Stochastic Structural Dynamics

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Lecture No. # 33

Probabilistic methods in earthquake engineering-2

So, we have been discussing application of probabilistic methods in earthquake engineering problems, specifically we are focusing on a problem due to vibrations, because during earthquakes, the earthquake ground motions are modeled as stochastic processes. And we are considering how we can apply principles of random vibration analysis to these problems.

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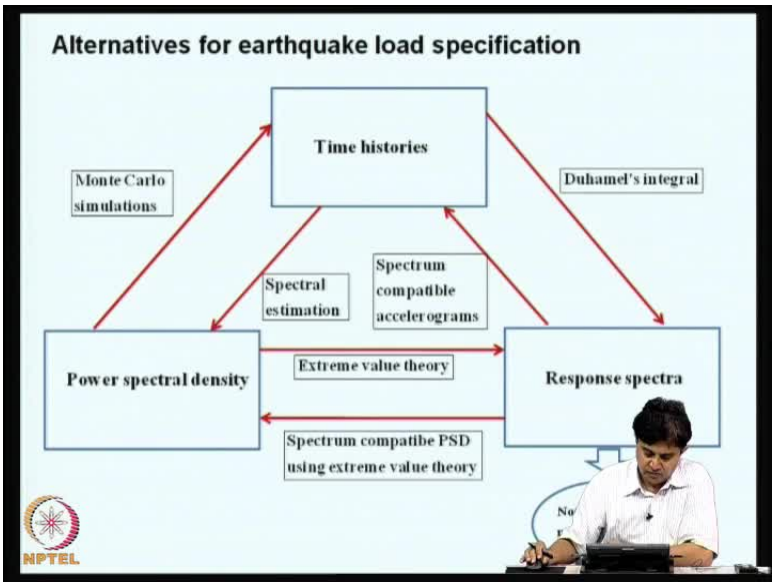

So, in the previous lecture, we discussed three alternative modes of representing earthquake ground motions; one is through time histories, this is basically a deterministic approach; the other through a set of response spectra; a third alternative is to represent through power spectral density function.

We discussed in the previous lecture, how these are three alternative representations for ground motions are related to each other. For example, if we are given a time history of

ground motion, we can use principles of analysis of single degree freedom system should dynamic excitation and derived the response spectra. Similarly, if we are given a set of response spectra, it is possible to generate accelerograms which are compatible with response spectra; this as spectra I did not discuss, because this is basically a deterministic, it has deterministic flavor.

So, we are focusing on application of probabilistic methods. So, this aspect was not discussed, but on the other hand, if we are given power spectral density functions, we can use Monte Carlo simulation methods and derive a set of time histories compatible with this power spectral density function.

Similarly, if we are given time histories, we can use statistical estimation methods and estimate power spectral density function. So, these two loops are covered in this discussion in this course. Similarly, if we start with power spectral density function, we can derive the associated response spectra by using extreme value theory, that is associated with probability distribution of extremes of responses using level crossing statistics, and so on and so forth. Similarly, the same theory can be used to derive a compatible power spectral density, if we start with specified response spectra.

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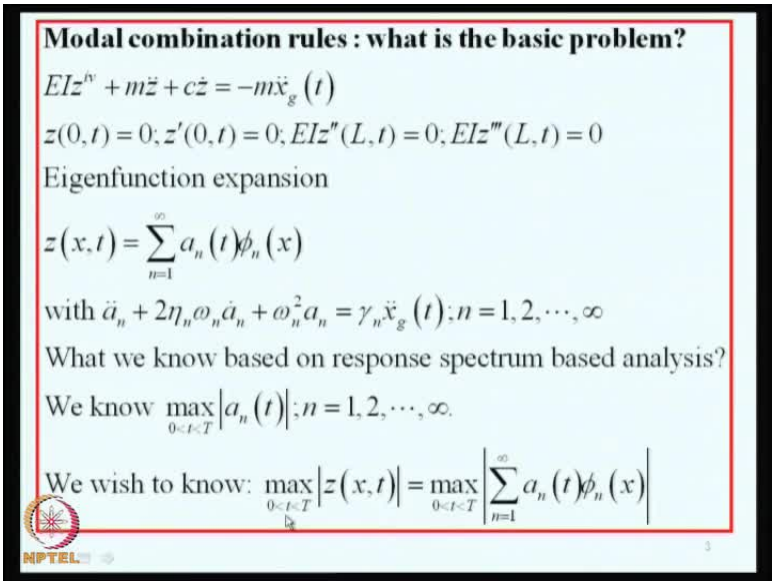

Now, one of the problem that arose when we discuss the application of response spectrums is methods to earthquake engineering problems, is the problem of what is known as model combination rules.

So, to quickly recapture what the problem is, we consider a cantilever beam subjected to base motion, x g double dot of t is the earthquake induced ground acceleration. And we use an eigen function expansion, and represent the solution in terms of the eigen functions and the natural function frequencies, and we get a set of uncoupled single degree freedom systems.

Now, our objective is to determine the maximum value of a desired response quantity; for example, displacement over a given duration. So, what we are interested is, in determining the maximum value of this summation - the absolute value of this summation. If x g double dot of t is specified in terms of a response spectra, what we know would be essentially maximum values of response of single degree freedom systems.

So, since the generalized coordinates are essentially a set of single degree freedom system, so from the given response spectra, we would be able to deduce the maximum value of each of the values of this function a n of t.

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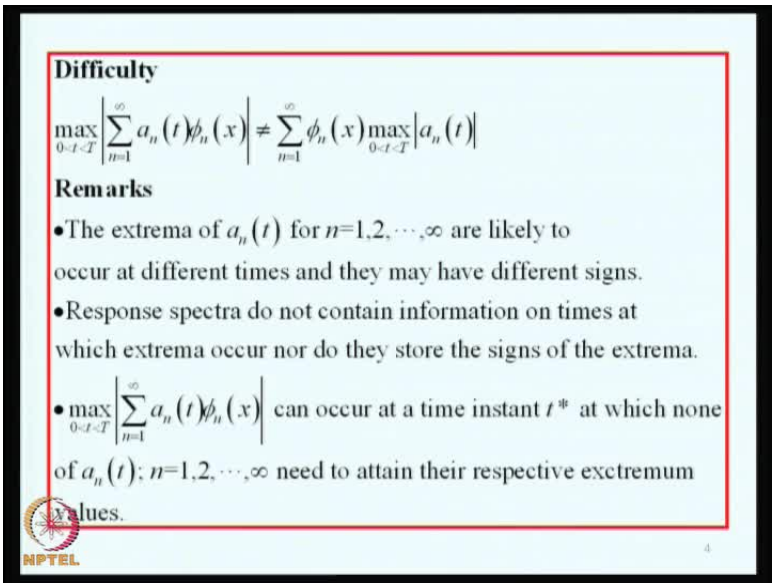

Now, the problem of model combination arises in determining the maximum value of - this absolute value of - this sum, in terms of the known maximum values of sum of the terms inside this summation. These two terms are not equal, so there is a problem there.

Now, it is obvious that the extrema of a n of t for n equal to 1 2 3 etcetera are likely to occur a different times, because each a n of t has different natural frequency, different damping and different model participation factor; therefore, there is no reason to expect that all mode generalized coordinates will peak at the same time.

And similarly, the maximum values may not have the same sign; for one of these generalized coordinates, the peak could be on the positive side; other could be negative side, and so on and so forth.

The specification of response spectra do not contain information, on time set which extrema occur, nor do they store this signs of the extrema. So, this information is lost; therefore, response spectra is not a complete specification, some information is lost; therefore, there will be some compromise in applying this tool.

Now, moreover, if t star is a time a twice this maximum value of this sum, maximum of absolute value of the sum occurs, then this t star need not coinside with any of the time instances, where a n of t for n equal to 1, 2, 3, reach the respective maximum values. So, there is a basic difficulty here.

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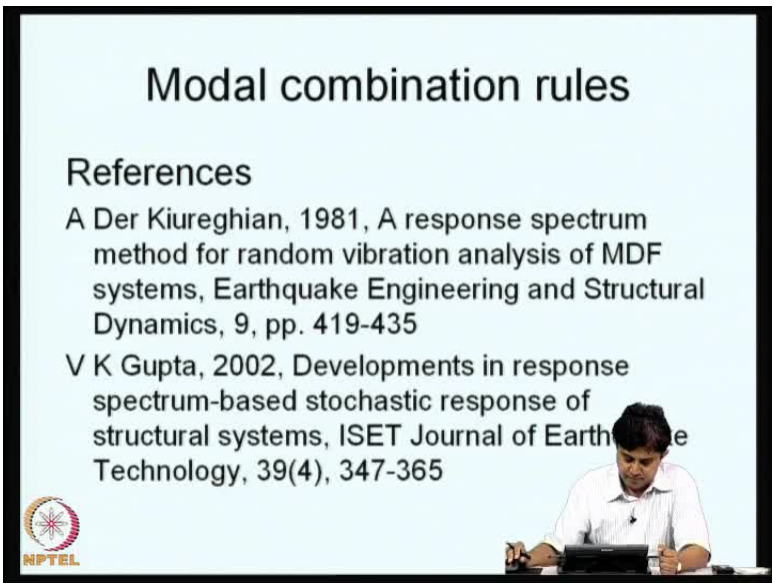

Now, the discussion on model combination rules; I am using two references, one is the prepare by A Der Kiureghin, which appeared in earthquake in engineering structural

dynamics; the other one is the review paper, which appeared in ISET journal of earthquake technology.

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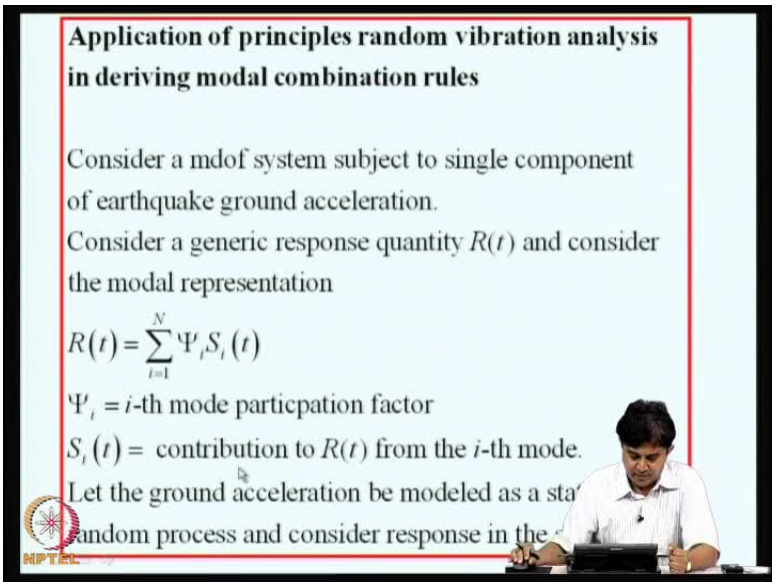

Now, we will discuss now the application of principles of random vibration analysis in deriving model combination rule. Now, let us consider a multi degree freedom subs system, subject to single component of earthquake ground acceleration. Now, let us consider generic response quantity R of t, and consider the model representation R of t is psi i S i of t, from i equal to 1 to capital N. Now, R of t could be displacement, stresses, reaction transfer or any other there function, which is actually taken to be a linear function of the system states.

This psi i is the ith mode participation factor associated with the response quantity R of t, and S i of t is a contribution to R of t from the ith mode, that is actually psi into S i will be the contribution to R of t from the ith mode. Now, let us model the ground acceleration to be a stationary random process with 0 mean. And we will also assume that, later that, it is a Gaussian random process, and let us consider the response R of t in steady state.

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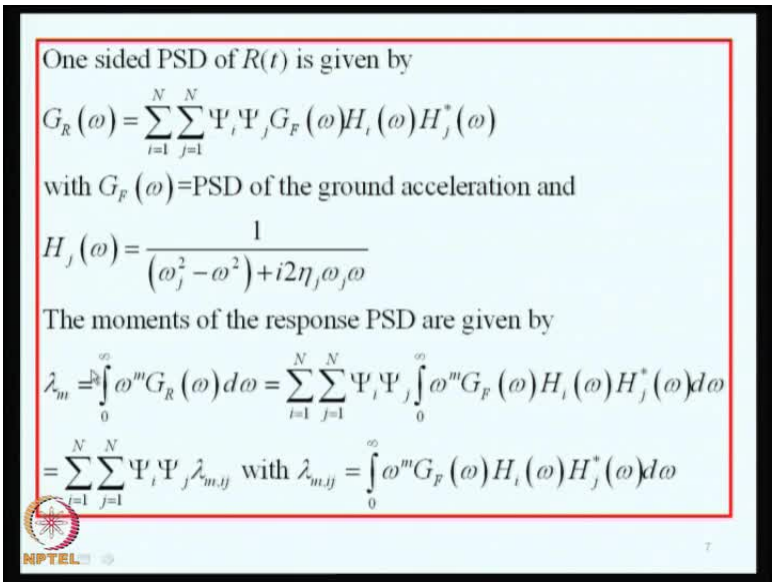

Now, we consider the one sided power spectral density function of R of t given by, G R of omega; this we can obtain in terms of the transfer function of the higher generalized coordinate, and G F of omega is the power spectral density function of the earthquake ground acceleration. H J of omega is the transfer function for the J generalized coordinates. Now, if you recall the extremes of response R of t, are characterize in terms of moments of the power spectral density function.

So, let us now write the expression for the nth moment of the power spectral density G R of omega and that can be written in this form. And if we utilize now the representation for G R of omega, in terms of participation factor and transfer functions of the generalized coordinates, we get this expression. Now, if lambda m which is a mth order spectral moment of the response quantity, it can be expressed in terms of mth order spectral moment of generalized coordinate i and j, because they appear in pair here; so, lambda m i j is this integral.

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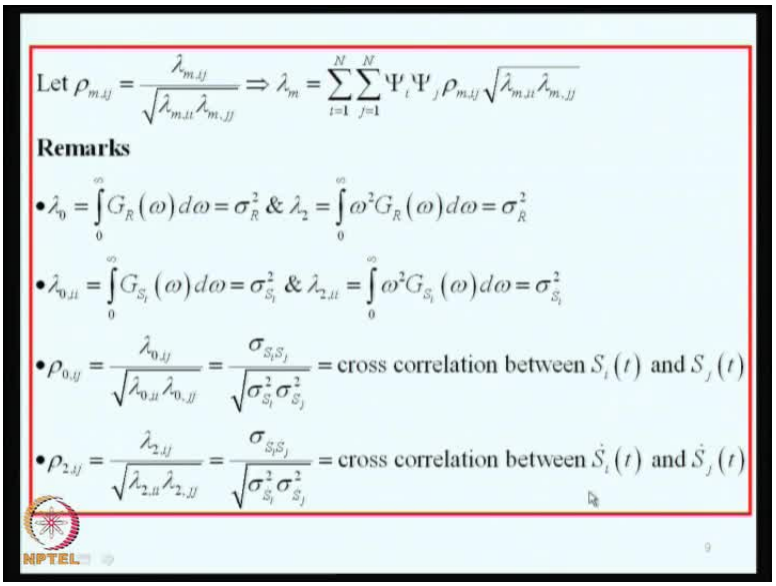

Now, what we do is, we normalize this lambda m i j as shown here; we divide lambda m i j by square root of lambda m i i and lambda m j j, and in terms of this normalized quantity, the mth spectral moment can be expressed in this form. Now, if m equal to 0, it is nothing but the area under the power spectral density function, which is nothing but the variance of the response process. Similarly, the second moment is the variance of the derivative process in the steady state.

Similarly, lambda 0 i i is actually the variance of the ith generalized coordinate in steady state; lambda 2 i i is the variance of the time derivative of the ith generalized coordinates in steady state. Now, just we can associated with m equal to 0 in 2, we can define now lambda row 0 i j as lambda 0 i j divided by this square root. And if we now interpret these moments in terms of with their associated variances and cross variances, you can convince yourself, that this row 0 i j is the nothing but cross correlation between S i of t and S j of t. Similarly, row 2 i j is the cross correlation between S i dot and S j dot; this i j indicate the indices applied on the generalized coordinates.

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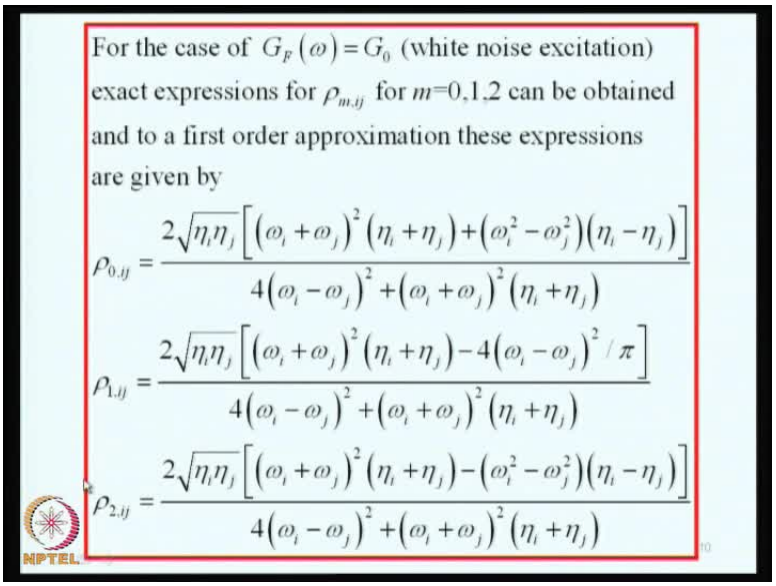

Now, if we simplify the problem by assuming that G of omega is a white noise, in which case power spectral density function will be constant, and we can derive the exact expressions for this spectral moments, for m equal to 0, 1, 2. And to a first order approximation, we can actually show that, row 0 i j, row 1 i j and row 2 i j are given by this these expression. The exact expressions are much longer than this; there is an approximation here in deriving this.

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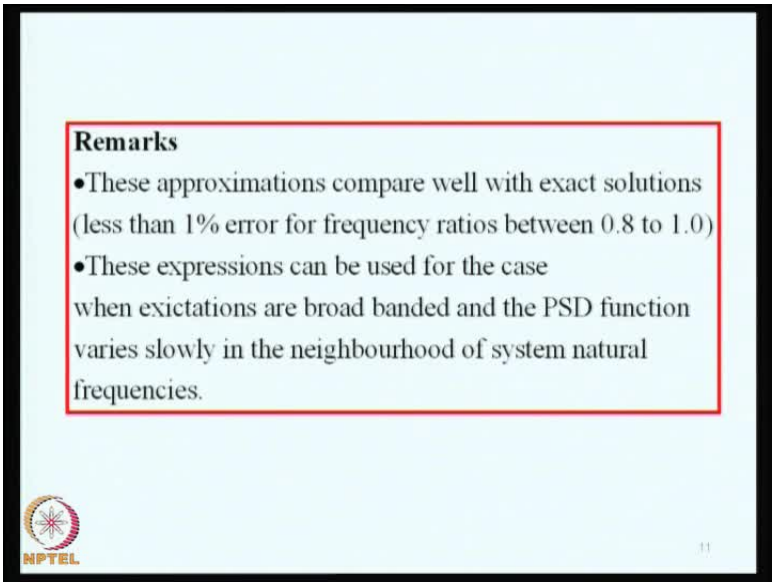

Now, these approximations compare well with exact solutions; think that error is less than about 1 percent, for frequencies between ratios between 0.8 to 1.

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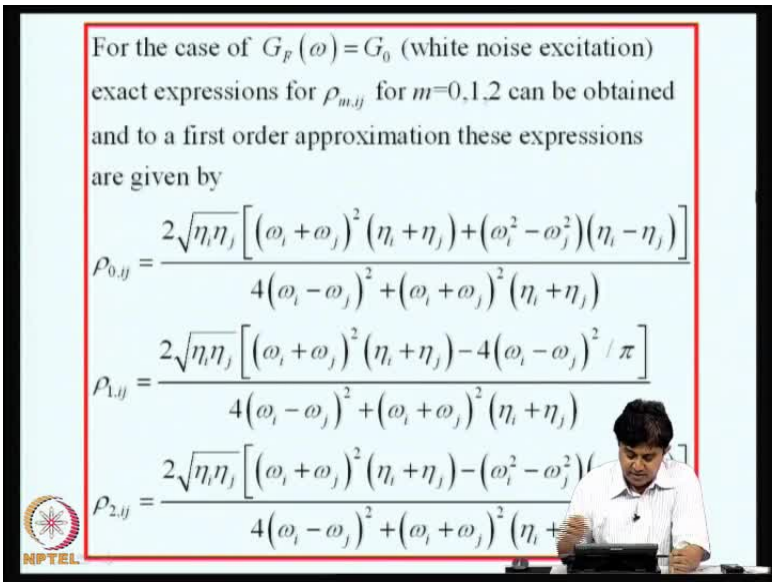

Now, we are assuming these expression are obtained under the assumption, that the ground motion is acceleration is modeled as a whit noise excitation. We can continue to use this expressions, if we make the approximation, that the power spectral density function of ground acceleration which may not be white correspond to white noise process, but if it is slowly varying in the neighborhood of system natural frequencies, we can still approximate this moments by using these expression; and indeed that is approximation that we are going to make in the subsequent analysis.

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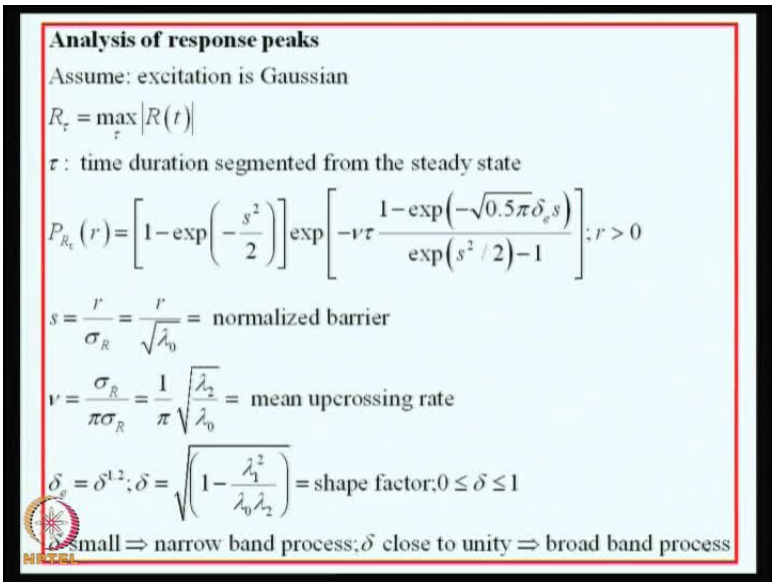

Now, let us now consider the problem of analyzing the response peaks. Now, in the here now, you bring in the assumption that the excitation processes is Gaussian, and I am interested in R tau which is absolute value of R of t over a duration tau; and this tau is the time duration segmented from the steady state; t is not from 0 to tau, but it is a segment of length tau in the steady state.

We already discussed how to derive the probability distribution of the peaks of this kind - maximum value of this kind - and I will not repeat that exercise, I will simply recall the expressions. And these expressions are slightly more generalized than the expression that I provided in the discussion in the earlier lectures, but these generalizations, think if you can refer to the papers by Der Kiureghin which have provided, you will be able to understand what this generalizations are.

Now, here we are introducing a variable s, which is the state variable normalize with respect to the standard deviation. So, this is the normalize barrier and new is the up mean up crossing rate given in terms of the spectral moments, and this delta e is related to this delta which is the shape factor that lies between 0 and 1. If delta is 0, this process is narrow banded; and if delta is 1, the process is broad banded, close to unity means broad banded process.

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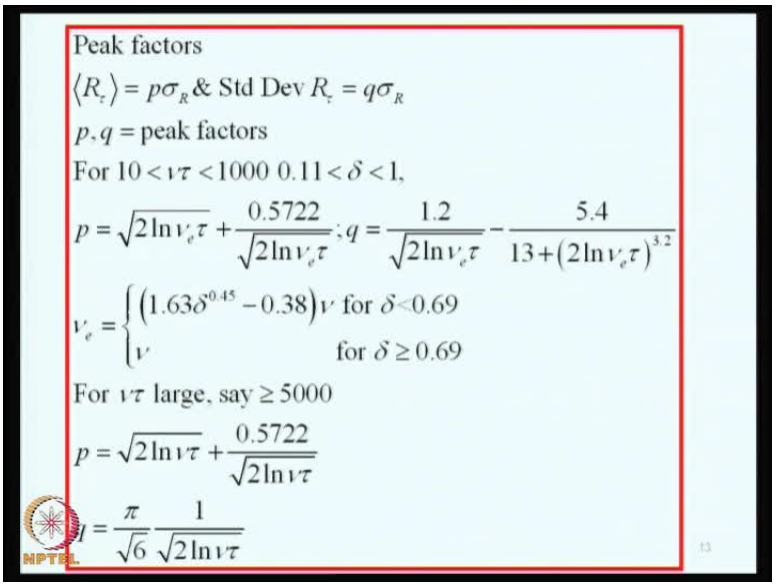

Now, we are interested for example in the p expected value of R f tau and will express this as a factor p into the standard deviation of R - R tau. So, that write it as p sigma r; in standard deviation R tau, I write it as q into standard deviation of R of t; sigma R is a standard deviation of process R of t and this p q are known as peak factors.

So, there is information available on how these peak factors behave and some of this I have provided in the earlier lecture. So, these are slightly more generalized; p peak factor for the mean is given by this, the peak factor for standard deviation is given by this; for new tau between 10 to 1000 and delta from 0.11 to 1, and for large new tau, we get these approximation which I think was these, where we saw in the previous lectures.

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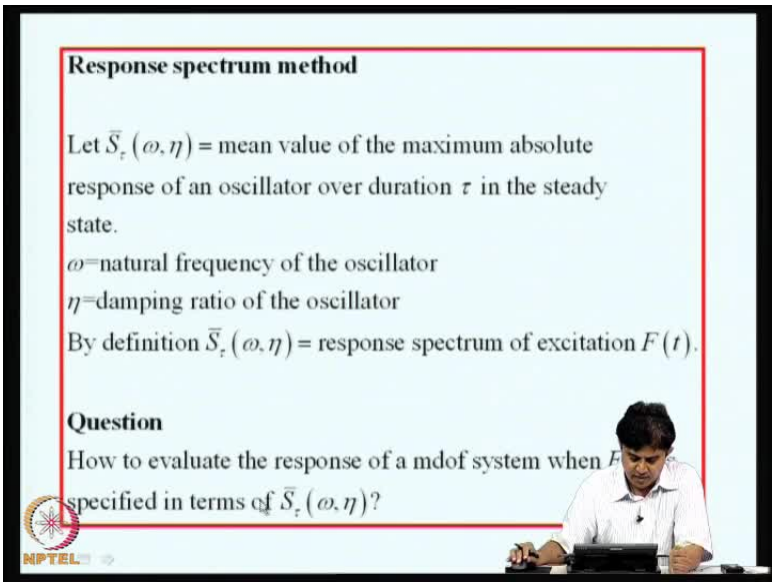

Now, till now whatever have discussed in simply the application of random vibration principle, where simply recalling what we have studied in some form earlier. Now, let us now consider the problem, where the excitation processes describe in terms of response spectrum and not the power spectral density function. So, we introduce the notation, S tau bar as the mean value of the maximum absolute response of an oscillator over duration tau in the steady state, were omega is an natural frequency of the oscillator and theta is the a damping ratio of the oscillator.

So, if we recall the definition of the response spectrum for relative displacement, that we discussed in the previous lecture, it is clear this quantity S tau bar in nothing you can interpreted as the response spectrum of excitation F of t.

Now, let as now consider how we can apply principles of random vibration analysis, if the excitation is specified in terms of this response spectrum; it is not the power spectral density function, it is a response spectrum; that is how to evaluate the response of a multi degree freedom system, when F of t is specified in terms of this spectrum, which is the response spectrum and not the power spectral density function.

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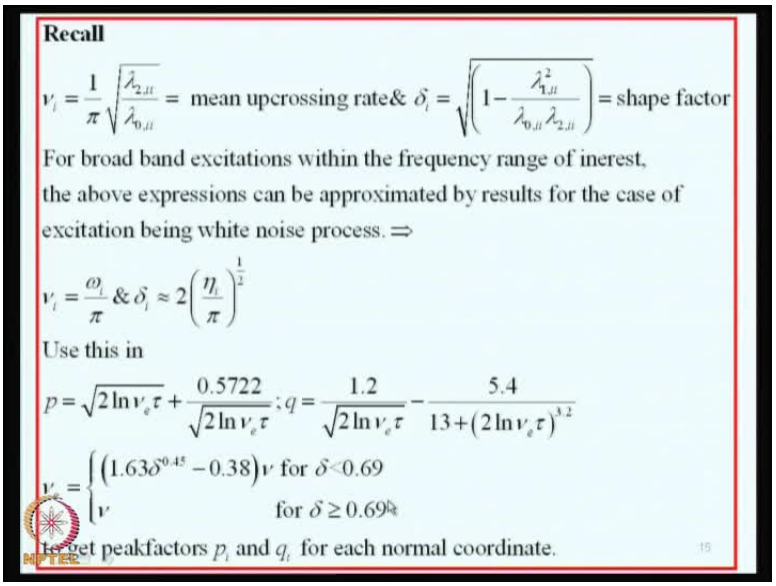

Now, we quickly recall, nu i is the mean up crossing rate in terms of the second and zeroth order spectral moments, and this is the shape factor. Now, we will assume that the excitation is broad banded. So, for broad band excitation, within the frequency range of interest, the above expressions can be approximated by results for the case of excitation being white noise process, this approximation will make; and for once this approximation accepted, we get simplified values for nu i and delta i. Now, if I now use this simplified versions into the formula for peak factors, we get peak factors for the ith and ith generalized coordinate, where i runs form 1 to n.

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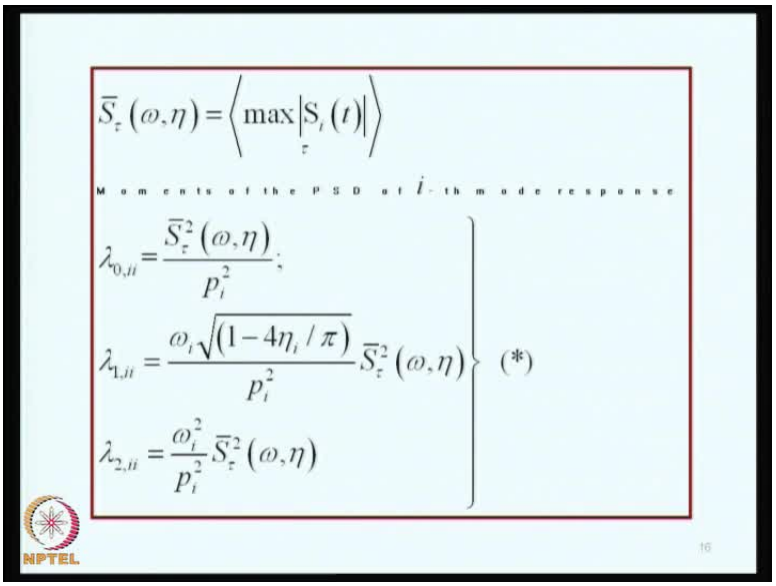

Now, the definition is expected value of maximum of S i of t over tau, I call it as S tau for omega comma eta. Now, the moments of the power spectral density function, it can be derived in terms of the given response spectrum. Recall that P i is the peak factor; therefore, lambda 0 ii which is nothing but the variance is this peak value divided by the associated peak factor; this is square, because this variance. Similarly, I get lambda 1 ii and lambda 2 ii in terms of the given the quantity S bar of tau and the peak factors P i's.

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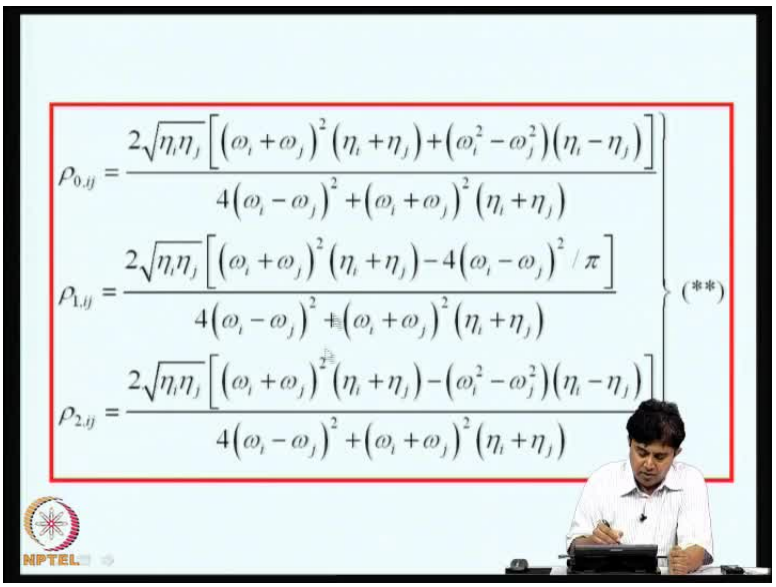

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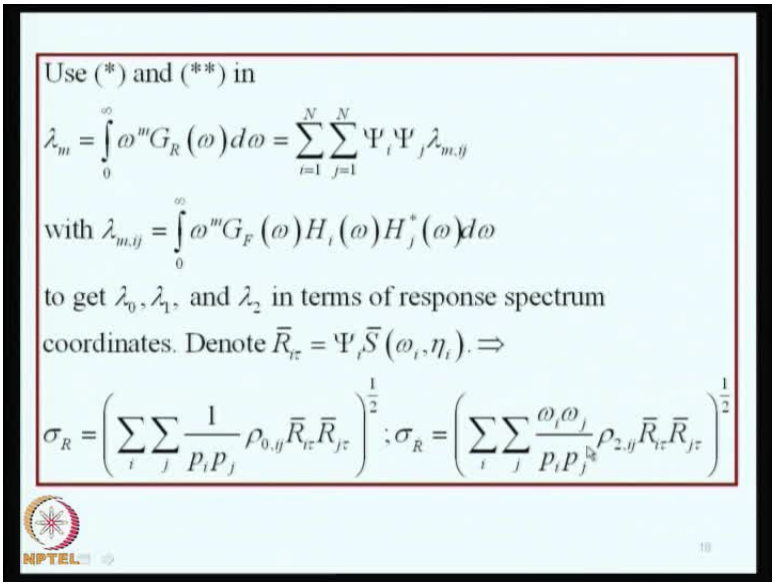

Now, we have already shown this expressions. Now, we are going to utilize them and go back to our expression for the spectral moment - mth order spectral moment - for the response process R of t, where there is summation n implied. We were discussing about the spectral moments of the generalized coordinates. Now, we have to go back to the sum; so, lambda m is this and lambda m ij is the mth spectral moment associated with ith and jth generalized coordinates.

Now, we can use this j formula star n double star to get lambda naught, lambda 1, lambda 2, in terms of response spectrum coordinates. Now, if I denote R i tau bar as psi i into S bar omega i eta i, I can show that the stand deviation of the response quantity R is given by this; P i P j are the peak factors for ith, and mean of ith and jth generalized coordinates; sigma R dot, similarly I get in terms of this. This row 0 ij and row 2 ij, we have now expression in terms of system natural frequencies and more shape etcetera, damping etcetera.

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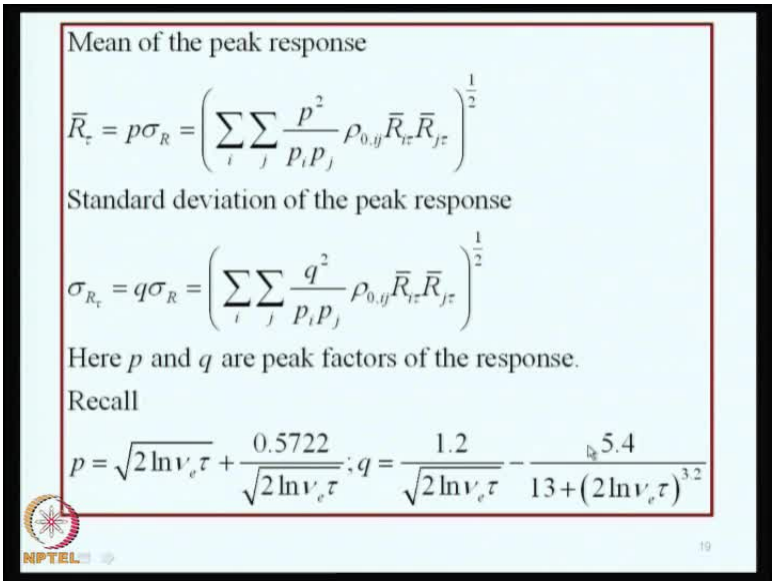

Therefore, once I get the standard deviation of the response quantity R of t, now I will multiply this standard deviation by the peak factor associated with the response quantity R of t. So, if I take this p inside, I get the ratio p square by P i P j row 0 ij and these quantities. Similarly, the standard deviation of the p corresponds can be derived, instead of the peak factor p, use the peak factor q and we get this formula. This are the formula for the peak factors p and q, for the response quantity R of t.

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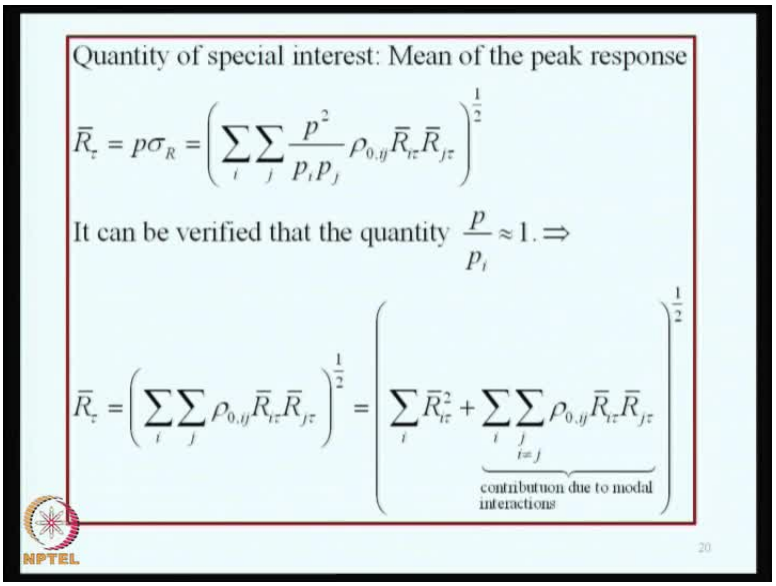

Now, quantity of special interest for us is the mean of the peak response; so, that is p into sigma R. Now, it can be verified, that the ratio P by P i is approximately equal to 1 over the frequency, I mean, range of parameter range that we are interested in. Consequently, you can approximate this ratio is P square divided by divided by P i P j as unity; this is an important assumption in the development of this combinations rule that we are discussing. If we accept that, then we have this R bar of tau as row 0 ij summation row ij row 0 ij R i tau bar R j tau bar.

Now, there are no peak factors that appear in these terms. So, we can separate the now the terms corresponding to i equal to j and i not equal to j. If we write that, the first set of terms is summation over i and R bar square i tau, and the other terms which is the double summation, where summation is over i and j such that i is not equal to j. This indicates a contribution due to modeling interactions; if we ignore model interaction, this will be our approximation; if we include model interactions, they have to combine these two.

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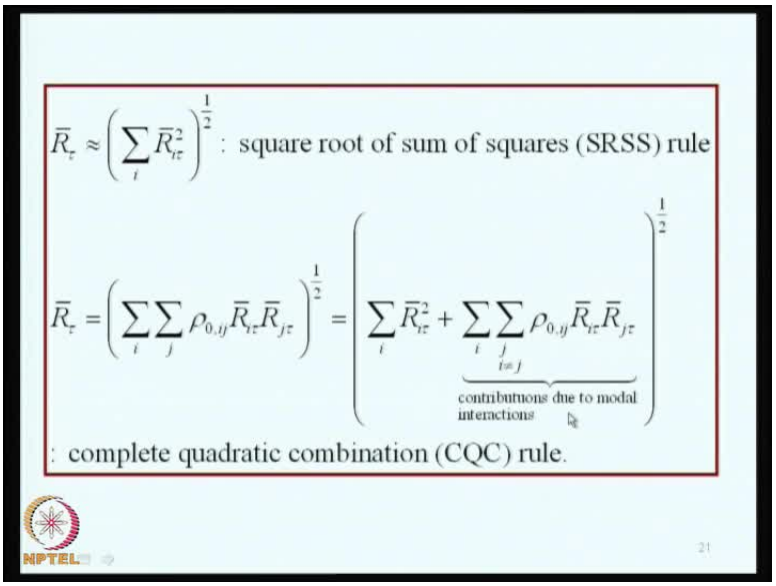

Now, if we ignore the model interactions, we get a rule known as square root of sum of squares are known as SRSS rule. So, this is how this model combination rule that will use. We simplify the maximum of the individual generalized coordinates, and square them and add it out; add all of them, takes square root; so, this sum of squares root of sum of squares method.

Now, we can get an improvement to this rule by including the terms which contribute to the model interaction; and if we do that, then this rule is known as complete quadratic combination rule. So, these two rule are some of the popular model combination rules, that is widely available in software. So, the point that you must understand is the geniuses of this rules lie in the application of principles of random vibration analysis.

Although in a practice, one may not model earth quake ground acceleration, random process and perform a full-fledged relay ability analysis, but still even for a deterministic analysis, some of the rules that are followed, there in have basis in linear random vibration principles; so, that is the point being made here.

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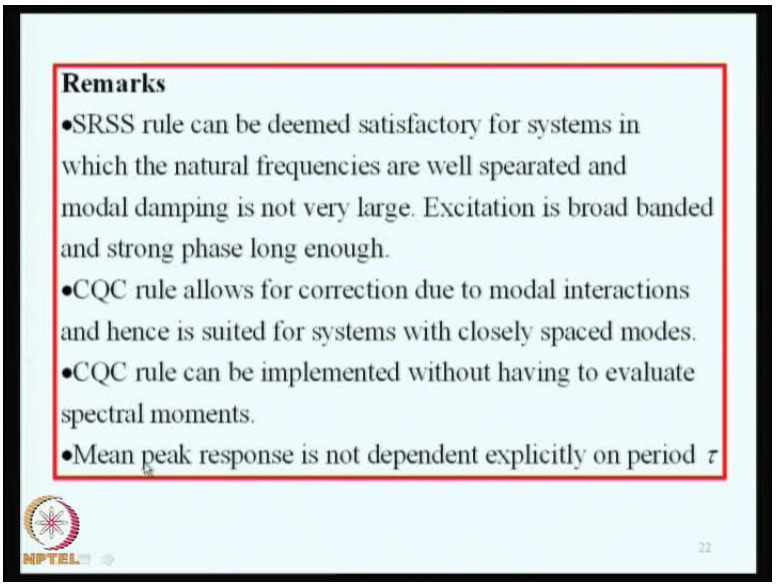

If you remarks, this SRSS rule can be deemed satisfactory for system in which the natural frequencies are well separated and model damping is not very large. And of course, we also want that excitation is broad banded and the strong phase of the ground shacking is long enough, because we are assuming the excitation to be stationary white noise.

So, that requires broad band excitations and excitation to fairly long, so that thus the stationarity assumption is justified. Since we are ignoring model interactions, we require that the natural frequencies are well separated and the modal damping is not very large. If modal damping becomes large, that will itself induce interaction between laboring modes. So, if these conditions are satisfied, SRSS rules provides reasonably good solution.

On the other hand, the CQC rule allows for correction due to modal interactions and hence is suited for system with closely spaced modes. The way we have derived this CQC rule, it can be implemented without having to evaluate the spectral moments; so, that is a advantage of making the simplifications on the nature of excitation, bandwidth, etcetera. Now, mean peak response is not dependent explicitly on the period tau, so that is the another feature of applying CQC rule.

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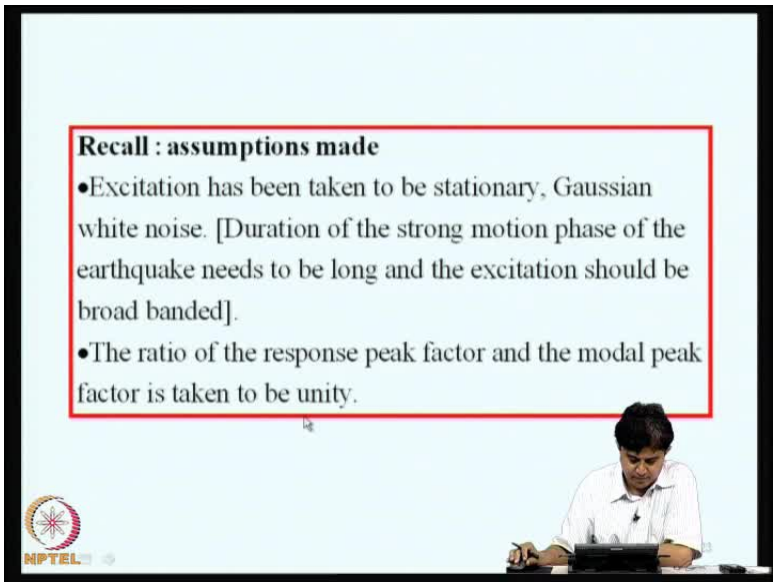

Some of the assumptions can be recalled again; we are assuming that excitation is taken to be stationary, Gaussian, white noise. So, that duration of the strong phase motion phase of the earthquake needs to be long and the excitation should be broad banded. The ratio of response peak factor and the modal peak factors is taken to be unity; this also has to be born in mind.

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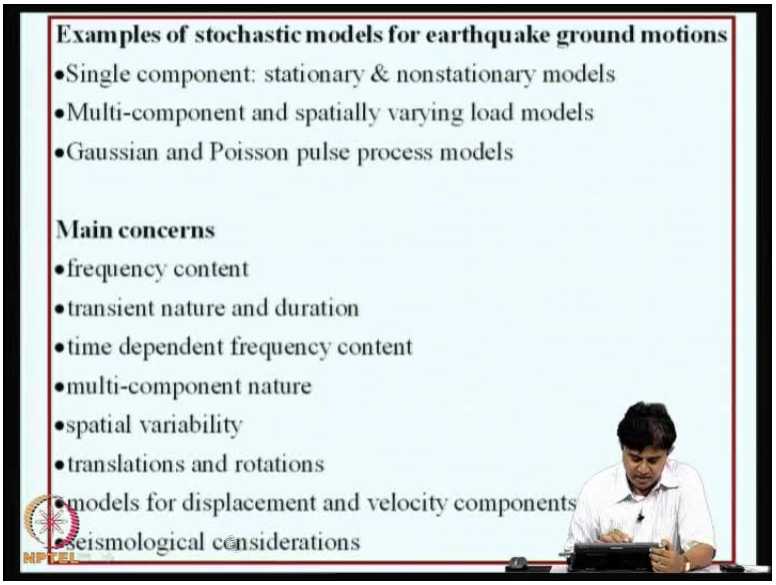

Now, we begin the discussion on the next topic, how are stochastic models arrived for earthquake ground motions. So, I will present a sequence of examples with reasonable as

increasing level of complexity, so that you get flavor of how stochastic process theory has been applied in modeling earthquake ground motions.

So, we will begin with discussion on single component; we will assume to start with the earthquake ground acceleration is a stationary random process. And we will consider the question how to characterized its power spectral density function. Next, we know that earthquake is a transient phenomena; therefore, the use of a stationary model is not entirely satisfactory. So, what is a influence of the transient nature of the excitation in arriving a stochastic model for excitation?

This non stationarity as we will see shortly can be in terms of amplitude modulation or it could be in the frequency content. The frequency content of earthquake ground acceleration in different times could be different, because the earthquake ground acceleration is a consequence of summation of a different kinds of waves. And each of these waves have characteristic properties; therefore, given that these wave arrive at different times, the ground acceleration will be dominated by different types of wave types at different times, and consequently, there will be a non stationarity in frequency contents also.

The earthquake ground acceleration at any point is a vector. So, it can be resolved into three components of translation and three components of rotation; therefore, if we are interest in characterizing the effect of earthquake ground motion at any point, we need to construct a vector random process model. So, that is the next level of complexity that we can consider.

Similarly, as earthquake was propagate through the earth's crust, the ground acceleration would vary from point to point; it would not be the same at all the points on the earth's crust. And we need to worry about the influence of this spatially variability on the load models and that takes us to random field models for earthquake ground acceleration.

There are two approaches to modeling earthquake ground motion: one is assuming that earthquake motion is a Gaussian random process, the other alternative is to use what is known as Poisson pulse process models; this we have discussed briefly when we discuss theory of random process. I will return to some of the basic issues during these discussion.

So, what are the main concerns in modeling? We would like to capture correctly the frequency content in the ground motion, that depends on the actually the local soil conditions; the frequencies typically vary from up to about 30 to 35 hertz. And the next is a transient nature and the duration; the earthquake does not last forever, typically 30 to 30 second to may be 120 second; one could experience strong ground motion and this has to be reflected in our model.

As I was already mentioning, there is issues about time dependent frequency content, that also needs to be included. Then, questions about multi component nature of the ground motions and spatial variability, that needs to be considered. And question on modeling translations as well as rotations; the rotations are not widely discussed, but translations are discussed.

And when I talk about translation, we have model for not only acceleration, but also we need to consider models for associated displacement and velocity components. Now, we should also think of what are the underline seismological considerations in the arriving at the power spectral density functions models, should be using a set of instrumental records. And then, make a model through statistical methods using statistical principles or should we also include to certain extend the physics of propagation from focus to the side. So, these some of these issues will touch upon as we go along.

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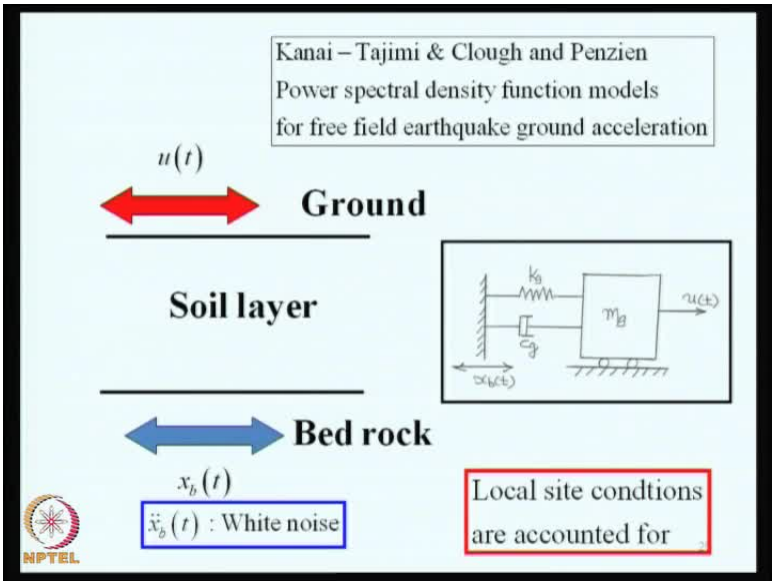

One of the conceptually simple model for earthquake ground acceleration is the provided by Kanai and Tajimi. Here, we consider the soil layer at a given site; and we assume that, at the bed rock level, the earthquake induced ground acceleration is a white noise process. And we will model this soil layer as a single degree freedom system, and the absolute response of this mass is taken as the so the called free field ground acceleration. So, this is the simple mechanistic model and it basically takes into account the local site conditions in arriving at the ground acceleration.

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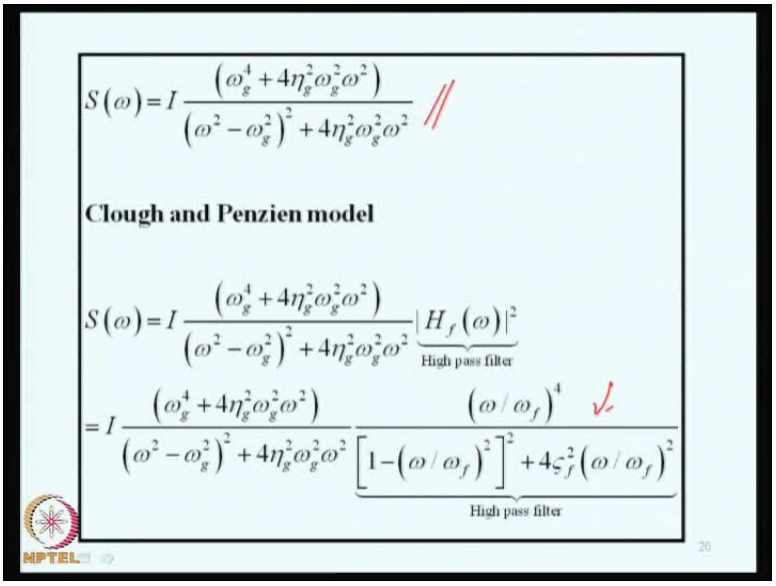

So, this have discuss at various points in the course. So, I need not have to get into all the detail; so, by applying methods of linear random vibration analysis, we can derive the power spectral density function for the absolute acceleration at the ground surface; and this is the so called Kanai Tajimi power spectral density model - this one.

We also saw that, this power spectral density function, this model has certain difficulties as omega becomes small, especially in specifying displacements and velocities. To circumvent that, Clough and Penzien propose that, the ground acceleration be further be pass through high pass filter and that filter characteristic is shown here. So, the ground acceleration power spectral density thus now becomes gets filter through another mathematical entity; this is the high pass filter transform function.

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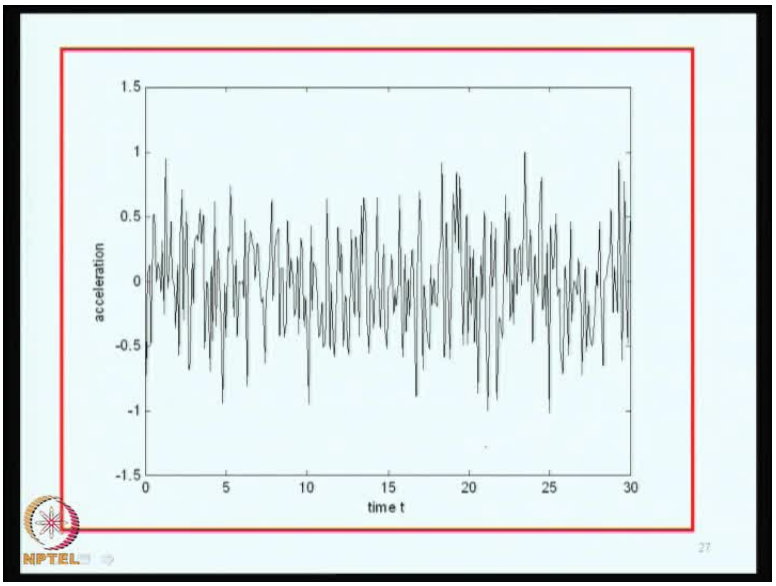

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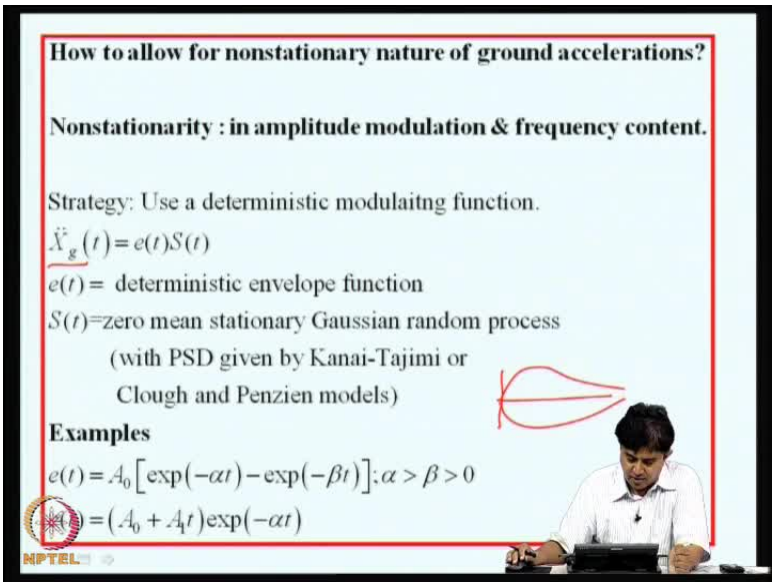

So, if you use this model, they ground acceleration schematically looks like this; this is a sample of a stationary random process; and the question is, how do you introduce non stationarity?

Now, non stationarity as I was mentioning, there are two issues that we need to consider: one is the non stationarity in amplitudes, the other one is frequency content; for time being if we focus only on amplitude modulation, we can if use this strategy where we consider the ground acceleration to be product of e of t into S of t, where e of t is a

deterministic envelope function, and S of t is a 0 mean stationary Gaussian random process, which could be the actually the Kanai Tajimi or the Clough and Penzien random processes.

So, in the literature, there different models have been proposed for the this envelope; some of them are here, this is A naught exponential e raise to minus alpha t minus e raise to minus beta t; this is combination product of A naught plus A 1 of t into exponential minus alpha t. So, there are many more such envelopes in the existing literature, say, they are basically capture the non-stationary trend in the earthquake ground motions.

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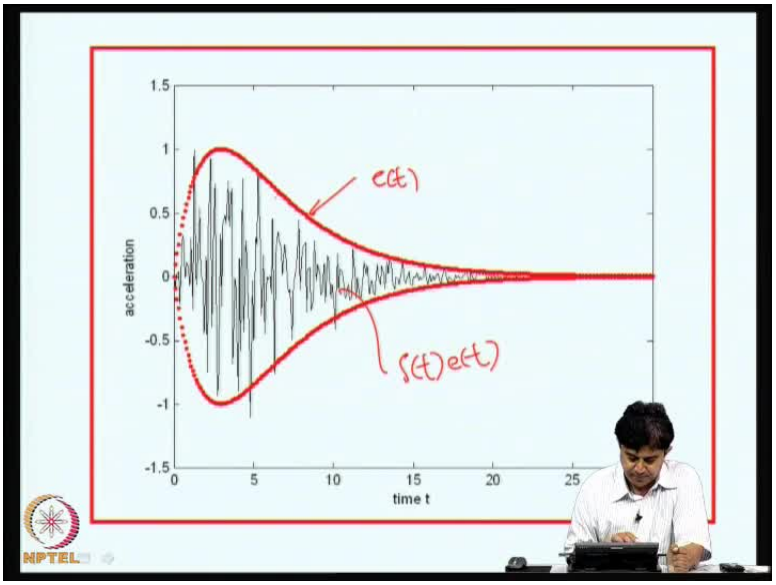

So, one of that typically is shown here; this is the e of t and this is S of t into e of t. So, if we generate S of t following say Kanai Tajimi model or Clough and Penzien model, we will be able to take into account the local soil conditions; and this envelope ensures that, we are capturing the transient nature on ground motion.

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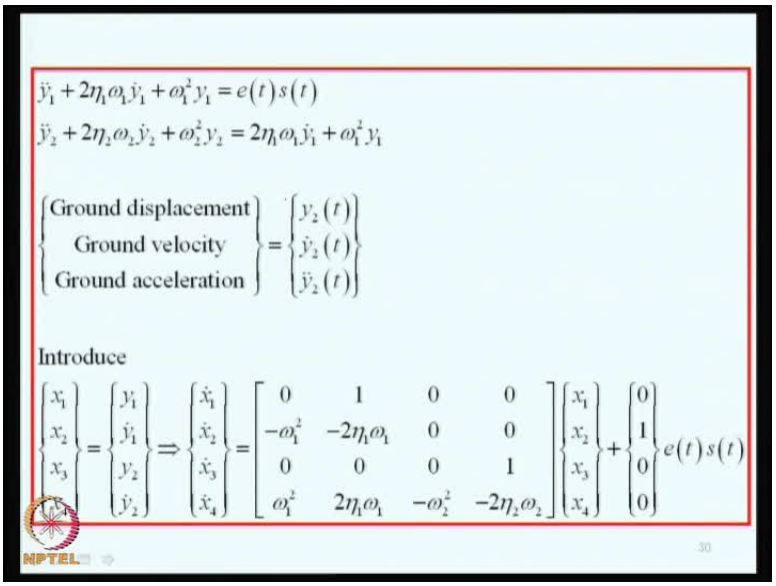

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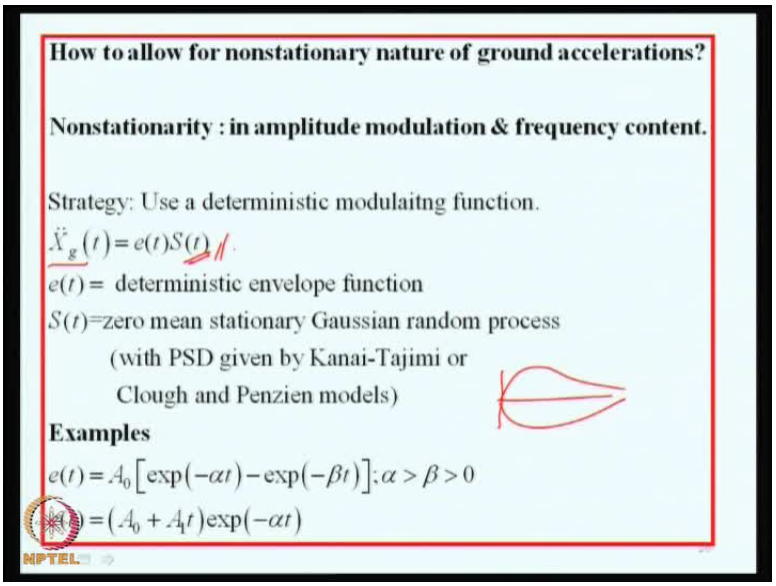

I also discussed in alternate approach to simulate ground displacement, velocity and acceleration. In the model that we discuss that so far, if we are using Fourier series representation to simulate samples of S of t, how to simulate the ground displacement and velocity associated with this model is not very clear.

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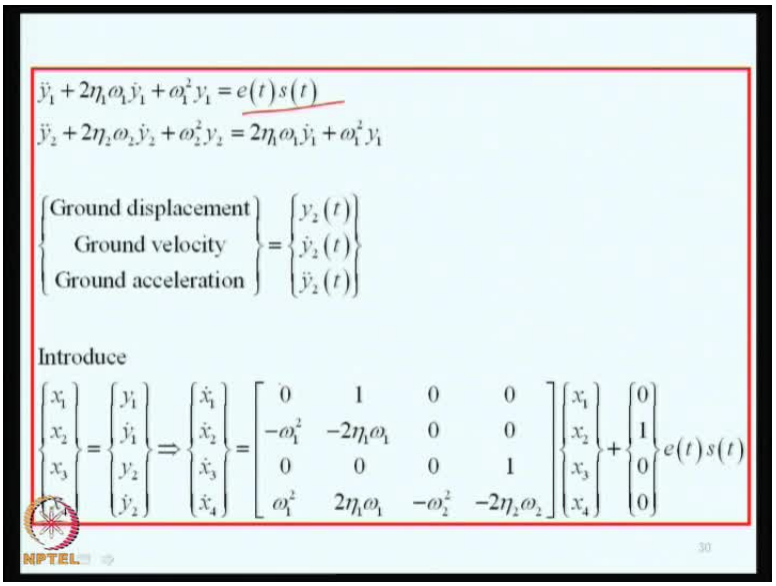

So, what we could do is, we start with the equations for the soil medium and the high pass filter, and right the time domain equations. So, what we assume is head the bed rock level, the excitation is modeled as a envelope multiplied by a white noise process. So, in the one of the previous lectures, we discussed how this set of differential equations can be interpreted as a set of stochastic differential equations, and how we could obtain samples using I taylor expansion bastes numerical schemes.

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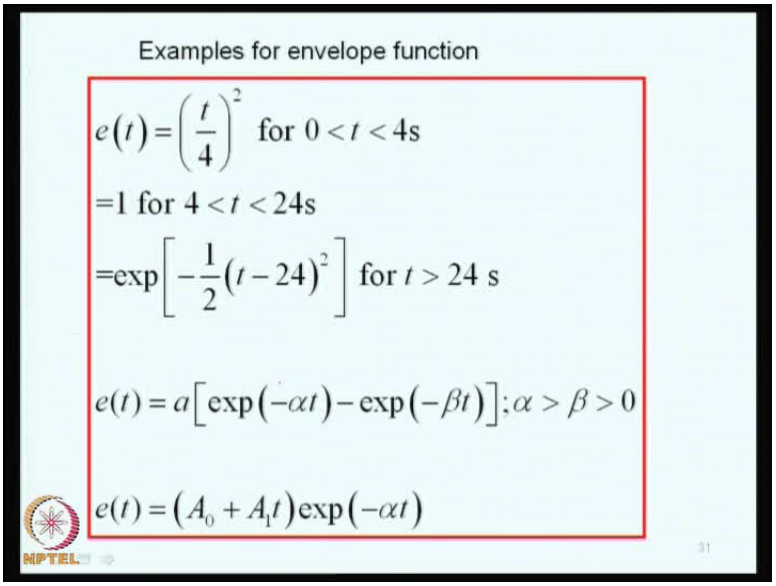

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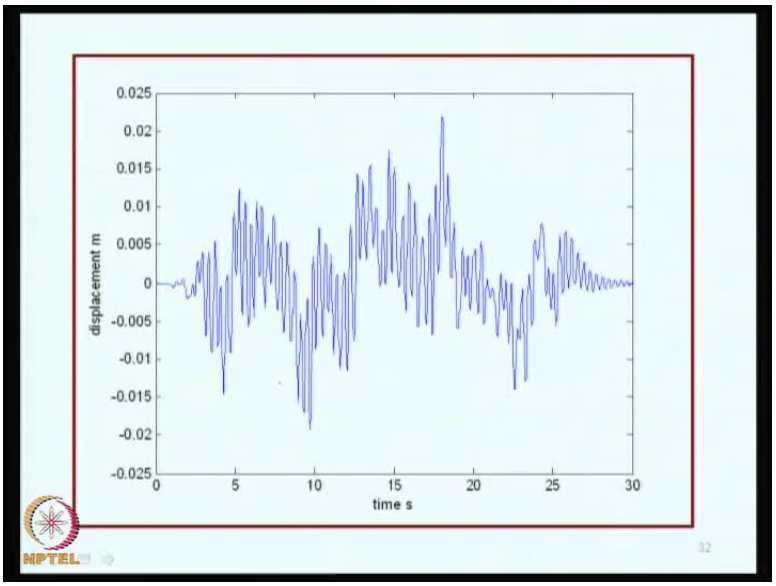

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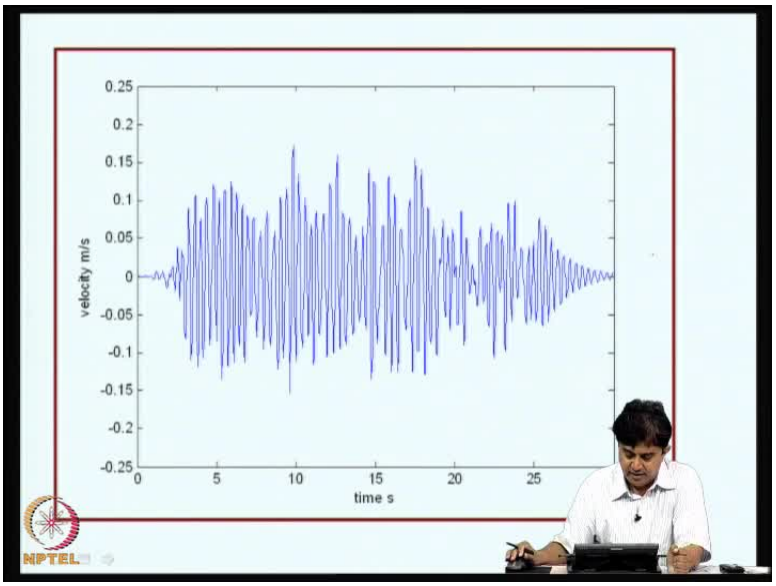

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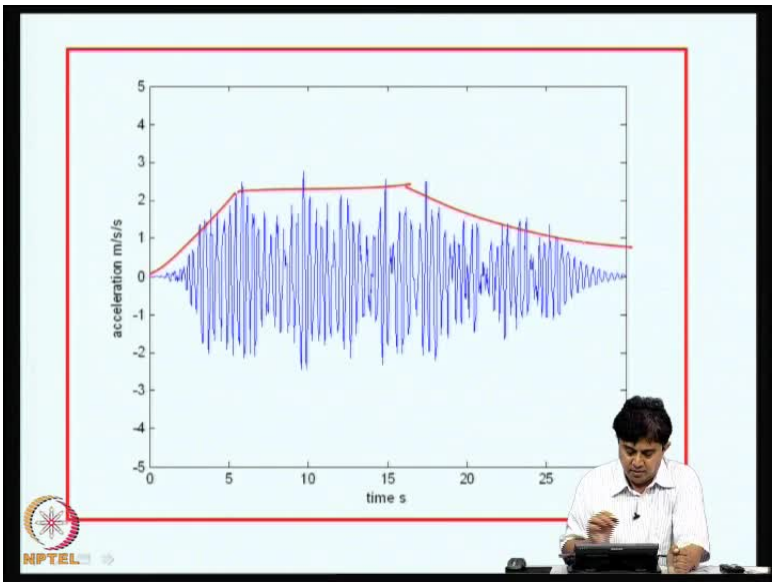

So, based on that, I showed the some of these samples; this is the sample of your displacement. We can clearly see there is a non-stationary trend, this is velocity, this is acceleration; so, they envelope here is something like this.

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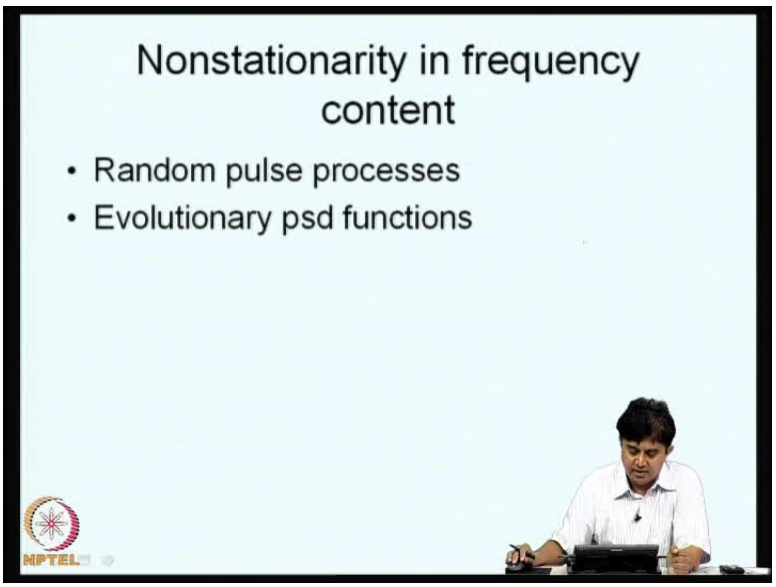

Now, how about the problem of treating non stationarity in frequency content. So, here, we will discuss what are known as random pulse processes, and we will show that, such processes have what are known as evolutionary power spectral density function; and they will be successful in capturing non stationarity in both frequency and amplitude.

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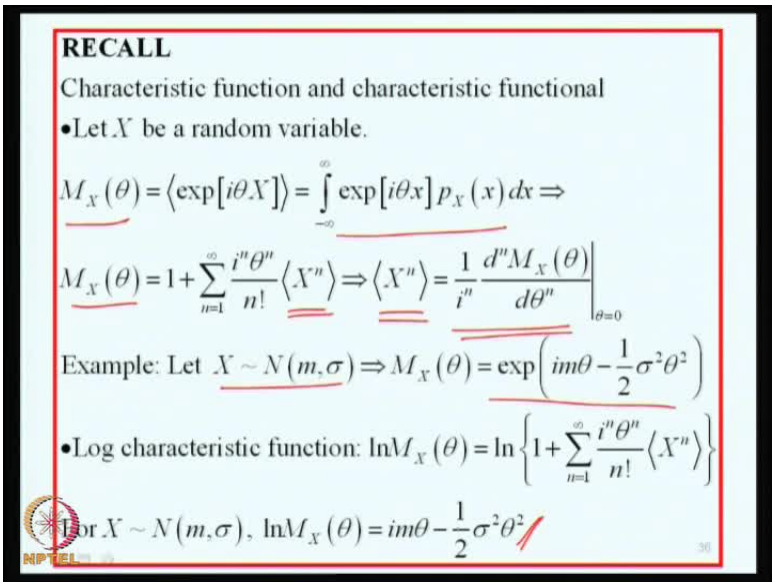

Now, we need to recall some of the concepts that we discussed. When we introduce probability and random processes, there were few thing that where postponed during that discussion. Now, it is a time to return to some of that, and we will quickly see there, what mathematical tools we need to analysis a random pulse process.

So, the concept of characteristics function and characteristics functional is central to this discussion; so, we will quickly recall this notion. So, let x be a random variable, and we define the characteristics function as the expected value of e raise to i theta x and this is this expectation.

Now, if you expand i theta x in series, we can show that this M x of theta is related to the moments of the random variable x, and using if we are given this characteristics function, we can evaluate the moments using this form; so, they are also moment generating functions.

So, for example, if X is Gaussian random variable with mean M and standard deviation sigma, we have shown that the characteristics function is given by this. And we also defined what is known as log characteristics function, which is logarithm of the characteristics function and this is return in this form. And for a Gaussian random variable, the log characteristics functions is a defined through this expression.

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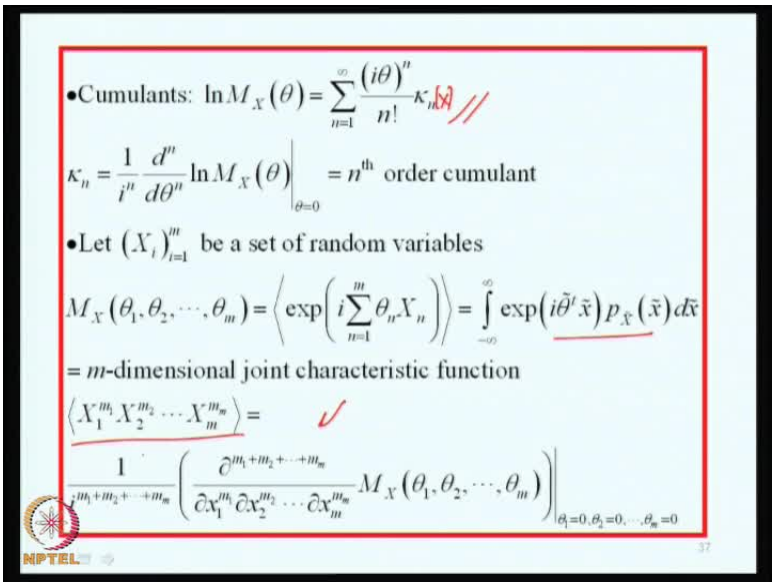

Now, if we write the log characteristic function for a general variable, this will write it as k kappa n of x; this kappa n of x which are the terms appearing in this kind of representation for the log characteristic function are known as cumulants.

Now, we will see is more on this shortly, but we will now generalize the notion of characteristic function in the log characteristic function for set of random variable. Suppose if I consider M random variables X 1, X 2, X m, then the Mth order joint characteristic function is defined as the expectation of e raise to i summation theta n x n. So, this can be expressed in terms of m l t dimensional integral expectation of this quantity.

So, this is m-dimensional joint characteristics function; no, if are given m-dimensional characteristics function, we can evaluate moments of this kind, in terms of the joint characteristics function using this formula; this also can be proved easily by, if we expand e raise to this terms inside the parentheses in a series and use the relations as shown here, we will get the required moments.

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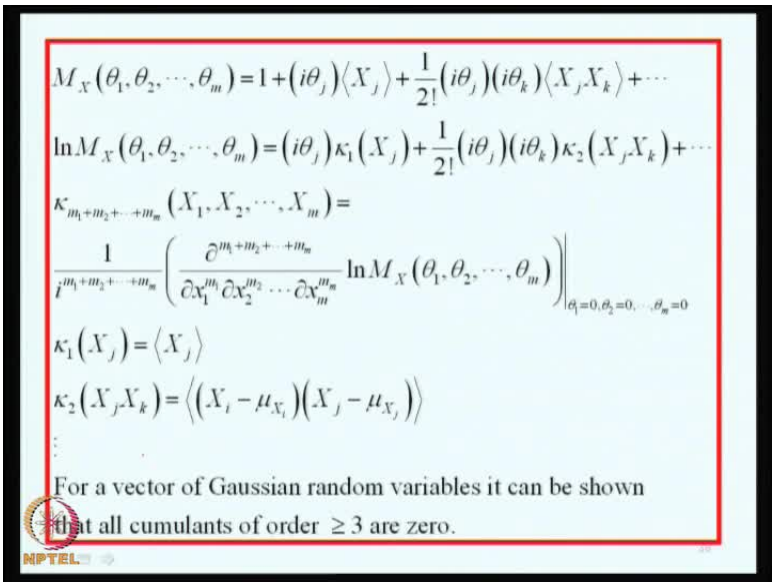

Now, we can follow this, we can write the joint characteristics functions in terms of the moments. So, this will include first order, second order and mth order joint moments of all this random variables, and it is a series. Now, if I take the logarithm of this, I get the log characteristics function, and this can be return in yet another series, where is kappa's are the cumulants, and cumulant of the order m 1 plus m 2 plus so on and so forth; m m is given through this relation.

And we can show that, first cumulant is the mean value; in the second order joint cumulant is actually the covariance. And we can derive the higher cumulants which will be related to a the moments of this random variables x 2 x 1, but these relation are more complicated. But for a Gaussian set of Gaussian random variables, it can be shown that all cumulants of order greater than 3 or equal to 0 in the first cumulant is the mean and second cumulant is the covariance.

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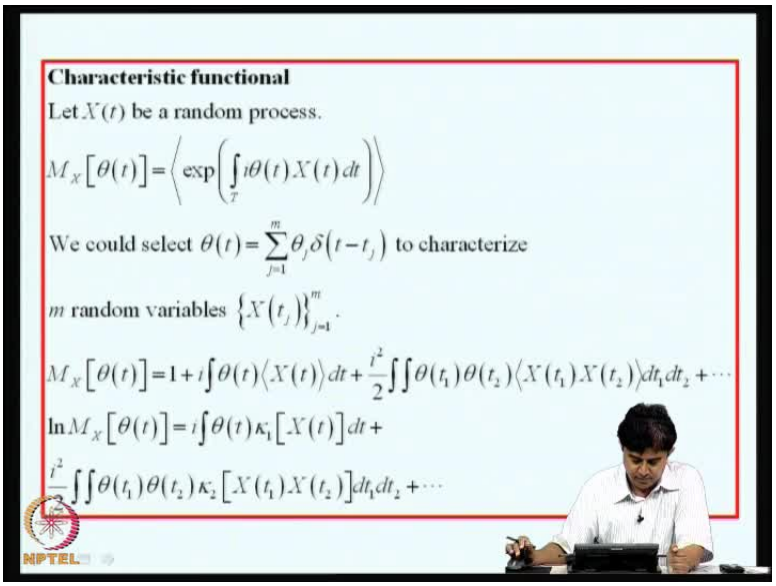

Now, how do we generalize the notion of a characteristic function to a random process? Suppose X of t is a random process, and we defined what is known as characteristic functional, where i denoted by M x of theta of t, which is actually the expected value of integral i theta of t x of t dt.

Now, of course, if I take theta of t to be a sum of direct delta functions, we will get the joint characteristics function of the m random variables a stationed at t equal to t 1, t 2, t 3, t n, but I am not interest in this; I am interested in a general form of the characteristics function, that is called characteristic functional. So, here, again I can expand this e raise to this quantity in a series, and we get these expressions and we can take logarithm of this; this will help us to define the first cumulant and second cumulant, and so on and so forth.

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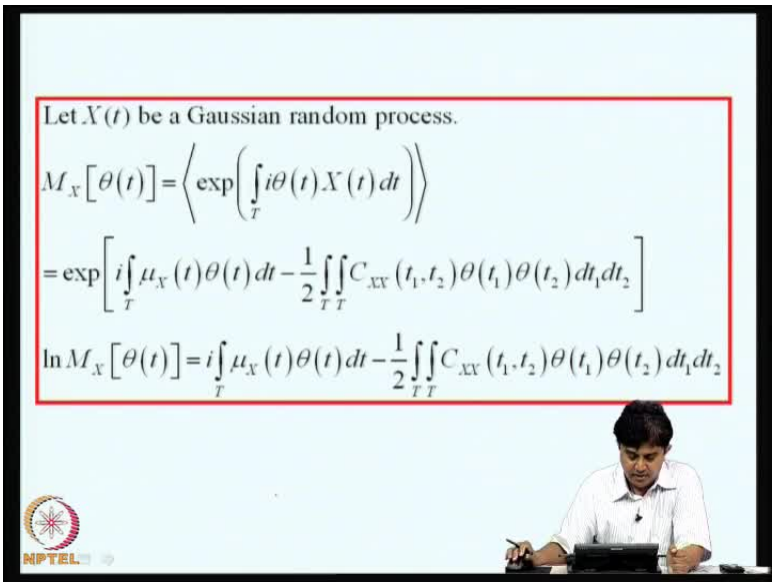

For a Gaussian random process, we can show that the characteristic function is given in terms of the mean and the covariance function; and this is the characteristic function. The logarithm of this is again expression terms of the mean and the covariance function, this can be verified; this is conceptually simple, but some more t ds to write down.

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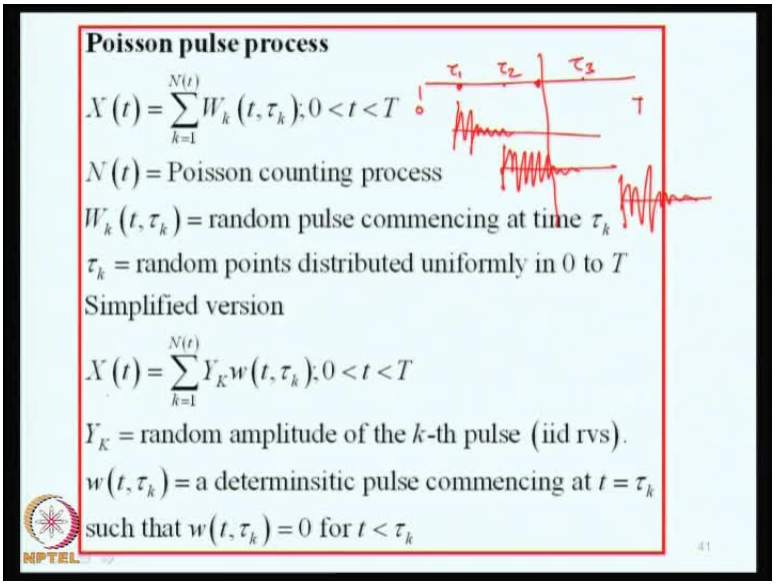

Now, we will now consider a random process x of t, we call it as Poisson pulse process, where this summation is from K equal to 1 to N of t of certain pulses, W k t comma tau

k, where n of t is a Poisson counting process; w k t comma tau k is a random pulse commencing at time tau k; tau k are random point distributed uniformly in 0 to t.

That means, we have a time access, say 0 to capital T. And suppose there are say some randomly placed points, we assume that the process x of t is a consequence of a pulse that arrives at, say, this is tau 1, this is tau 2, this is tau 3. The pulse arrive the tau 1 is, this is my w 1; the pulse arriving at tau 2 is another random process; tau 2, tau 3 is a another pulse. So, at any time t, there will be the X of t that we see, will be the sum of the effects due to the various pulse that have arrived up to this point.

Now, w k t comma tau k is a random pulse commencing at time tau k. Now, a simplified version of that would be, we can assume that this shape of this pulses are identical, but they are amplitude is modulated by a random variable Y k; the shape is the same and arrival times are still random. So, in that case, Y k is the random amplitude of the kth pulse and Y k take to be i i d - set of i i d random variables - and w of t comma tau k is deterministic pulse commencing at t equal to tau k, such that, w of t comma tau k is 0 for t less than tau k; that means, till the time tau k is arrived at this, w is 0. Now, with this description, I am now interested in characterizing the random process x of t; how do we do that?

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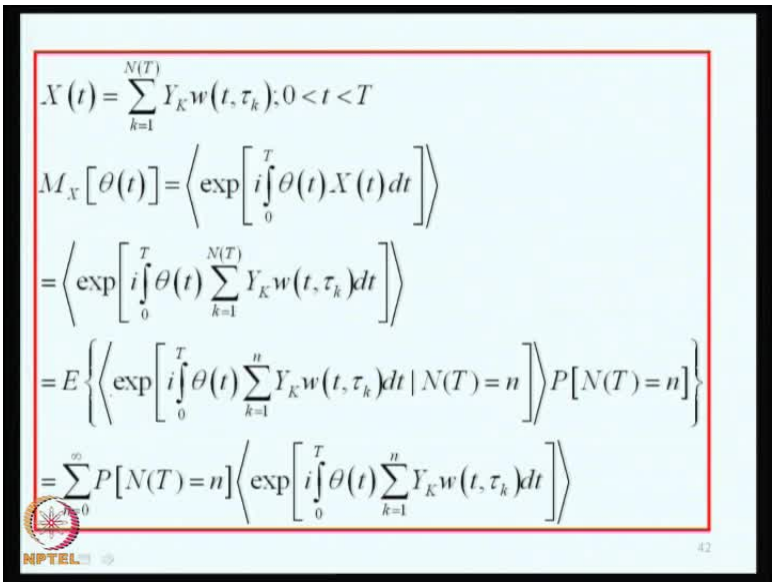

Suppose I look at characteristic functional, this is expected value of i integral 0 to T theta of t X of t dt. So, for X of t, I write now this expression and I am taking expectation;

what I will do is, first I will condition on N of t being equal to n and then multiplying by probability of N of t equal to n, and take this m x of theta is a double expectation.

So, if we do that, the first the conditional expectation can be evaluated and it can be some later over this these probability; and this is what we get here. So, the probability of N of t equal to n and expected value of this quantity, where this N of t is conditioned at n.

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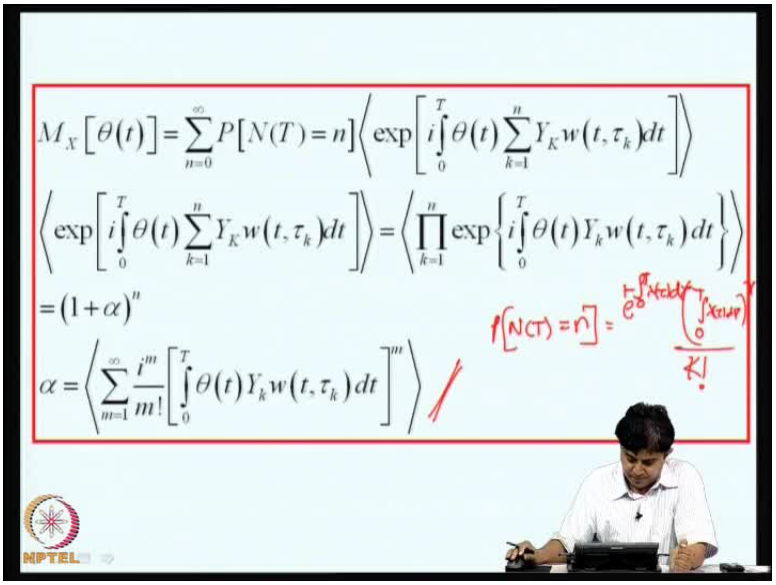

Now, for probability N of t equal to n, I can write now the Poisson model for that; we get this expression as shown here and we introduce a notation alpha, where alpha is this particular quantity. So, in this steady, P of N of T equal to n, we are assuming the arrival rate to be… and integral lambda of tau d tau 0 to t to the power of k by k factorial. So, this is the model that we are assuming for Poisson model with a time depended arrival rate.

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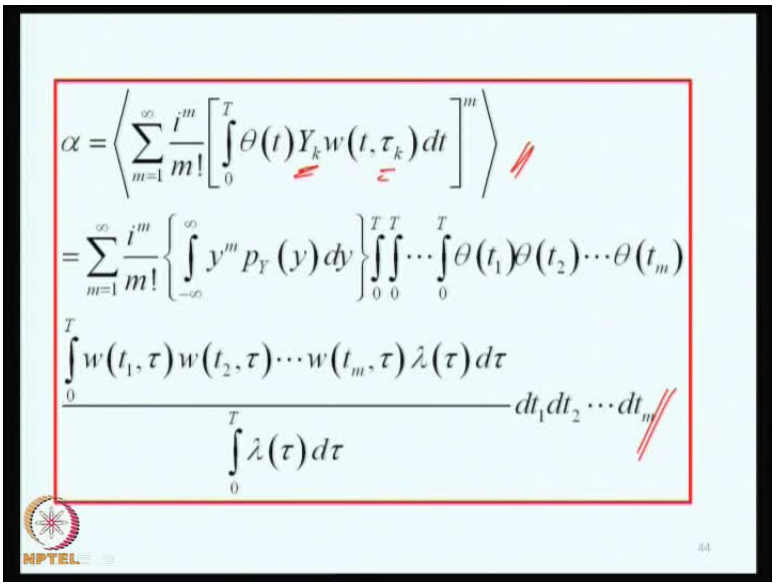

Now, this alpha is given by here and this can be simplified, this is an expectation. Now, what is random here, are the random variable Y and this tau k; and if we interpret them carefully and allow for the faceted Y k's are i i d and tau k's are Poisson points, we get this expression.

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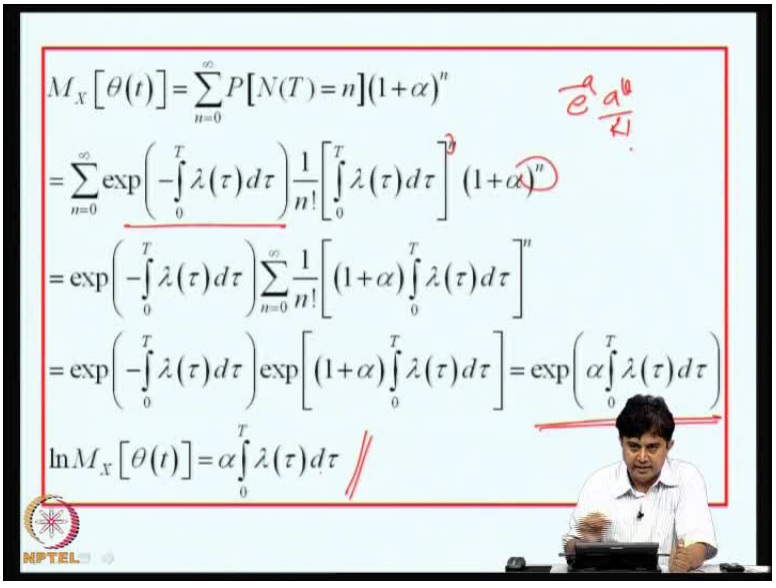

So, we will return to the definition m x of theta of t; this is probability of N of t equal to n into 1 plus alpha to the power of n. So, now, N of t equal to n is nothing but this exponential minus 0 to t lambda of tau d tau 1 by n factorial and 0 to t lambda tau d tau;

this is likely raise to minus a to power of k by k factorial into 1 plus alpha 2 the power of n.

So, we can rearrange this terms a bit; this term is independent of nth - first term that is taken out. And we take this 1 plus alpha inside this, because that is the power n that is common to both these expressions. And if we simplify, we can show that the log characteristic function is given by this and the characteristics function is this. So, the log characteristics alpha into 0 into t lambda of tau d tau.

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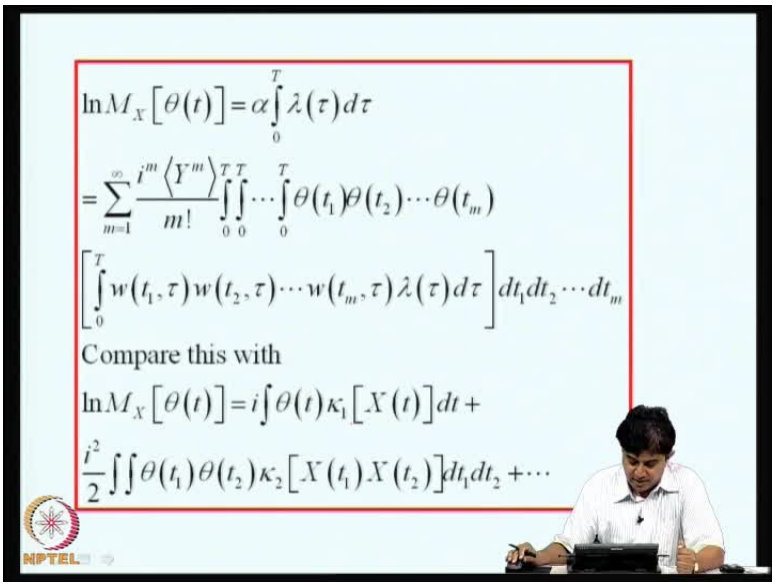

Now, we are interested in finding the cumulants. So, we look at this log characteristics functional bit more carefully and we substitute for the expression for alpha in this. And if we compare this with the series expansion for the log characteristics function, we will be able to identify the first and second order cumulants.

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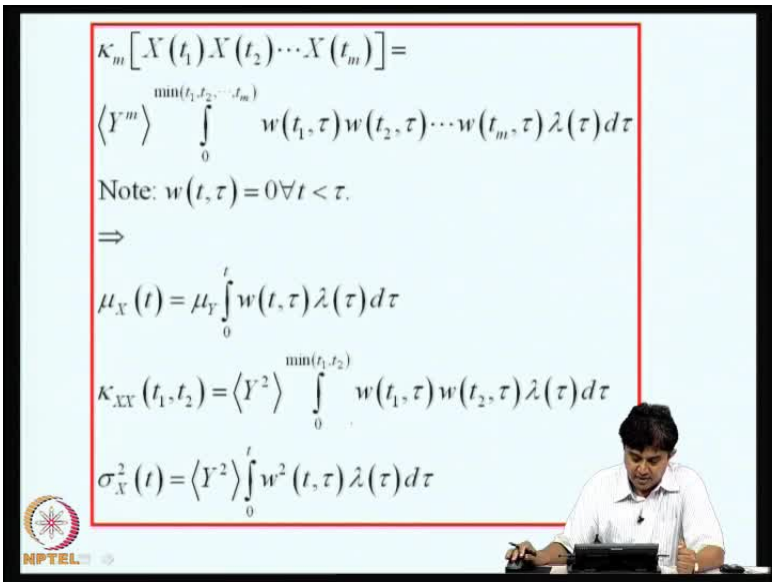

And you can show that, the mth order cumulant is given in terms of the random variable y and this pulse in this form. Now, the integral upper limit is minimum of t 1 comma t 2 comma t m, simply because of this definition; you can verify that it is indeed true.

And from this now, if we look at the mean the first cumulant, it will be given by this; and second cumulant is given by this. And from this, if I now consider second cumulant is nothing but the covariance; so, if you from this, we can compute the variance and we get in this form.

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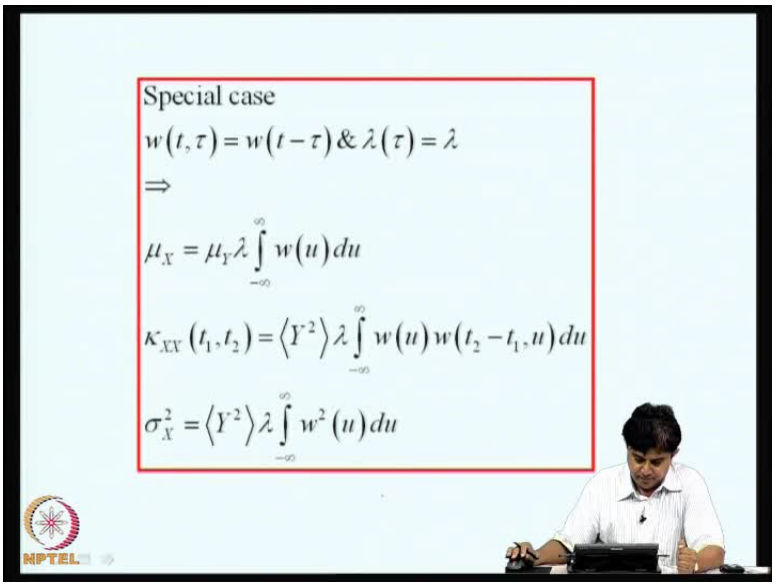

If we take special case of w of t comma tau to be a functional time difference tau, this simplifications and arrival rate to be constant; the formulary leads to certain simplifications. I leave this as an exercise, you can show that these simplified results are obtained under these assumptions.

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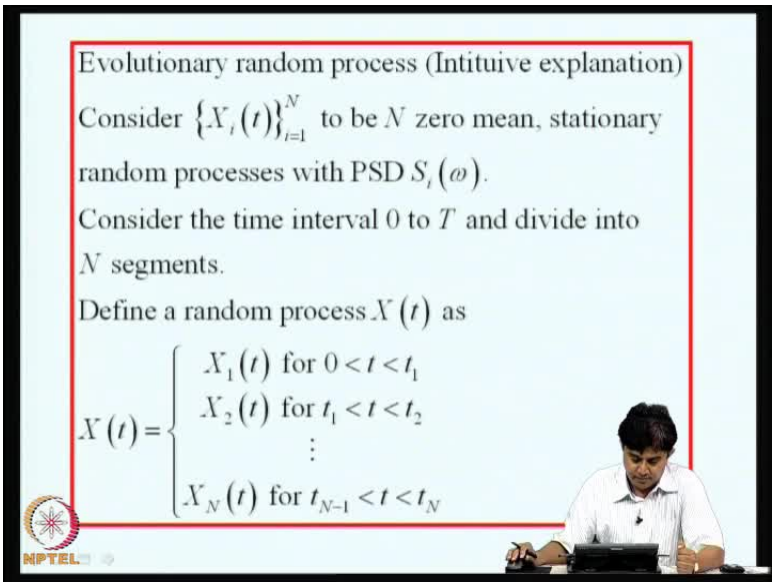

There is yet another concept that we need to discuss, before we consider modeling the non stationarity in frequency content in ground accelerations, that is motion of evolutionary random process. So, I will begin by offering an intuitive explanation for this; so, let us consider a set of random processes X of y of t, say, i from 1 t n; that is, X 1, X 2, X 3, X n. Let us take that, these each of this random processes have 0 mean and their stationary, and the power spectral density function is S i of omega, for x i of t.

Now, let us consider a time interval 0 to T and divide into N segments. Now, I will define a random process X of t as X of t equal to X 1 of t, if t is between 0 to t 1; otherwise, X 2 of t, it is between t 2 t 1 to t 2, and so on and so forth. So, X of t is defined in terms of capital N number of segments, where each segment in one random process X 1 of t. Since properties of X 1 X 2 X 3 X N are all different, it is clear that X of t is a non-stationary random process.

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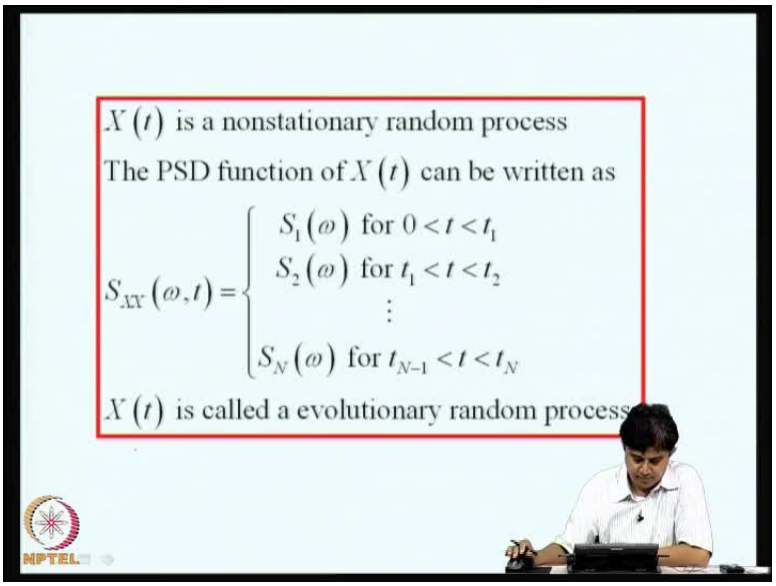

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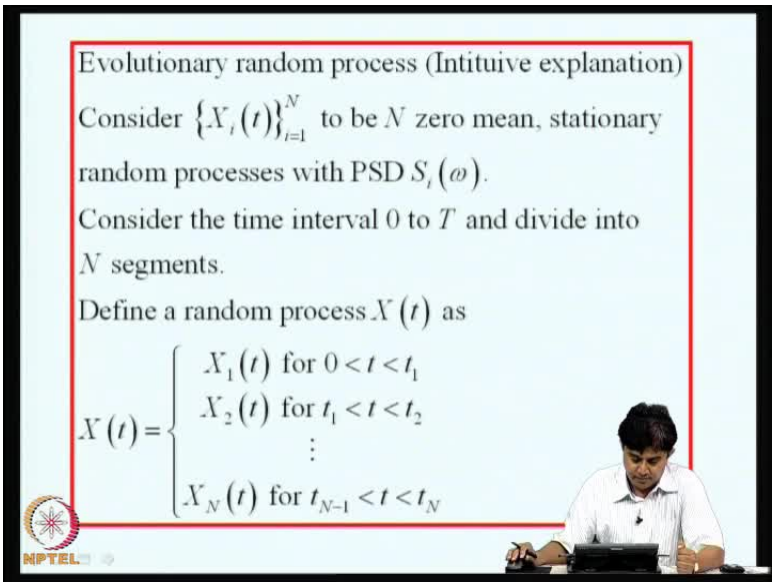

Now, if we look at the power spectral density function, since X of t is equal to X 1 of t over this duration, I can notionally say that the power spectral density function of X of t; if you are interested in the time duration 0 to t 1, we will be S 1 of omega. And if you are interest in t 1 to t 2, it is S 2 of omega, and so on and so forth.

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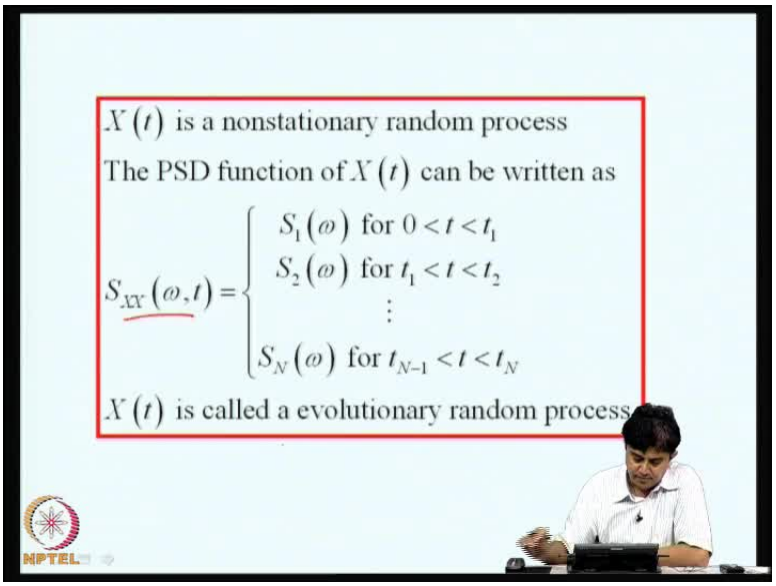

So, I can say that, the power spectral density function is a time dependent function; this is equal to S 1, if t is between 0 to t 1; if it is S 2, if it is t between t 1 to t 2; and so on and so forth. Such a process we call it as evolutionary random process.

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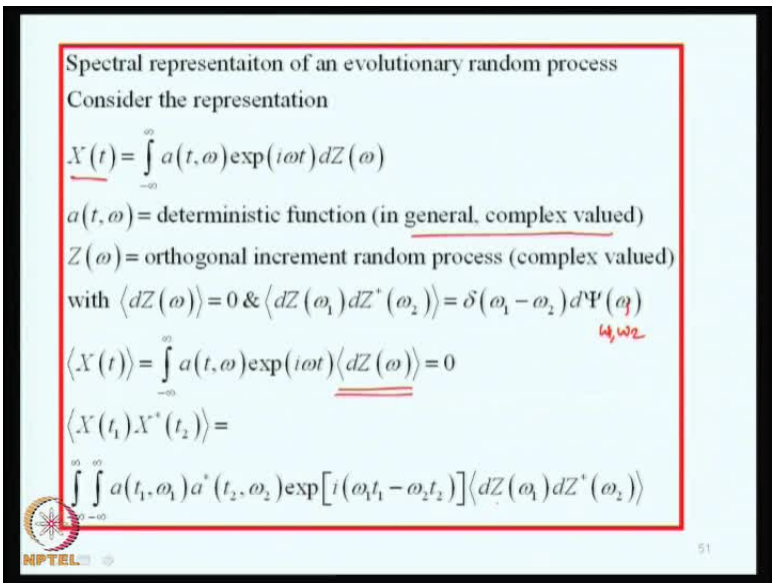

We can generalized the notion of this evolutionary random process, where the number of segments becomes very large and we need to develop certain alternative formulations. So, we consider now the representation kind of a Fourier representation, but in terms of

Riemann (( )) integral. We consider X of t to be a of t to be omega exponential i omega t dZ of omega.

This a of t comma omega is a deterministic function in general complex valued. And Z of omega is a orthogonal increment random process, it could be complex valued, with expected value of dZ of omega is 0, and dZ of omega 1 into dZ star of omega 2 is direct delta of omega 1 minus omega 2.

So, here, we can write this as omega 1 comma omega 2. Now, what is the mean of X of t? If you flow this, mean of X of t is 0, because mean of this increment is 0; and if we now consider the covariance X of t 1 into x star of t 2, I am assuming the X of t less complex value. We can write this in terms of this, we get this expression.

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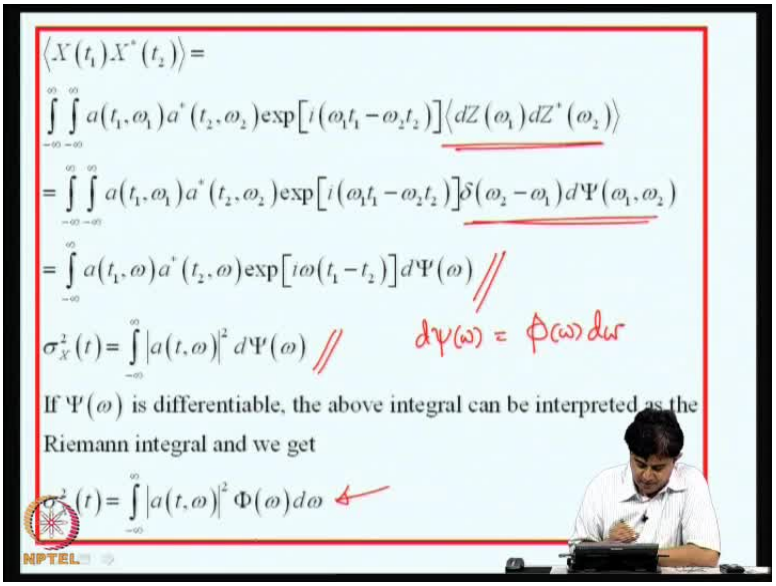

And if we now write for this quantity, the direct delta function and d psi of omega 1 comma omega 2, we can we get this simplified expression, because there is a direct delta function; one of the integration can be done easily.

Now, if t 1 equal to t 2, I get the variance and I get this expression. Now, if this capital psi of omega is a differentiable function, then the above this integral can be taken a Riemann integral. And I can write this as, d psi of omega can be return as phi of omega into d omega; if I do this, we get this expression. And now, phi of omega, for example, if a is not function of time, phi of omega is nothing but the power spectral density function.

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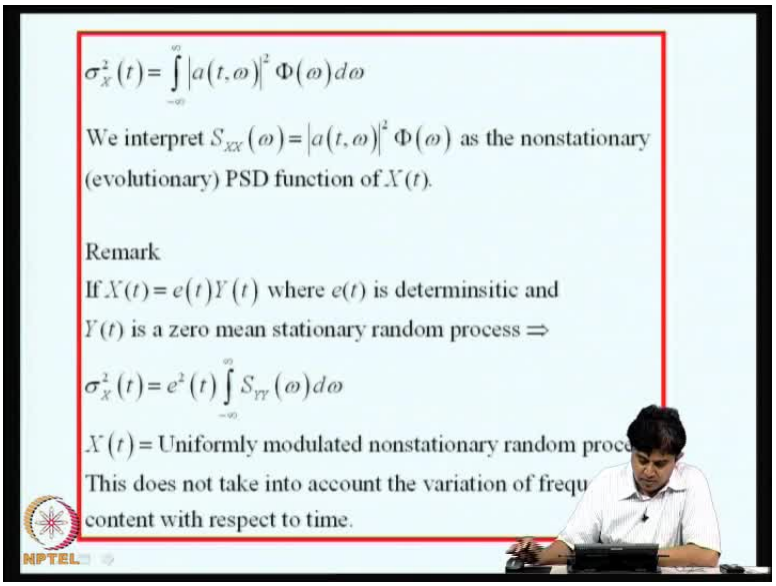

Now, we interpret this function modulus of a of t comma omega whole square into psi of omega is the non-stationary or the evolutionary power spectral density functional X of t. Now, we have already seen random process models, where X of t is obtained a e of t into Y of t, where e of t deterministic and Y of t is a 0 means stationary random process.

If we now use this concept here, we get sigma X square of t is e square of t minus infinity to infinity S y of omega d omega. If e square of t is unity, it is independent of time; then, of course, we recover the stationary random process. In this case, where e of t is a functional time, we call X of t to be uniformly modulated nonstationary random process. Here, the non stationarity is in nonstationary feature that we are capturing is the amplitude modulation in x of t but not the frequency variation.

So, you want to capture the time varying nature of frequency content, we have to use this kind of model. This are evolutionary power spectral density function model. I will show you in the next lecture, that the Poisson pulse process, if you look at its power spectral density function, it has evolutionary - it is actually evolutionary power spectral density model; so, that I will show in due course.

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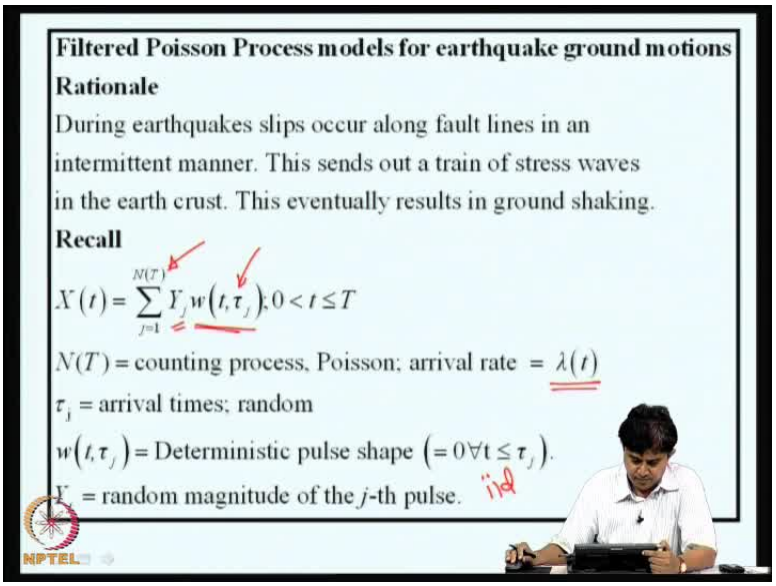

Now, equipped with these basic mathematical tools, now we will return to the problem of modeling earth quake ground motions. And we propose that earthquake ground motion we modeled as filter Poisson process, that is the Poisson pulse process that we are discussing just now.

The rational here is that, during earthquakes, slips occur long fault lines in an intermittent manner; this sends out train of stress waves in the earth crust. And this at any ground point on the crust, this waves arrive as how the slipping process proceeds; these are waves arrive at different times. This super position of different waves results in the ground shaking.

Now, we can propose a model, therefore, that X of t which is a ground motion observative at a given side, is superposition of several stress waves with random amplitudes arriving at random time instance. And the number of stress wave that arrive is modeled as a Poisson counting process. So, n of t is a counting process, it is Poisson, and arrival rate for purpose of discussion, we take it to be function of time; tau j are the arrival time which are again random. We take this pulse w t comma tau j deterministic pulse shape; for it is equal to 0, for T less than are equal to tau j, Y j is the random magnitude of jth pulse; and typically, we will take Y j to be a sequence of i i d's.

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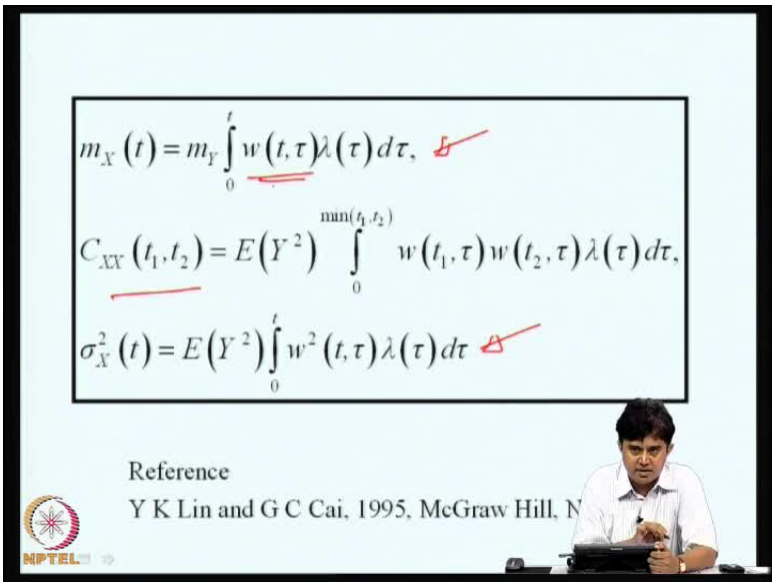

We have shown now through the use of characteristic functional, what is the mean and what is the covariance of this pulse process. So, this is the expression for the variance. Now, as I said, I will be showing shortly that this process actually captures the, not only captures the modulation non stationarity in amplitude modulation, but also the time varying nature of the frequency content.

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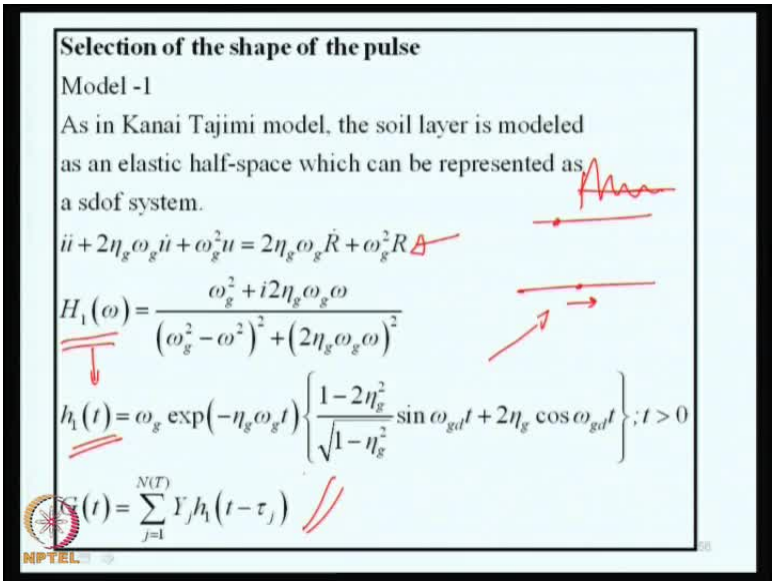

Now the question is, how do you select this pulse shape? So, what we could do is, we can use different models; I will discuss one of the models in today lectures and other

models will take up the next lecture. So, what will do is, we will assume that the pulses are output of Kanai Tajimi single degree freedom model; that means, as in Kanai Tajimi model, the soil layer modeled as an elastic half space, which can be represented as single degree freedom system.

So, the governing equation for transmission of a single pulse is given through this and the transfer function is H 1 of omega. And the Fourier transform of this will give me the impulse H 1 of t, and I will assume that, the pulse shape w is actually this impulse response; that means, what I am assuming is, at the bed rock level, I apply unit impulse; and whatever I see here is a acceleration, I will take that as a pulse. And this will convolve with whatever waves that are arriving there, and we will get the ground motions through this model.

So, in this case, the ground motion j of t is given j equal to 1 to n of t y j h 1 t comma tau j. Now, we can improve upon this and we will consider some of this improvements in the next lecture. So, at this stage, we will conclude the present lecture.