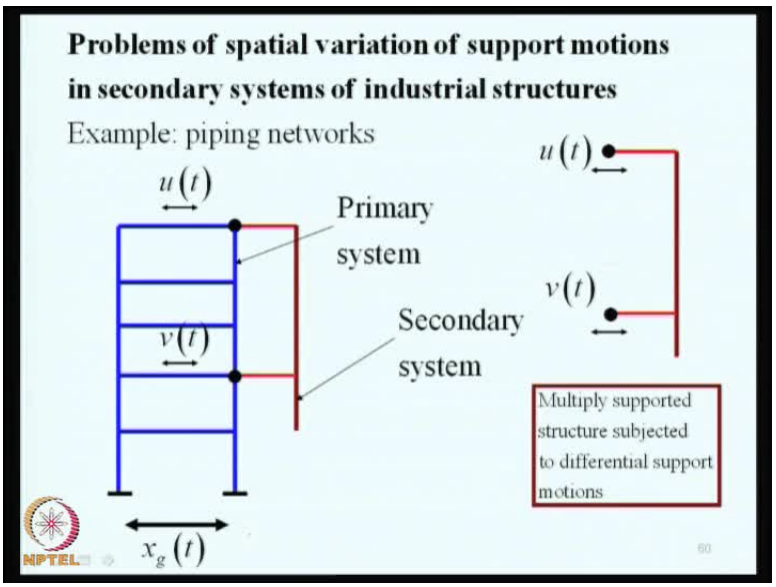

Now, before we leave this topic of spatial variability, I would like to just briefly highlight problems of spatial variation of support motions in secondary systems of industrial structures. So, for example, I was telling piping networks. For sake of discussion, let us consider a building in which there is a piping. This is a piping structure,

which is supported at two points on a primary system. So, this blue structure is the building. And, this structure suffers the support displacement x g of t. Due to this, the floors suffered the displacement, u of t and v of t. If the mass of this piping structure is relatively small with respect to the mass of this structure and under other certain considerations, we can assume that there is a kind of uncoupling that is possible, where I will consider the piping structures separately and analyze its response for these floor displacements. So, again, the point that that is being made is that this structure now is multiply supported and subjected to differential support motions.

Now, the definitions of these support displacements have to be arrived at carefully. For that, we may have to consider the possible dynamic interaction between the secondary system and the primary system. It is not that while finding u of t, v of t, we entirely ignore the presence of this secondary system. So, there lies certain complicating features.

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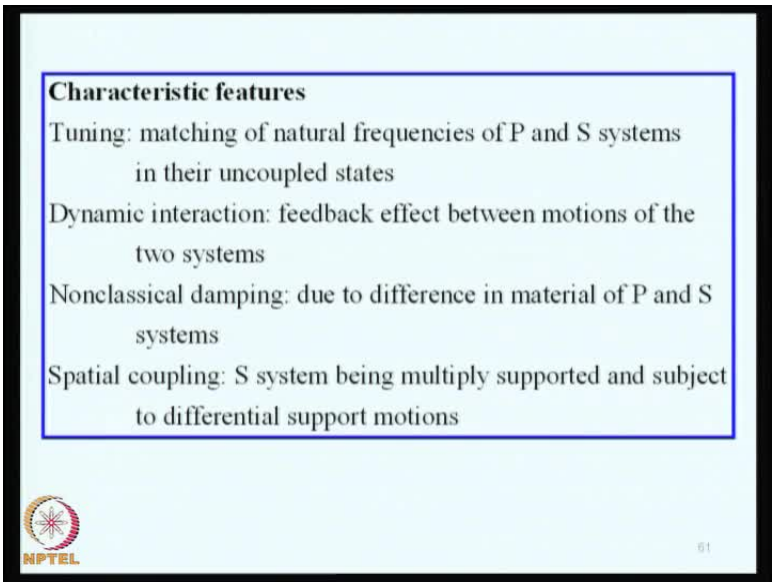

So, the characteristic features that we need to take into account are – first one what is known as tuning. Here the natural frequencies of P and S systems in their uncoupled systems come close to each other and there is a significant dynamic interaction. That leads to feedback effect between the motions of the two systems. It can be present under resonant or even when tuning is not perfectly present; still there can be a feedback. Now, the primary system in this particular example (Refer Slide Time: 52:29) could be a civil structure made up of say concrete; it could be a reinforced concrete structure. The piping

is the metal structure. The energy dissipation characteristics between these two systems are quite different from each other. And, the assumption of classical damping model for these types of structures may not be admissible; and, one may have to deal with known classically damped systems in arriving at motions, u of t and v of t.

Finally, there is a (Refer Slide Time: 53:01) spatial coupling – the secondary system being multiply supported and subjected to differential support motions. So, that means this piping system (Refer Slide Time: 53:10) responds to the floor displacement at this level as well as at this level. That induces certain spatial coupling in the system. Now, this problem is quite different from the study of spatially extended structures in a land-based structures like bridges and large dams, where the support motions are essentially characterized by the phenomena associated with wave passage in the earth medium; whereas, here the support motions that reach the supporting points of the secondary system – these excitations propagate through a manmade structure and they can be quite complex. It depends on the natural frequencies and more shapes of the structure and it can be quite diverse. So, the model combination rules that we derived based on arguments for land-based structures need to be carefully looked into before we can apply for this class of problems.

I would not go into details of this; I leave it as a thought. There is vast literature available on this. I thought it is useful to mention this, because we are talking about multiply supported structures. So, at this point, we will conclude this lecture and we will consider further applications of random vibration analysis in the remaining lectures.