Stochastic Structural Dynamics

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Lecture No. # 31

Monte Carlo Simulation Approach -7

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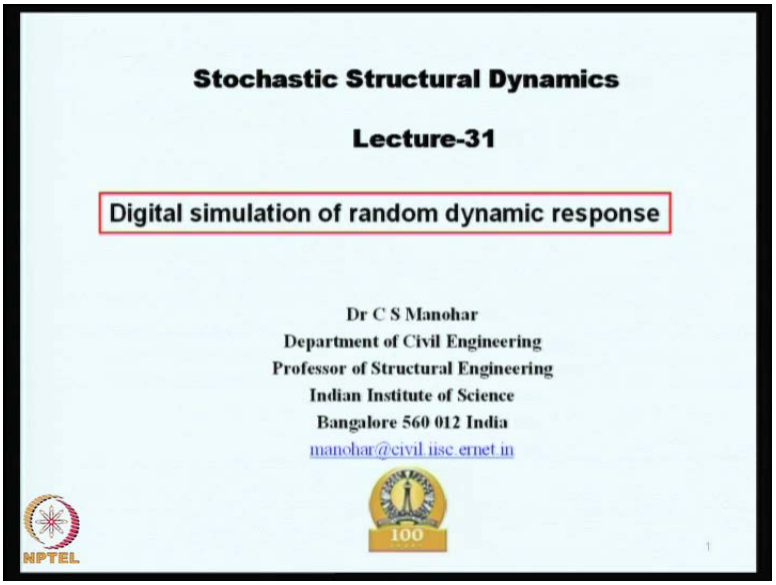

We have been discussing Monte Carlo simulation methods for analyzing the response of randomly excited systems.

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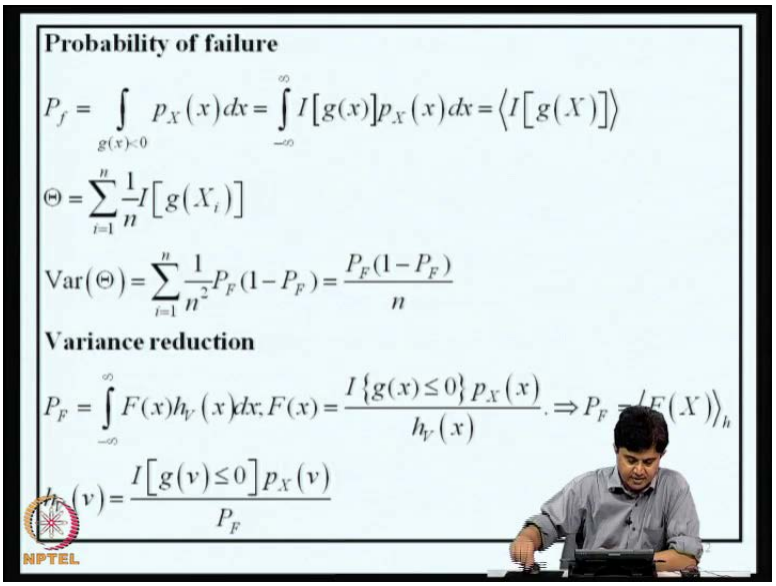

So, in the previous lecture, we considered problems of evaluation of probability of failure; and the probability of failure was expressed with respect to a performance function, g of X less than or equal to 0, x is the vector of basic random variables in the problem and this is a multi-dimensional integral over an irregular surface. This probability of failure, can be expressed in terms of an expectation of an indicated function as shown here, and one can obtain an unbiased consistent, estimated with minimum variance for this probability of failure. And variance of this estimator, we showed, that it is an inversely propositional to n there, where n is a sample size. And we discussed the issues related to reduction of this variance; if we follow this estimator, the only way we could reduce the variance is to increase the sample size, but on other hand, if you modify the definition of the estimator, then we may be able to reduce the variance by certain alternative routes.

So, that led us to the discussion on problem of variance reduction, where we showed that probability of failure can be evaluated as an expected value of a slightly modified function, with respect to a important sampling probability density function h; and if this importance sampling density function happens to coincide with a certain form shown here, then the sampling variance in fact goes to 0. This is an ideally important sampling density function, but this is hard to construct, because the knowledge of probability of failure, that is a vary quantity that we are looking for, is involved in the definition of this ideally important sampling density function.

But on the other hand, the fact that such an ideal important sampling density function exist, motivate us to search for certain sub optimal solutions, where the variance can be reduced for a given sample size n. if you can modify the estimator, we can reduce the variance to a value lesser than this variance, corresponding to the estimated theta; so, that is the basic problem.

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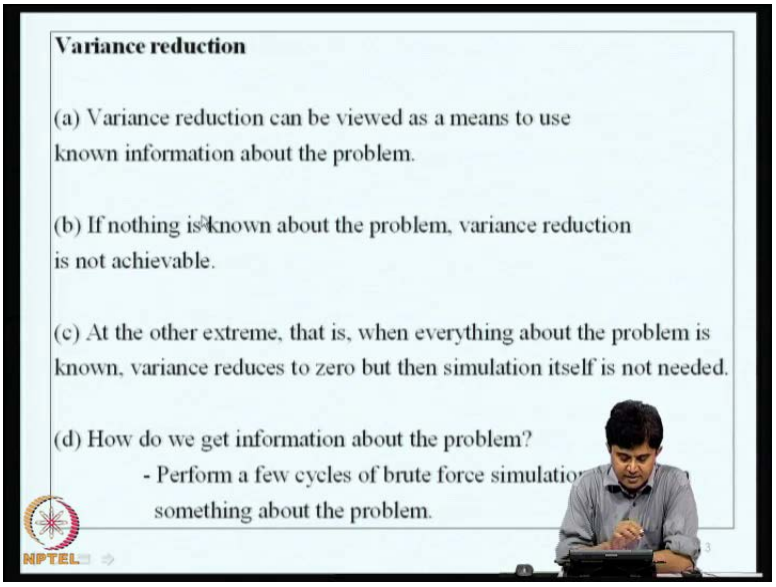

In fact, variance reduction can be viewed as a means to use known information about the problem, in formulating that estimator. In fact, if nothing is known about problem, no variance reduction is possible. On the other extreme, when everything about the problem is known, that means, the probability of failure exactly known, the variance reduces to 0, but then simulation itself is not needed.

So, the problem is now therefore, how do we get information about the problem, so as to construct a suitable important sampling density function or how do we generate information about the problem on hand and I use it on the fly and modify the estimator suitably. So, the basic idea is to perform a few cycles of brute force simulation and learn something about the problem, and utilize this knowledge in constructing the estimate of probability of failure.

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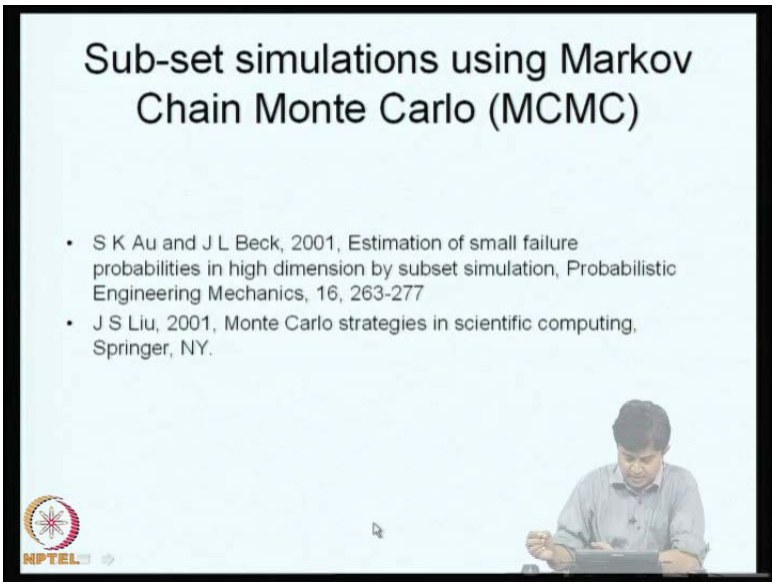

There are a few techniques in the existing literature, which work on this principle or which work based on methods that I have motivated by these facts; one of the methods is what is known as subset simulations and subset simulation using Markov chain Monte Carlo was something that I started discussing in the previous lecture.

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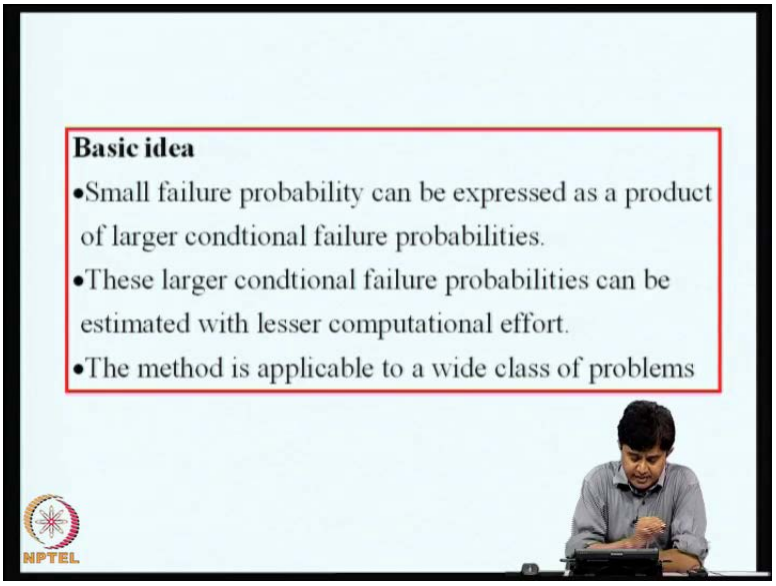

So, we will continue with that; the basic idea here is, that small failure probability can be expressed as a product of larger conditional failure probabilities. These conditional failure probabilities can be estimated with lesser computational effort. The way this

method has been formulated in the existing literature, the method is fairly widely applicable to problems of statics, dynamics, non-linear mechanics, non-Gaussian random variables, so on and so forth.

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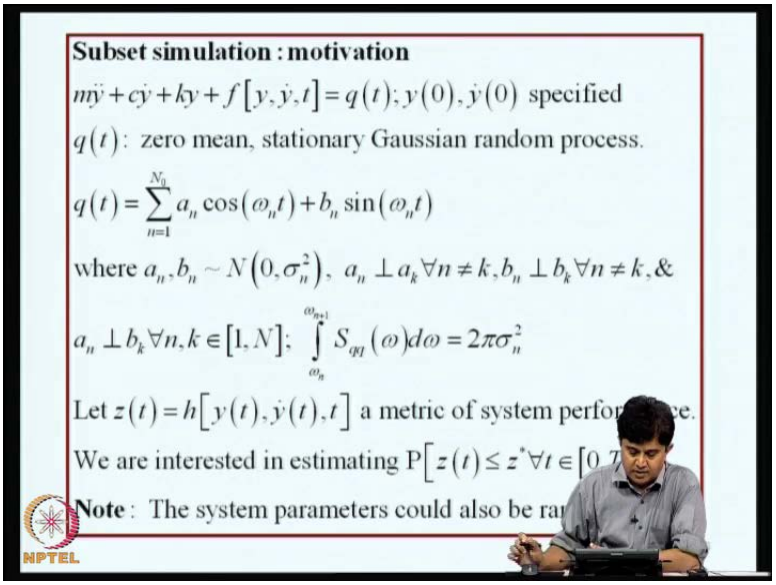

Now, let me again clarify the motivation behind the development of this method, in the context of non-linear dynamical system driven by random excitation. So, we will consider a non-linear system, my double dot plus cy dot plus ky plus f of y comma y dot comma t is equal to q of t, where q of t is a 0 mean stationary Gaussian random process. You started with, we will assume that, this random processes means where periodic and we will be able to write this in a Fourier series with random coefficients, where a n b n are normal random variables, whose variances are adjusted to match the information from power spectral density function.

Now, if we define z of t, which is a non-linear function of y y dot and a function of time, to be a metric of system performance, we are basically interested in estimating the probability that, this metric - the value of this matrix - stays below a permissible value z star for a duration 0 to capital T. So, in this problem, we can also note that, the parameter m c k and any of the parameter associated with definition of f, could themselves be random variables and that we denote as theta.

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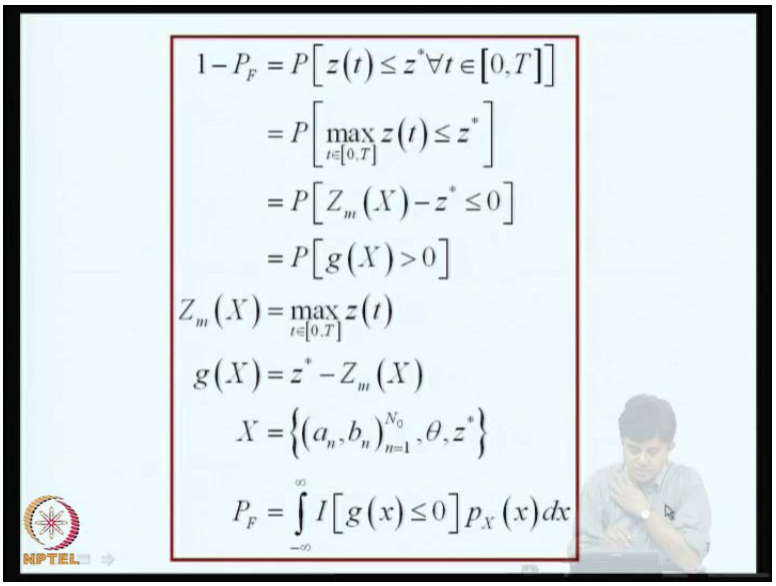

So, the problem on hand is, the reliability of system is probability that, this metric of a performance assumes value less than critical value or a time duration. Now, this is a problem in time variant reliability, and we convert this into a problem in time invariant reliability, by replacing this by the maximum this event, by the event that maximum of z of T stays below z star and this maximum is taken for the duration 0 to capital T. So, if this maximum is defined as z M of X and minus z star less than or equal to 0 is the failure event.

So, we define this z star minus Z m of X as a performance function, X is the, capital X is the set of random variables relevant to the problem; it involves the parameters associated with excitation, system parameters and also the matrix of performance. So, the probability of failure that we are looking for, is given by the expected value of the indicated function, g of X less than or equal to 0. So, this is a multifold integral; the size of this integral is equal to size of this random vector X.

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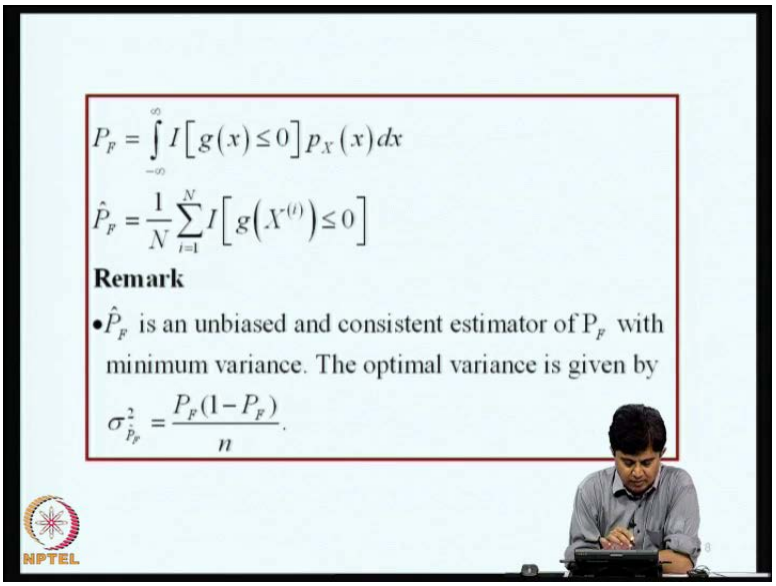

Now, an estimated for this probability of failure using brute force Monte Carlo given by this (Refer Slide Time: 07.12), where ith sample drawn from this probability density function. Now, we have already seen that, P F hat is an unbiased and consistent estimator of P F with minimum variance and the optimal variance is given by this, where the variance is, now as I have already said is inversely proportional to the sample size.

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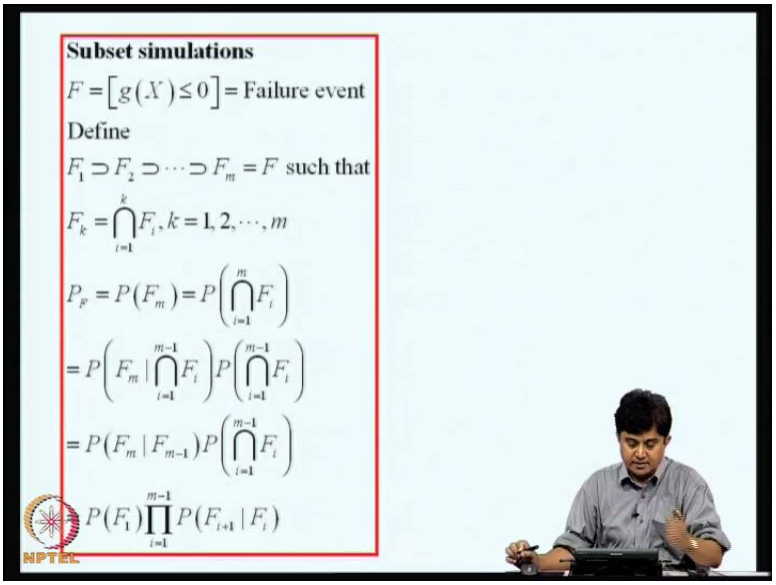

The idea of subset simulations is, to consider this failure event and define a sequence of events F 1, F 2, F m which are related as shown here, where F 1 is a subset of F 2, and so

on and so forth, so that the intersections of all these sets is the… F k is the intersection of F i up to value k.

Now, probability of failure is therefore probability of F m, where m division have been made, and this is probability of intersections of all these and this we express in terms of conditional probabilities, this I have explained in the last class. So, the probability of failure is obtained as, product of several fairly large probabilities.

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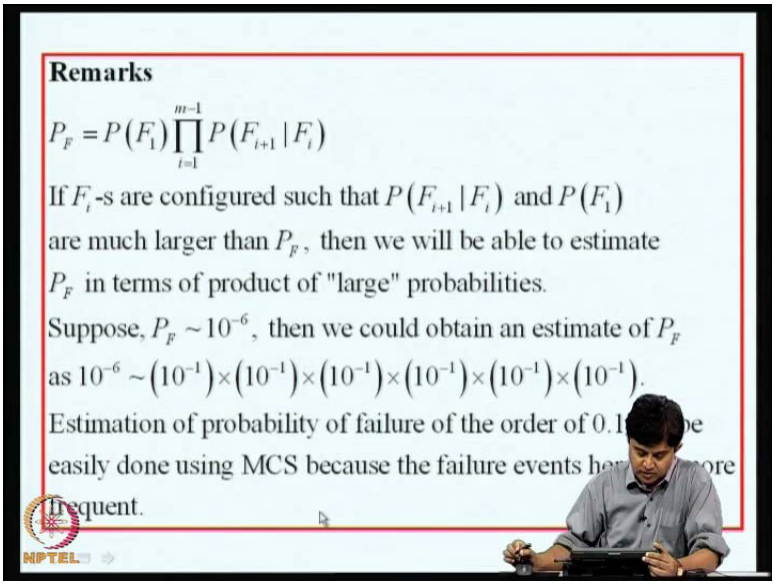

So, if F i's are configured, such that, P F i plus 1 conditioned on F i and P F of F 1 are much larger than P F, then we will able to estimate P F in terms of product of large probabilities. For example, a number like 10 to the power minus 6 can be obtained as product of 10 to the power of minus 1 multiplied 6 times; that means, each one of this number in the parenthesis is the probability of failure associated with one of this conditional events. So, this is a basic idea of subset simulation.

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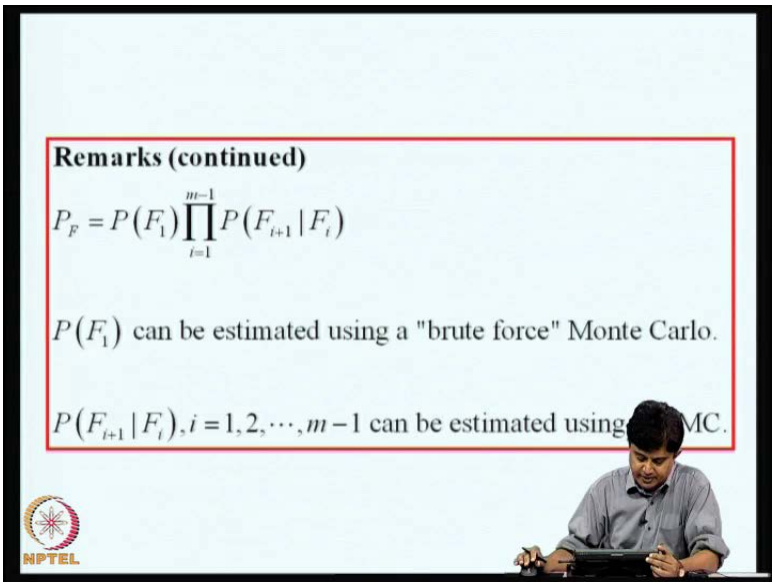

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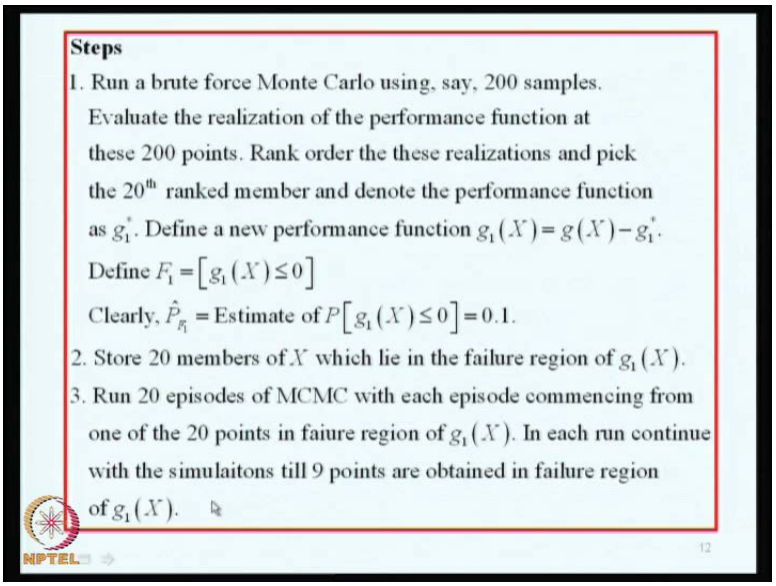

And we will now describe the details of this method in this lecture. So, what we do is, we run a brute force Monte Carlo simulation using, say, 200 samples; I use this number 200 to illustrate the method, this 200 has no specific sanctity beyond this. Now, for every 200, in every member of this set of samples, we evaluate the realization of the performance function. Then, we rank order these performance functions, and pick the twentieth ranked member and denote this number, that is, the performance function as g 1 star. We introduce a new performance function g 1 of X is g of X minus g 1 star.

Now, we define the event F 1 as g 1 of X less than or equal to 0. Now, clearly, P F 1 hat is estimate of probability of g 1 of X less than or equal to 0 is 0.1, because we have taken 200 samples and this g 1 star is defined with respect to a twentieth sample. Now, that would mean, there will be 20 points in the failure region, where failure region is defined with respect to the event F 1. So, in the region F 1, there will be now 20 points at the end of this first step. Now, we store all these numbers, which lie in the failure region of g 1 of X.

Now, we run 20 episodes of Markov chine Monte Carlo, with each episode commencing from one of the 20 points in the failure region of g 1 of X. In each run, we continue with simulation till 9 points are obtained in the failure region of g 1 of X. So, at the end of this, we will have 200 points now in the failure region of g 1 of X.

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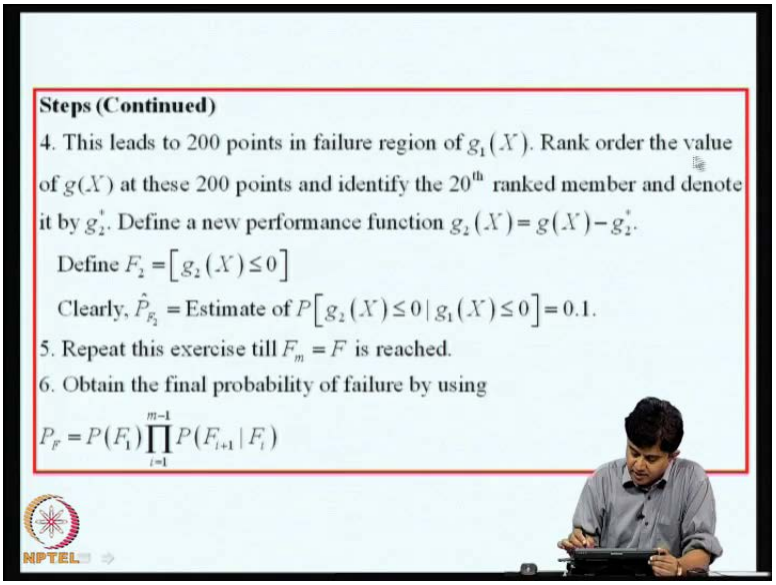

Now, we rank order the value of g of X at these 200 points and identify again the twentieth rank member and denote this by g 2 star. We define a new performance function, g 2 of X is equal to g of X minus g 2 star; we define now the event, F 2 g 2 of X less than or equal to 0; again, P F 2 will be the estimate of g 2 of X less than or equal to 0 conditioned on g 1 of X less than or equal to 0; this number is 0.1, again because we have taken 200 points and g 2 star is defined with respect to the twentieth member in the ordered list.

Now, just as we defined F 2, now we will define F 3, F 4, etcetera, and we will repeat this exercise till F m equal to F is reached. So, at every stage, I will be getting one of these probabilities - these conditional probabilities - of F i plus 1 conditioned on F I, and at the end of this, we find out the requisite probability of failure, by multiplying P of F 1 into these conditional probabilities, m minus 1 conditional probabilities, probability of F i plus 1 conditioned on F i.

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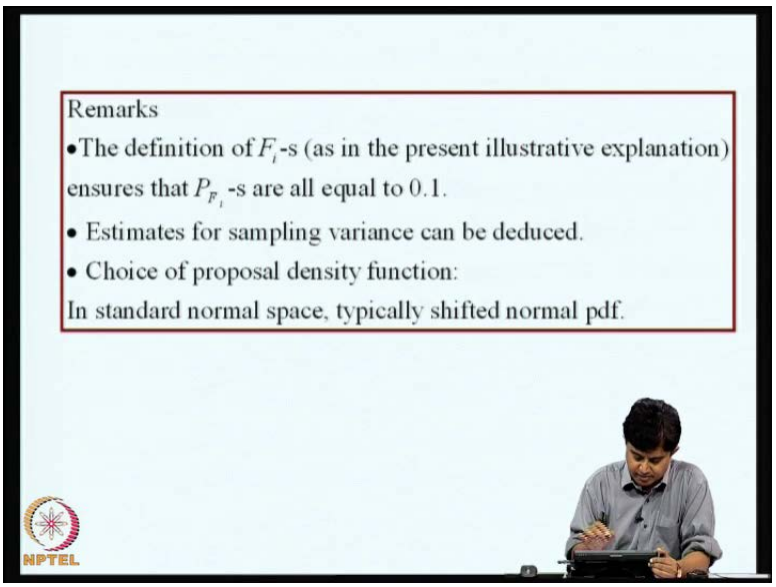

So, in the present, the way I have illustrated, the illustrative explanation is provided, the definition of F i's ensure that, all P F i's are equal to 0.1, except the last one which could be a different number.

Now, in the existing literature, there are results which provide estimates of sampling variance following this procedure; and also in the Markov Chain Monte Carlo, one need to select the proposal density function, and typically, if you are working in standard normal space, we select the proposal density function to be a normal probability density function centered at the point, where we commence our Monte Carlo simulations - that is MCMC simulations.

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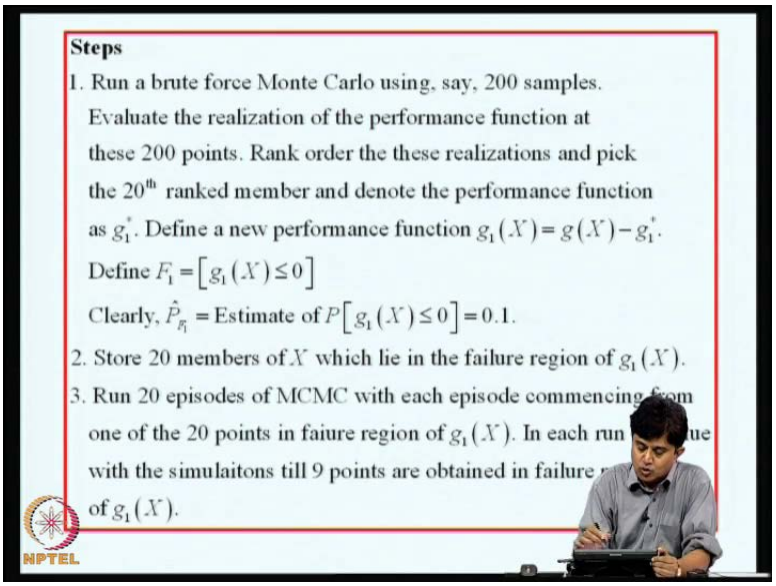

So, the results that we get, are controlled by several factors, namely, these 200 samples I have mentioned, this 200 is a number; so, how much it can be? 100, 500, 1000, and that is one of the factors that will be controlling our answer. And similarly, what is F m? How do you define F 1, F 2, F 3, F m? This probability of failure as 0.1, we are selecting the 20th rank member in a set of 200 samples; we could as usual select 15th rank or 10th rank or 30th rank, so on and so forth, in which case these probabilities will be different; so, that is another factor. And similarly, this Markov chain Monte Carlo also would involve choice of proposal density function, and proposal density function if it is normal, the choice of the mean value, the choice of the covariance matrix, etcetera. So, there are several parameters, that affect the performance of this method.

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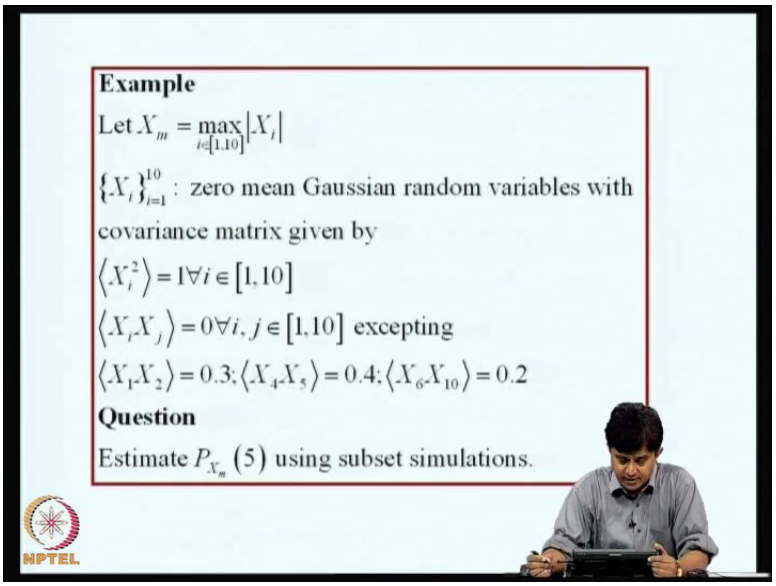

We will now consider two examples to illustrate the method of subset simulations: one is X m is maximum of X I, where i takes values from 1 to 10. For purpose of illustration, we take this ten random variables to be 0 mean Gaussian random variables, with covariance matrix given by, expected value of X i square is 1 for all I; that means, in the covariance matrix along the diagonal, the entries are 1. There are few of diagonal terms which represent the cross covariance, which are nonzero; so, X i X j is 0 for all i j, excepting X 1 X 2 which is 0.3, X 4 X 5 which is 0.4 and X 6 X 10 which is 0.2. You must notice that, this covariance matrix is symmetric; therefore, if I say expected value of X 1 X 2 is 0.3, it automatically implies that expected value of X 2 X 1 is 0.3; similarly, member 5 4 is 0.4, the member 6 10 is 0.2.

Now, the question that we will ask is, we would like to estimate, the probability that the highest value is less than or equal to 5 using subset simulations. Now, we can quickly recall, that we have already discussed, how to find maximum of sequence of independent random variables, we have also derived the probability density function for maximum of n random variables, when the nth order density function is known; therefore, there is an exact solution available for this problem

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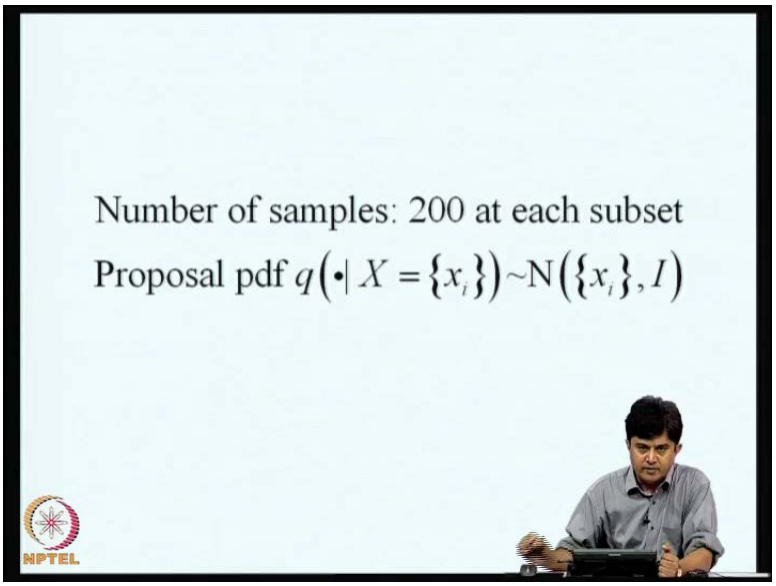

So, what we do is, we select a 200 samples for each subset simulation, and we take the proposal density function to be normal, where the mean is centered at x i, which is ith sampling point, and covariance in this example, we have taken it to be an identity matrix.

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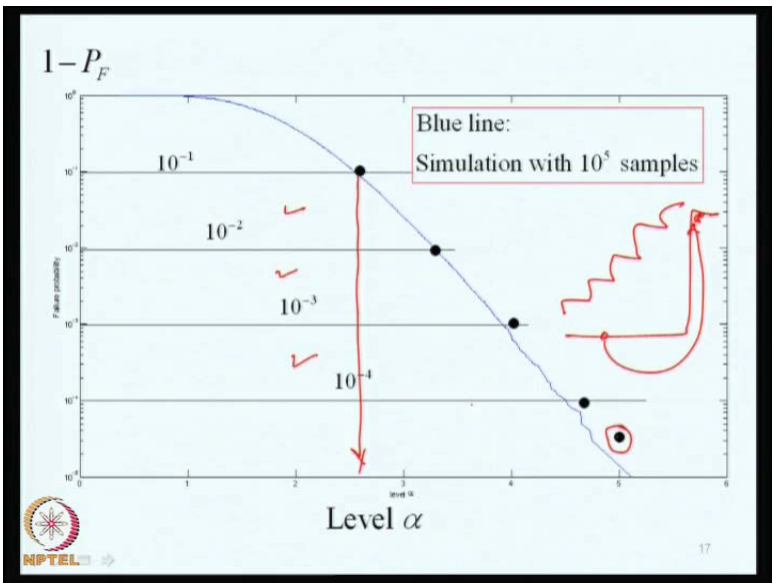

So, in this graph, what we do is, we start with the first simulation using 200 numbers and we arrive at this value of alpha; this will be the value corresponding to this number; we will define g 1 star and get this probability of failure; then these are subset simulations 1, 2 and 3; and at the end of it, the twentieth member is 0, so that means, the twentieth

member is already in the safe region, and therefore, we find out the final point using the last set of 200 numbers.

The blue line is a result of a brute force Monte Carlo simulation with 10 to the power of 5 samples. So, at every step, we are moving by a factor of 0.1, and in the end, we multiply all that and reach this point; this is like climbing a staircase; instead of climbing a large staircase, we are taking small steps and reaching this point from this point. So, this is a basic principle of subset simulations.

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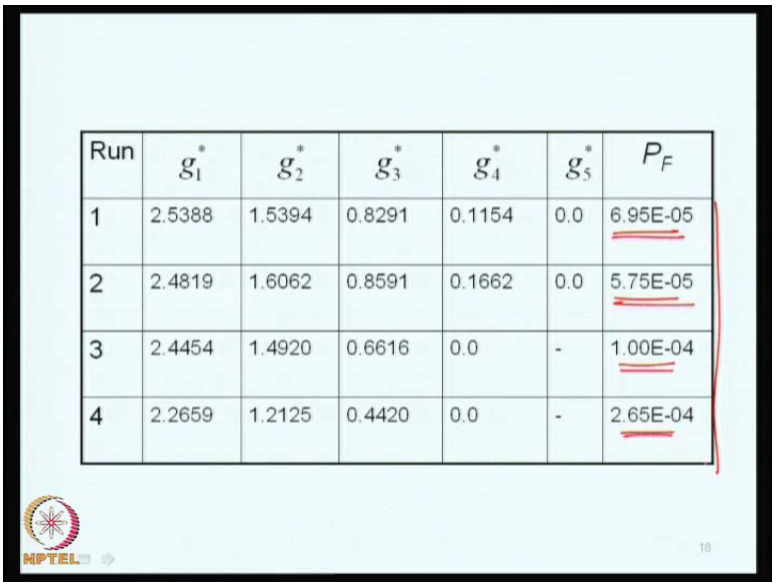

So, the some details the subset simulation was done for 4 episodes - distinct episodes - and in the first episodes, the g 1 star we shown here, g 2 star is this, g 3 star is this, g 4 star is this, and g 5 has become positive and this is the probability of failure (Refer Slide Time: 17.28). And similarly, in the second episode, we get this number and third episode we get this number, and so on and so forth; the point that is being made here is that, its estimate of probability of failure is subjective sampling fluctuations, and if complete description of the solution is needed, one need to probably establish the confidence limits on these numbers.

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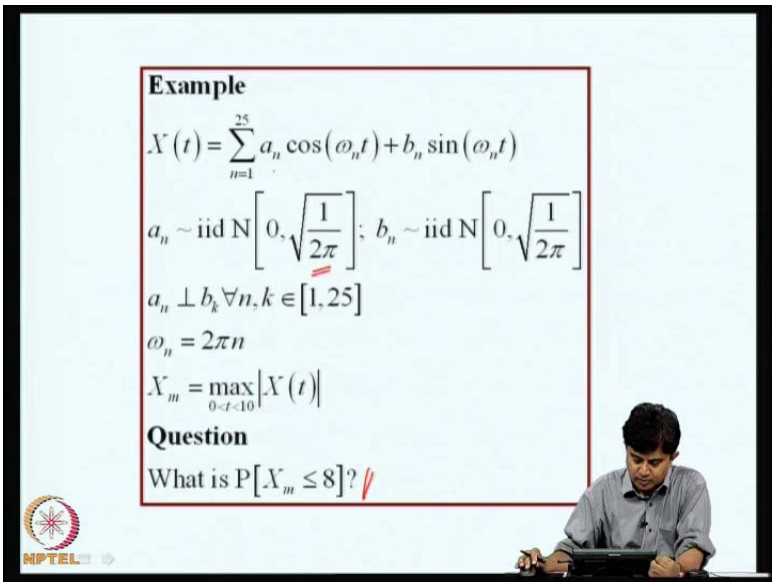

Another example, we take a random process X of t, we take it as a n cos omega n t plus b n sin omega n t, where a n b n are taken to be iid sequence of normal random variables with 0 mean and standard duration as shown here, and b n is also iid sequence and we assume that a n and b n b k are independent; and omega n is taken as 2 pi n, which would mean that, omega 1 is 2 pi and we are interested in finding the maximum value of this random process, over a time duration t to 0 to 10.

Specifically, we are asking the question - what is the probability that X m is less than or equal to 8? Now, this example would serve us as an illustration on, how to use subset simulations, when we are dealing with response which is a random process. So, this discussion is relevant to all situations, where the power spectral density function is available, either through a response analysis or via modeling exercise and we are interested in doing estimating probabilities of this kind based on that.

So, this Fourier series representation essentially assumes that, they have to auto covariance function is periodic and that amounts to certain approximations in discretizing the power spectral density function, but as N tends to infinity, this method is likely to give acceptable results.

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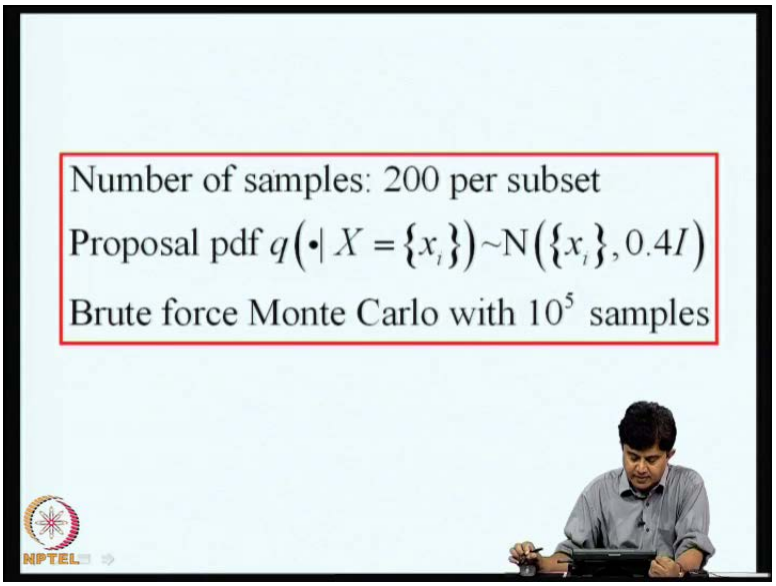

So, we have implemented the subset simulation on this, with 200 samples per subset, and again, we take the proposal density function to be normal, centered around the point in the failure region and we take the covariance matrix to be 0.4 into identity matrix. We have also performed a brute force Monte Carlo simulation with 10 to the power of 5 samples, so that it this forms are basis on which, I mean we can compare these two results.

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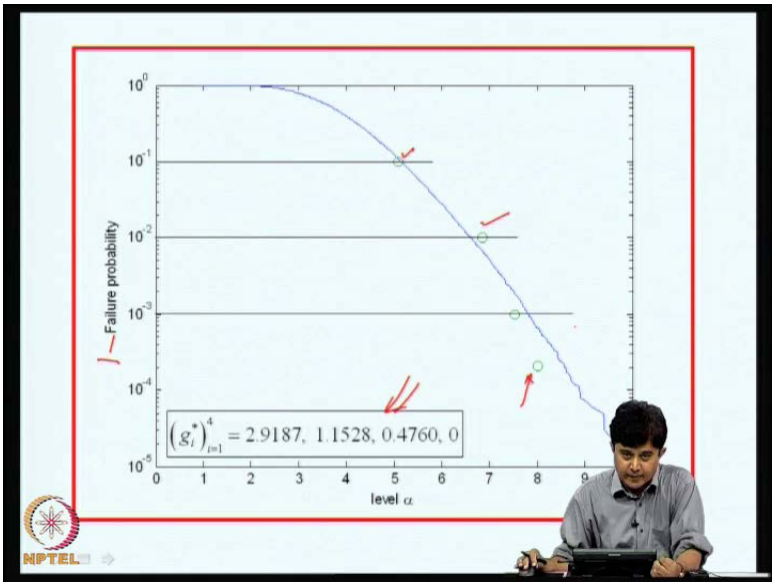

So, the results are as shown here. Blue line here is the result from brute force simulation with 10 to the power of, say, 5 samples, and these numbers are probability of F 1, this is F 2 conditioned on F 1, F 3 conditioned on F 2, so on and so forth; this is the final probability. This is probability F 1 F 2 F 3 and this is F m - probability of failure associated with those events; this is actually 1 minus probability of failure, so this is actually the reliability of the system. So, in this calculation, the g 1 star g 2 star where obtained as shown here and these numbers are subjective sampling fluctuations; this is the realization that we got, when this simulation was performed.

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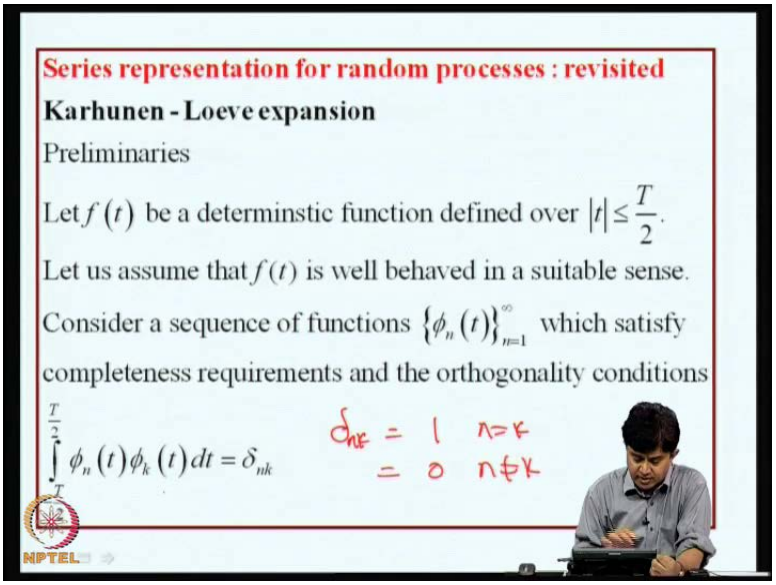

Now, we will close the discussion on subset simulation and we move on to another aspect of Monte Carlo simulations. I talked about problems of spatial variability in structural mechanics, and I oppose that, as one of the difficulties that we need to deal with, along with the difficulty that I mentioned about calculus of Brownian motion process.

So, now, I will address the second difficulty, namely, how to deal with random variability in space. So, as I prelude to that, we will discuss certain ways of representing random processes, and then take up the problem of dealing with structural dynamics with parameter and certainties, where uncertainties are spatially varying.

So, we start the discussion, by considering what is known as Karhunen and Loeve expansion. A few preliminaries to begin with, as I prelude to this discussion, we will

consider a function f of t to be a deterministic function, defined over an interval minus T by 2 to plus T by 2. Let us assume that, f of t is well behaved in a suitable sense and we consider a sequence of functions phi 1, phi 2, phi 3, phi n, n going to infinity, which satisfy completeness requirements, and the orthogonality conditions integral, minus T by 2 to plus T by 2 phi n of t phi k of t dt is chronicle delta nk; so, delta nk is, 1 for n equal to k and 0 for n not equal to k. So, this is the orthogonality relation satisfied by the function phi n.

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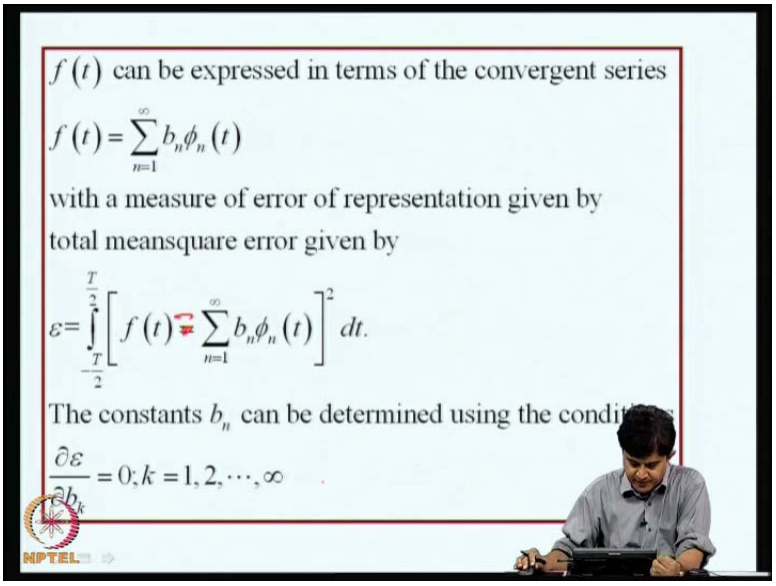

And we express f of t in a convergence series, f of t is equal to summation n from 1 to infinity b n phi n of t; phi n of t are known, b n`s are the unknowns. Now, the error of this representation, a measure of that can be the difference between these two, and we square that and integrate over the time period of interest. So, this can be viewed as, the total mean square error for the time period of interest.

Now, the constants b n which is unknown can be evaluated by minimizing this error, with respect to b n; that means, dou epsilon by dou b k equal to 0, for k equal to 1, 2, 3, 4 to infinity, so on up to infinity.

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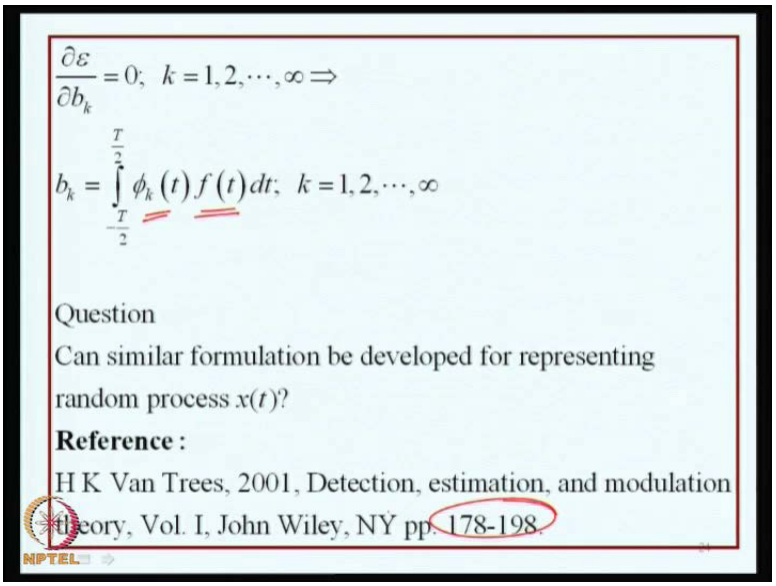

So, a simple calculation will show, that b k would be minus T by 2 plus T by 2 phi k of t f of t dt, k running from 1 to infinity; so, f of t is known, phi k of t is known, therefore, this quadrature can be performed; so, I get b k. So, this provides a means of representing a function f of t, in terms of set of orthogonal functions; and coefficient which can be evaluated, based on the knowledge of the function and the basis functions chosen.

Now, the question we ask is, can similar formulation be developed for representing random process x of t? This is the question that we will consider presently, and a reference for this discussion is the book by Van Trees - Detection, estimation and modulation; it is a 4 volume book and I am referring to the first volume, and these are the page numbers, where you will find the matters that I am discussing.

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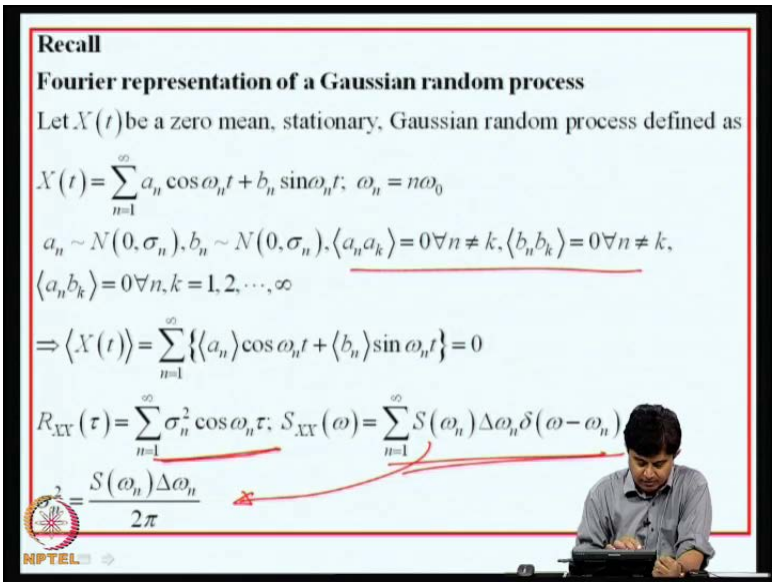

So, we will begin by recalling, this is not the first time that we are representing a random process in a series, we already use Fourier representation for a Gaussian random process; if you recall, we consider X of t to be zero mean, stationary, Gaussian random process, which is defined as, n to 1 a n cos omega n t plus b n sin omega n t, where omega n is n omega naught, where a n and b n are normal random variables satisfying these requirements.

So, now, the expected value of X of t is 0, that is what we started with. And if we compute the auto covariance of X of t, we get it to be a periodic function, because it is expressed as sum of cosine terms; and associated with this, we get the power spectral density function, which is the sequence of direct delta functions, and this is S of omega n can be related to a continuous power spectral density of X of t, if it is I mean, when it is available.

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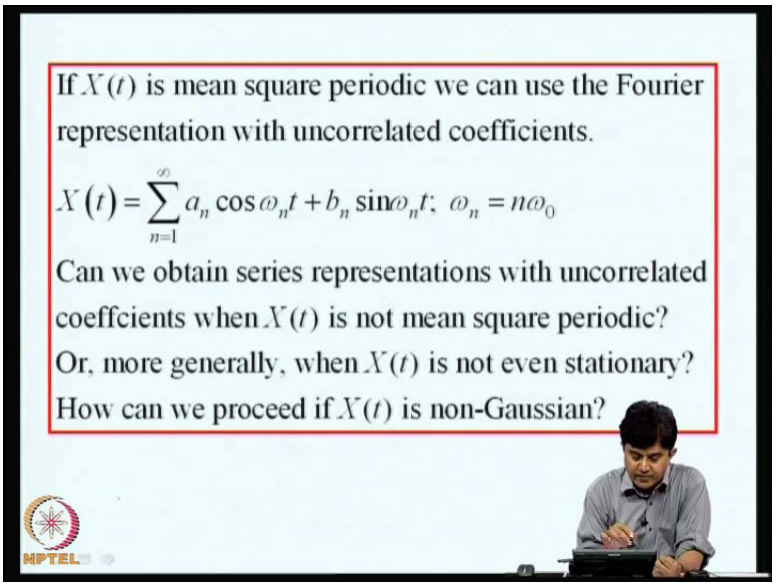

Now, if X of t is mean square periodic, we can use a Fourier representation with uncorrelated coefficients; that means, if I you use this representation and the auto covariance of X of t is periodic, then a n and b n will be uncorrelated. Now, the question is - what happens to X of t if R xx of tau is not periodic? That we can obtain series representations with uncorrelated coefficients, when X of t is not mean square periodic; or more generally, when X of t is not even stationary, how do we get this type of representation?

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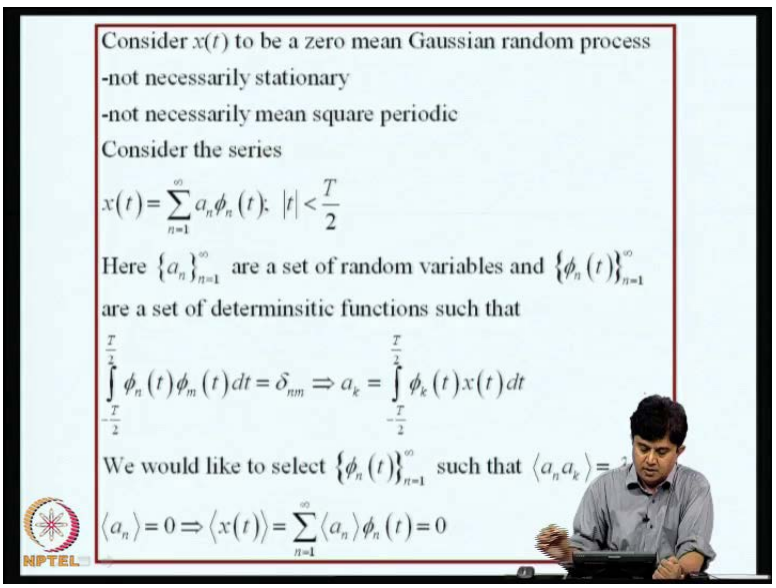

Furthermore, what happens if X of t is non-Gaussian? How do we obtain series representations? So, these are the questions that we would like to consider. Now, let us start by considering a Gaussian random process; so, let us consider x of t to be a 0 mean Gaussian random process, which is not necessarily stationary to start with and not necessarily mean square periodic.

Now, consider the series x of t, n equal to 1 to infinity, here, a n phi n of t. Here, this a n a 1, a 2, a 3, a infinity are set of random variables, and phi n of t are a set of deterministic functions. These functions are such that, they are orthogonal over the time duration of interest; that is, integral phi n t phi m t d t is equal to 0, when n not equal to m; and it is normalized to be 1, when n equal to m. Based on this, we can obtain in an expression for a k, in terms x of t and phi k of t as we have done for the deterministic function just a while before.

Now, the question that we want to ask is, if x of t is a random process, it is immediately evident that a k is a random variable, because it is a transformation on a random process. Now, we would like to select this phi n, and this family of functions phi 1, phi 2, phi 3, phi etcetera, such that, these random variables a n a k are orthogonal; in the sense, expected value of a n a k is 0, when n naught equal to k, and it is some constant lambda n, when n equal to k.

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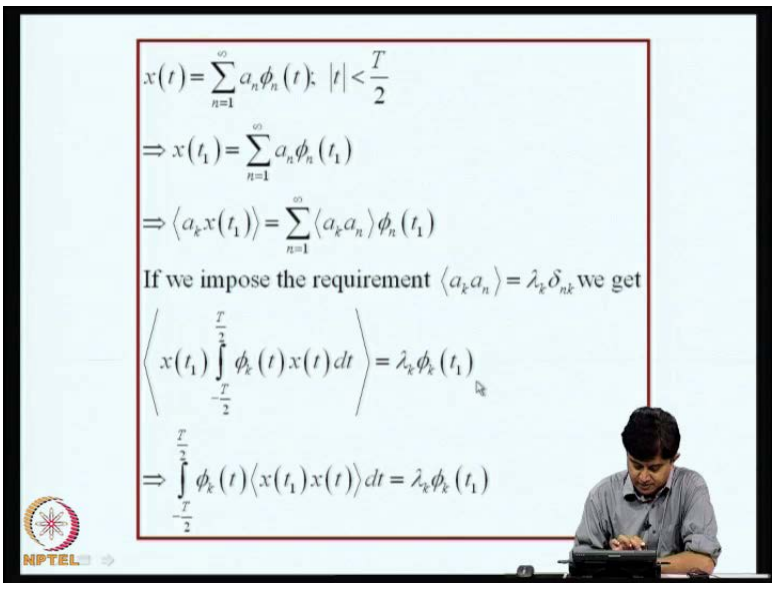

Now, if you take expected value of a n to be 0 for all n, then expected value of x of t is 0. So, one of the, if expected value x of t is 0, we can impose the condition expected value of a n is 0. Now, we have this representation, x of t is n equal to 1 a n phi n of t; therefore, t equal to t 1, it is x of t 1 summation a n phi n t 1, I will multiply both sides by a k and take the expectation.

Now, on the right hand side, I get expected value of a k a n phi n of t 1. Now, I we want a n a k to be orthogonal, that would mean, this summation will yield a non-zero value, only when n equal to k. Now, for a k, I will express in terms of x of t, using the formula that, that is, this formula that we derived just now.

So, on the left hand side, I have expected value of x of t 1 into this integral phi k t x of t d t; on the right hand side, I have lambda k phi k t 1. Now, we can take the expectation of operator inside and x of t inside the integrand, and I get the equation integral phi k t expected value of x of t 1 x of t dt is equal to lambda k phi k of t 1.

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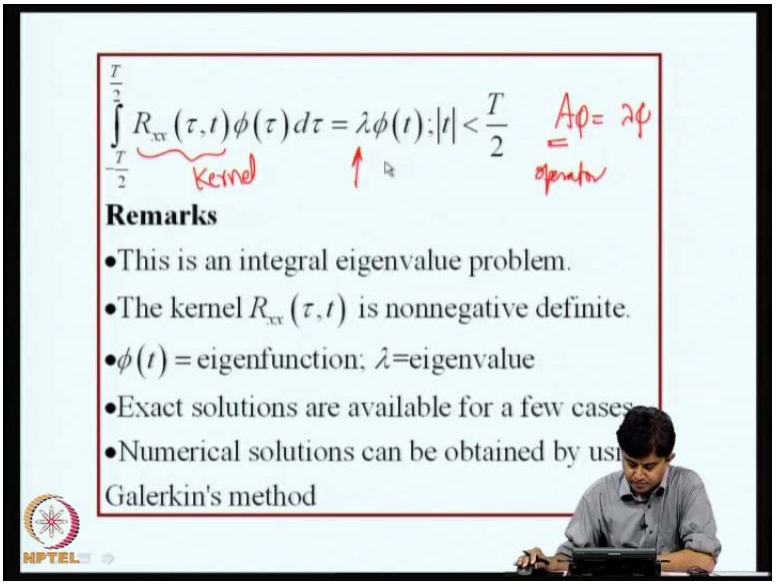

Now, this can be written in a more standard form as, integral minus T by 2 to plus T by 2 R xx of tau comma t phi of tau d tau is lambda phi t, for t between minus T by 2 to plus T by 2. So, what is that we are looking for? We are looking for phi of t, which are, which ensure that our random variables a 1, a 2, a 3, a n, etcetera are all mutually orthogonal. This if you see, is an integral Eigenvalue problem; this is of the form, where this A is now an integral operator; this function is known as kernel.

Now, this given the properties of the covariance, this is associated with the auto covariance x of t; therefore, it is non-negative definite, and the function phi of t is a Eigen function and this lambda is a Eigenvalue. Now, to find phi of t, we need to solve this integral equation. Depending on the definition of this covariance matrix, it is possible to solve this integral equation exactly in some instances, but more often we need to use numerical methods; so, methods like Galerkin's method can be used to approximately solve this Eigenvalue problem.

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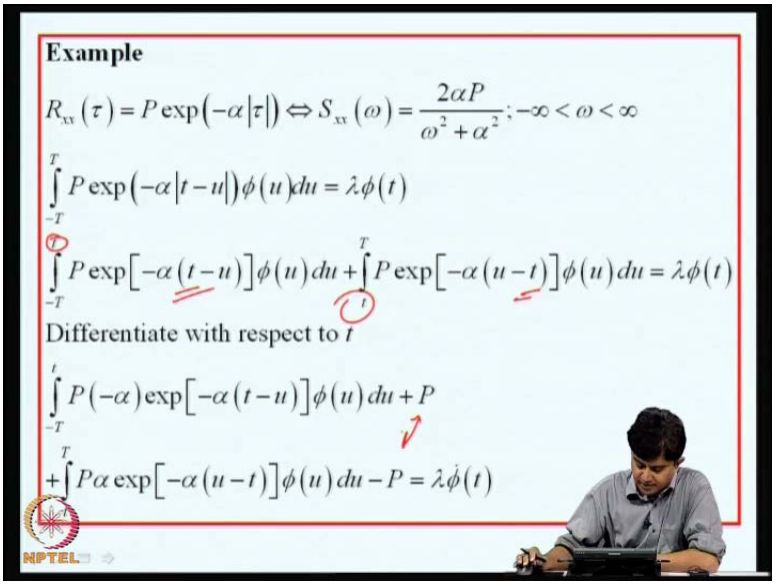

I will consider a few examples and show how this integral equation can be handled. So, let us consider a case in which the auto covariance function is, P exponential minus alpha mod tau; associated with this, we have the power spectral density, which is 2 alpha P divided by omega square plus alpha square. So, the integral equation that we need to solve is now, minus T to plus T, I am not taken interval to 2T instead of T, this is p exponential minus alpha t minus u phi of u d u is lambda phi t.

Now, what we will do is, we want to get rid of this modulus function in the integrand; therefore, we spilt this integral, I will write this as minus t to lower case t into this integrand plus t to capital T. So, whenever u is between minus T to plus t, this modulus sign is not needed, because already positive; in the region where u is between t to capital T, it will go with a minus sign, so that adjustment has been made. So, this is lambda phi of t; now, we differentiate this function with respect to time t. So, the quantity is to be

variable t appear at few places, that is what you should notice at few places; so, we need to take care of this, while differentiating; so, first we differentiate the integrand, and then the limits following the Leibniz's rule and we get this equation.

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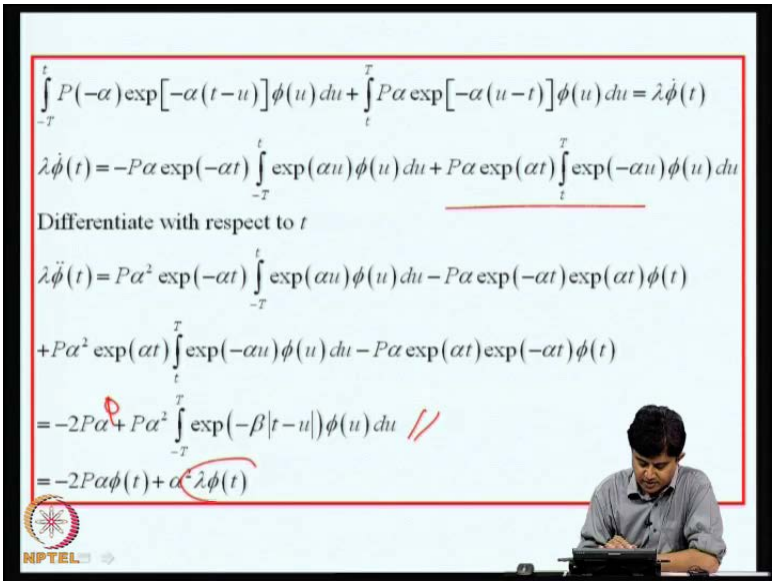

Now, this P this P gets cancelled and we get a slightly simplified version - this is this (Refer Slide Time: 32.40); this is what I will rewrite it as. Now, again I will differentiate once more, this become lambda phi double dot; and this integration can be carried out, and some simplifications can be done, we get this expression, minus 2 P alpha plus P alpha square this integrand, but this integrand is nothing but lambda phi t, because that is the definition of our Eigenvalue problem; so, that I can write it in this; instead of this integral, I can write this function.

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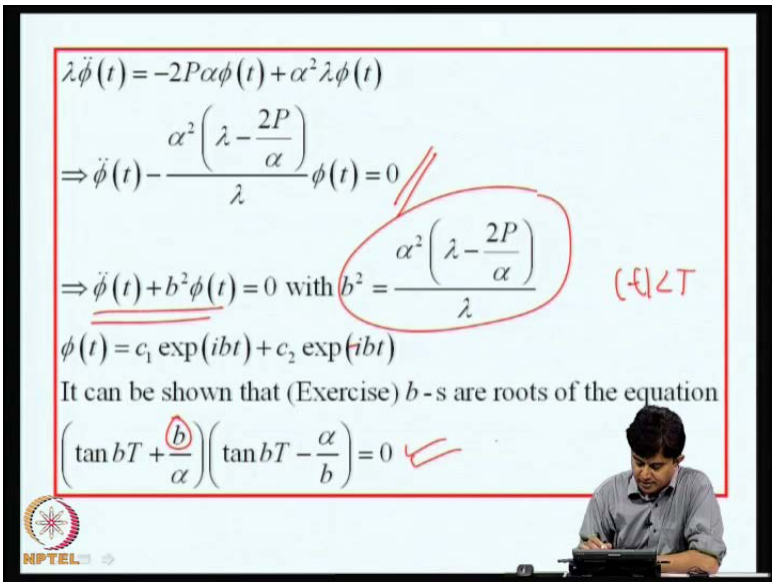

So, what we have done is, essentially we have reduce the integral equation into a differential equation, which is given by this. So, we have the problem, phi double dot of t b square phi of t equal to 0, where T takes values from minus t to plus t, it is a bounded value problem, it is an Eigenvalue problem; and we can get the solution as, phi of t c 1 solution exponential ibt plus c 2 exponential minus ibt; I will not get into the technique of solving this, you need to substitute this into the integration equation and do certain simplifications.

In any case, we will be able to show, which I leave it as an exercise, that the characteristic equation governing the Eigenvalue parameter b is given by this, and also, you need to consider different cases of possible values of this function, alpha and lambda and P etcetera, and it will turn out, that this is the only feasible solution that we are interested in. And solving that… this is a transcendental equation, which has to be solve numerically to find the Eigenvalues b 1, b 2, b 3, etcetera.

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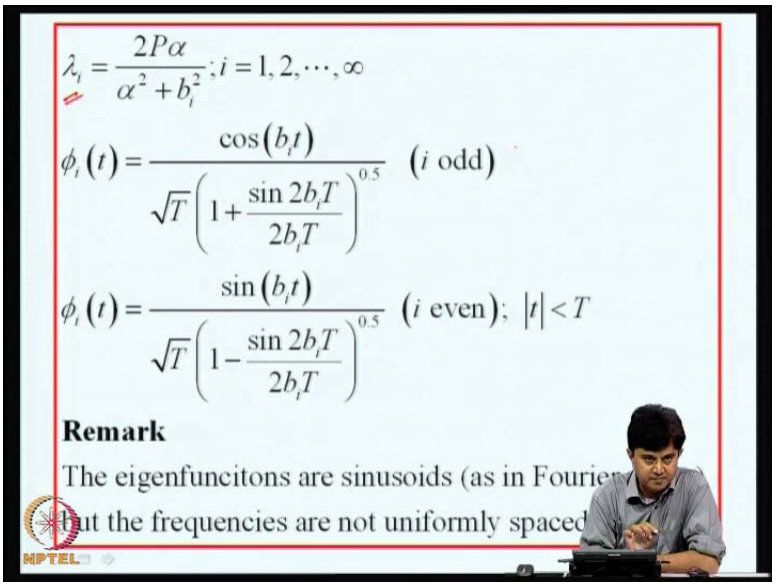

And once you do that, we can define the original Eigenvalue lambda i, in terms of b i, and it turns out, that this Eigenvalue is nothing but the ordinate of the power spectral density function, evaluated at b i.

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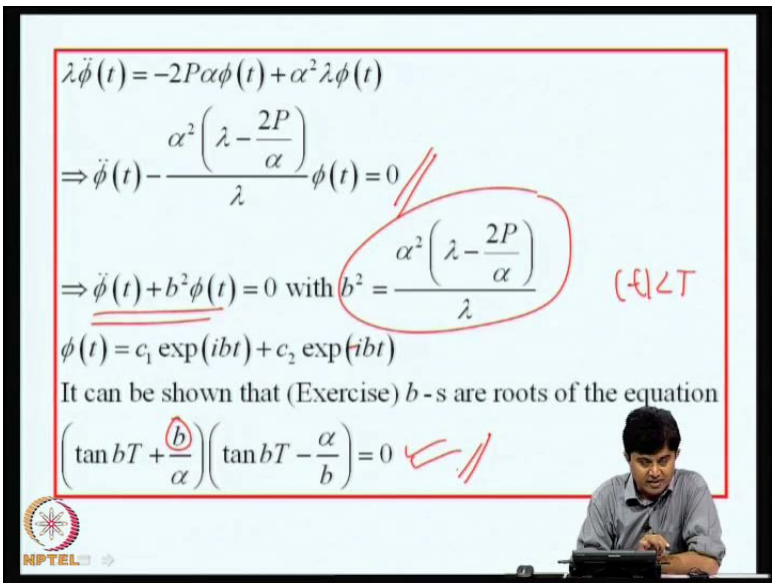

Associated with this, I get the Eigen functions; interestingly, they are cosine and sine terms. Whenever i is odd, I get a cosine term; whenever i is even, I get a sin term. This is an interesting result, if you compare this now with the Fourier series representation, we see that the basis functions here, are again sinusoidal - sin and cosine functions - just like

as in Fourier series. But the important difference is that, this b i which are the frequencies are not uniformly spaced, they are obtained as roots of this characteristic equation. So, this process need not be mean square periodic.

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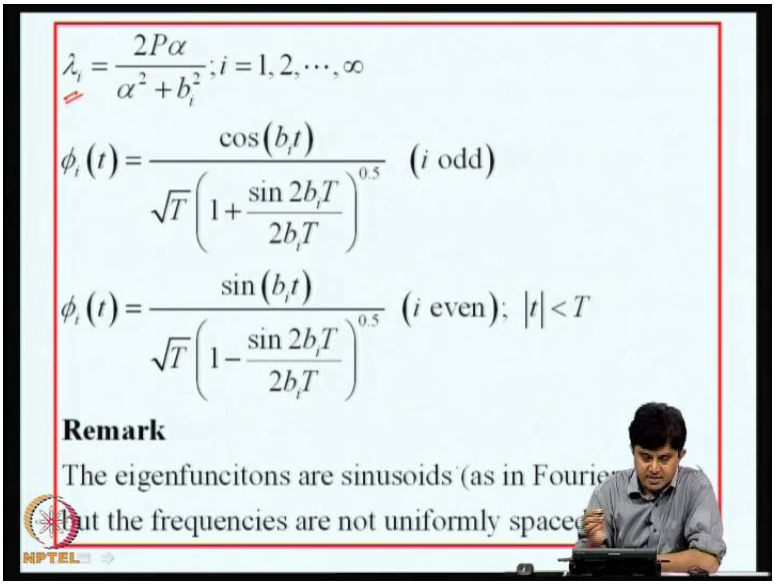

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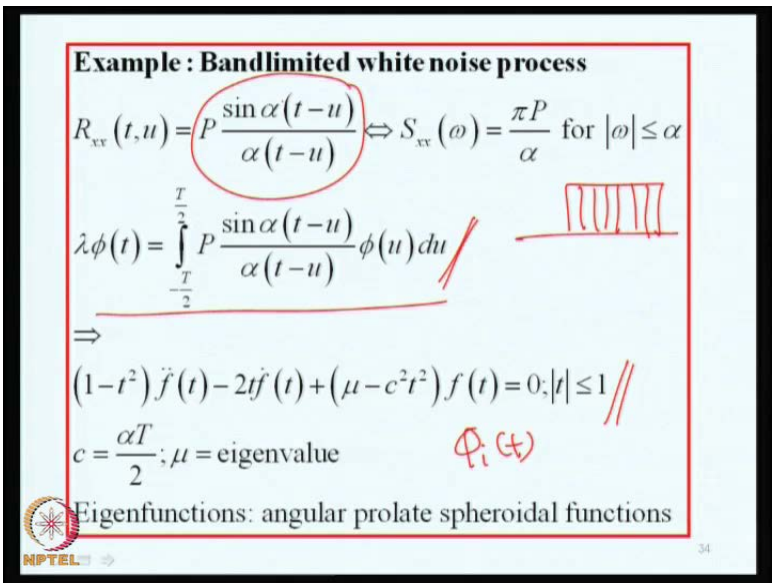

Another example we consider, a band limited white noise process, where the auto covariance function is given by sink function; that is, P sin alpha t minus u divided by alpha t minus u; and associated power spectral density is the constant function, which is pi P by alpha, whenever omega is less than or equal to alpha; otherwise, it is 0.

So, the integral equation to solve the Eigen value problem is given by this, I do not want to get into the details of this, but I would like to mention that the differential equation associated with this, we need to follow the same approach as we did in the previous example, we have to differentiate this twice and use the definition of the Eigen value problem; if you do that for the Eigen functions, we get this second order differential equations, with time varying coefficients. This is the special class of differential equations and they admit exact solutions, and these Eigen functions are known as angular prolate spheroidal functions. These are tabulated and their properties are known.

But the point that I am making here, is that, the phi i of t here are not sinusoidal; so, in that way the function is, representation is quite different from the Fourier series representation. In a Fourier series representation, if you were to discretize the power spectral density function and use a Fourier series, then it will turn out that the auto covariance function will be periodic, whereas a given auto covariance functions here is not periodic; so, that is the issue, that is being considered in this discussion.

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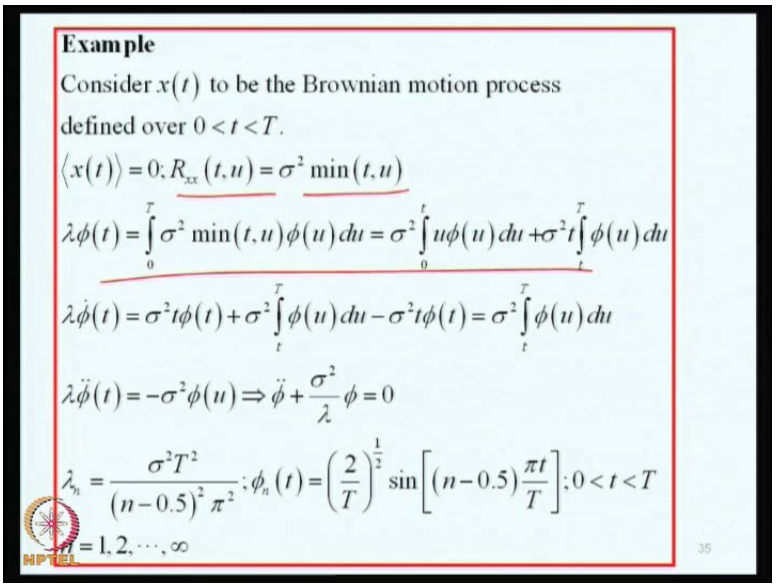

Now, how about a non-stationary process? We can consider x of t to be the Brownian motion process, defined over the period 0 to capital T. Let us assume that mean is 0 and we know that the auto covariance function of a Brownian motion process is given by, sigma square minimum of t comma u.

Now, how do we obtain the Eigen functions, which helps us to expand x of t in terms of orthogonal random variables? So, this is the Eigen value problem that we need to solve. So, we follow the same steps, we differentiate this twice and convert this problem into a differential equation; here, we get a simple problem, phi double dot sigma square by lambda phi equal to 0; and this Eigen values are given by this and the Eigen functions here are sinusoidals. So, this again is something like a Fourier series, but there are no cosine terms that has to be born in mind, but the frequencies are uniformly space, but in a slightly different manner.

So, this would help us to represent the Brownian motion process, which is a non-stationary random process. This representation is in the sense of a mean square, I mean the convergence of this representation is defined, with respect to mean square convergence; and you can see the details, by looking at the book by Van and trees.

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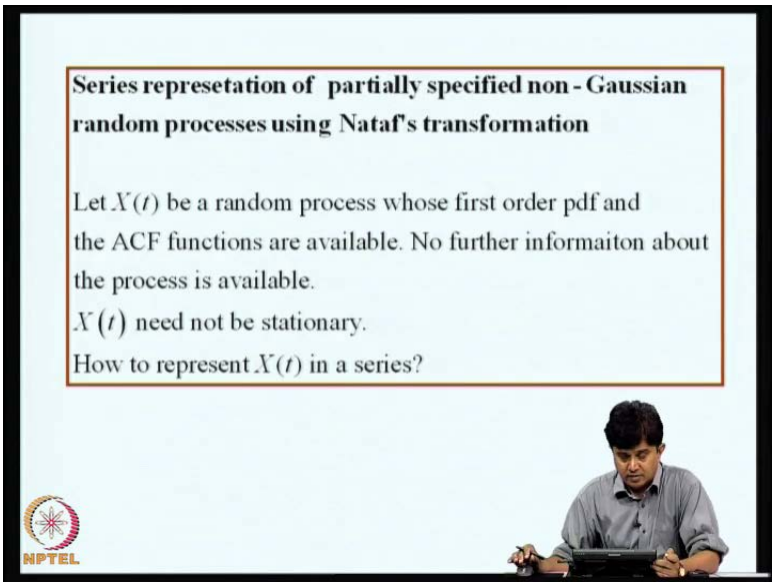

Now, how about the non-Gaussian random processes? So, let us consider the problem of series representation of partially specified non-Gaussian random processes using Nataf's transformation and the Karhunen and Loeve expansion, that we developed now. So, let X of t be a random process, whose first order PDF and auto covariance functions are available. No further information about the process is taken to be known. X of t need not be stationary. So, the question is - how we can use the series representation in representing X of t?

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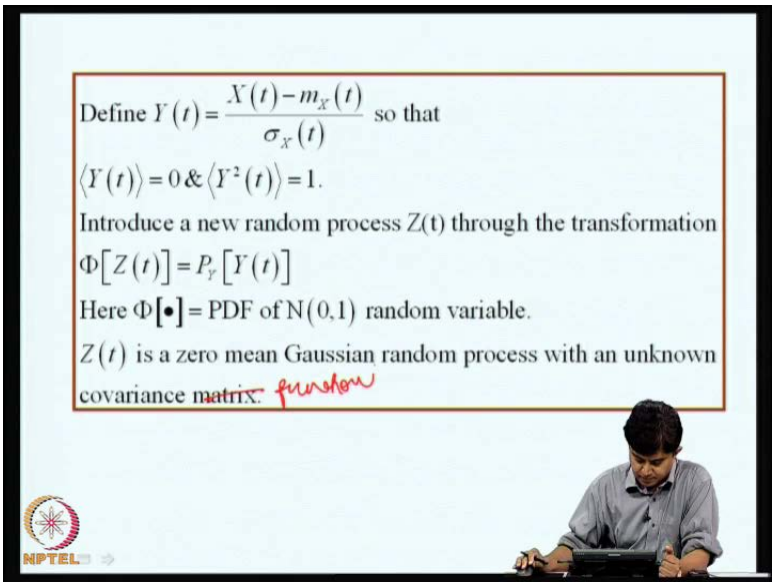

So, what we do? We begin by subtracting the mean and dividing by the standard deviation, so that the process Y of t will have 0 mean and unit standard deviation. Now, we introduce a new random process Z of t, which is a Gaussian random process with 0 mean and unknown covariance function, through this transformation phi Z of t is P Y Y of t; this is Nataf transformation that I have discussed earlier. Here, this phi is the probability distribution function of N(0 , 1) random variable, and Z of t is a 0 mean Gaussian random process with an unknown covariance function actually

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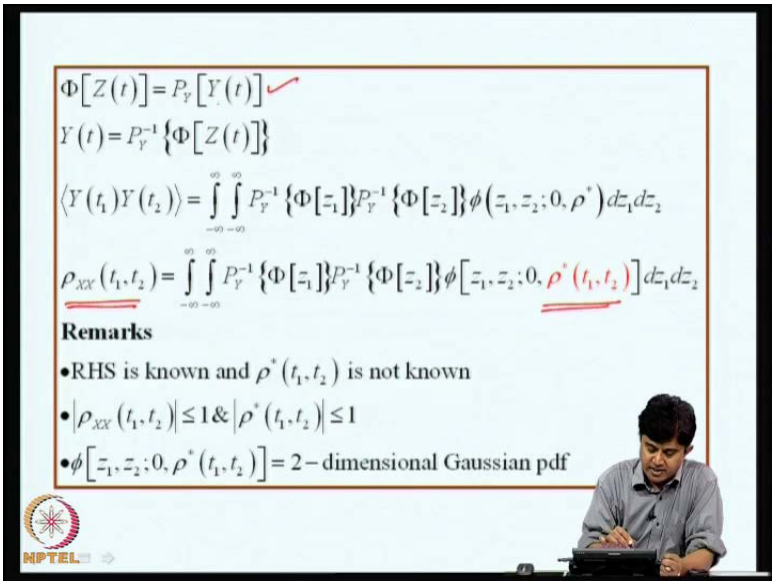

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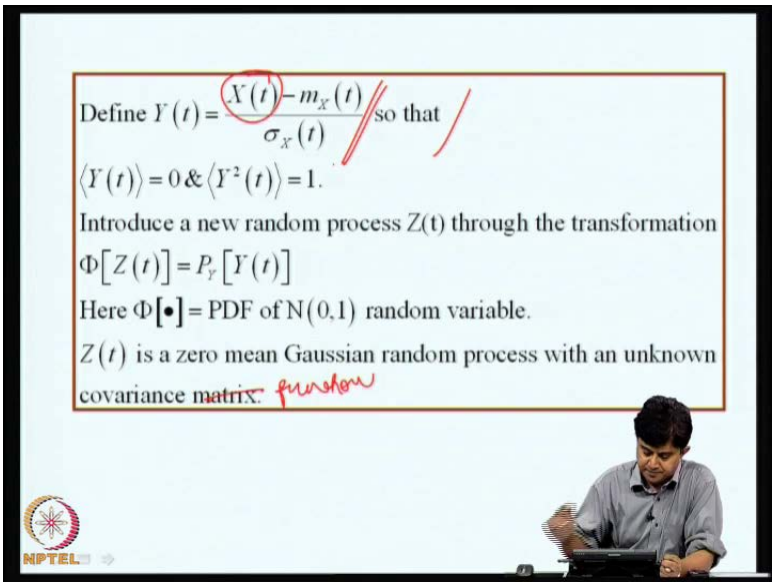

So, we have already develop the recipe for finding out the unknown; what is not known is the covariance of process Z of t and that can be related to the covariance of the given random process; this is unknown that we need to determine, I already discussed on how to solve this equation. So, if you follow this discussion, what it means is, we will be able to find a Gaussian random processes Z of t, in the sense that, we able to determine its covariance function; so that, using this transformation, we can simulate Y of t or characterize Y of t; and once Y of t is known, we can get X of t using this relation.

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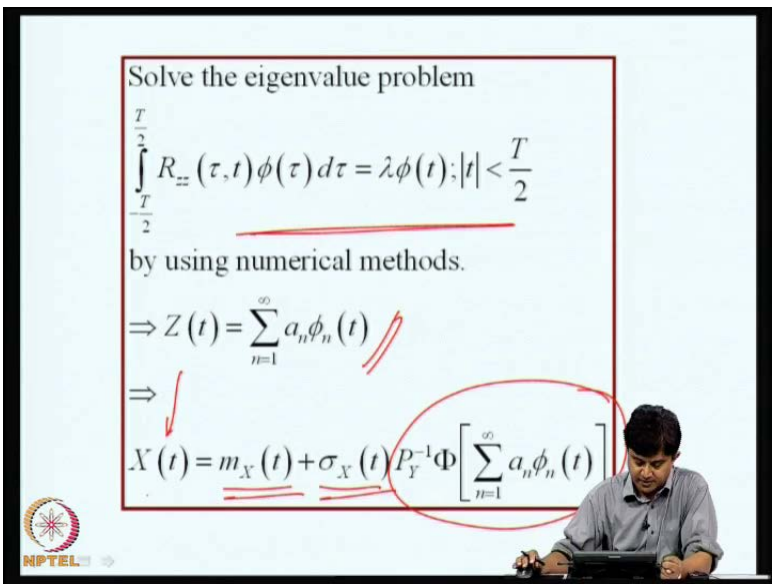

Now, the idea here is, we will use the Karhunen and Loeve representation to solve for to represent Z of t; so, what we will do is, we will solve this Eigenvalue problem, get hold of this basis functions, and this has to be done using numerical methods and details of which are again available in the existing literature; the idea of this discussion is not get into technicalities of solving integral equation, but to highlight what is possible using these tools.

So, at the end of this, we can obtain a series representation for the Gaussian random processes Z of t, and was this is done, we get the representation for X of t, in terms of its mean, a standard deviation and this non-linear transformation on a series. So, the idea here is that, a random process X of t has now being represented, in terms of a set of orthogonal random variables. So, this process is called as discretization of random process X of t.

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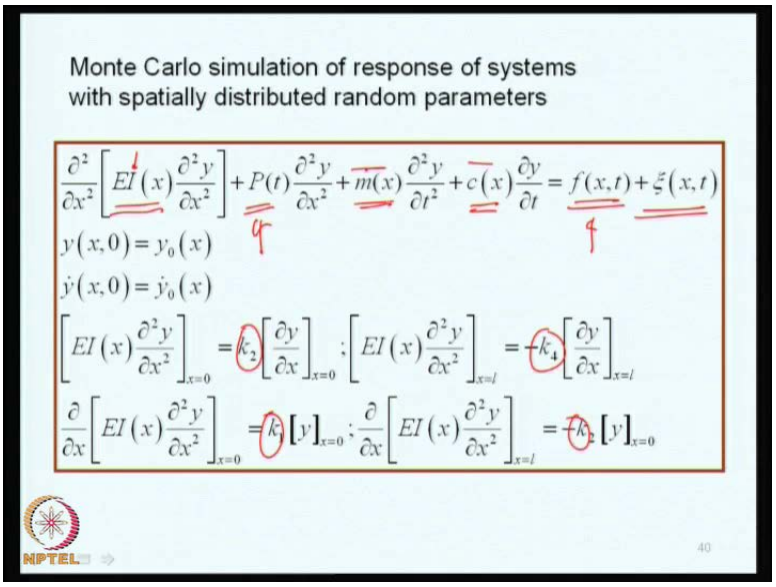

Now, equipped with this tools of representing Gaussian and non-Gaussian random processes, we can now consider the problem of Monte Carlo simulation of response of systems, with spatially distributed random parameters. I have shown this view graph earlier; this is the problem of a Euler Bernoulli beam, with randomly varying flexural rigidity, randomly varying mass per unit length, and randomly varying damping parameter; the system has random excitations here, here, and a modeling error term

here(Refer Slide Time: 42.36); and there are other uncertainties, in specifying the boundary conditions and they appear in terms of this spring constant k 1, k 2, k 3, k 4.

Now, if we are defining the performance of this beam, in terms of, say, want my (( )) at given point on the beam or reaction transport to the supports or maximum displacement etcetera, we have dealing with a time variant problem, where the random process y(x,t) itself is the function of the time varying random processes P of t f of x comma t, and spatially varying random fields EI m of x and c of x how do we proceed.

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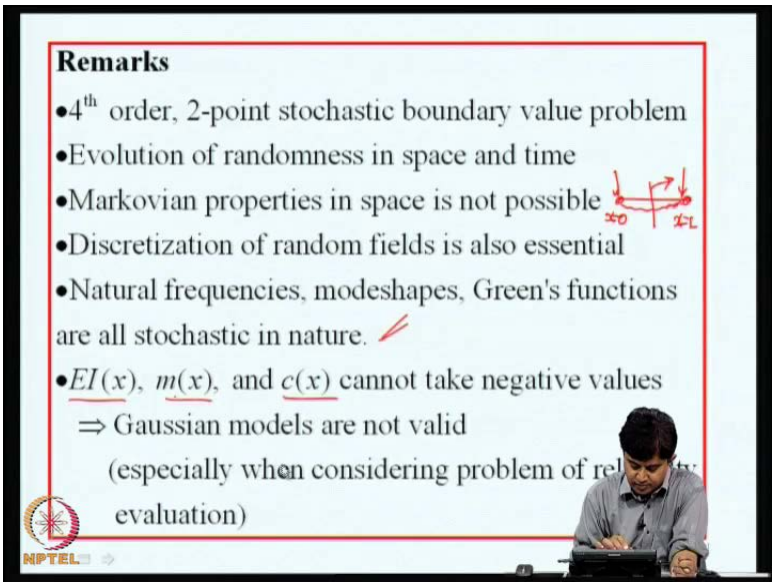

So, what we have on hand is a fourth order, 2-point stochastic boundary value problem. See the moment the coefficients becomes random; then, mathematical nature of the problem changes substantially and there will be many interesting consequences, so that we begin by noting that, this is a 2-point stochastic boundary value problem and there is evaluation of randomness in space and time.

We cannot use Markovian properties as far as evaluation in space is considered, because it is a boundary value problem. We have a state defined here at x equal to 0, and at the end of the beam, at x equal to l, the system states are again specified; therefore, the evolution for a given time in t of the solution trajectory cannot be Markovian, because at any given value of x, there is a condition ahead, which needs to be satisfied, and that makes Markovian property; this kind of random processes will not have Markovian properties.

Now, in order to tackle these problems, it is important that the random fields EI of x m of x, c of x, etcetera are to be discretized. By that, what we mean is that, these random fields need to be represented in terms of a set of random variables; this is quite similar to the problem of discretization of the displacement field itself. Displacement field is a continuous function of x and t in x; we replace the representation of u of x comma t, in terms of some shape functions and the nodal value of u at a few specified points. So, the evaluation of displacement along x is replaced by a few values at the specified nodes.

Now, on account of the randomness in stiffness and densities, the natural frequencies mode shapes Green's functions, etcetera, all becomes stochastic in nature; and characterizing these properties itself is a major problem in stochastic structural dynamic. And I am not going get into that, the discussion here is focused only on numerical methods for reliability assessment, and the way we have formulated the problem; for example, in determining a reliability with subset simulations, we need to characterize the problem in terms of a set of random variables, and that is what we are trying to do.

Now, this quantity EI of x m of x and c of x cannot take negative values; this is another constraint that we have, because these have certain physical meaning, they cannot be negative. So, we cannot use Gaussian models for characterizing these quantities, especially in problems of reliability evaluation.

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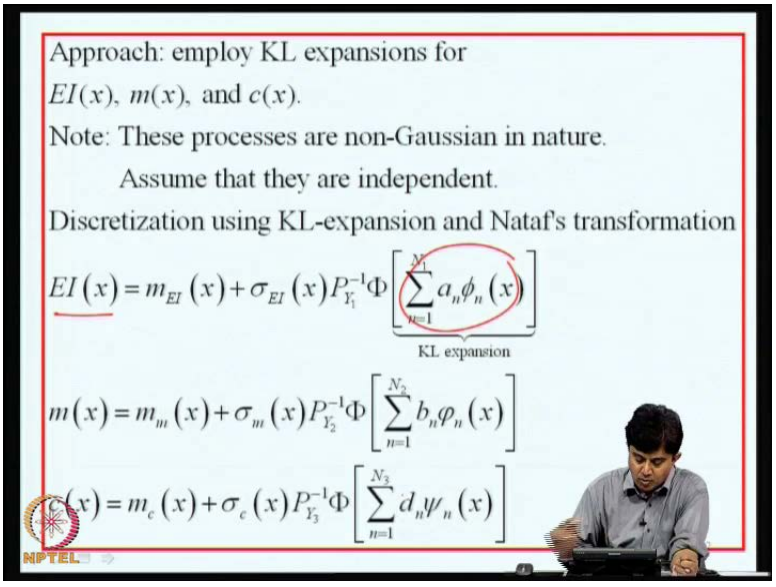

So, how do we proceed? So, we basically would like to use Karhunen and Loeve expansion for representing EI m of x and c of x. Now, these processes are non-Gaussian in nature. And for purpose of discussion, presently, we assumed that they are independent. So, what we do is, we use Nataf's transformation and KL expansion for the associated Gaussian random process, and we discretize EI of x as shown here. We first define y of x by removing its mean and dividing by standard deviation; and then, we introduce the associated Gaussian random process, and that Gaussian random process is represented in terms of Karhunen and Loeve expansion. So, EI of x now thus gets represented, in terms of n 1- number of random variables, namely, a 1, a 2, a 3, a n 1.

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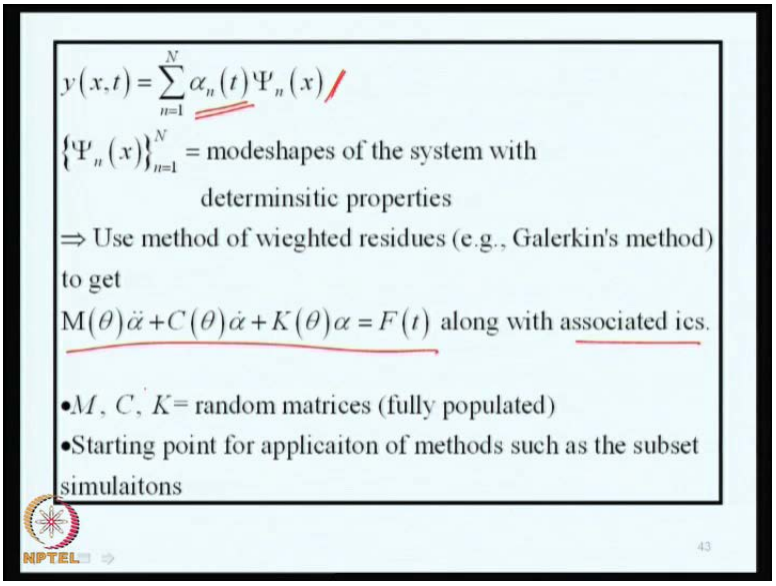

Similarly, mass can be discretized in terms of other random variables b 1, b 2, b 3, b n to c of x in terms of some other set of random variables. The moment this representation is made, we can now go for a series representation for the displacement field itself; this step can here, phi n of x is taken to be global travel functions, but they need not be global. You can use finite event method, and use interpolation functions and nodal values, etcetera, the standard finite element formulation can be used, but for purpose of illustration here. I am using the global shape function, this could be, for example, modes shape of this system with deterministic properties.

Next, we can use the method of weighted residuals like Galerkin's method and get a set of equation for these generalized coordinates alpha n; and we get this matrix equations as

shown here, M alpha dot plus C alpha dot plus K alpha equal to F of t along with the associated initial condition. The theta that we see here are the random variables, which are obtained from discretizing quantities, that vary in space; therefore, this M C K are now random addresses, they are fully populated. And we know the functional dependency of elements of these matrices as random variables, and therefore, we will be able to simulate the samples.

So, equipped with this, F of t is the random process in time. Now, we are ready to launch our, you know, Monte Carlo based reliability analysis; we can use, for example, subset simulation and consider various performance functions like maximum displacement, maximum stress, etcetera, and be able to tackle the problem.

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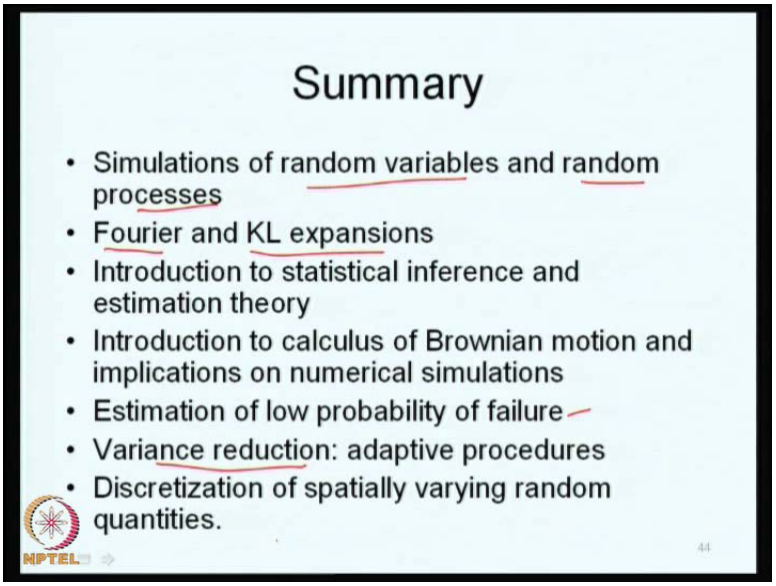

With this, we are going to close the discussion on Monte Carlo simulation methods for studying randomly parametered systems with random excitations, but before that, we can quickly summarize, what all we did during the course of last few lectures. We begin by developing methods for simulations of random variables and random processes.

So, we started by discussing scalar random variables; then, we generalize it to vector random variables considered non-Gaussian random variables. We considered partially specified non Gaussian random variables, Nataf's transformation, and so on and so forth; and we also discuss Rosenblatt transformation and methods like that.

Subsequently moved on to simulations of random processes. Here, again, we considered scalar Gaussian random processes, vector Gaussian random processes, scalar non Gaussian random processes, vector non Gaussian random processes, stationary random processes, non stationary random processes, and so on and so forth.

We also discussed the Fourier and KL expansion based methods for representing random processes, Fourier series methods. I have discussed at several points during the course of these lectures, and the Karhunen and Loeve expansion methods was discussed during today's lecture. We have also been introduced to the methods of statistical inference and the estimation theory, and the peculiarities associated with calculus of Brownian motion and their implications on numerical simulations of system driven by white noise. I have been clarified, and we developed this Cyto series expansions and strong forms of discretize version of this stochastic differential equations, and we illustrated a few examples.

Subsequently, we consider the problem of estimation of low probability of failure. So, if probability of failure is of the order of 10 to the power minus 5, we showed that, if you want estimate of probability of failure with coefficient of variation above 4.1, we would need about 10 to the power of 7 samples; and the computation then becomes exorbitant. And therefore, we need to consider the problem of variance reduction here. We discuss the problem of one of the tools among many that are available in the literature, namely, the subset simulation, which is essentially adaptive, important sampling procedure that has been discussed.

Finally, we raised, we considered the problem of systems with spatially varying random properties and we considered the question of discretization of the spatially varying quantities using Karhunen and Loeve expansions. So, with this, we will close the discussion on Monte Carlo simulations. And in the next course of lecture, we will consider some applications of stochastic structural dynamics in problems of earthquake engineering, wind engineering failures due to fatigue, and so on and so forth; so, at this point, we will conclude this lecture.