Stochastic Structural Dynamics
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Lecture No. # 29
Monte Carlo Simulation Approach-5

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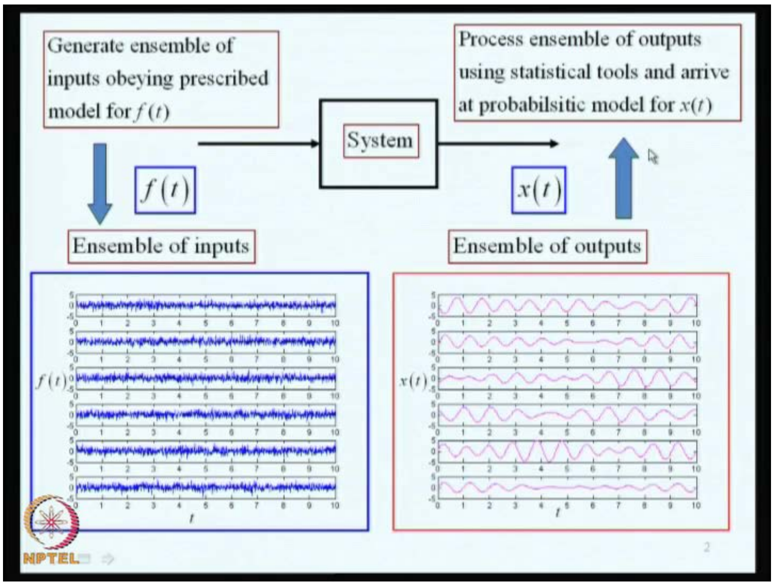

We have been discussing the application of Monte Carlo simulation procedures for simulating response of random randomly excited dynamical system. So, we will continue this discussion; so, the basic framework of our solution strategy is displayed here. This is the dynamical system which is excited by random excitation; so, there is a ensemble of excitations. And the complete description of f of t is assumed to be available, and based on that, using a methods that we have already discussed, we can generate an ensemble of time histories, samples of f of t which are compatible with a given probabilistic description of f of t.

Then, for each sample of this excitation, we will use deterministic solution procedures and produce a corresponding response time history. So, associated with each sample of input, there will be a sample of output and we get an ensemble of response time histories, and this ensemble of response time histories now need to be processed using methods of statistics, to arrive at probabilistic description of x of t. We will estimate, for example, mean of the response, covariance of the response, power spectral density, probability distributions density functions, first passage time, extreme value, so on, and so forth, using suitable estimators.

So, this is the framework; we have now describe how to stimulate samples of f of t compatible with the given description of joint density of f of t. And we have already discussed, how to calculate the dynamic response of a system given the sample of an excitation; that is the deterministic analysis of response that we have discussed. And we also discussed how to characterize estimates of different character properties of x of t and what are the properties of these estimators and sampling distributions, and things like that we have discussed.

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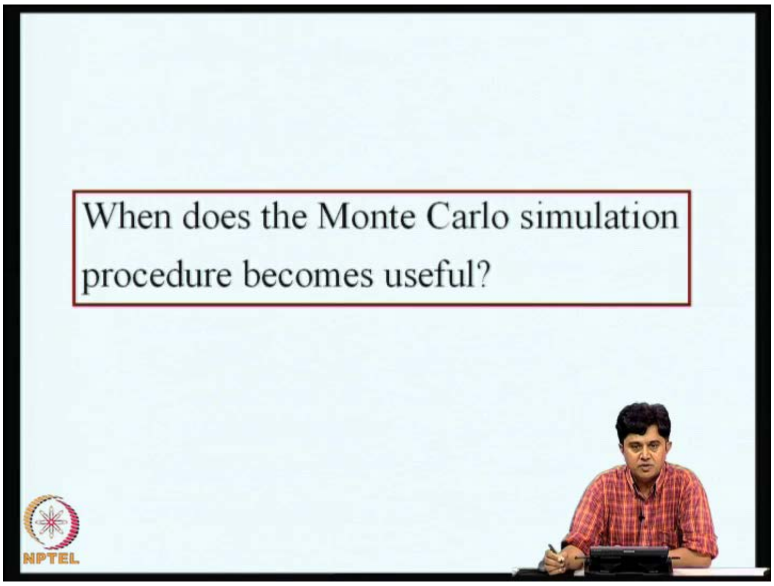

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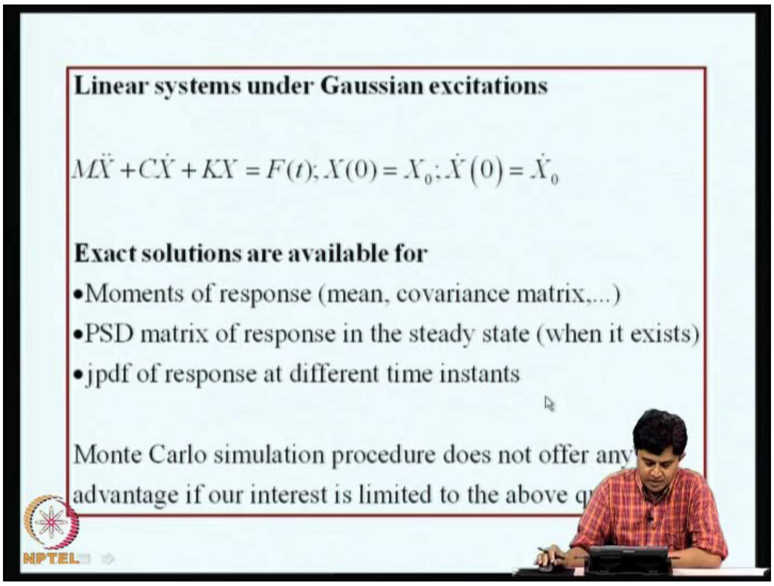

Now, let us proceed, first at the outside we will ask, when does the Monte Carlo simulation procedure becomes useful? Let us start by considering linear time invariant systems under Gaussian excitations. So, this is the governing equation MX double dot plus CX dot plus KX equal to F of t, which specified initial displacement and velocity vector; X is a vector, this for this represents a multi degree freedom system; F of t is a Gaussian random process, it can have a non-zero mean, it can be non-stationary.

Now, if we are interested in characterizing the response in terms of its moments, like for example, mean, covariance, etcetera or PSD matrix of the response, if a steady state solution is possible; that is, when F of t is stationary and we look at steady state response, we have a an exact solution for this PSD. Similarly, if you are interested in joint probability density function of response vector at different time histories, we know that the system is linear and excitation is Gaussian; therefore, this joint probability density function is also going to be Gaussian.

So, if you are interested in only these quantities moments or power spectral density or joint probability density functions, actually there is no need to use Monte Carlo simulations to solve this problem. Actually, it is a wasteful approach compared to the elegance of an exact solution; it does not offer any advantage, if our interest is limited to this kind of response quantities.

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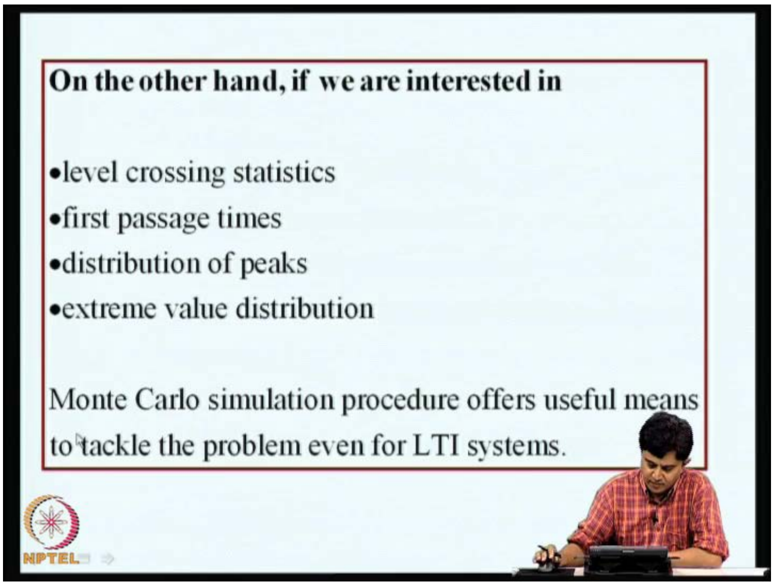

On the other hand, if we are interested in say level crossing statistics or first passage times or distribution of peaks or extreme value distribution, even for linear time invariance systems under Gaussian excitations, we do not have an exact solution to probabilistic description of these quantities. We have used certain heuristic arguments and postulated models; for example, we assumed that, if levels are high, we can model the number of times level is crossed using Poisson models and first passage times become exponential, and peaks are value distributed for narrow band processes, and so on and so forth.

Similarly, extreme value distribution, we are had at a gumble model for the extreme value of the response, but all the solution that we develop are essentially approximate in nature; and they are not robust in terms of variations in, say, the excitation characteristics or damping characteristics in the system. So, in such cases, of course Monte Carlo simulation procedure offers useful means to tackle the problem, even for linear time invariant systems.

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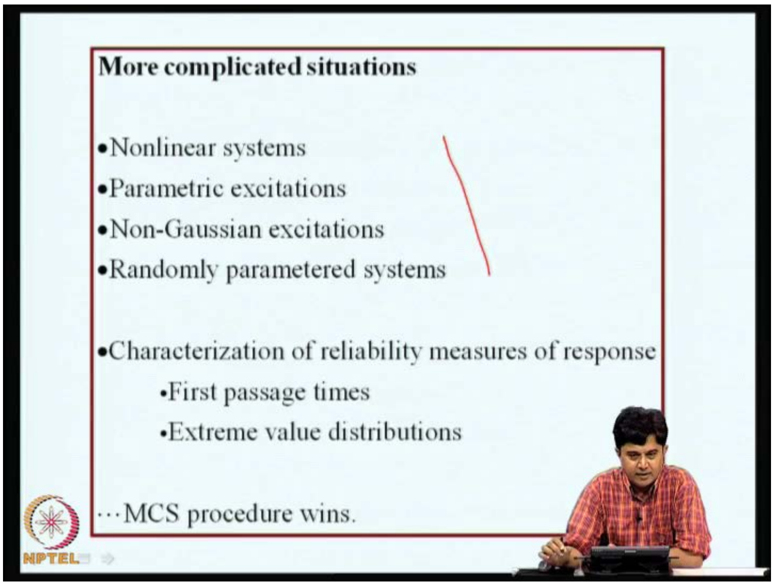

If we have to, if we encounter more complicated situations, for example, system behavior is non-linear or there are parametric excitations or excitations are non-Gaussian in nature or system parameters are uncertain, therefore, we use random variables or random process models for even to describe the system parameters. In all these cases, the system cannot be analyzed exactly and invariably one has to go for an approximate analytical solution or the Monte Carlo simulation procedure.

Similarly, just to re-iterate, if characterization of reliability measures of response like, first passage times or extreme value distributions are of interest, the Monte Carlo simulation procedure becomes, I mean, it wins over the other methods. So, this is the broad context in which we are interested in Monte Carlo simulations.

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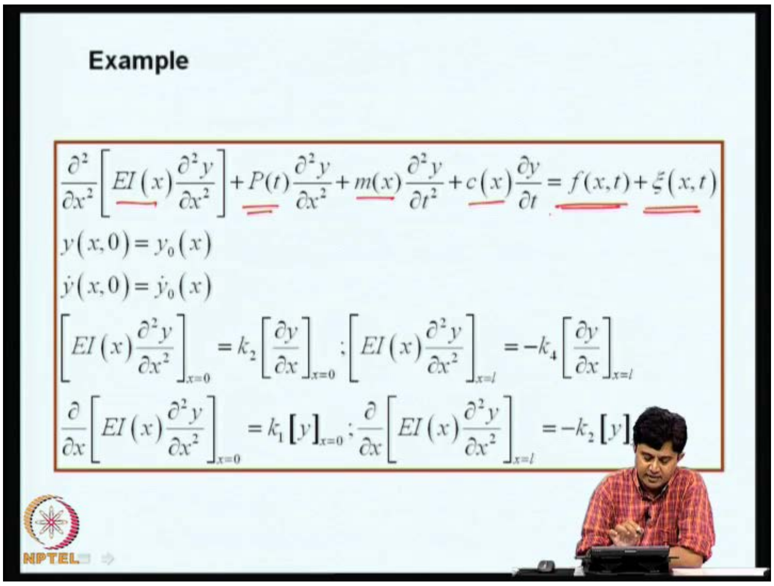

Just to get a feel for the kind of problems, where we could apply Monte Carlo simulations? We will consider a inhomogeneous Euler Bernoulli beam, which is excited by a force which varies both space in time, and it is also excited by a parametric excitation P of t; and we assume that, the flexural rigidity, mass for unit length and damping coefficient, for all specially varying. And also the boundary conditions, we do not have perfect hinge or perfect free and conditions; there are some you know flexibility at the joints and they are captured through spring constants k 1, k 2, k 3, k 4.

Now, the load f of x of t could be, for example, wind load and this can be random in nature; it can vary randomly in space and time. The parametric excitation could be random. This psi of x comma t is a quantity that represents modeling error and this also could be randomly varying in space and time. The quantity EI of x, m of x, c of x vary in space special coordinate x, and they can also be random and they can be modeled as random field. If there variation with respect to x is not very significant, they can as well be modeled as random variables.

So, this is a stochastic boundary value problem is a stochastic partial differential equation and an exact solution to this, even to character mean of the response is not available. But on the other hand, we can simulate samples of f of x comma t psi of x comma t EI of x, m of x, c of x, etcetera, and use deterministic procedures to find out the response of this system for each realization of these stochastic quantities. And we can produce ensemble of response quantities and that ensemble can further be processed to get descriptions of interest.

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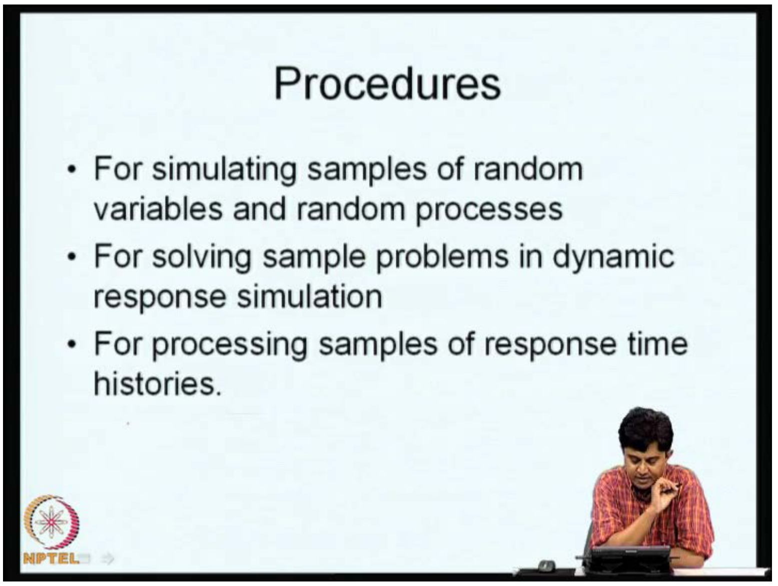

Now, we are already covered in this course, procedures for simulating samples of random variables and random processes, compatible with the given probabilistic description of these quantities. Similarly, we have described the procedures for solving sample problems in dynamic response simulation; that means, if the beam is excited by a dynamic excitation, we already develop the necessary procedures to produce the response time histories using model expansion, and so on, and so forth. Similarly, also covered the procedures for processing samples of response time histories. So, basically in implementing Monte Carlo simulation procedures, we could ask, where does actually the difficulties arise?

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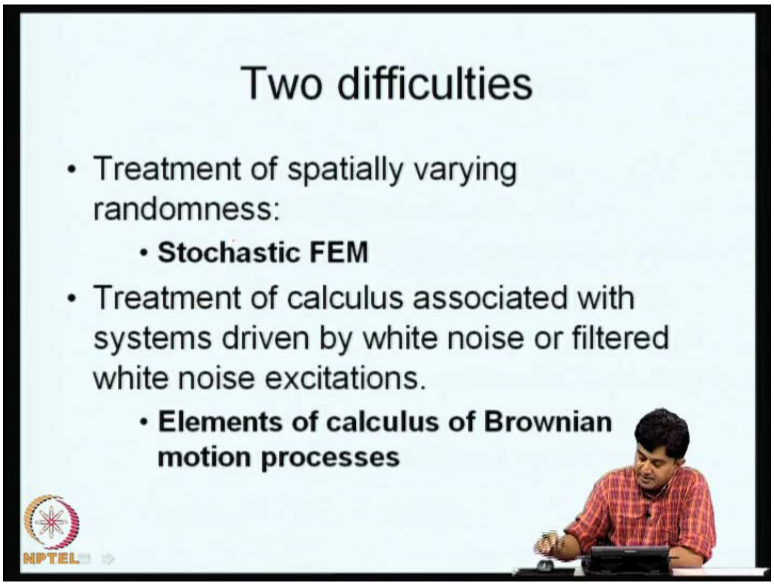

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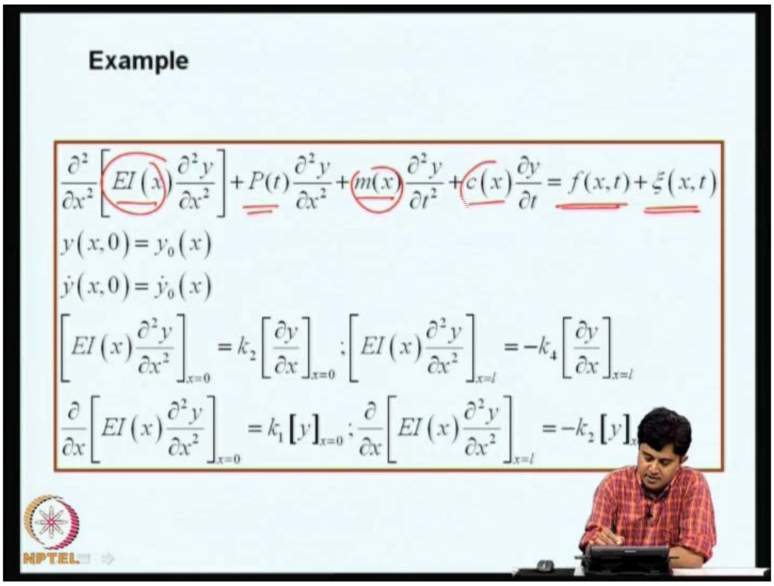

Given the coverage of topics in this course, there are two difficulties that still remain: one is treatment of spatially varying randomness. So, I talked about flexural rigidity being function of x, m of mass per unit length being function of x, etcetera and this quantity appears as coefficients in the governing differential equations. And they these quantities could vary in space and that this type of problems we are not yet discussed; and one need to generalize the say the finite element method for handling inhomogeneous beams, to take into account the additional complexities of spatial variations - the spatial random variations - in system parameters, that takes us to the topic of stochastic finite element method.

On the other hand, there is yet another difficulty that is associated with treatment of calculus associated with system driven by white noise or filtered white noise excitations. So, what we would do in this lecture is, we will cover the second aspect and we will consider how to simulate - numerically simulate - samples of response of systems, which are driven either by white noise or by filtered white noise; and that would require preliminary discussion on elements of calculus of Brownian motion processes.

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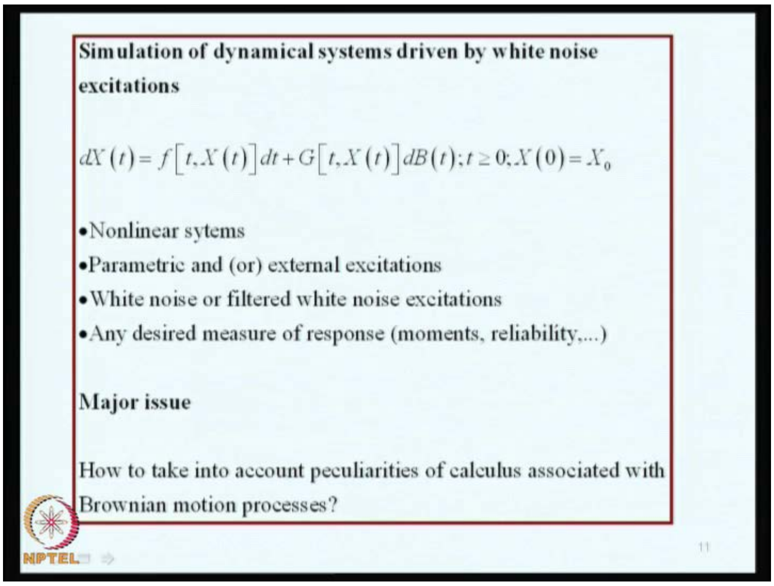

So, this is what we are going to do in this lecture. So, the topic for today lecture is, simulation of dynamical systems driven by white noise excitations. So, the governing equation is written in the form, dX of t is f of t comma X of t dt G into dB of t with specified initial conditions; dB of t is increment of Brownian motion processes. The range of problem that are represented this through this equation is quiet wide; the system could be non-linear, the excitations could be parametric or external; and this excitation could be straightaway white noise processes or they could be filtered white noise excitation. If they are filtered white noise excitation, the state variable associated with these filters get augmented with the systems states, and X of t in this case represents an extended vector; and we could deal with any desired measure of response like moments, auto covariance, power spectral density, first passage time, extreme value, so on, and so forth.

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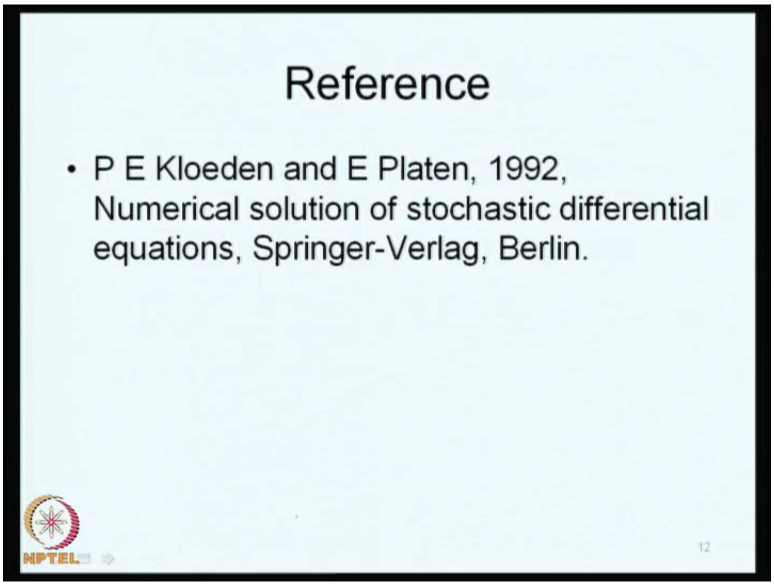

Now, what is the major issue? The major issue here is, how to take into account peculiarities of calculus associated with Brownian motion processes? If we quickly recall, we have already seen the issues associated with a random walk becoming a continuous time random process and we will return to that shortly. Before that, I would like to site this reference, this is book by Kloeden and E Platen, which describes extensively the numerical solutions of stochastic differential equations; and the lecture that I am presenting now, this will be a very for this lecture, this will be a very useful reference.

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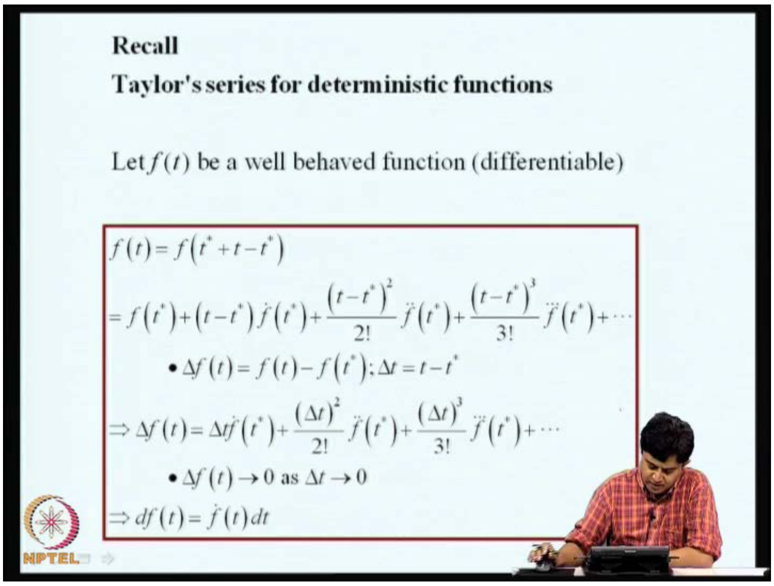

So, we will begin by recalling Taylor's series for deterministic functions. Suppose I consider a function f of t which is well behaved, that means, it is sufficiently smooth, that is, we can differentiate it as many times as we wish, we can considered f of t and write it as f of t star plus t minus t star. And we can expand this function around t star and we write it as f of t star t minus t star into f dot t star plus delta t square by 2 factorial f double dot etcetera; this is the familiar well known Taylor's series expansion for f of t.

Now, I will denote, I will take f of t star to the left hand side and denote f of t minus f of t star as delta f of t, and I will call t minus t star as delta t. So, I can rewrite this in terms of these increment, delta f of t is delta t f f dot t star plus delta t square by 2 factorial f double dot, etcetera. Now, if we take now the limit of delta t going to 0 and delta f of t going to 0, and use that here, we get the well-known result df of t is f dot of t into dt, because delta t square delta t cube, etcetera; all of all of them would go to 0.

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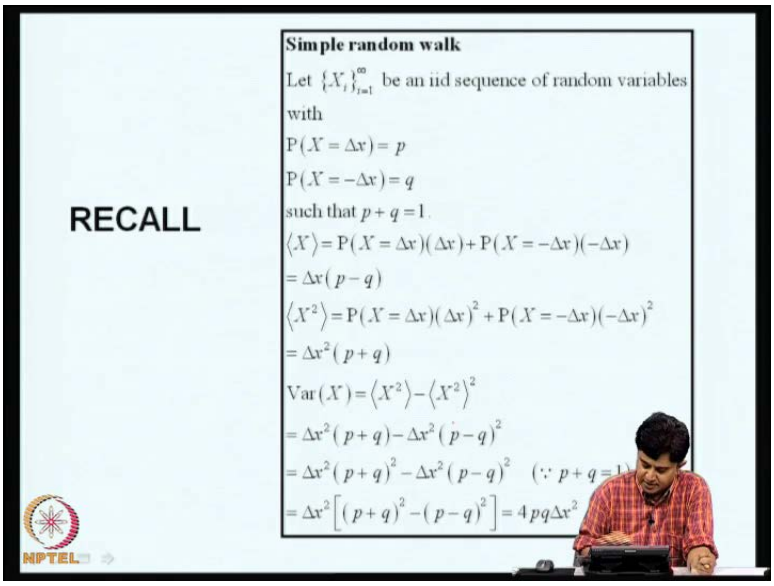

Now, this would not happen, if f of t is a say a Brownian motion process or a function of a Brownian motion. So, what happens in that case? So, we will quickly recall; this is again a recall of what we have already discussed. This is simple random walk, if X i i equal to 1 to infinity is an iid sequence of random variables, and each random variable taking two possible values delta x and minus delta x with probability P and 1 minus P, we can evaluate the mean, that the delta x minus delta x into p minus q, and mean square and the variance. So, this we have done; the variance of this two state random variable is 4 pq delta x square.

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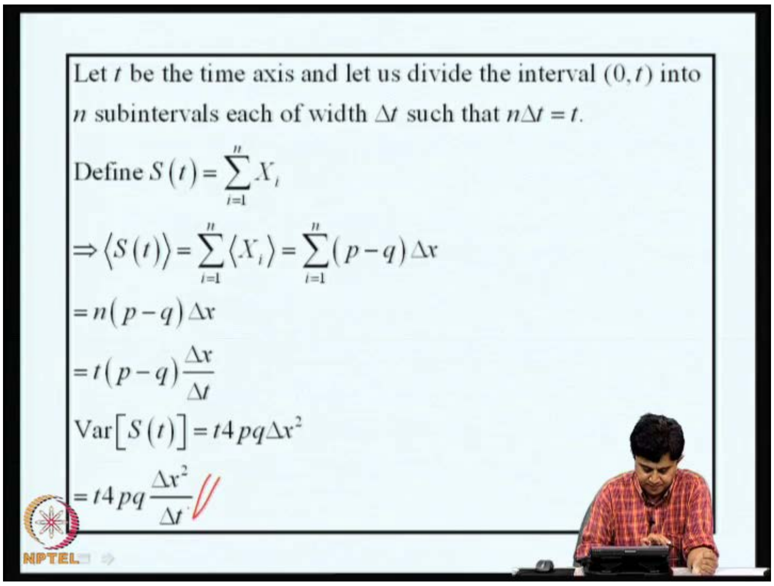

Now, let us assume that, there is a time axis and divide the interval 0 to t into n subintervals, each of with delta t, so that n delta t is t. Now, I define a random process S of t as a sum i equal to 1 to X i. Now, the expected value of S of t we can evaluate, and we can show that, it is t into p minus q delta x by delta t. Similarly, variance can be shown to be given by this quantity.

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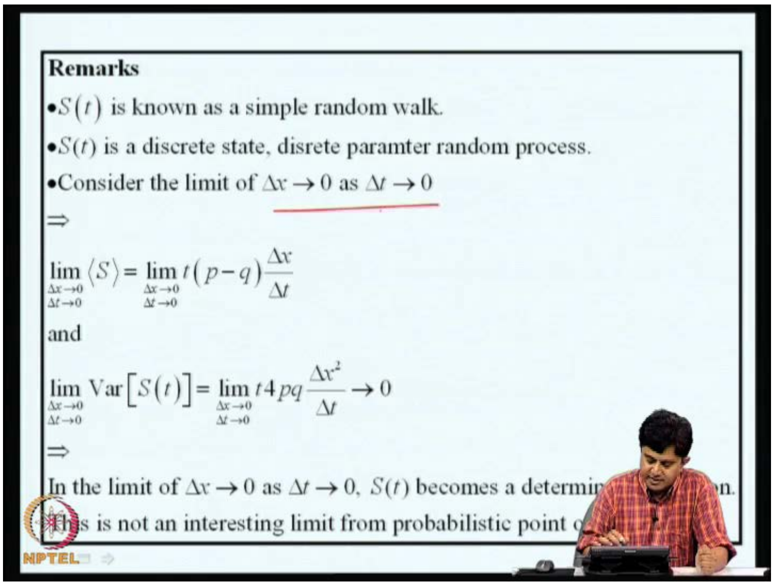

Now, this is the simple random walk process. Now, it is a discrete state discrete parameter random process. Now, I would like to consider the limit of delta x going to 0, as delta t going to 0; this is a typical situation that we envisage in a deterministic science. So, what happens to the expected value of S? It becomes a quantity delta x by delta t, which is exhaust acceptable meaning, but variance of S of t goes to 0; and this is not an interesting limit, that means, under this particular limiting operation, S of t becomes a deterministic function is the variance is becoming 0.

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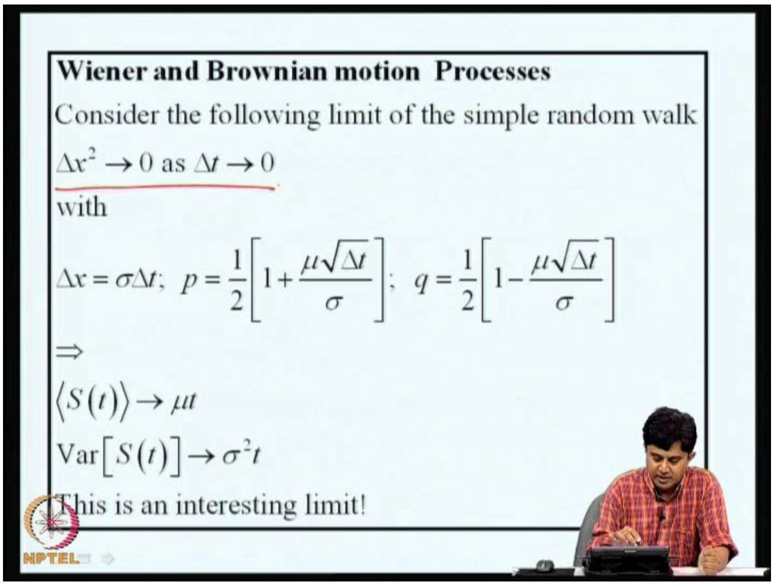

So, it is not a interesting a limit from point of view of probabilistic treatment of random walk. Now, on the other hand, if delta x square goes to 0, as delta t goes to 0, then what happens? The variance of S of t now goes to a quantity, which is non-zero; and by minor adjustment of this terms, we can show that, mean of S of t become mu t and variance becomes sigma square of t, and this is a quiet an interesting limit. This resulting random process is a Wiener process that we have seen; and for this Wiener process, the complexity associated with the trajectories is that, as delta t goes to 0, delta x square goes to 0. So, this is, this will create certain additional complexities, when we have to deal with calculus of these random processes.

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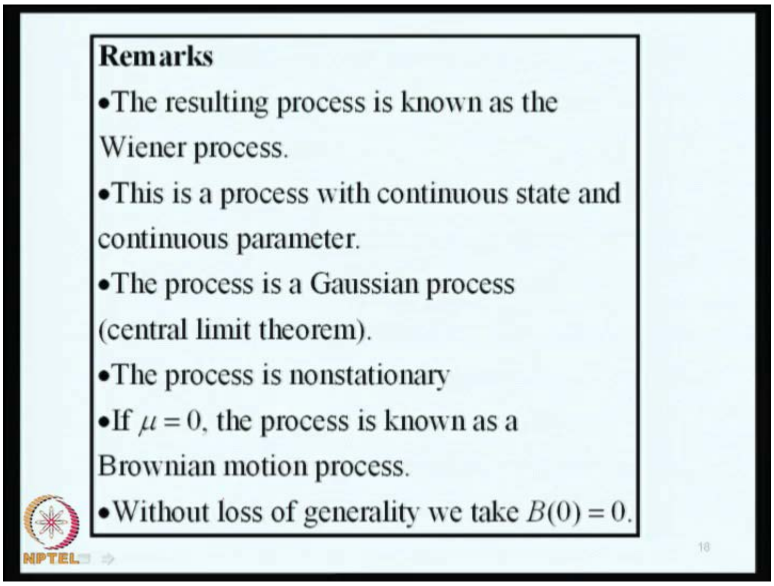

So, S of t under their specific limiting condition as we know is a wiener process. This is a continuous state and continuous parameter random process. It is a Gaussian random process, because there is addition of iid sequence of random variables, and we invoke central limit theorem and that process is Gaussian.

Now, the process is non-stationary; you can easily see that, because mean is function of time, and variance is also a function of time. And if mean is 0, the wiener process is known as a Brownian motion process. So, we in the further discussion, we deal with Brownian motion process.

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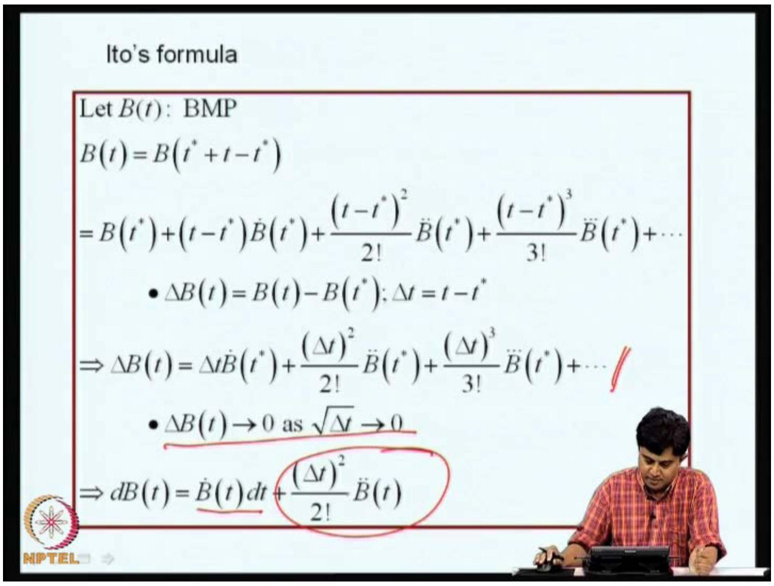

Now, let us try to now consider the Taylor's expansion of b of t; just as we did for f of t, if we now repeat those steps, if a B of t is a Brownian motion process, I can write it as B of t star plus t minus t star. And following the usual steps, I can write this step as well, and introduce delta of B of t is B of t minus B of t star, and delta t as t minus t star.

Now, I can again write this step as well, which is quite similar to what we did earlier. Now, we have to now use this limit, delta B of t goes to 0, as square root delta t goes to 0; it is not that, as delta t goes to 0, delta B of t goes to 0. So, consequently what happens? dB of t will now be no longer just this term; B dot of t dt as was the case for f of t, but now we get a new term, delta t square by 2 factorial B double dot of t. This additional term introduces various complexities, when it comes to numerical solution of the governing differential equation.

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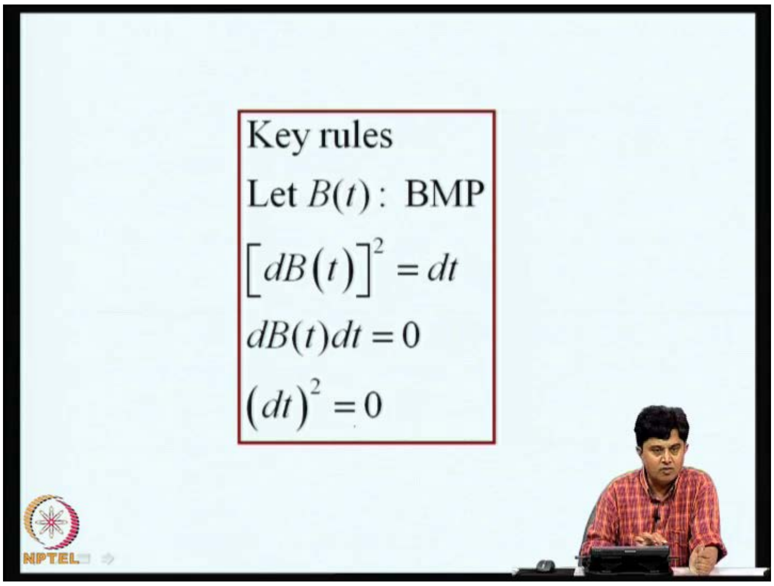

Now, I will states some rules and I will prove it in due course. But to begin with, we can follow these rules, see accept these rules and see what will be the consequence, and then, we will see how can be shown. The rules is square of dB of t is dt, dB of t dt is 0 and dt square is 0; so, what does that mean?

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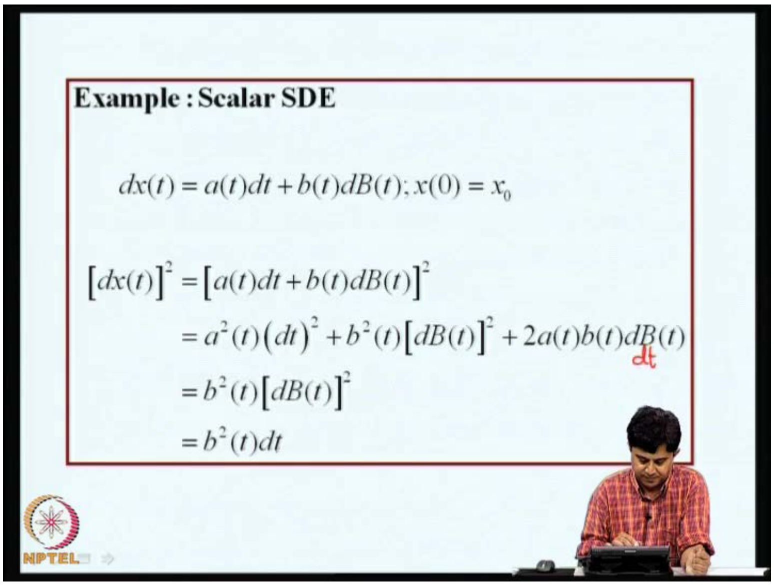

Let as consider a stochastic differential equation which is scalar, dx of t a of dt plus B of t dB of t and x of 0 is x naught. Now, if I square dx of t whole square, I get a dt plus B dB t whole square; this is a square dt square plus B square dB t whole square plus 2 a b dB t. Now, we follow the rules dt square is 0, dB of t whole square is dt, and dB t into dt is 0; so, I get B square of t into dt.

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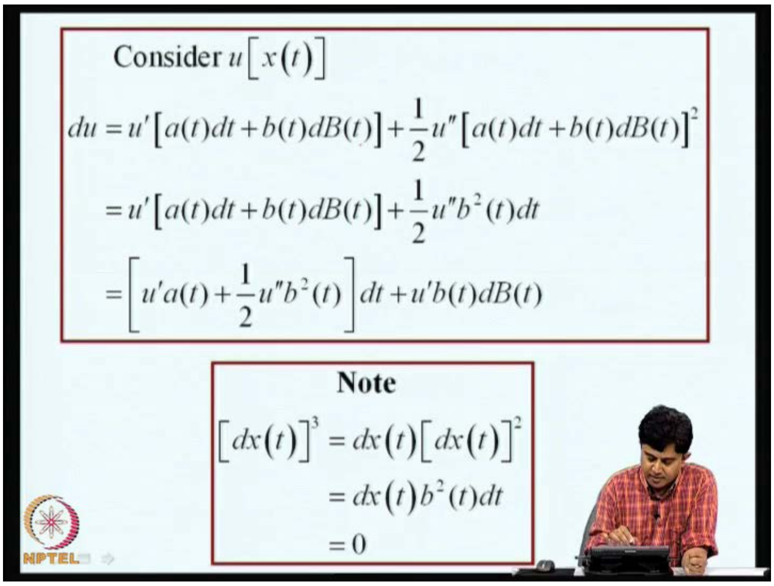

Now, let us consider a function u of x of t; what is d u? d u is u prime into x dot, which is this plus half of u double prime into x dot square, because that is what I have to… see, this is what we have shown here; so, this is this similar logic is being used here. So, if I now substitute for x dot in terms of x dot is a dt plus b dB t, and then carry out this operation and use those three rules, I get d u as u prime a of t plus half u double prime b square of t dt plus u prime b of t into dB of t.

So, this is known as Ito's formula. Now, what happens should dx of t whole cube? Just should make sure that this is 0, as we have been telling, we can quickly do this calculation dx of t into dx of t whole square, and dx of t, this is b square of t dt; just now I have showed dx of t whole square is b square of t dt and this is 0.

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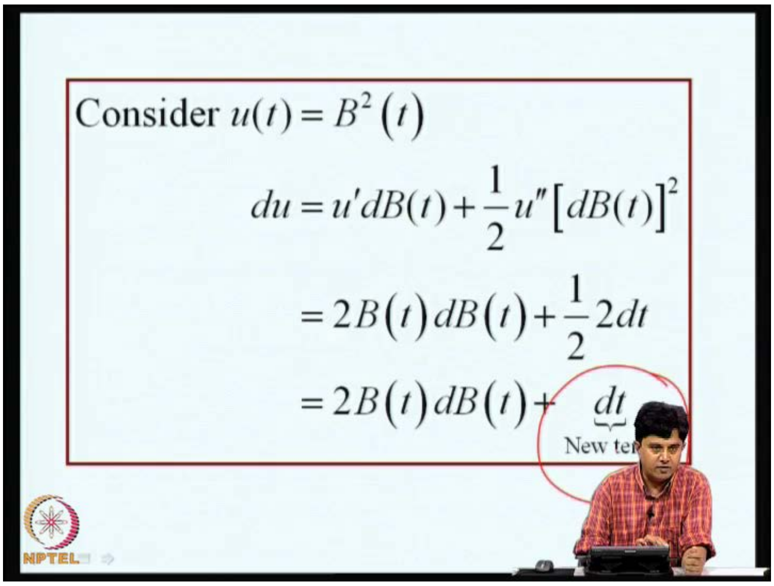

Now, as an example, let us consider u of t is B square of t. If I now follow this rules Ito's formula, I get d u as u prime dB of t plus half u double prime dB of t whole square. And now, if I use the fact that u is B square, I get now this as 2B of t into dB of t plus d t. So, this is a new correction, because of peculiarities of parts of B of t.

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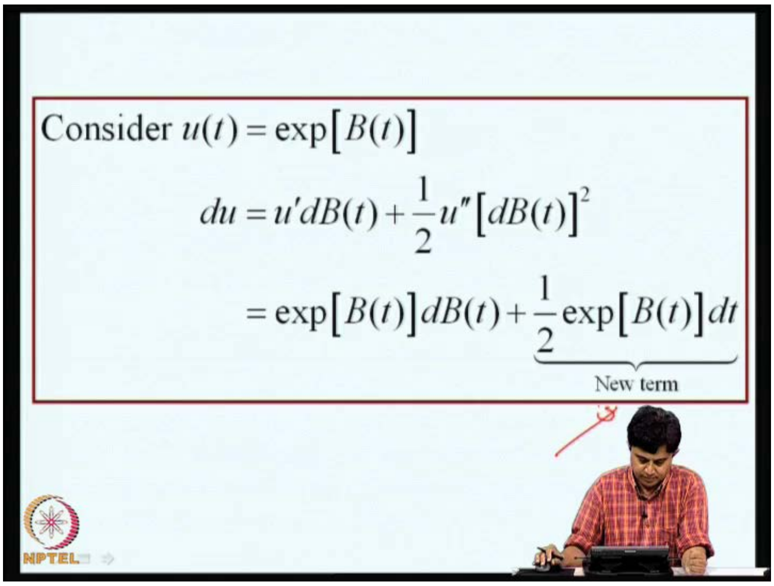

Similarly, if I consider now u of to be exponential of B of t, what will be d u? u prime dB of t plus half u double prime dB of t whole square. Now, u prime is exponential B t dB t plus half-exponential B of t into d t; so, this again is a new term.

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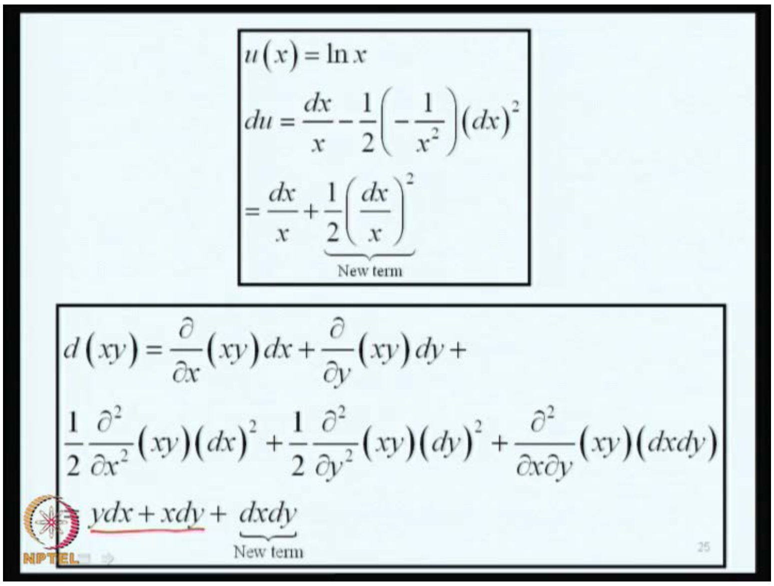

Similarly, we can use rules of this calculus. Now, if u of x is ln x according to the stochastic calculus, du will not be simply dx by x but there is an additional term, that will be, if you include that it will be a dx of x plus half of dx by x whole square.

Similarly, what is the product rule? d of xy is dou by dou x of xy d x plus dou by dou y of xy dy plus these additional terms, which we have to include now. And if I include this and use the rules that we have been talking about, we get in addition to the traditional y dx plus x dy, there is a new term which is dxdy; this has to be go on in mind.

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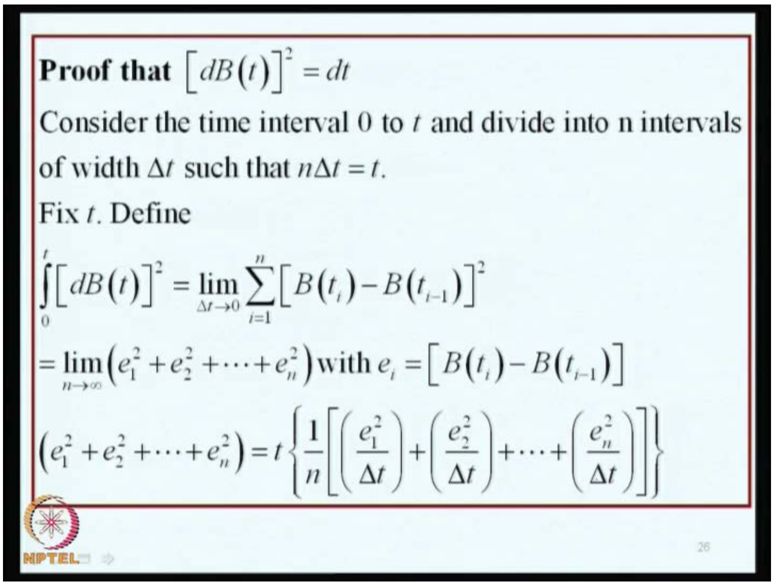

Now, I stated those three rules, we can quickly see if we can offer a simple proof for those things. Suppose I consider how to proof that dB of t whole square is dt, now what I will do is, I will consider the time interval 0 to t and divide it into n intervals of width delta t, such that, n delta t is t.

We will fix t and now let us define 0 to t dB of t whole square, as this summation limit delta t going to 0, i equal to 1 to n, dB t is increment of Brownian motion process. So, I will write it as B of t i minus B of t i minus plus minus 1 whole square. Now, I will define e i as B of t i minus B of t i minus 1; and I will rewrite this sum as limit of n tends to infinity e 1 square plus e 2 square plus, so on and so forth, e n square.

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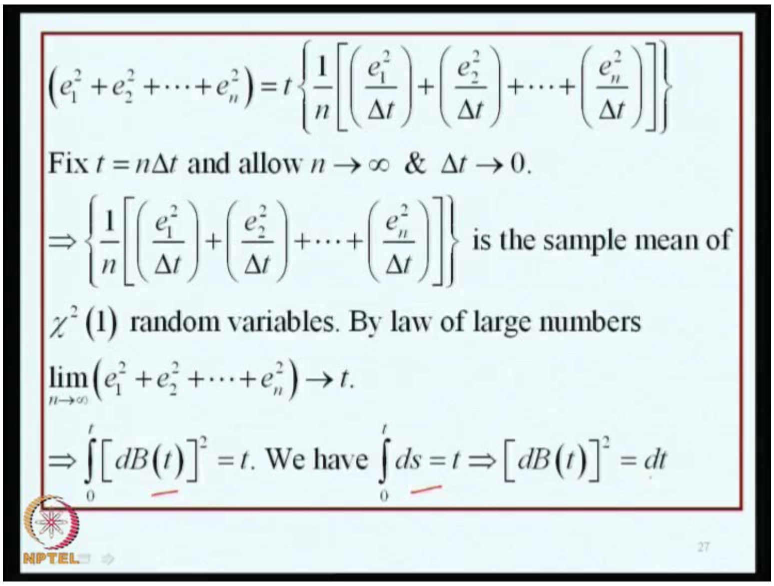

Now, I will rearrange the terms, I will multiply and divide by t, and in the denominator, I will write for t, n delta t. And I will rewrite this summation in this form; this is the form. Now, we will fix t to n delta t, and allow now n to infinity and delta t to going to 0. Now, we know that, e 1 is a Gaussian random variable - e 1, e 2, e 3, they are all Gaussian - and this actual this term inside the brace represents sum of squares of Gaussian random variables; and we know that, it is a chi square random variable and we can verify the parameter for that is 1. And if I now use law of large numbers, we can show that this limit is actually e 1 square plus e 2 square plus ,and so on, e n square actually t.

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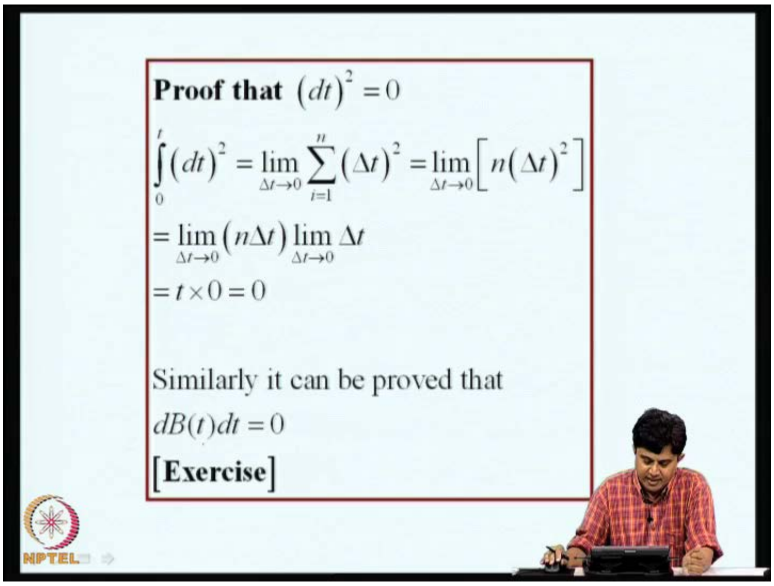

So, consequently, I get 0 to t dB of t whole square is t, but we know that 0 to t ds is equal to t. So, if you compare this with this, we can now infer dB of t whole square is dt. Now, how do you show that dt square is 0? This is straight forward; you write, you follow the same logic, write 0 to t dt square as limit, delta t going to 0, i equal to 1 to n delta t square; and if this becomes n delta t whole square, delta t going to 0; and this can be written as limit of delta t going to 0 n delta t into delta t, n delta t is t, and the other delta t goes to 0; therefore, answer is 0.

So, you can similarly show that dB of t into dt is 0; this I leave it as an exercise. So, we have we have now a glimpse of the consequence of, you know, delta x square going to 0, as delta t goes to 0, in a simple random walk leading to the Brownian motion process. So, that has to be gone in mind, whenever we deal with Brownian motions.

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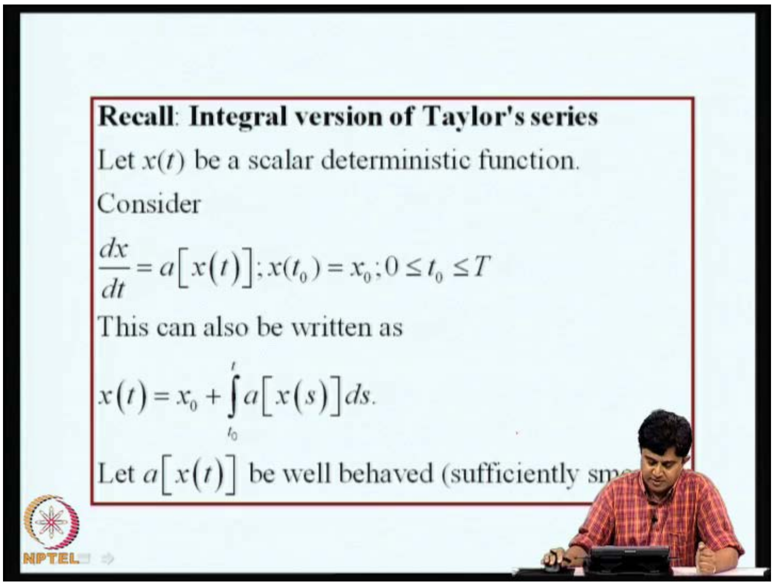

Now, our objective in discussing this is to develop numerical schemes for solving stochastic differential equations. So, if we now go back to deterministic differential equations, we know that the schemes for integrating these ordinary differential equation - deterministic ordinary differential equations - essentially I develop using Taylor's series expansions; we represent that, we use that Taylor's series expansion and truncated as certain order, and that leads to various types of integration schemes.

So, this we need to now extend that logic to deal with stochastic differential equation. So, to facilitate that, we will quickly recall Taylor's series, but in a slightly different form, whereas integral version of Taylor series. Now, let x of t be a scalar deterministic function, now consider the differential equation d x by d t is a of x of t, and the condition on x of t is given a t naught be x naught, and t naught itself can take values from 0 to capital t. This can also be written as x of t as x naught, which is at t equal to t naught, and then, integral of t naught to t a x of s ds, because dx is a x of t into dt. So, if I integrate that, this is a derivative and this is a time length. So, you should give the change in x of t over from t naught to t. Now, we will assume for purpose of discussion the d of x of t is well behaved; that means, it is sufficiently smooth, which means, that you can differentiate it as many times as you wish.

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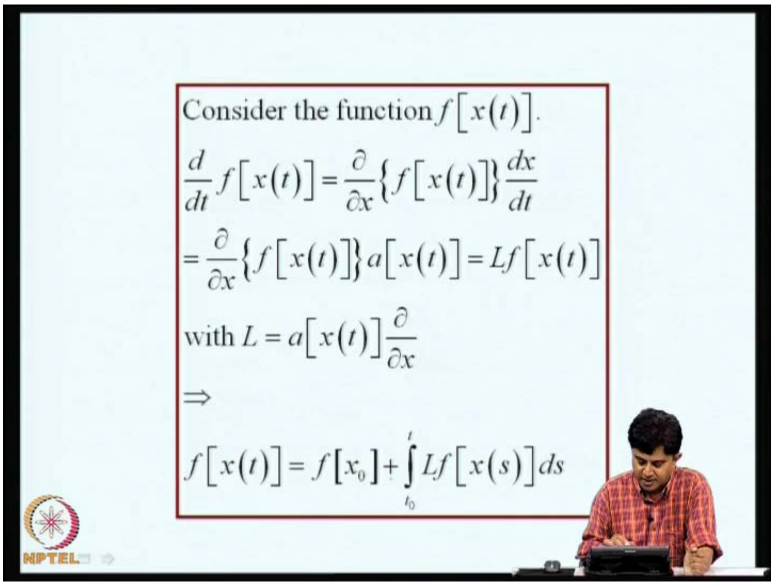

Now, we will begin by considering the function f of x of t. d by dt of f of x of t is dou by dou x of f into dx by dt, that is, dou by dou x into f, dx by dt I will write now, a of x of t. So, I will call this as L into f, where L is a operator a into dou by dou x; so, therefore, f of x of t can be now written as f of x naught plus t naught to t L f into ds.

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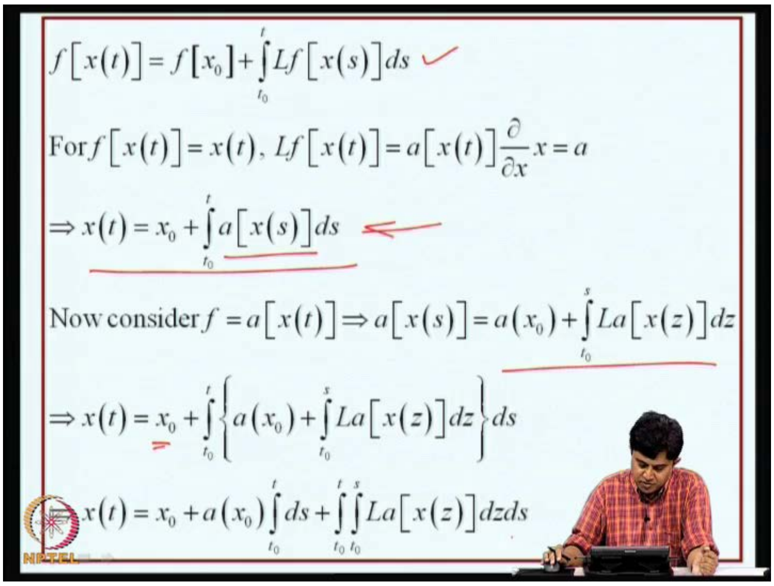

Now, if f of x of t is x of t, we should recover what we are already seen; L f will be a, L f will be a and we recover from this equation, that is, from this equation we recover what is already derived by an alternative argument.

Now, what will do now is, for this integrant, we will now write a Taylor's expansion, I mean, this expansion. So, if I take now in this formula f to be a of x of t, I will get a of x of s to be a of x naught plus t naught to s La dz. Now, this I substitute into this formula and I will get x of t is x naught, that is, this x naught, plus t naught to t, that is this; for a of x of s is here, I will write this expansion. So, I get a of x of t naught to s L a x of z d z ds. Now, I can rearrange this terms and write it as, x of t is x naught plus a of x naught t naught to t d s plus these double integral.

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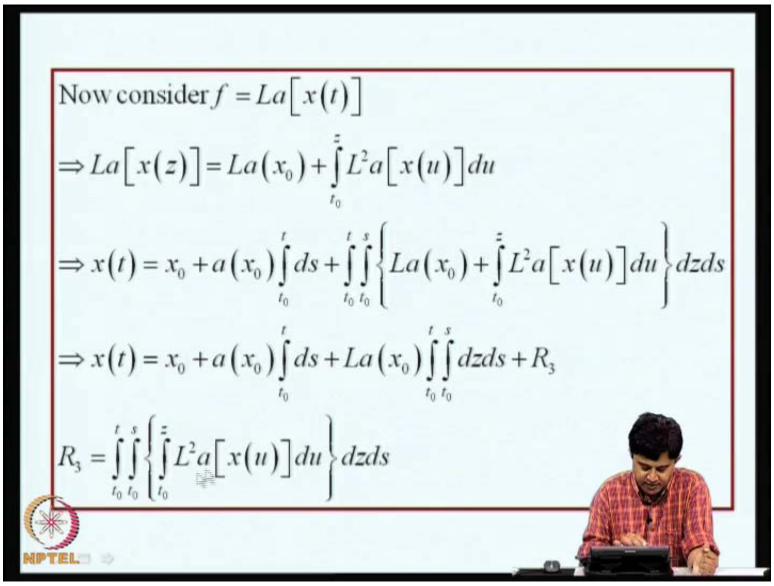

I can proceed further; and now, consider f to be La and again use this formula and represent La as La x of z as La x naught plus t naught to z L square a du. So, this I can substitute in x of t. And if I repeatedly do that, I will get now at this stage x of t is x naught plus a x naught t naught to t ds plus this triple integral, and that can be simplified and I can write it as x naught plus a x naught t naught to t ds plus La x naught into these double integral t naught to t t naught to s dz ds plus a remainder term; remainder term R 3, which is this integral. Of course, now, I can proceed further by taking L square a to be f, and again use this kind of representation for L square a and go on substituting into this.

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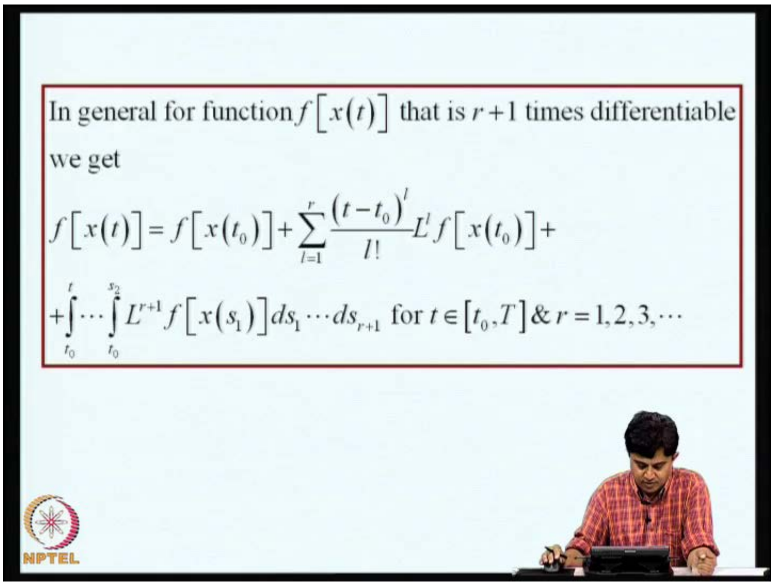

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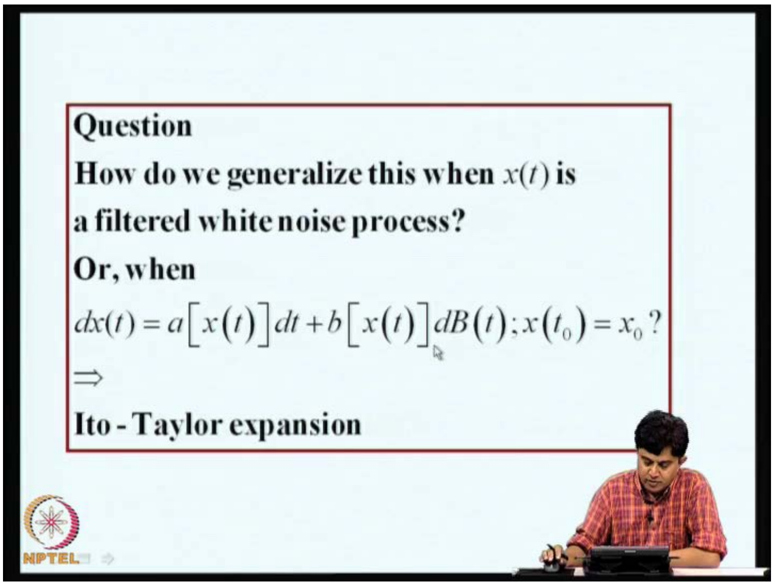

So, that finally leads to the so called Taylor's series expansion; it is an integral form, which is f of x of t given by this. This will be the starting point for developing numerical schemes for solving differential equations, and the question that we need to ask now is, how do we generalize this, when x of t is a filtered white noise or a white noise process itself? Or, when x of t is governed by this stochastic differential equation?

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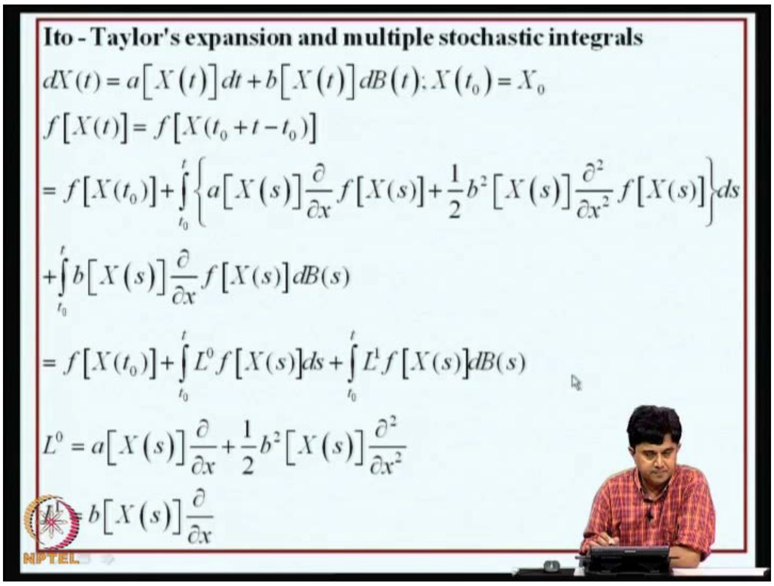

So, question therefore we need to ask is - what is the generalization of Taylor series, when x of t is governed by this s d? So, how do we proceed? So, this we will again rewrite this SDE, that is, dX of t is a X of t plus b dB t and initial condition is t naught. And consider f of X of t is f of X t naught plus t minus t naught, and now, I will write expansion for this varying in mind the Ito's formula.

So, this should be f of X of t naught plus t naught to t plus this term inside the brace, this is not just the first term as shown here, but there is a second term, into ds plus this additional term that we get, b of x of s dou by dou x f x d b; this is this follows from the Ito's formula. Now, I will introduce two operators: L naught and L 1. L naught is a dou by dou x plus half b square dou square by dou x square and L 1 is b dou by dou x. And with this notation L naught and L 1, I can write f of X t in the form of f of X of t naught plus this integral L naught f ds plus L 1 f d bs.

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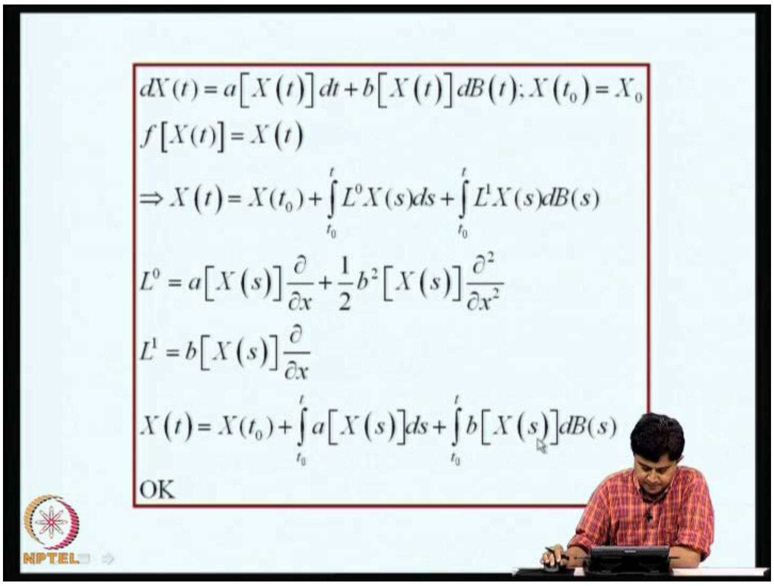

Now, if f of X of t is now X of t, I can I need to recover what I know very well is the answer. So, X of t is X of t naught plus t naught to t L naught X plus t naught to t L 1 x dB s. Now, I know, I can operate now L naught on X, L 1 on X; if I do that, I get X of t is X of t naught plus t naught to t a ds plus t naught to t b dB s, which is nothing but equivalent of this s d; this actually the stochastic integral that we are interested in.

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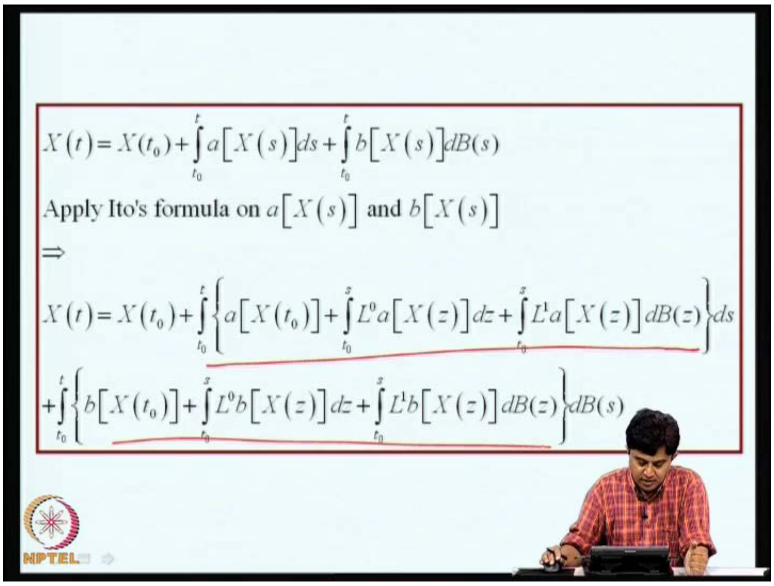

Now, let us begin by considering X of t in this form, x of t naught plus integral a d s plus integral b dB s. And we will now apply Ito’s formula on a and b, a of X of s and b of X of s which are in the integrands of these two integrals. If I do that, I will get now X of t to be X of t naught, t naught to t, and for a of X of s, I get these terms inside the brace as the application of Ito's formula on these leads to this term. Similarly, the application of Ito's formula on b leads to this second term.

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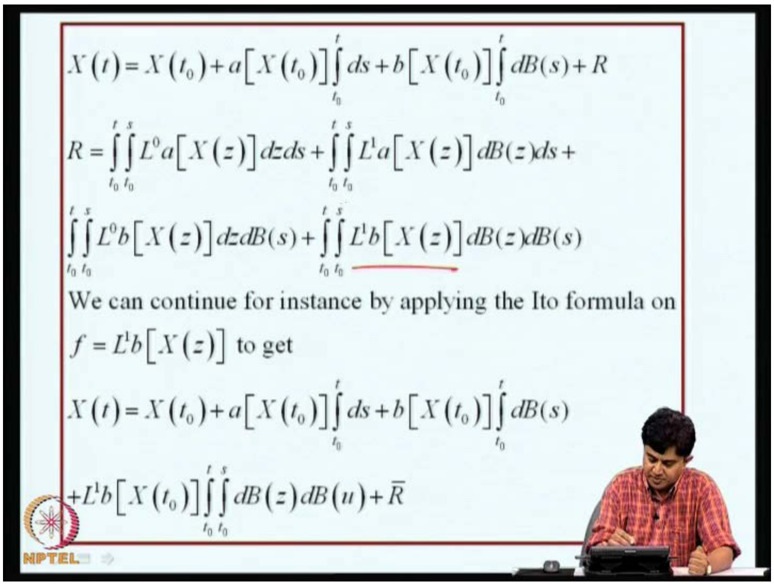

Now, I can proceed further, I am basically following what I did for deterministic Taylor's expansion with additional care, additional steps to take care of the peculiarities of Brownian motion processes. So, X of t can be written as X of t naught a into t naught ds plus b into t naught dB s plus this remainder term, which has L naught a dz ds plus double integral L 1 a dB z ds plus L naught b dz dB plus L 1 b dB z dB s.

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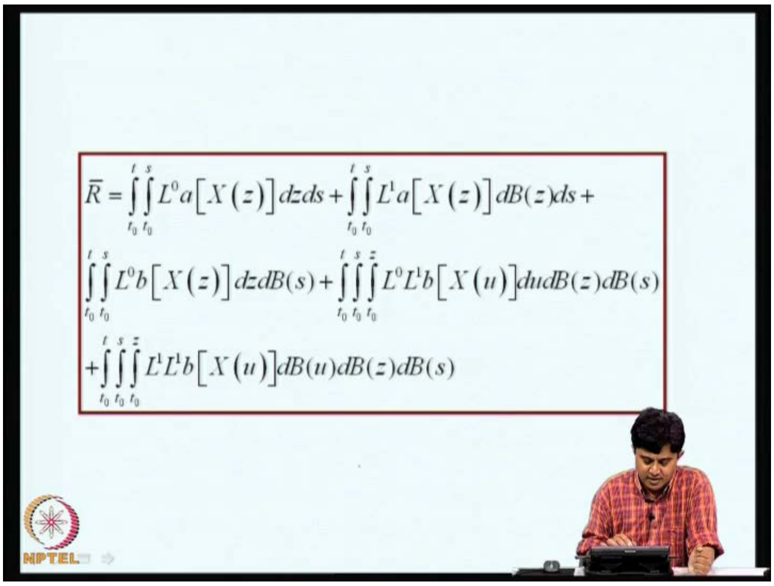

Now, we can now continue to apply the Ito's formula on the integrands in the remainder term; for example, if we now consider L 1 b X of z and apply Ito's formula on that, I can get for this term like this plus a remainder, which will observe these terms plus the remainder from representing this, the terms other than this which are already included here. The R bar in fact we can derive to, you can show that, it consists of double integrals and triple integrals of this kind.

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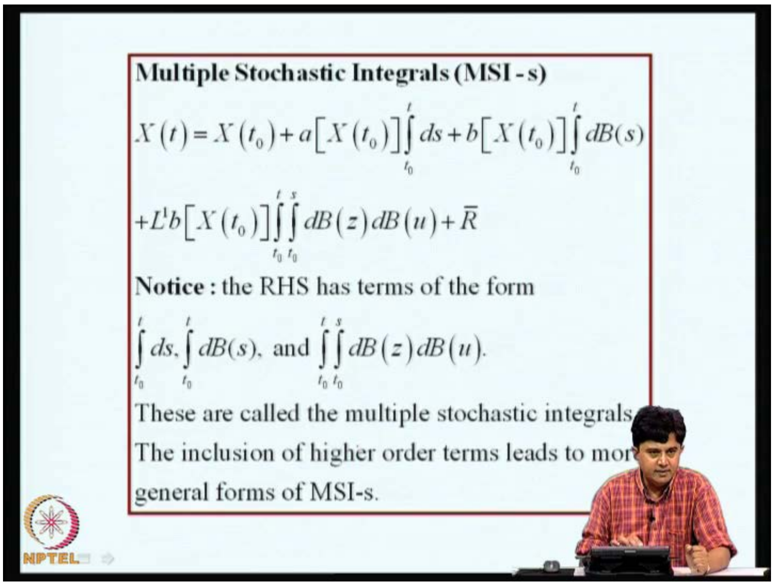

Of course, on this L 1, L 1, b, I can again apply the Ito's formula and go on doing this, and a more general representation for Ito's taylor expansion can thus be found. But what is important to note here is, if we know focus attention on the expansion at the second term level, I have X of t naught plus a into t naught d s plus b into t naught t dB s plus L 1 b into a double integral t naught t t naught to s dB z dB u.

Now, the right hand side has these integrals, t naught to t ds t naught to t dB s and this double integral t naught to t t naught to s dB z dB u; these are all random variables. The first term of course is not random variables, but these two are random variables. These are called multiple stochastic integrals. The inclusion of higher order terms of course leads to more general forms of MSI. MSI-s is multiple stochastic integrals. So, Ito Taylor's expansion thus on the right hand side we will go on getting this MSI, which are random variables.

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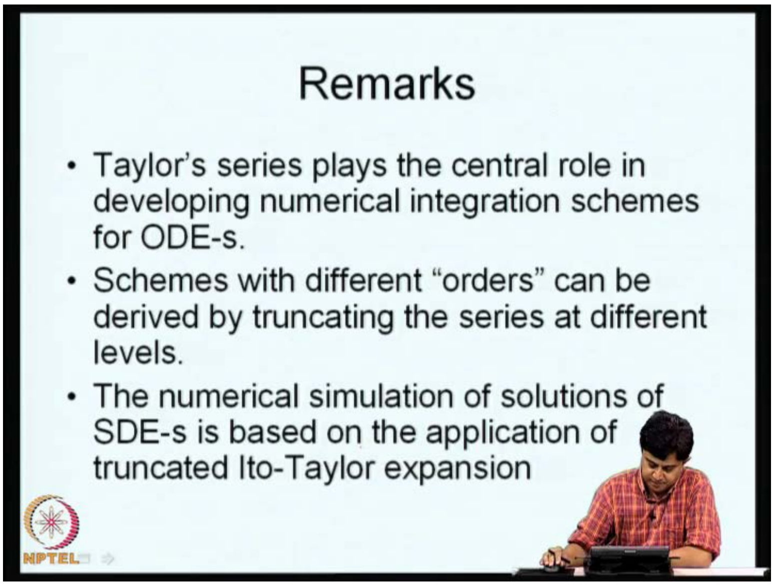

Now, we can make a few remarks at this stage. The Taylor's series plays the central role in developing numerical integration schemes for ODE; this is true for deterministic set of ODE-s anyway. Schemes with different orders can be derived by truncating the series at different levels; this is true for deterministic ordinary differential equations. The numerical simulation of solutions of stochastic differential equations is based on the application of the truncated Ito Taylor expansion; just as in deterministic case, where using Taylor expansion - truncated Taylor's expansion - here we have to use the truncated Ito Taylor expansion.

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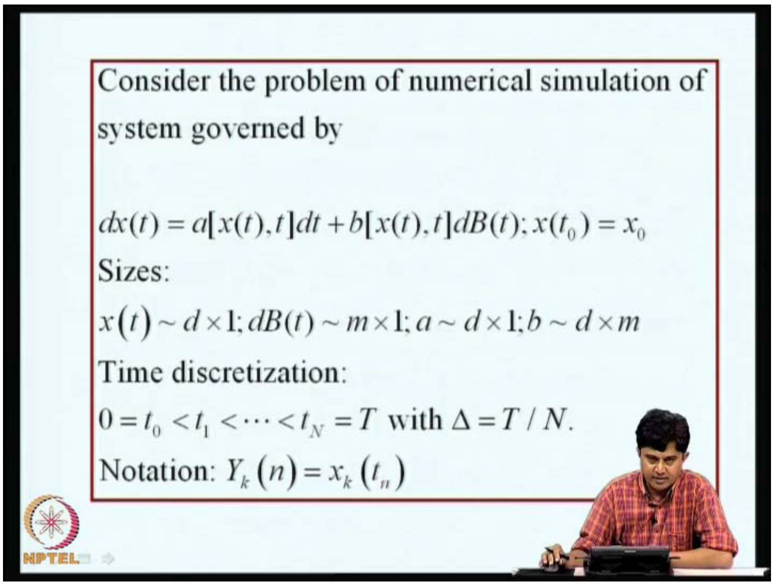

Now, I will state the problem and I will provide the recipe for the numerical scheme to integrate. I will refer you to the book by Kloeden and E Platen, which contains all the necessary steps. So, the idea of this discussion is to give a favor of the issues involved than getting into all the details of various steps involved in developing integration schemes.

So, let us consider the problem of numerical simulation of the system governed by this s d e; dx is a dt plus b dB t, initial condition is t naught is x naught. Now, I am now going to consider vector s d e s; so, x of t is d cross 1, dB of t is m cross 1, a is d into 1, b is d into m. What we do is, we discretize a time interval 0 to capital t into capital N intervals; we will assume that these time instants are spaced uniformly at time step of delta, which is capital T by N, and I will use the notation Y k of n to denote the k th component in vector x evaluated at time t n.

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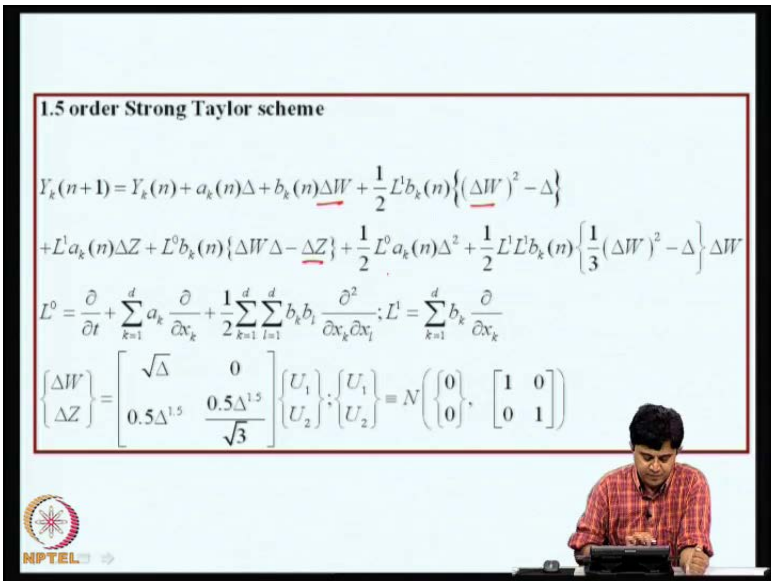

Now, using Ito Taylor's expansion and truncating at suitable level, we can develop different numerical integration scheme. And one of the integration scheme is so called 1.5 order strong Taylor scheme, you know, has this prescription; this is this can be derived by applying the Ito Taylor's expansion. What we need to notice here is that, on the right hand side, there are now some random variables; this the delta W is the random variable, this originates from multiple stochastic integrals, this delta W here, and there is a delta Z here, and on the right hand side, we are getting delta W whole square and so on so forth. It would mean that, these random variables are non-Gaussian, but this delta W itself are obtained as transformations or Gaussian delta w; and delta Z, there are obtained transformation on two standard normal random variables u 1, u 2; that is 0 mean and unit covariance matrix.

As a let me emphasis again the details of this formulary here can be worked out by application of Ito Taylor's expansion. And the objective of this discussion is not to demonstrate the velocity of these derivations, but to illustrate how it can be used, and to point out the integrals involved in developing these schemes.

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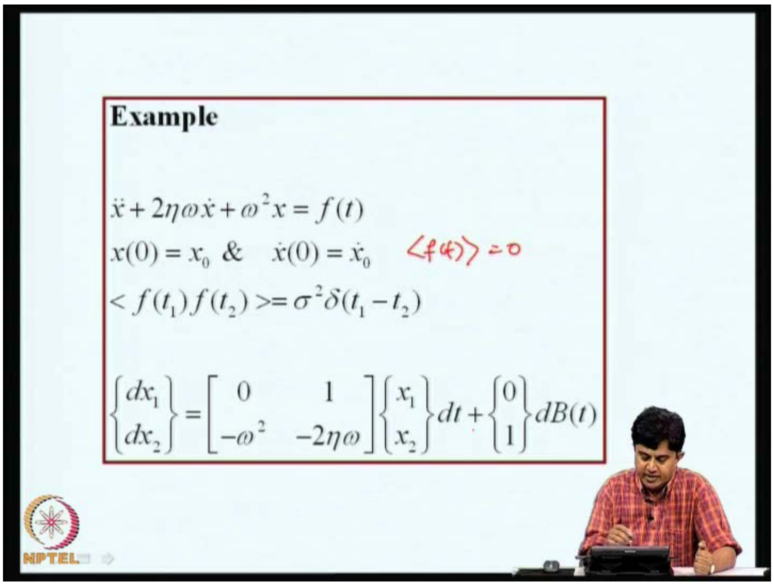

Now, I will now consider a sequence of the examples. So, let us begin by a single degree of freedom linear time in variant system under random excitation f of t, and let us assume that f of t, mean of f of t is 0 and auto covariance is sigma square delta of t 1 minus t 2, that would mean f of t is 0 mean, stationary, Gaussian white noise process.

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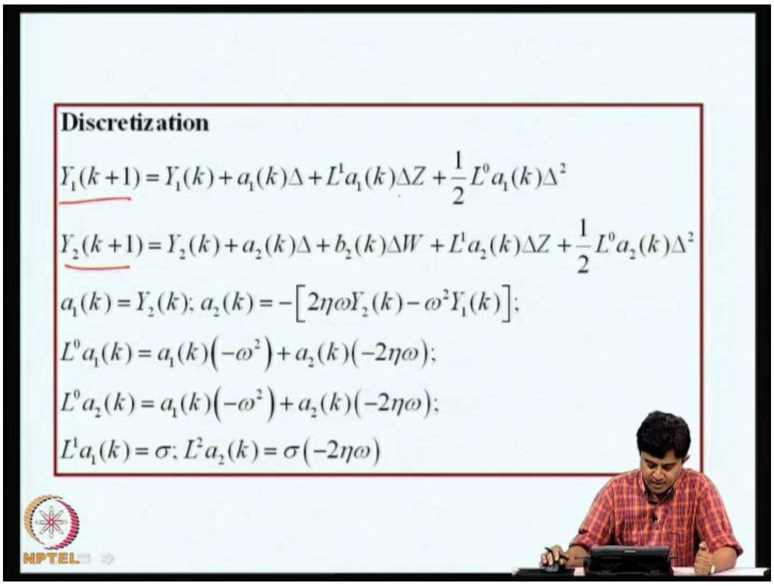

Now, this second order differential equation can be recast a set of first order differential equations and they can be interpreted as stochastic differential equations; and the d f of t f of t is a white noise, we need to write it as increment of Brownian motion process. And based on this representation and by using this integration scheme, we can get now a discrete map, Y 1 is x of Y 1 of k plus 1 is x of t evaluated at t is equal to t k plus 1.

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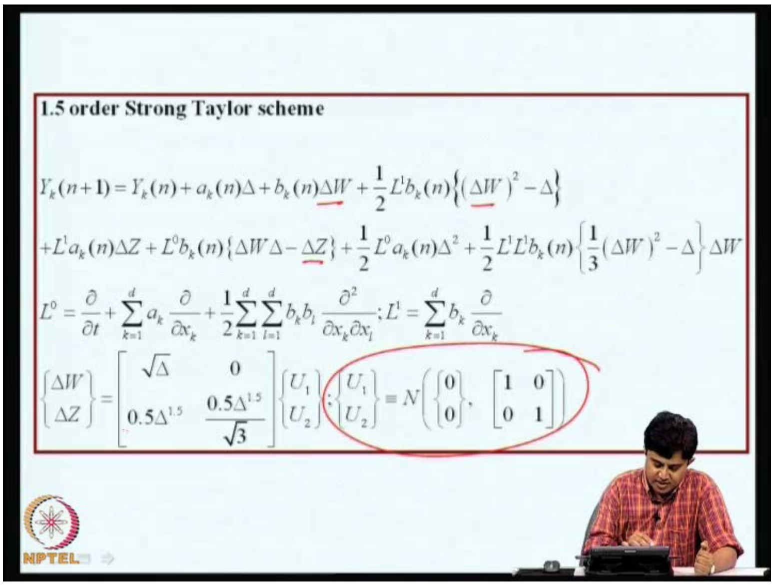

So, similarly, this is velocity, this is displacement; and we get these maps, and we can be we can begin with k equal to 0 and recursively evaluate this quantity, and obtain the evaluation of displacement and velocity as the function of time. I leave the exercise of verifying the details of this steps; we know an exercise for you, but I will show some numerical simulation, you can easily write as simple computer program to implement this. You need to generate these random numbers delta W and delta Z, starting from the u 1 u 2, that we need to generate here; this can be done using the methods that I have already described. And we can use this transformation and generate delta W by delta Z and plug it here, we can run this maps, and we can get time histories of displacement and velocity.

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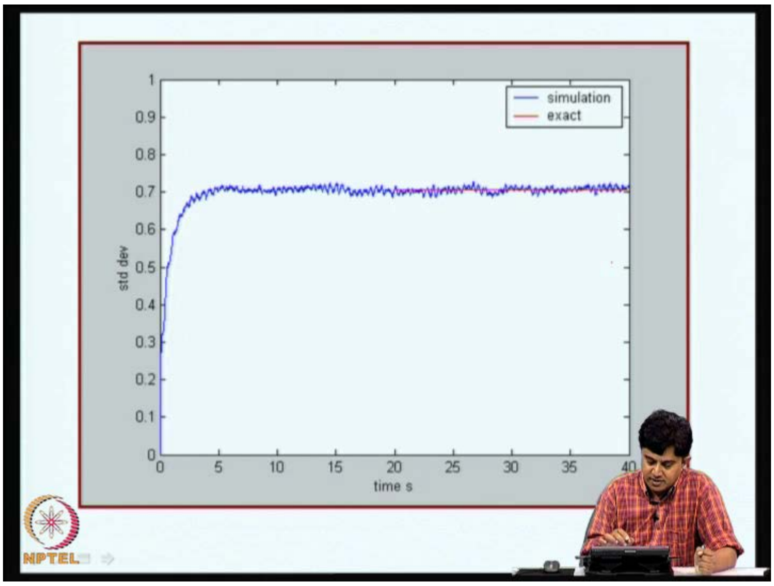

So, what we what we have try to do is, I have generated an ensemble of 5000 samples of x of t and x dot of t, and carried out ensemble average, averaging to say to estimate the standard deviation of the response. System is linear and it is driven by Gaussian white noise; and we know the exact solution for this case and that exact solution in steady state, I have shown only the steady state solution; the red line here is the exact solution, the blue line here is the estimate of the standard deviation, for time varying from 0 to 40. And as you will see, as we reach steady state, the simulated the estimate from simulations matches quite well with the exactly known solution. This is the result for velocity process; again, there is an acceptable match between simulations and the exact solutions.

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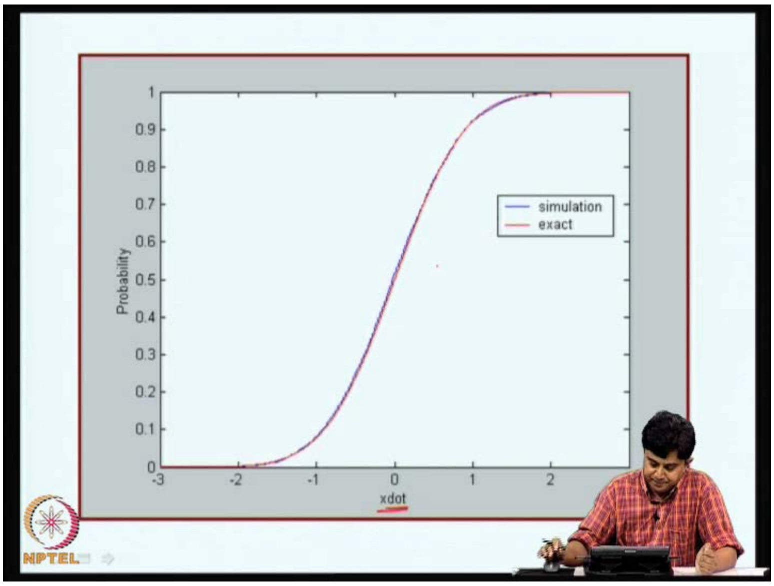

This is the result on probability distribution of velocity process, and this has this is represented in the by removing the mean and normalizing with respect to standard deviation. And we have shown here two graphs: the red one is the exact solution and blue is the simulation using this approach, and you can see that the mutual agreement is perfect.

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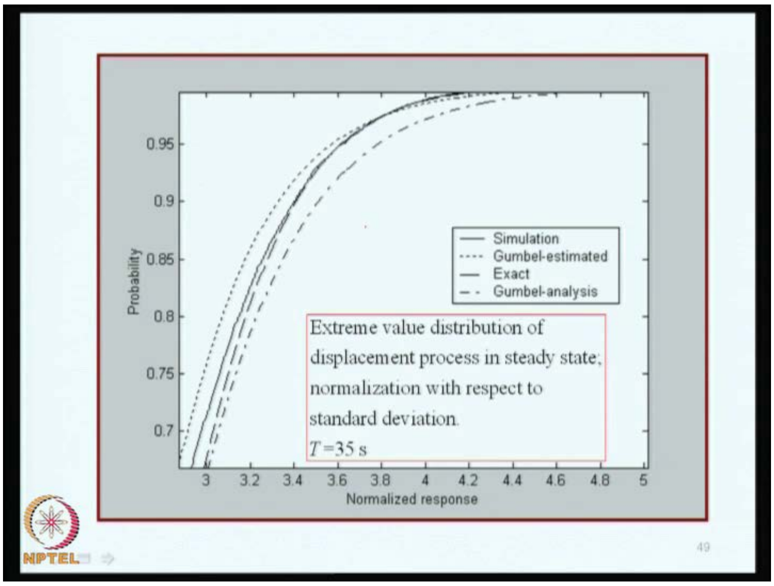

I have also shown some results here for extreme value distribution of displacement process in the steady state, and extreme values computed with respect to the time duration of 35 seconds. And on the x axis here, the states of extreme value x m which is the random variable; we normalize with respect to the steady state standard deviation of the displacement process. And on the y axis, we show the probability; there are various results shown here; the full black is the simulation using 5000 samples; the dotted line is the a Gumbel model fitted to the extremes. And this exact is the analytical solution; it is not a really exact, but on analytical solution; and if the analytical solution based on level crossing statistics in first persist time is simplified to get a Gumbel model, we get this fourth line, and all of them are shown here and you could see that considerable difference between the behavior of these four results.

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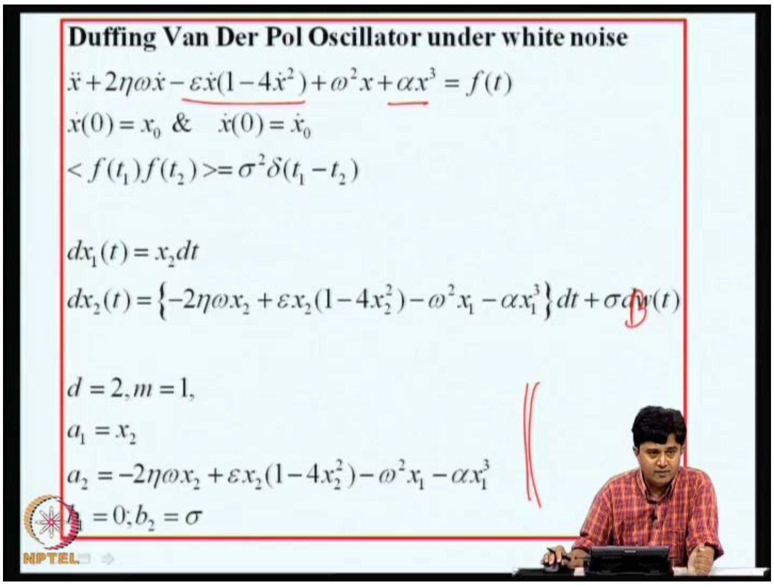

Now, let us move on to some non-linear oscillators. As I already said in the beginning, as the system becomes more complex, the Monte Carlo simulation methods you know play a crucial role in solving those problems. So, suppose let us consider now a non-linear single degree freedom oscillator, where there is non-linearity in energy dissipation as well as in stiffness; this oscillator known as Duffing van der Pol oscillator. If alpha equal to 0, we get the classical Van Der Pol oscillator, alpha and eta is 0; and with epsilon 0, we get the classical Duffing Oscillator.

So, again, I assume f of t to be a white noise, 0 mean Gaussian. And I recast this in the state space form, and interpret this as the Brownian motion the stochastic differential equation; and in the we can now convert this SDE into the map using the scheme that I have already provided, and in that, I will need this these quantities. And by applying that formula with this quantities, we will get the map that we should use for simulating samples of this system.

Now, if I take now epsilon to be zero, I have the system x double not plus two eta omega x dot plus omega square x plus alpha x cube is f of t. So, this are Duffing Oscillator under stationary white noise excitation. And during our discussion on Markov vector approach to solve non-linear oscillator problems, I have shown that, for this system, a exact steady state solution can be obtained for the probability density function of displacement and velocity; that is, the vector xx dot is a Markov vector. And for the transition probability density function, we can drive the governing Fokker Planck equation; and in that, if we consider only the steady state, that means, dou f by dou t is put to dou p by dou t is put to 0, we can solve that equation and we get the exact probability density function; from which, we can of course compute the exact variance of response displacement and velocity.

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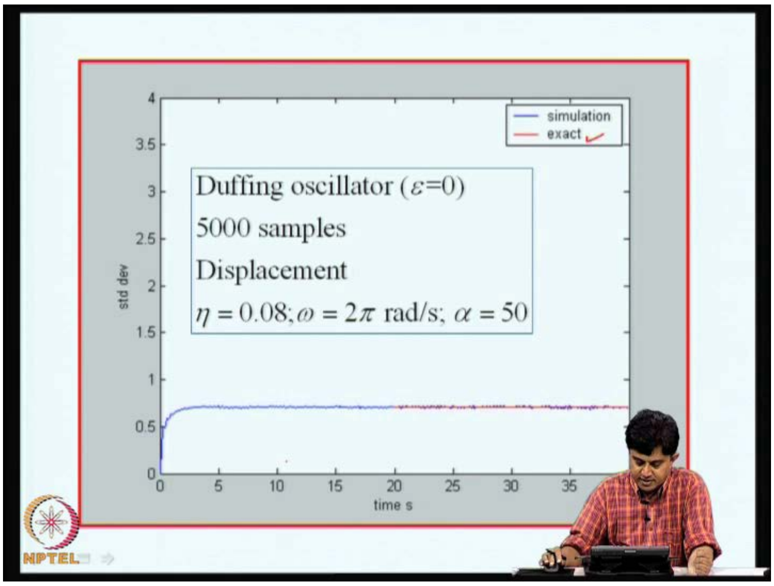

Now, again this what I wanted to show here; this is the displacement response for a Duffing Oscillator, and you see here, there is a red line and a blue line. Red line is the exact solution using Markov vector approach and the blue line is the simulation. So, you can see that, the simulations with 5000 samples estimate matches very well with the exactly known steady state solution.

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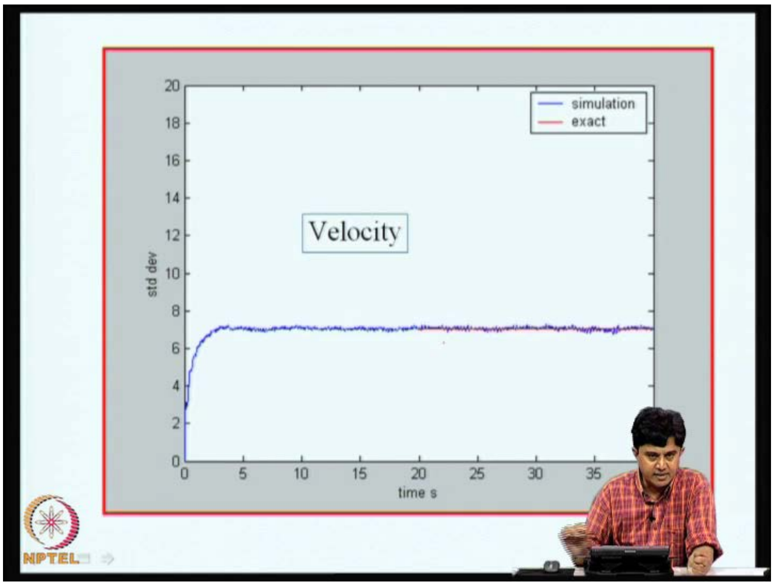

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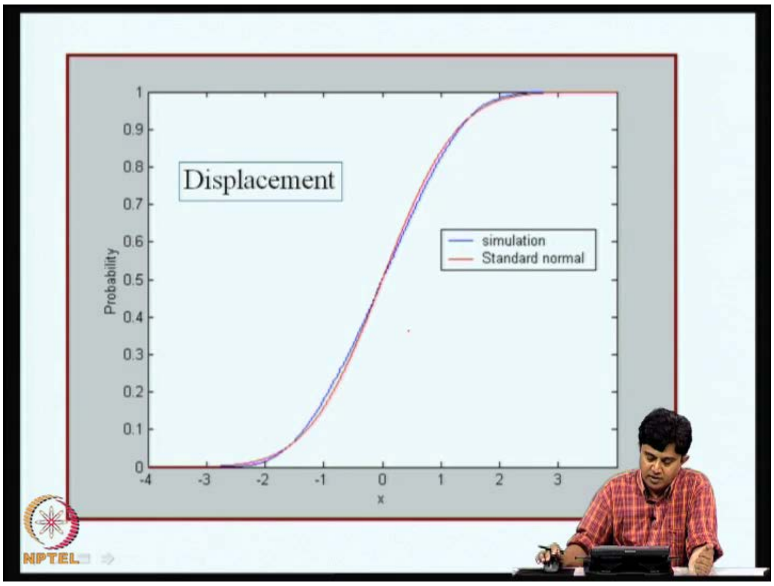

Similarly, this is the solutions for velocity process; again, there is a very good agreement between exactly known solution and the simulated response. This is the comparison of simulations on probability distribution function of the displacement in the steady state - blue line; the red line is the standard normal random variable; the displacement probability distribution is plotted on the… from the displacement process, we have removed the mean and divided that quantity by the standard deviation, and that is, in that sense, it is normalized; and when we the difference between the red and the blue line shows the influence of non-linearity for this problem.

It is not that these two lines are expected to match, their known to differ. But velocity process as have seen in steady state is a Gaussian random variable. So, here, we should expect a perfect match between simulation and standard normal random variable, and that is what indeed we see for this example.

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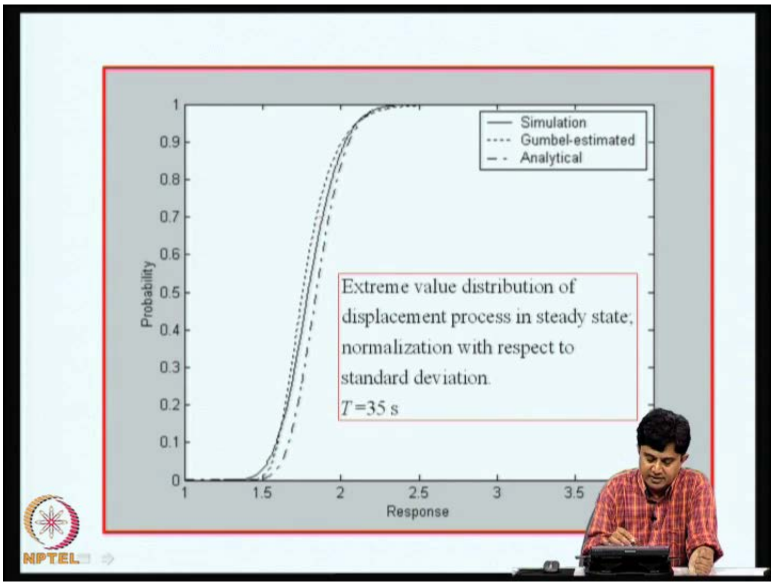

This is results on extreme values, simulation, an analytical solution based on level crossing statistics; and from the simulated extreme values, we have fitted a Gumbel model and we show the probability distribution for these three situations. Simulation is also approximate given the finite number of samples that we used; so, all three solutions are approximate in nature and this is how they compare with each other.

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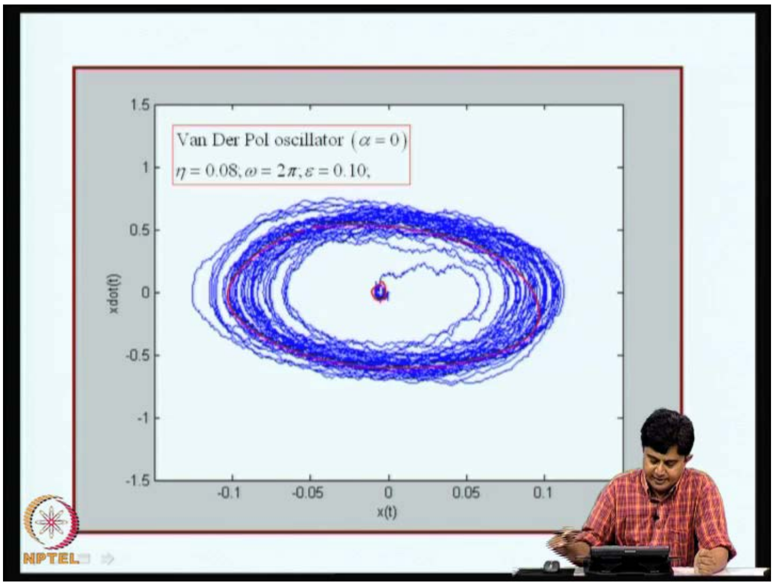

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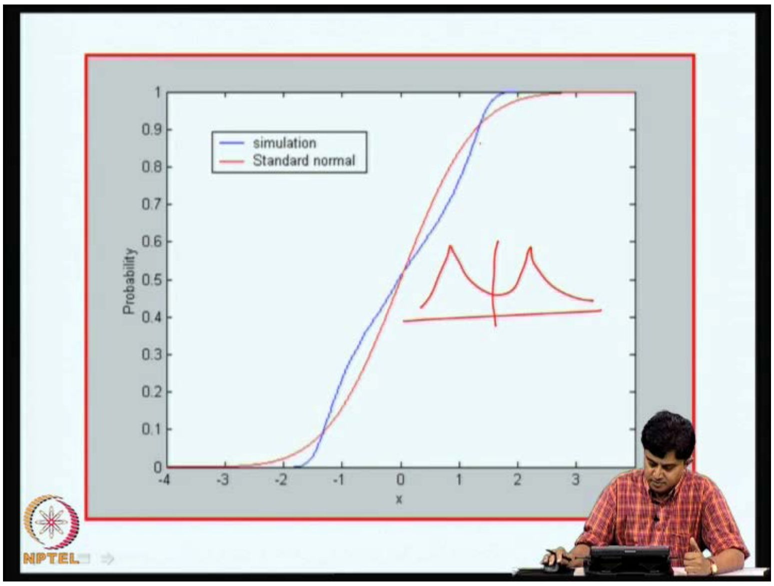

This is the phase plane plot for the case of a Van Der Pol oscillator; that means; I will now put alpha equal to 0 and I have taken epsilon as 0.1, Van Der Pol oscillator has a limit cycle and there is no excitation. And you can see that here, thus the this trajectory is initiated somewhere here and it moves towards the reminiscence of the limit cycle in the deterministic case. And because of the noise that we have applied, the trajectory over around the stable limit cycle and this produces a highly non Gaussian response. And you can see here that, when we compare the probability distribution function of the displacement in steady state with standard normal variable, we see that there is a significant departure. And in fact, if you plot the probability density function, the probability density function will be bimodal and it will appear like this; and in the probability distribution function, it will appear as this kind of a wavy blue line, that we are seeing here. This is that was on displacement, this is on velocity, the same features are observed here also.

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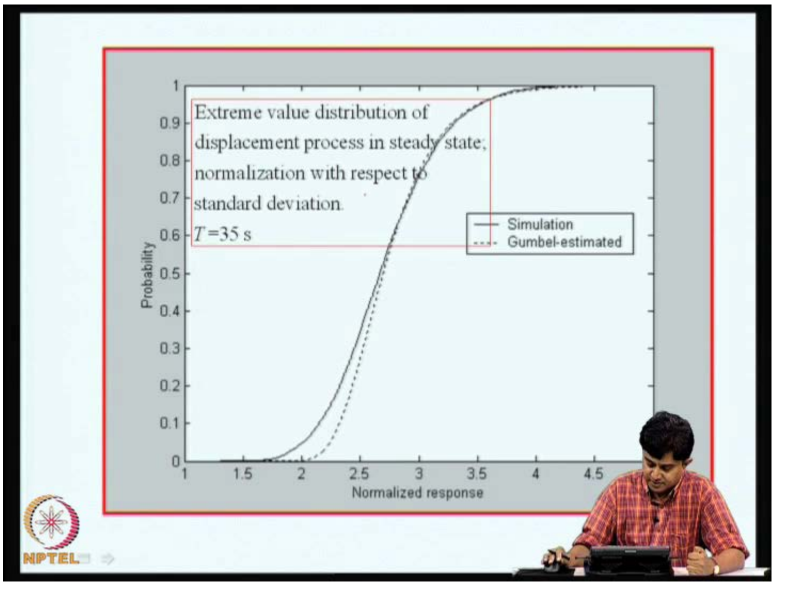

Again, results on extreme values comparing with simulation; there are no exact analytical solutions here for response process, because the Van Der Pol Oscillator under white noise does not belong to the class of problems for which there is an exact steady state solution available. So, we have to compare there is no basis for comparing in this case.

So, here, we are showing the simulation; black full line is the probability distribution function of the extreme response, on the 5000 samples that we obtained. And this Gumbel estimated, we assume hypothesis that the extreme value distribution is Gumbel and we estimate the parameters of Gumbel distribution using the samples.

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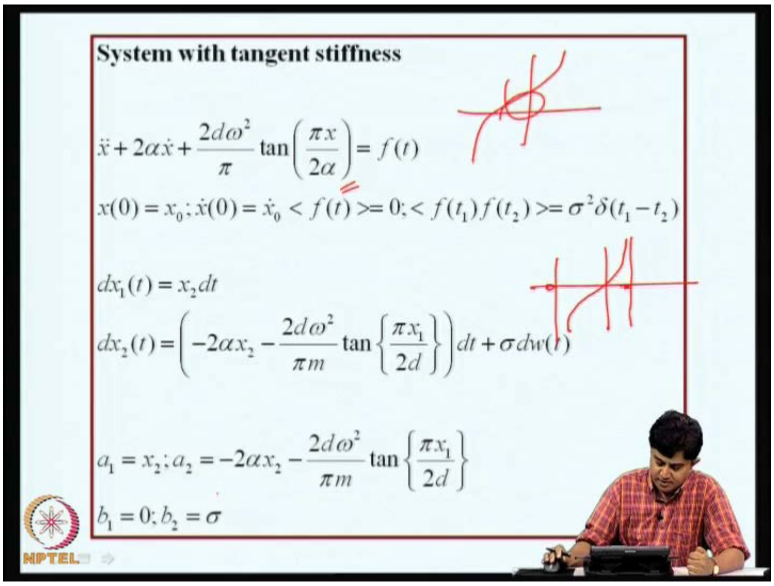

Now, we can give a few more examples; one more example is a single degree freedom system with a tangent stiffness. So, this system if displacement cross certain limits, the stiffness becomes very large; and it offers infinite stiffness, if displacement approach these values; that is what this tangent stiffness means; this system is now driven by Gaussian white noise as described here.

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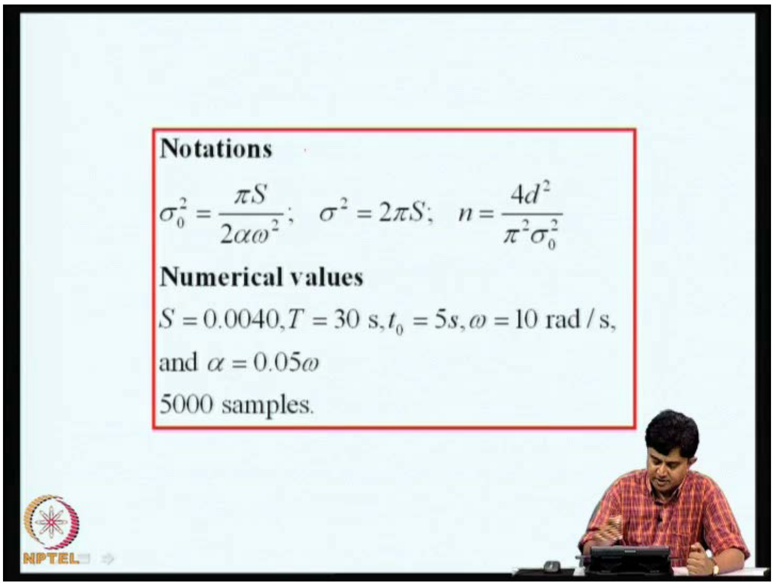

So, we converted into SDE, represented it as SDE, and apply the integration scheme and we again simulate. This problem is again belongs to the class of promise for which an exact steady state solution can be derived using a Markov vector approach; so, that is why this is specifically chosen as an illustration here. And these are the numerical values and certain notation, that we are using in developing the solution. There is a parameter n, which basically describes the distance; if displacement cross this, it encounters enormously high stiffness.

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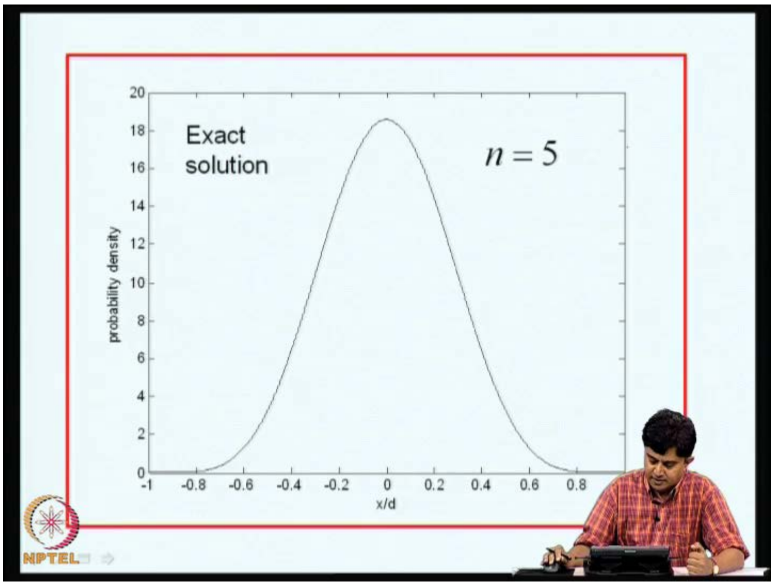

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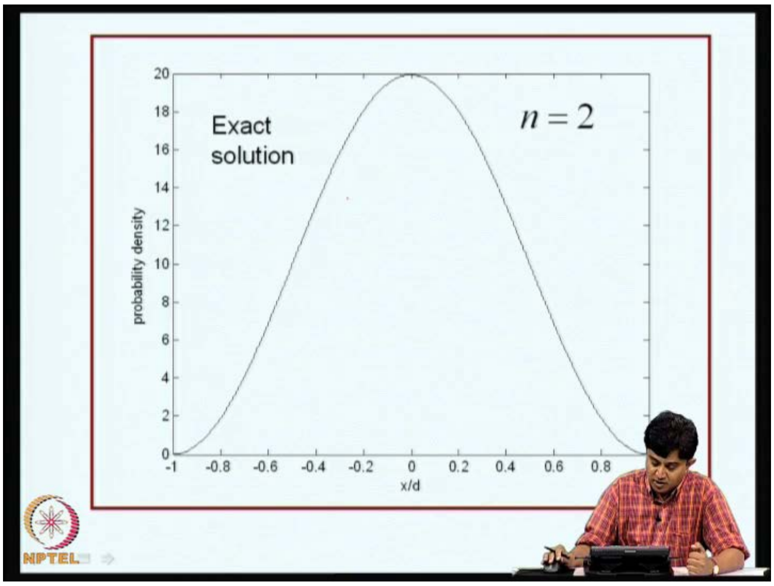

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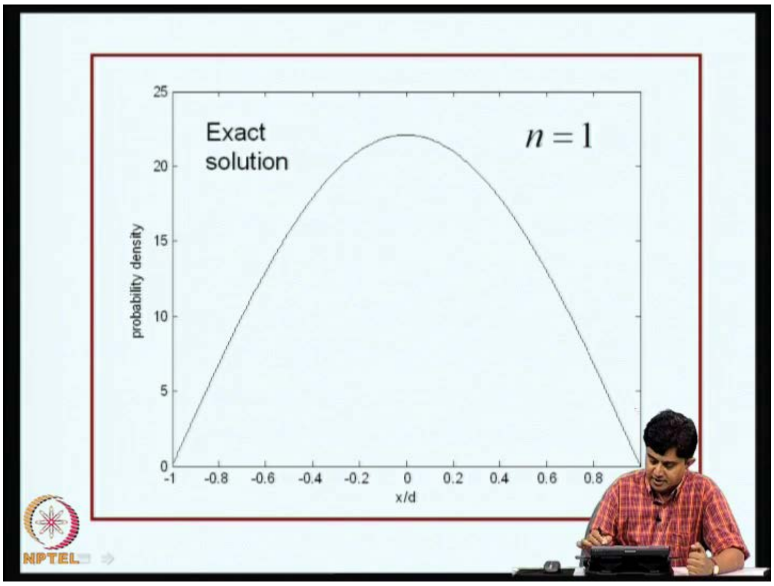

So, for small value of n, the distance is very large very small; for n equal to 5, the distance is very large. So, the response is almost Gaussian; this is the exact solution. As n becomes smaller, the response is now bounded between minus 1 and plus 1; and still smaller, it is almost highly non-Gaussian; these are the exact solutions.

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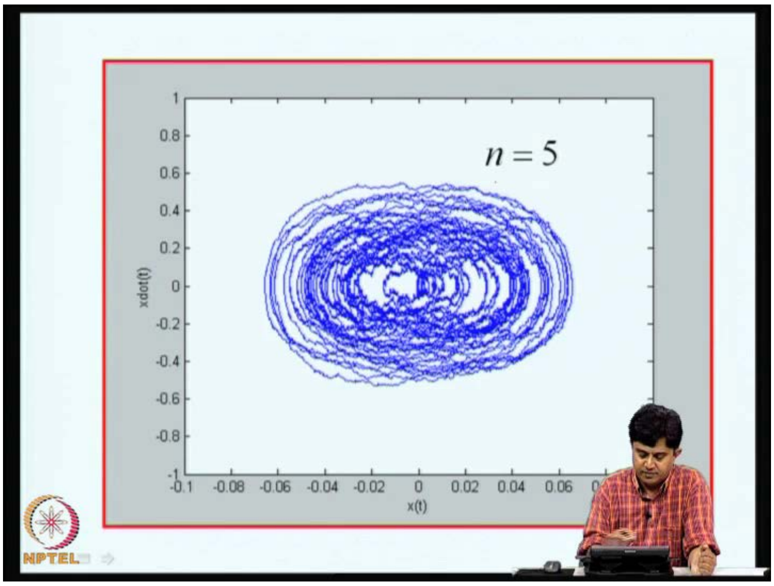

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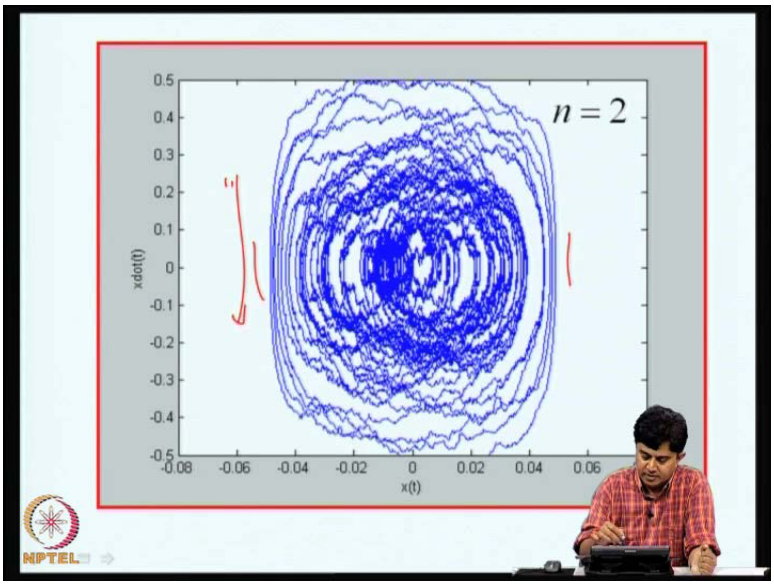

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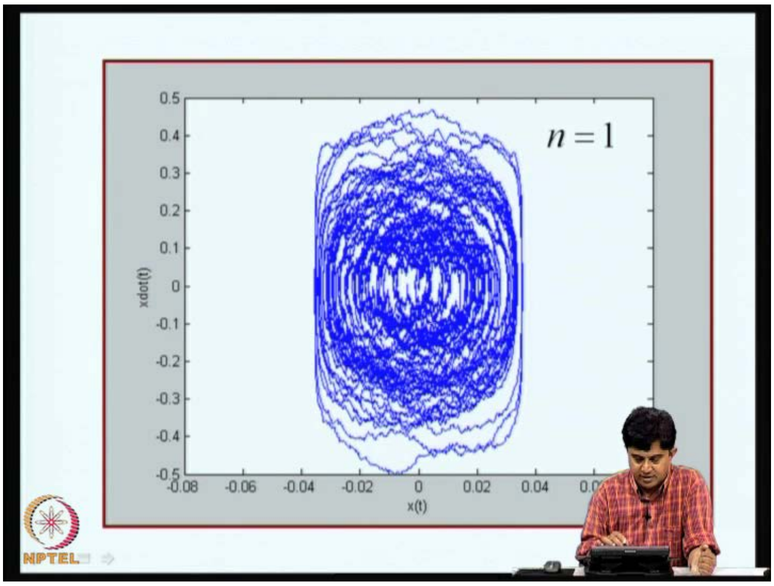

Now, let see what happens to simulations. This is a trajectory of simulated sample, velocity versus displacement for large n; so, it is almost diffuses throughout the field. Now, n equal to 2, it is now constraint between these two places; and in the velocity direction, of course, there is no problem, it takes larger range of values; and for n equal to 1, it is not severely constraint. So, we can see that, the such response of such systems, I mean is highly non-Gaussian.

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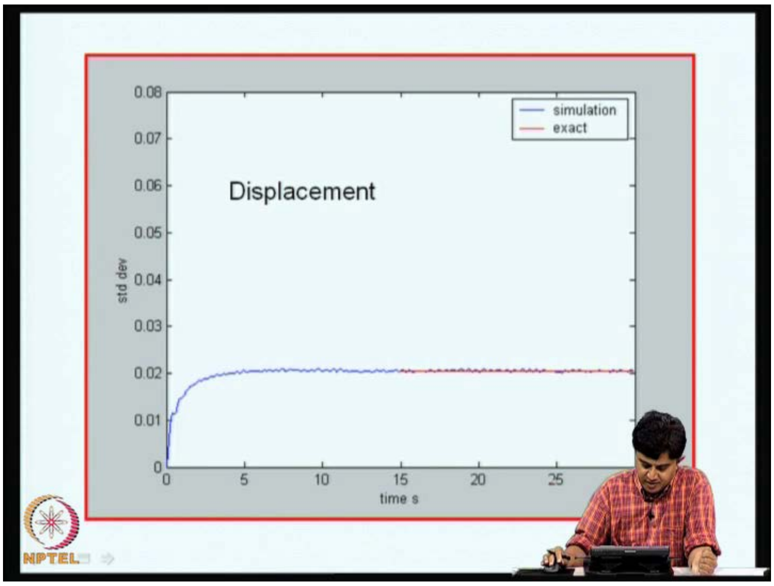

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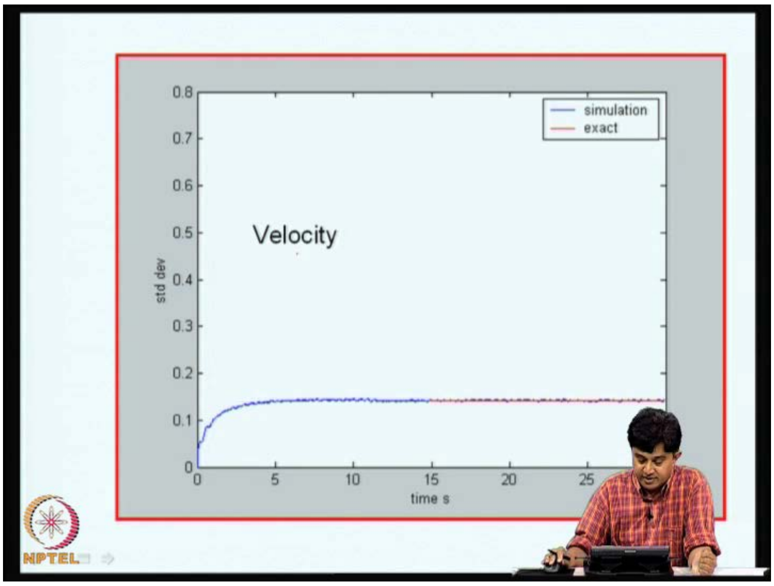

Now, again, since the steady state solution is exactly obtainable, we can now compare the estimate from simulations with the exact solutions; exact solution is the red line, blue line is the simulation. This is response displacement standard deviation verse time history; similar time history for velocity response, again the match between simulation and theory is acceptable.

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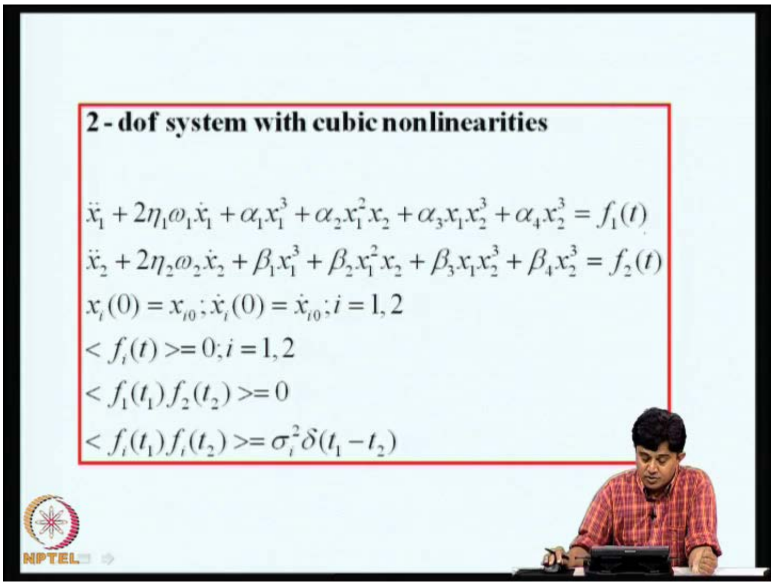

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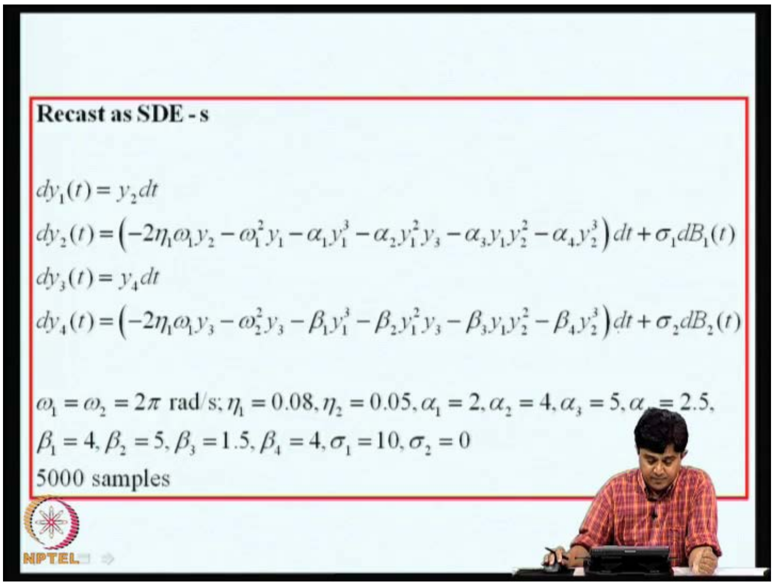

So, we can continue with this; we can consider, for example, now a more complicated system - 2 degree freedom system - with cubic nonlinearities and each of this system is driven by white noise, we can switch off one of them or keep of both of them on. So, we have now the tools to treat this problem; we convert them into a 4 cross 1 set of stochastic differential equations, discretize them, and we can run the map and obtained sample trajectories.

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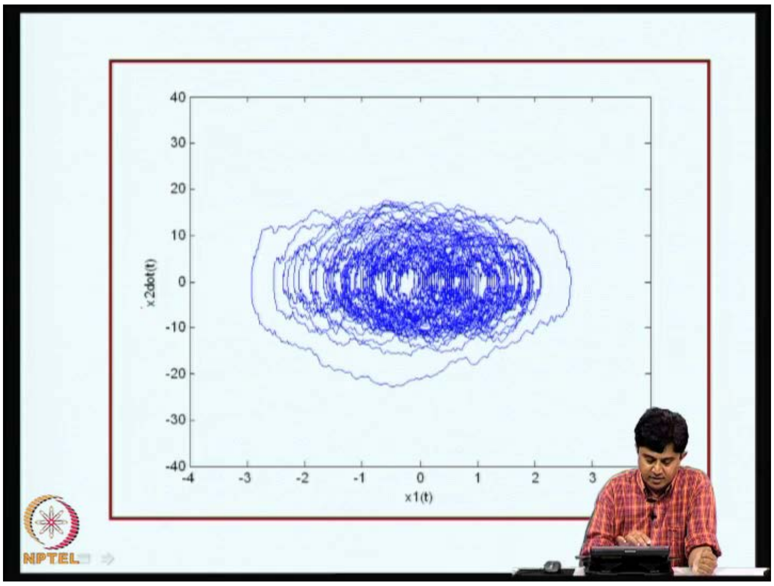

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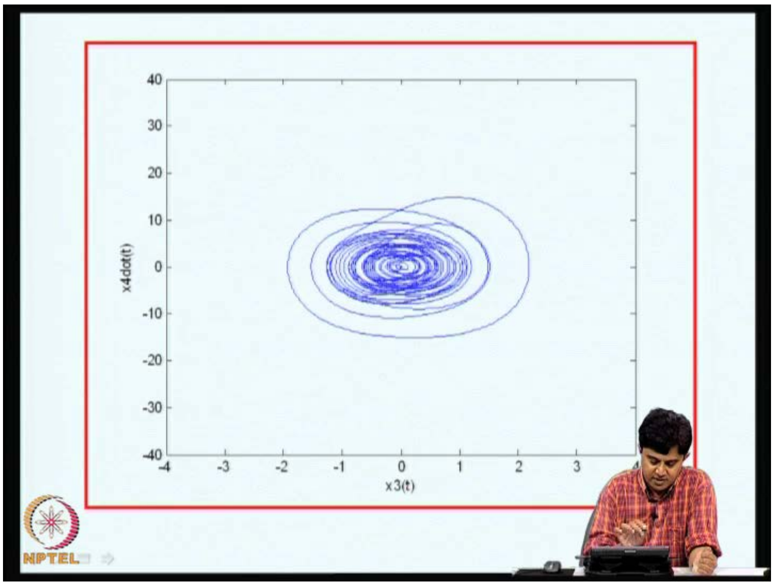

So, this is space plane; this is x 2 dot verses x 1 of t. This is x 4 dot x 4 verses x 3; that means, a phase plane plot for the first coordinate and the second coordinate. In this example, we are assuming that the second coordinate is not driven; therefore, this trajectory fairly smooth than the driven coordinate; this is quite raged; this is fairly smooth.

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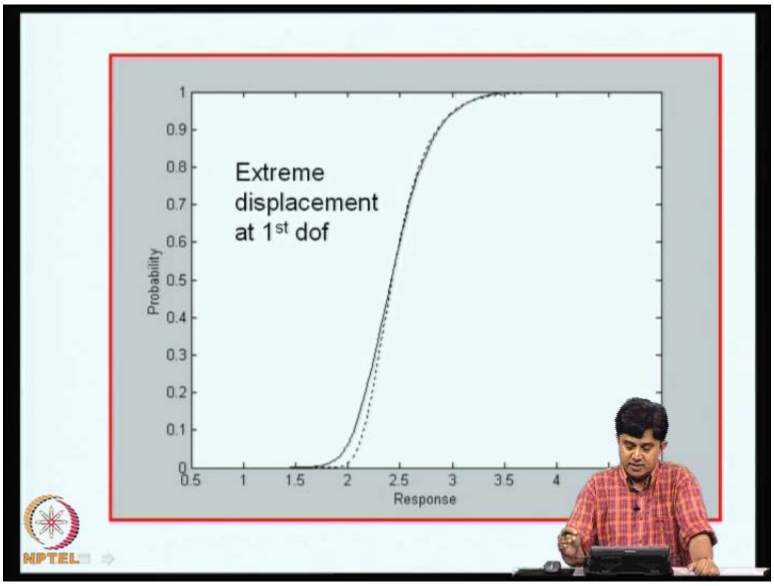

So, this is a result on extreme value response of the displacement process for the first degree of freedom. We could again of course plot at the time histories of evaluation of standard deviation, compute covariance, you can estimate power spectral density, you could do anything, but I am showing only limited results; the full line is simulation and dotted line is the Gumbel model for the simulated date.

So, what will do is, we will close the lecture at this stage. So, what we have done today is, that we have developed an integration scheme to handle certain types of stochastic differential equations. So, in the next lecture, I will present a few more examples; and then, we will consider a few questions on how to estimate power spectral density function and certain other response quantities. So, we will conclude the lecture at this stage.