Stochastic Structural Dynamics

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Lecture No. # 28

Monte Carlo Simulation Approach-4

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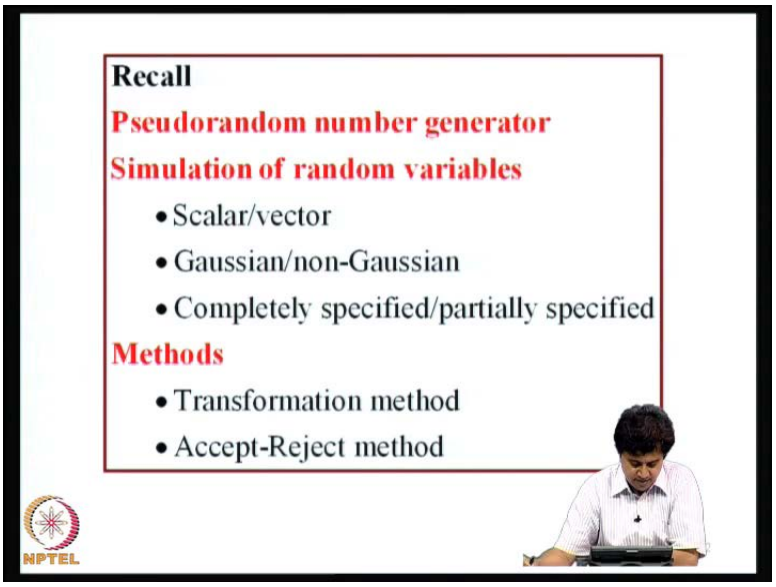

In the previous lecture, we discussed methods for simulating samples of random variables on a computer. So, we talked about pseudorandom number generators, that means, deterministic algorithms which would help you to simulate random numbers, which are distributed uniformly in 0 to 1. And subsequently, based on methods of transformation or accept reject methods, we are able to simulate samples of scalar or vector random variables, Gaussian or non-Gaussian random variables; and we also discussed what happens if random variables are completely specified and what happens if they are partially specified.

So, this is what we discussed in the previous lecture. Now, in the present lecture, we will now consider how to extend these capabilities to simulate samples of random processes.

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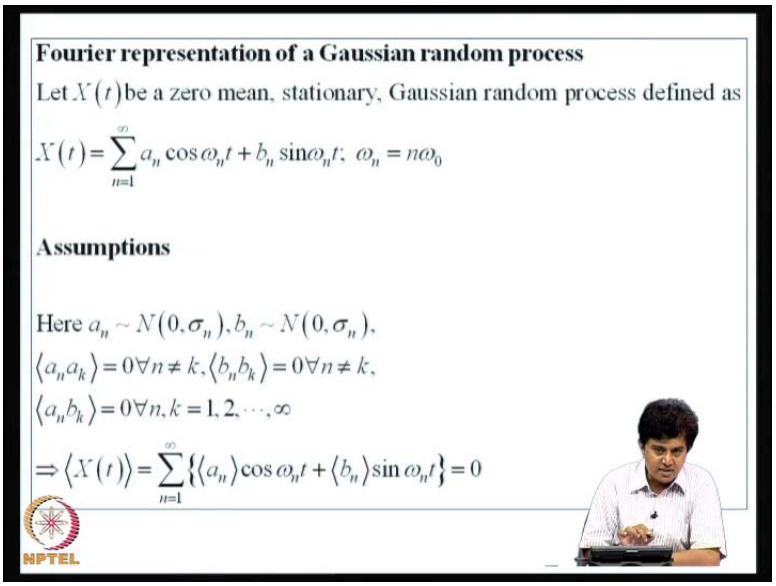

Now, we will go back to the discussion that we had earlier on Fourier representation of a Gaussian random process. So, we start with a 0 mean, stationary, Gaussian random process, and we define in terms of random variables a n and b n as shown here. Suppose X of t is n equal 1 to infinity a n cos omega n t plus b n sin omega n t omega n is n omega naught; we assume that, this a n and b n are random variables, and a n is normal with mean 0 and standard deviation sigma n, b n is normal with mean 0 and standard deviation sigma n; a n, a k are the independent for n naught equal to k; there uncorrelated, there Gaussian, therefore independent. Similarly, b n, b k are independent for n naught equal to k, and a n, b k are independent for every n and k.

So, now, first let us deduce what are the properties of this random process. What will be the mean of this process? Expected value of X of t is, you have to take expectation inside, expected value of a n is 0, b n is 0, therefore, expected value of X of t is 0.

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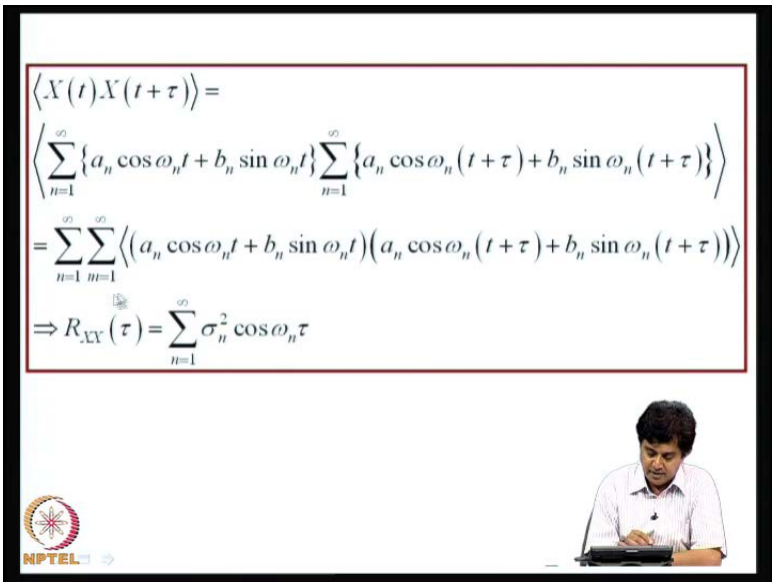

How about its covariance? Expected value of X of t and X of t plus tau. So, X of t is the first summation here, n equal to 1 a n cos omega n t plus b n sin omega n t; the second term n equal to 1 to infinity a n cos omega n t plus tau plus b n sin omega n t plus tau. Now, this becomes a double summation, and if we expand and multiply, and use the fact that, this assumptions on mutual dependents of a n and b n, a k and a n b k, etcetera, we can show that, this auto covariance is indeed a function of only the time difference; and indeed, we get R xx of tau has n equal to 1 to infinity sigma n square cos omega n tau.

So, therefore, it follows that x of t is has 0 mean and covariance function which is function of tau; therefore, it is a white sense stationary process, but since X of t is Gaussian, because a n b n are all Gaussian and we are doing a linear transformation on Gaussian random variables; X of t is also Gaussian. Therefore, X of t is also strong sense stationary.

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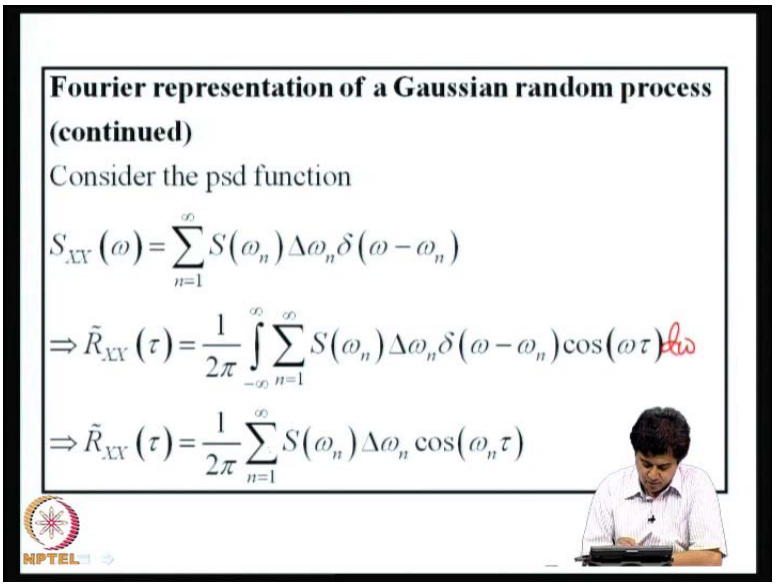

Now, let us consider we start the argument from a slightly different direction. Now, let us consider a power spectral density function, which is given by n equal to 1 s omega n delta omega n delta omega minus omega n. Corresponding to this, I get a covariance function which let be call it as R tilted R xx tau, which is 1 by 2 pi; and the corresponding transformation following the relationship between covariance and psd, and we can show that, R xx tilde is 1 by 2 pi n equal to 1 S omega n delta omega n cos omega n tau.

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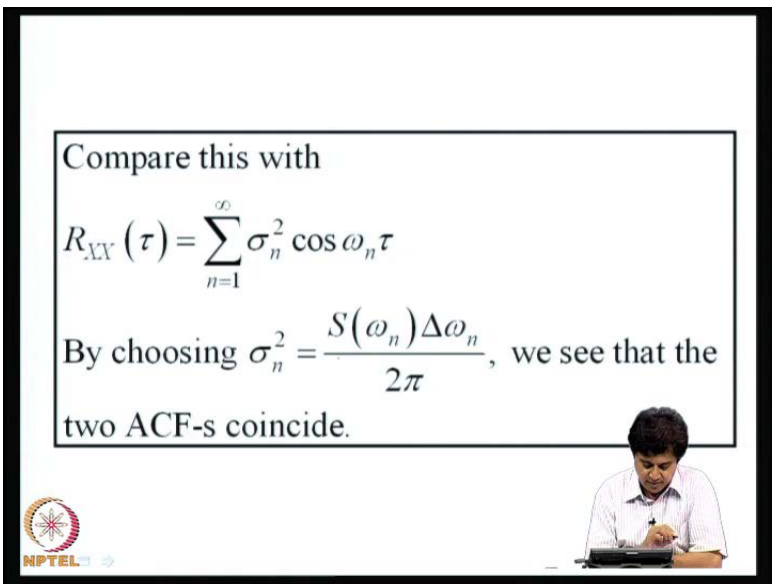

Now, let us compare this with the auto covariance of X of t, that we just now obtain. We had n equal to 1 sigma n square cos omega n tau; and in this case, if I put S omega n delta omega n by 2 pi as sigma n square, you can see that the two representations are identical.

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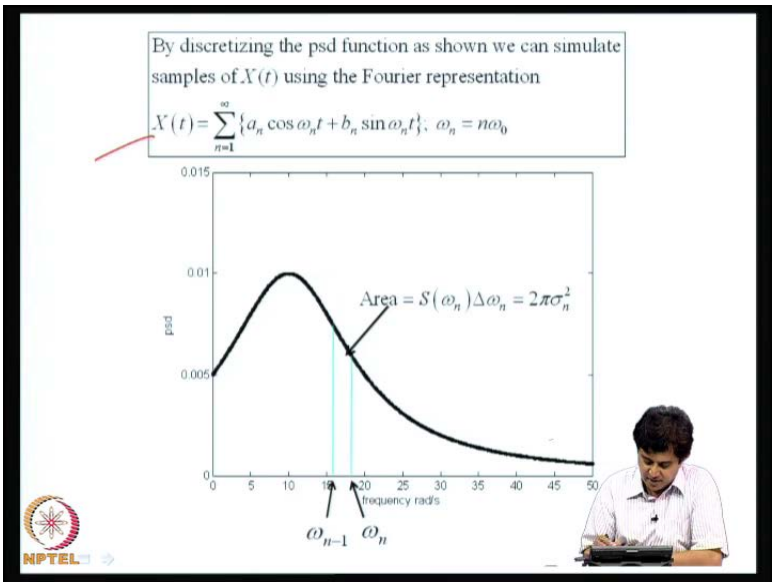

So, that would mean, if this is the target power spectral density function of X of t, we can discretize the power spectral density into some n intervals; and the area under this interval I call it as S omega n delta omega n. And if I consider now for each of these intervals corresponding to the two random variables a n b n and use this representation, we can show that samples of X of t, as n becomes large, we will be having power spectral density which is the continuous form of the specified power spectral density function. So, this gives as a approach to simulate samples of Gaussian random process is stationary 0 mean, which specified power spectral density function.

So, you start with the power spectral density function, discretize, and for each segment, you define two random variables which are independent and have the variance given by this area under the psd, and use that in this representation, and you will able to produce sample samples of X of t, which has power spectral density as given here.

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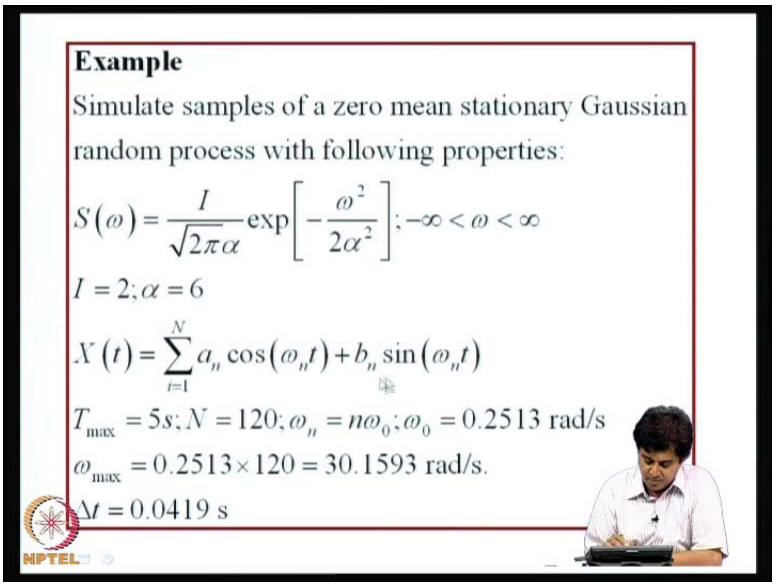

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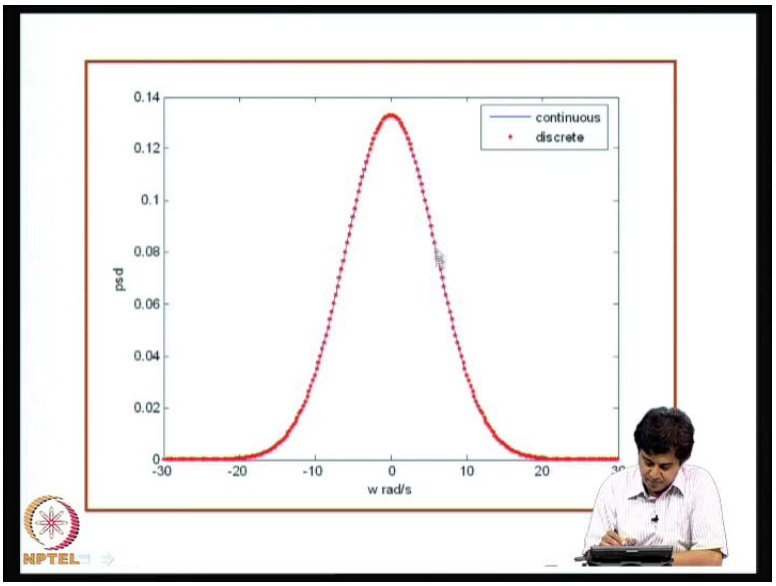

So, let us see an example, simulate samples of a 0 mean stationary Gaussian random process with properties that, S of omega is an as a as selected for illustration an exponential type of a Gaussian type of power spectral density function; this is almost similar to a Gaussian probability density function, except that, it is known a psd function; it is area under the curve need not be 1, it will be equal to the specified variance, and some numerical values I have provided for the parameters: I and alpha, which appear here. Now, X of t I will write it as, i equal to 1 to n a n cos omega n t plus b n sin omega n t. So, I am actually discretizing the power spectral density function into, each dot is a

point where I have discretized; and the parameters used are, we are simulating this for about 5 seconds, and I am retaining 120 terms and omega naught is taken us 0.2513 radian per second; the omega max is 120 points is about 30 radian per second, and delta t which will be function of 1 by omega max, is a 0.0419 seconds.

So, I simulate 240 Gaussian random variables with their mean being 0 and variance obtained from the target power spectral density function. You have to find area under those small intervals and you have to assign them as standard variances to the those one of, each one of those 240 random variables.

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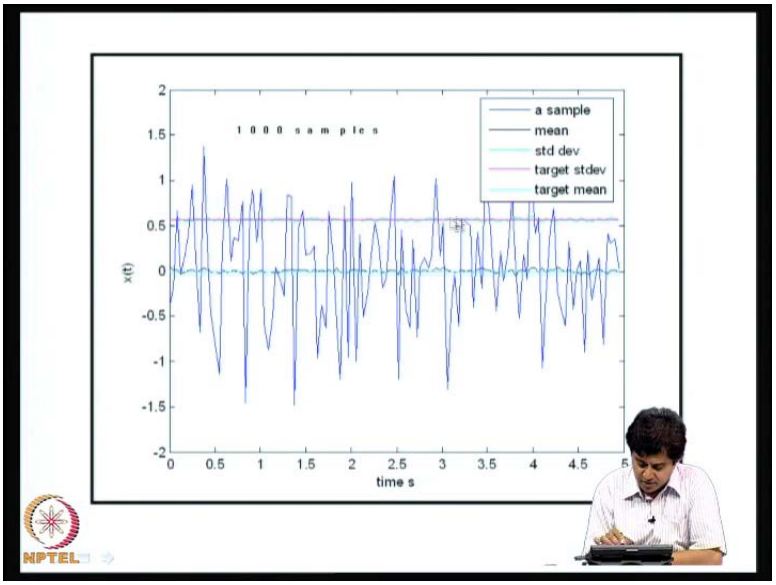

So, if I do that, one of the sample that I got is shown here - blue line; this is a sample and 1000 such samples were obtained, and the ensemble mean and stand deviation were computed. So, you see here, the target mean is 0 and the black line that you see houring around 0 is the simulated mean and this green line a pink line is a target standard deviation, and the green line that is fluctuating about this is the simulated standard deviation; so, things look quite.

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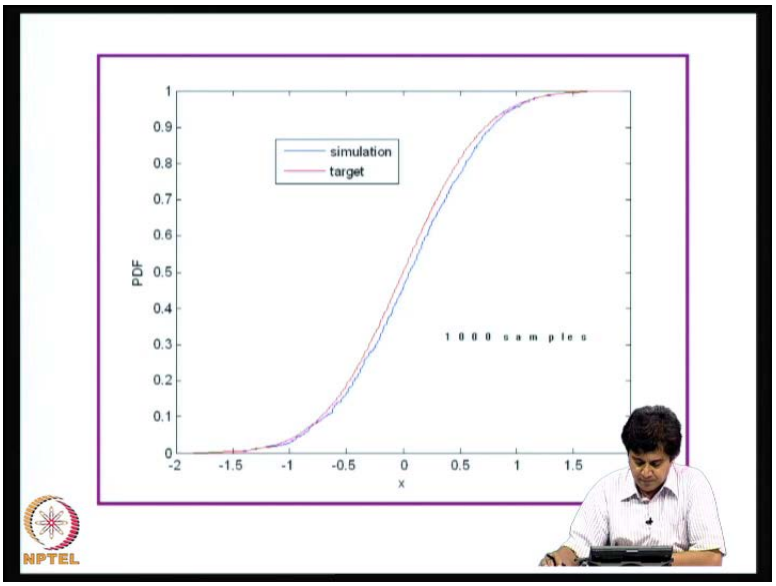

And the probability distribution function with 1000 samples; blue is a simulation and red is a target. There is reasonable agreement; again, we can do a hypothesis test to actually objectively access, whether that agreement is acceptable or not.

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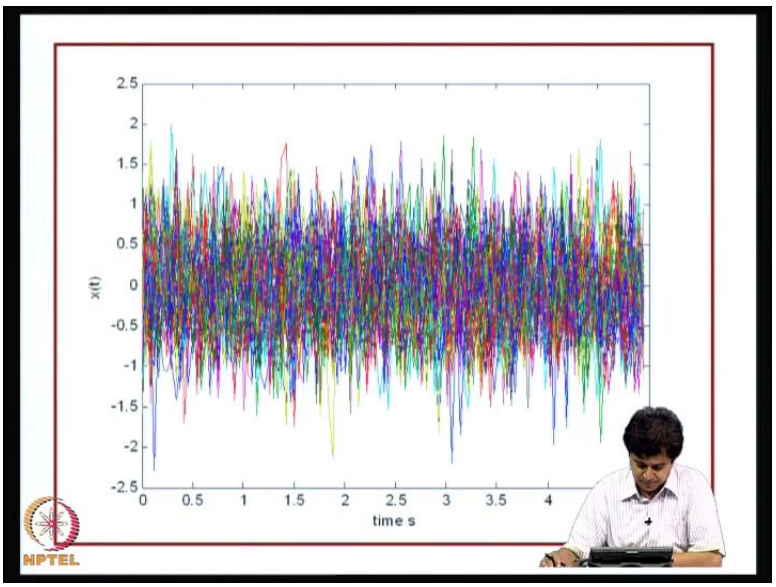

And this is the ensemble of the time history; all of them I have shown on the same well. This is again characterless, but gives a visual impression of what we are simulating.

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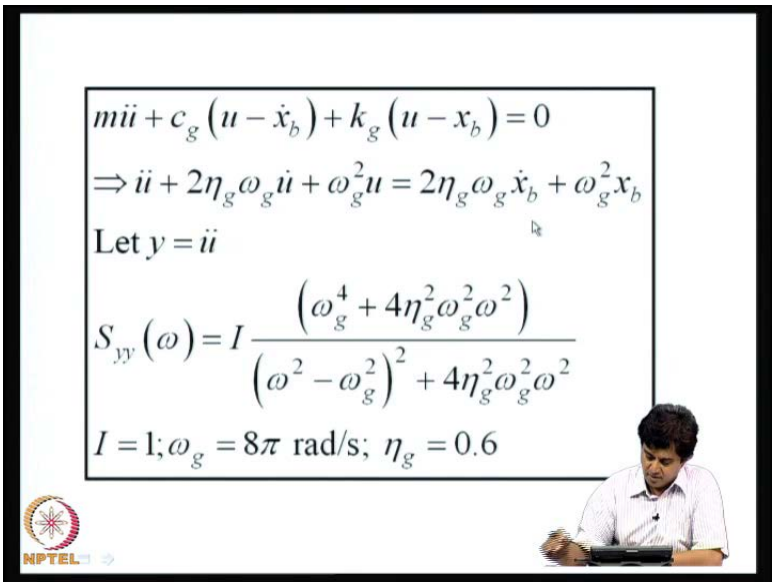

Another example, we discussed this Kanai-Tajimi power spectral density models, when we discussed a models for earthquake ground accelerations. Suppose I want to simulate now samples of earthquake ground accelerations, whose power spectral density function is given by the Kanai-Tajimi power spectral density function, we some of these parameters, I equal to 1, omega g is the ground natural frequency, eta g is the ground damping. So, the idea here is that, the soil layer is underlying a overlying a bedrock is modeled as single degree freedom system, with damping eta g and natural frequency omega g, and it is subjected to a white noise acceleration at the bedrock level. And the ground surface acceleration is modeled as output of this single degree freedom system in steady state to the white noise excitation applied at the bedrock level.

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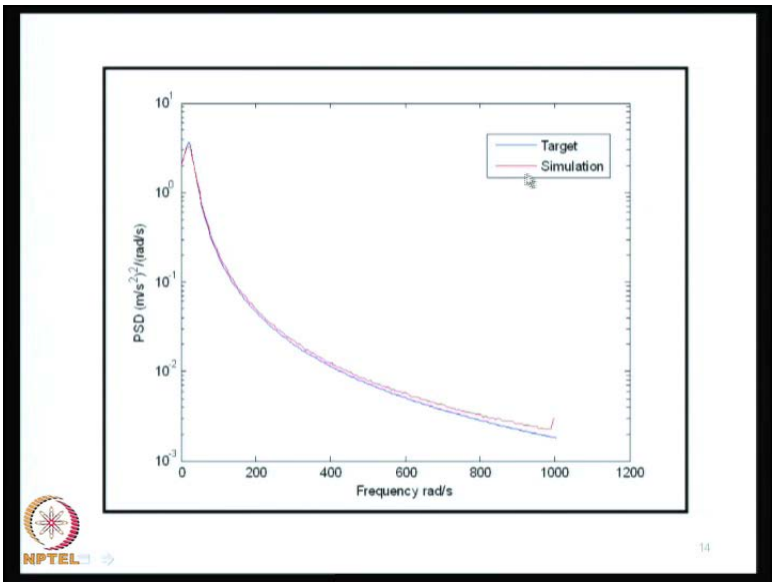

So, the target power spectral density function is the blue line axis, y axis is shown on large scale and x axis is on linear scale here. So, after doing I think 5000 simulation, we are able to estimate the power spectral density from the data and it is compared with the target, and again, we see good agreement between the 2. So, in a future lecture, I will describe how to actually estimate power spectral density from data, and what are the sampling distribution for power spectral density functions and how to test hypothesis, etcetera will do that later, but right now will be a happy with a kind of visual comparisons, and this that level, the results are satisfactory.

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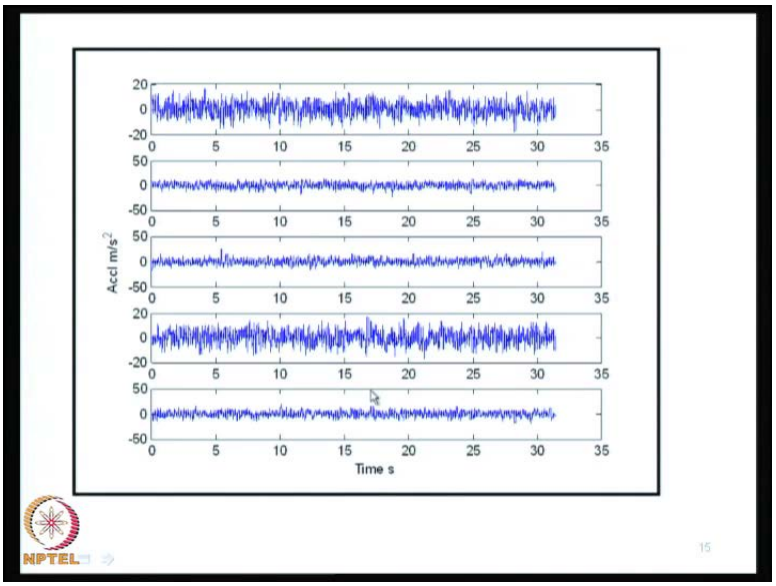

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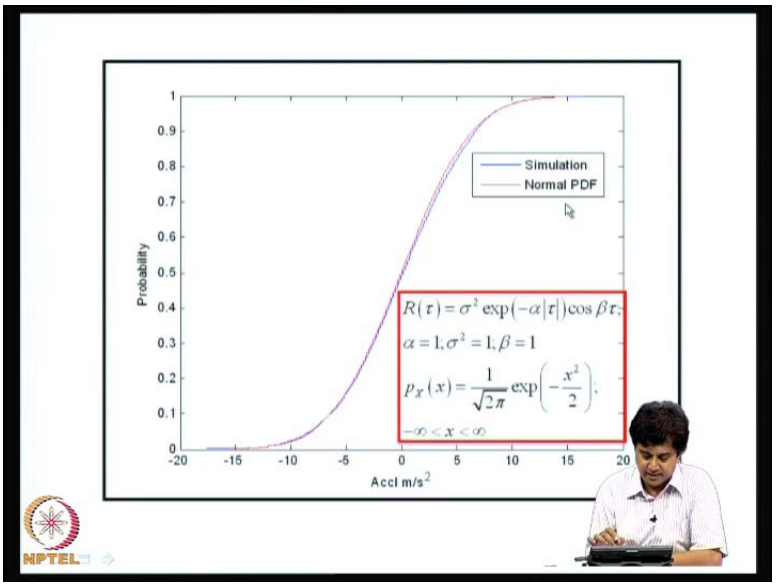

This is a display of the a few samples of the time histories. Again, to give a visual idea of the how the samples look, we are simulating samples of stationary random processes for a chosen time length. This is the an example where… this another example where the covariance function is a harmonically decaying - exponentially decaying harmonic - with certain parameter displayed here and target density function is Gaussian. And what is shown here is the results on probability distribution function; blue is the simulation, red is the normal, which is the target just has on a illustration.

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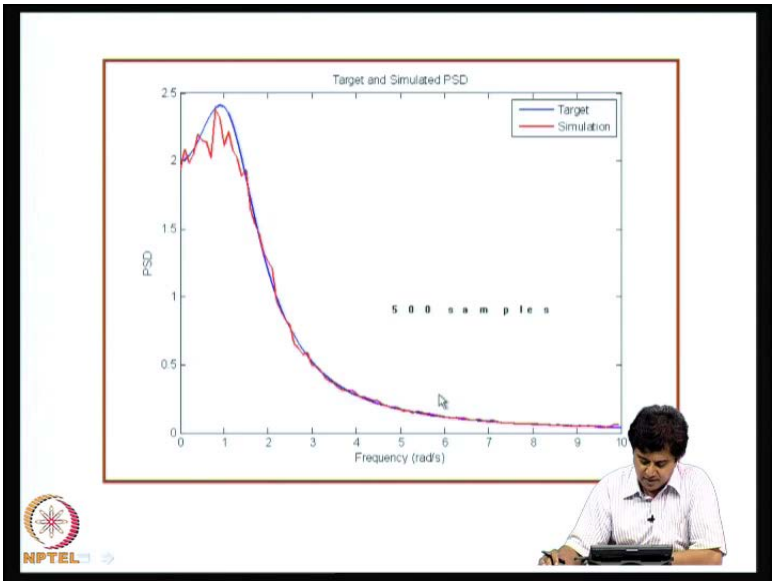

This is for the same example; this is auto covariance function. Its Fourier transform will give us a power spectral density function, and one sided power spectral density function is shown here; blue is the target and red is the simulations obtained using 500 samples. So, there will be fluctuations - sampling fluctuations - that arise, we you need to understand that.

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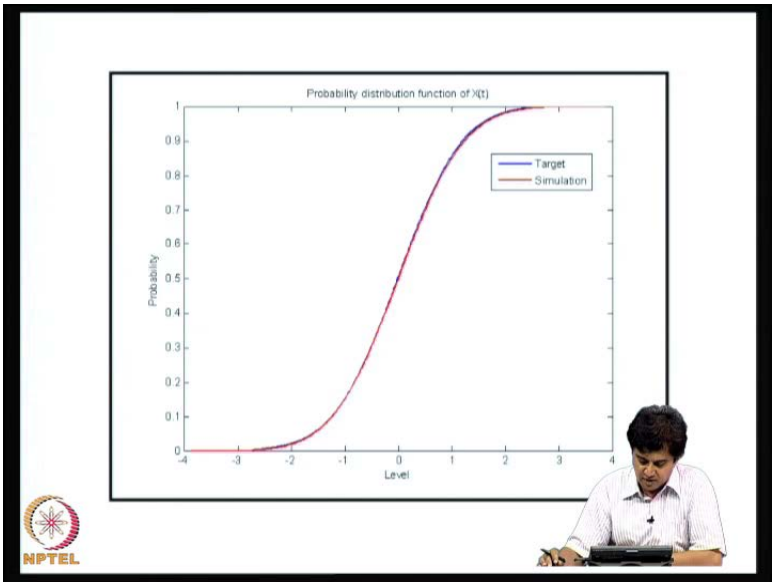

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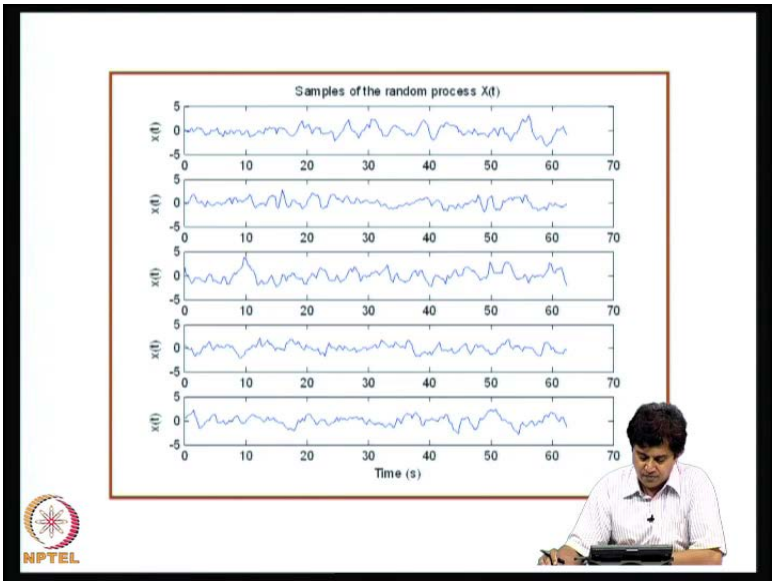

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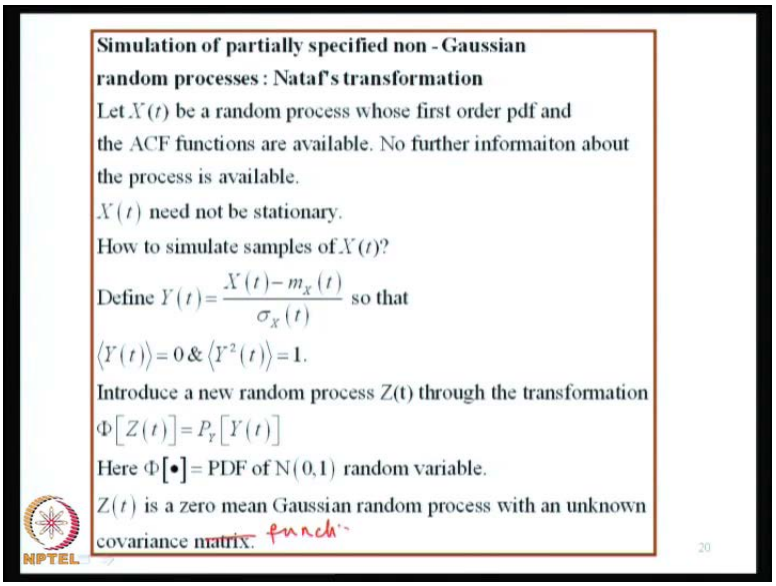

This is a result on probability distribution function; again, the comparisons are satisfied; some samples of X of t. Now, so simulation of a scalar Gaussian random process with, which is stationary with 0 mean and given auto covariance are equivalently the power spectral density function, seems reasonably straight forward. Non-stationarity can be introduce by multiplying stationary processes by suitable envelops, that is one of the common strategy is used; we will discuss that later.

But right now, we will move on to the problem of simulating samples of scalar non-Gaussian random processes. The description of the random process here is a limited to the power spectral density function or the covariance function - auto covariance function - and first order probability density function. So, here, we are considering, let X of t be a random process whose first order pdf and auto covariance functions are available; no further information about the process is available is taken to be available. X of t need not be stationary. So, the problem is how to simulate samples of X of t.

So, what we define? We follow the Nataf's of transformation method that I discussed in the previous lecture. So, first step is, we remove the mean and divide by standard deviation, so that Y of t has 0 mean and unit standard deviation. Now, you introduce a new random process Z of t through the transformation, phi of Z of t is equal to P y of Y of t, where phi of this argument is probability distribution function of a normal random

variable with 0 mean and unit standard deviation. Z of t is a 0 mean Gaussian random process with an unknown covariance matrix or the covariance function here.

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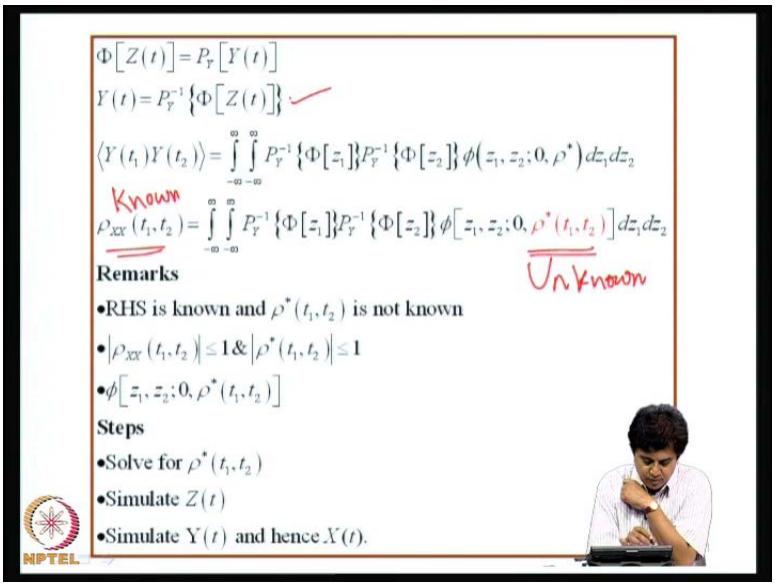

So, we follow the argument, we have shown in the previous lecture that, as for the first order probability distribution function is concerned, the properties are preserved; so, we not have to go through the step again. So, samples of Y of t can be simulated following this rule, one samples of Z of t are obtained. Now, I need the covariance function of Z of t; so, what I know is a covariance of Y of t. So, I will express this in terms of the covariance of Z of t, this rho star t 1 comma t 2 is not known; what is known is here, this is known, this is unknown.

So, as the prelude to the implementation of this method, we have to follow that approach, that I described in the previous lecture. We know, although with this is not known, we know that this lies between minus 1 and plus 1; the left hand side also this rho xx is also takes values in minus 1 to plus 1, this is known, this is unknown. So, what we do is, we find out the left hand side for various specified values of rho star from minus 1 to plus 1 and compute the corresponding rho; and among these computed values of rho, there will be one value which is the target value. So, we will be able to find out the corresponding rho.

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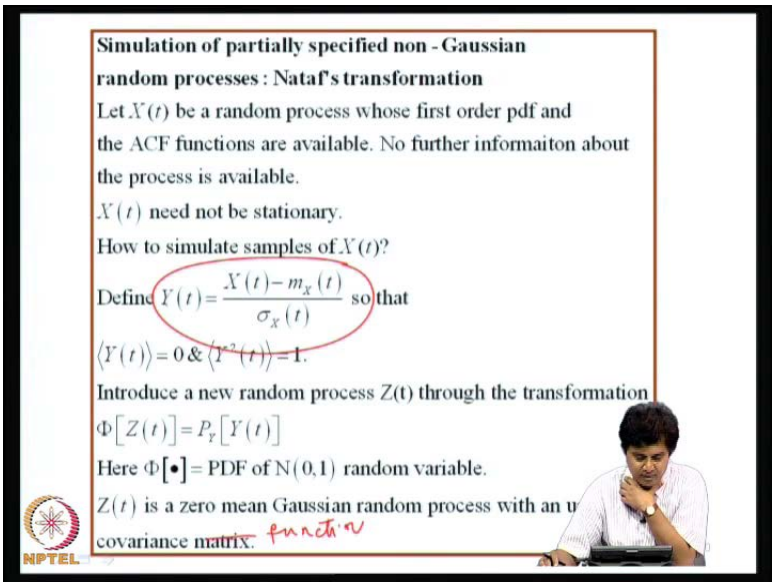

Now, this step can be implemented; so, this step, the steps involve in simulation would be solved for rho star t 1 comma t 2 and simulate z of t, and then simulate Y of t using this rule, and then go back and simulate X of t by using this rule. So, the computational steps involved are quite tedious, but in principle, it is dual.

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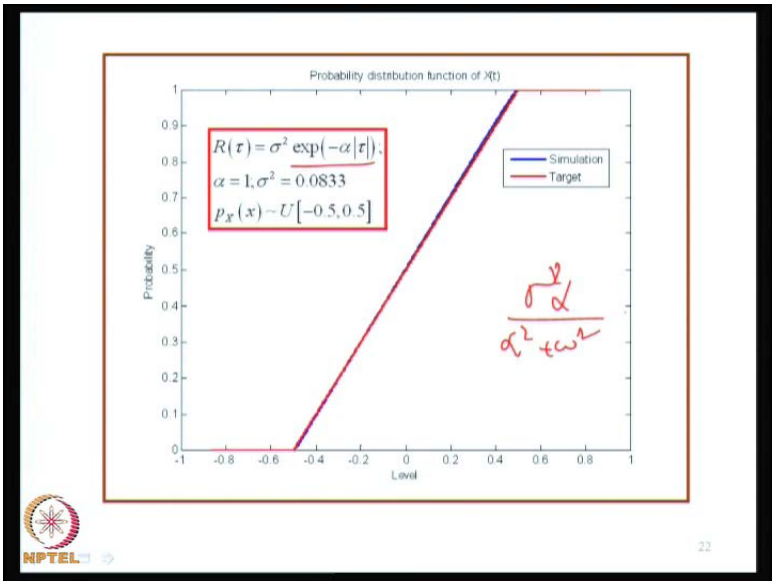

So, things work out become bit simple, if a process is stationary. So, indeed I start with the case of a stationary random process, whose covariance is sigma square exponential minus alpha tau and parameters of that is given; and I want the density function to be

uniformly distributed between minus 0.5 plus 0.5. So, first we have to find out rho star, and then do this transformation - Nataf's transformation - and this some results are displayed here; red is a target and blue is a simulation, and the agreement is quite satisfaction.

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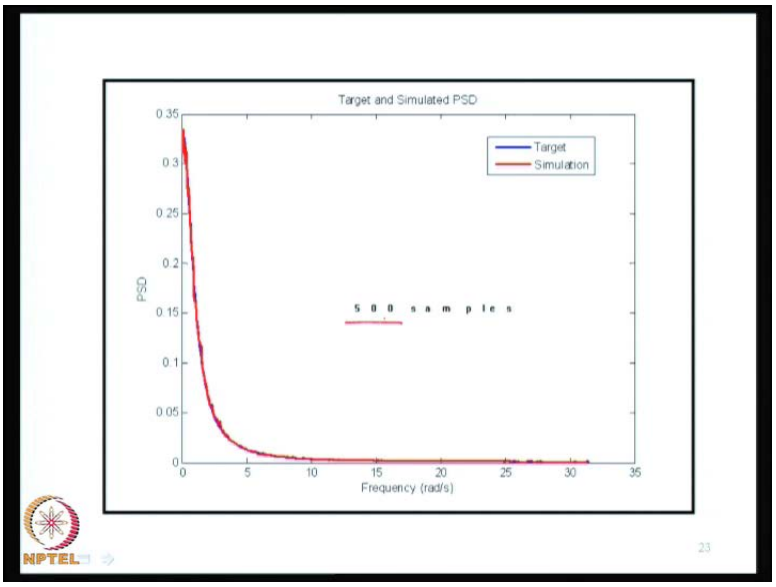

Now, this is a target power spectral density function, which is actually the Fourier transform of this, which is something like sigma square alpha by alpha square omega square, something like that. So, that is displayed here; the red and the blue, blue is the target and red is the simulated estimated one; they show quite a good agreement and we are unable to distinguish the two curves on these plot, and this is obtained with 500 samples.

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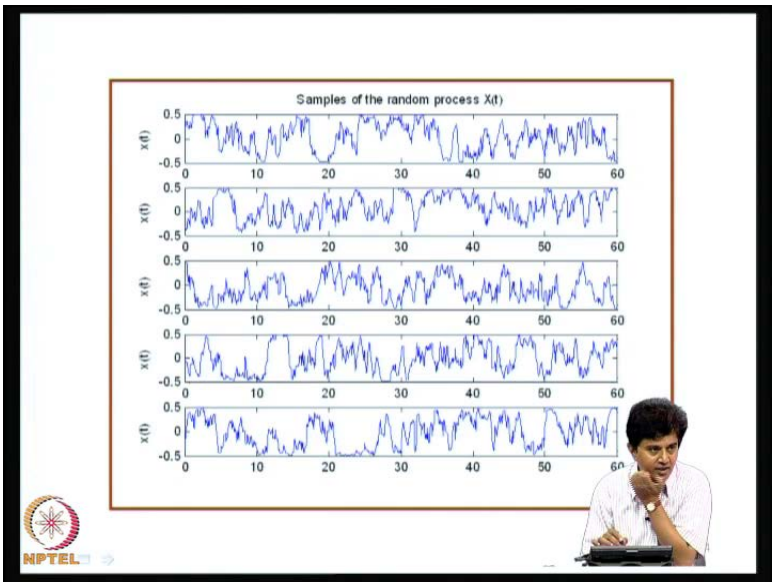

These are few samples of X of t, and notice that, that is uniformly distributed between minus 0.5 to plus 0.5 and it has 0 mean. And if you compute the power spectral density function or auto covariance function, it will match with the target values - target for values - that is specified in the statement of the problem.

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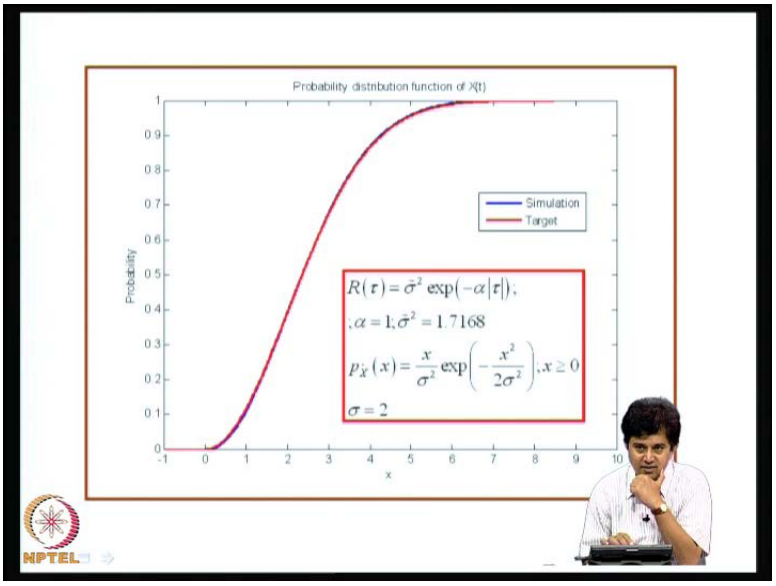

Same auto covariance function but will slightly different numerical values, but now the density function is not uniform, it is Rayleigh. So, mean would not be 0, the samples will have non-zero mean; and the parameters are specified here, sigma that appears in the

definition P x of x is 2. How do you a simulate samples of this process? So, again, we follow this Nataf's transformation method and go through that. Again, let me emphasize this description of the random process is not a complete specification of the random process; we are only telling the marginal density function and a covariance function. If the process is Gaussian, this is adequate; the first order covariance will be sufficient; for characterize second order properties and the first order property would give you the mean, second order is covariance; that is enough to completely specified Gaussian random process.

But we are talking about non-Gaussian random processes. So, this is not a complete specification. So, we are able to simulate samples, whose properties match with what has been specified; and other properties of this process, we really do not know what it correspond to it; it is left to the implicit modal that is contained in the Nataf's transformation.

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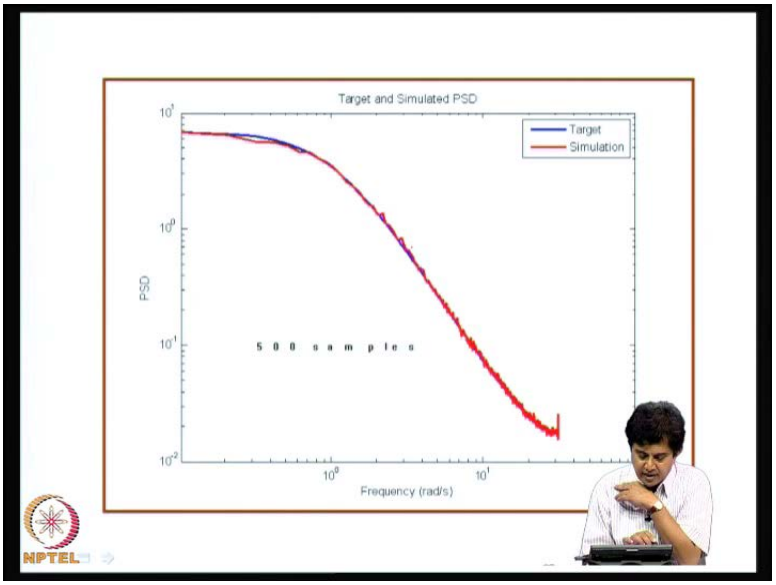

Now, this is the power spectral density function further Rayleigh random process. And the power spectral density function is shown on a log-log scale to make the comparisons clearer, and again, the agreement between target which is blue and simulation which is red is quite ok.

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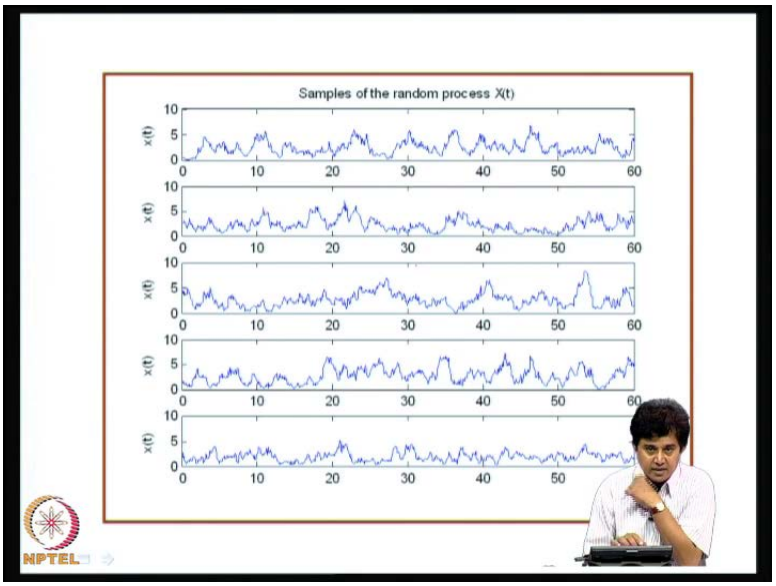

This is samples of a Rayleigh random variable and the psd that we mentioned that random process. And again, we can see that mean of these samples are non-zero, it is oscillating apart some non-zero value, which is what we should be expecting for this model.

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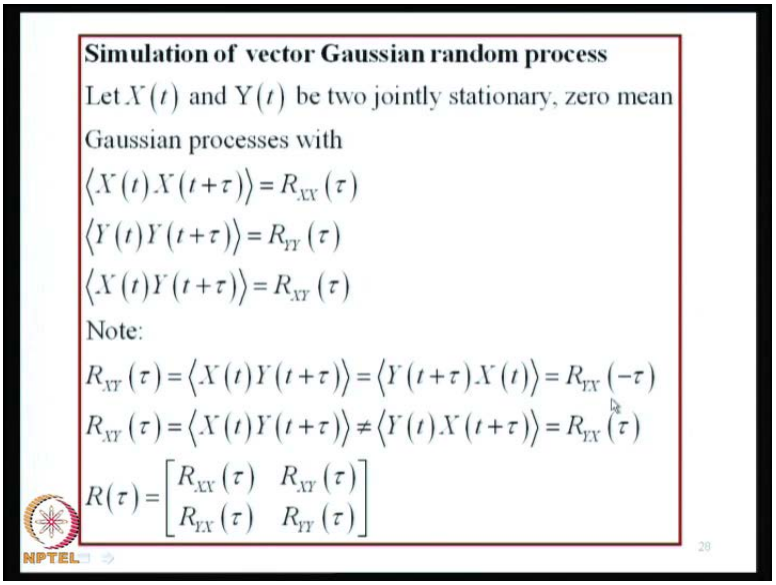

Now, scalar random processes we are able to simulate Gaussian, we have to discuss target power spectral density or target covariance, we are able to match. And for scalar non Gaussian, the target first order PDF and the specified PSD are auto covariance we

are able to match. Now, how about vector Gaussian random processes? So, to illustrate that, let us consider two random processes X of t and Y of t; this is specified to be jointly stationary and they have 0 mean. So, the covariance - auto covariance and cross covariance functions - completely specify the random process.

So, the auto covariance of X of t is expected value of X of t into X of t plus 2, which is R xx tau. Similarly, R yy of tau is defined like this; R xy of tau is the cross covariance function between X of t and Y of t which is expected value of X of t and Y of t plus tau. Now, let us look at R xy of tau, this is X of t in to Y of t plus tau. So, this is actually we can this is equal to Y of t plus tau in to X of t; therefore, this should be equal to Y of t plus tau in to X of t and this is nothing but R yx of minus tau.

So, R xy of tau is R yx of minus tau; so, it is not a symmetric function, but there exist certain skew symmetric of this kind. Now, R xy of tau is X of t into Y of t plus tau, but this is not equal to Y of t in to X of t plus tau; therefore, R xy of tau is not same as R y of x of tau. So, the R of tau is specified in terms of these functions; it is not symmetric, but there exist certain relationship between the two.

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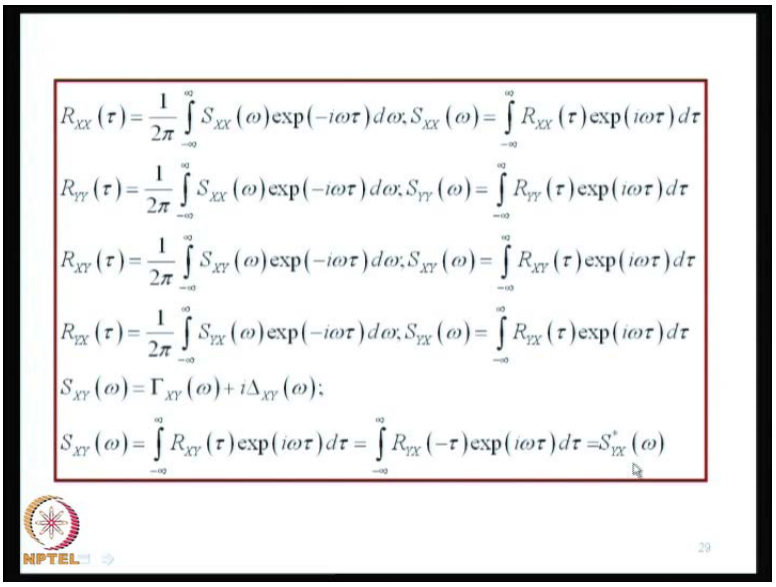

Now, how about the corresponding descriptions in frequency domain? So, the Fourier transform pair of R xx and S xx are displayed here; the R xx expressed in terms of S xx and S xx this expressed in terms of R xx. So, similarly R xy is shown here. Now, R yx of tau, we can write in this form and defined S yx of omega; and since R xy of tau is not

symmetric, they corresponding power spectral density function will be complex valued, and it has real part and an imaginary part. So, we can show that, since R xy of tau is R yx of minus tau, it follows that S xy of omega is same as conjugate of S yx of omega. So, these relations exist; there not four independent functions, but they are not symmetric either. So, there is a interdependence between the four quantities in frequency and time domains.

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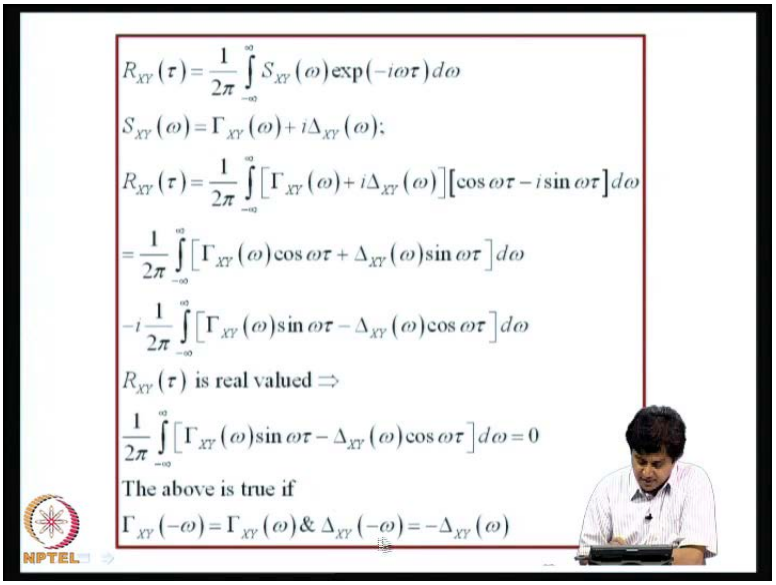

Let us examine this cross covariance and cross power spectral density function slighter slightly greater detail. So, S xy of omega is gamma xy of omega plus i delta xy of omega. Now, substitute this into the definition R xy of tau and I get this gamma plus i delta, and for e raise to minus i omega t, I will write it is as cos omega t minus i sin omega t; I can separate real and imaginary parts. Now, although S xy of omega is complex valued, R xy of tau is real valued. So, that would mean the imaginary part of this must be equal to 0; therefore, this second integral must be 0. From this, it follows that gamma xy of omega is an even function and delta xy of omega is an odd function, that ensures that R xy of tau is real valued.

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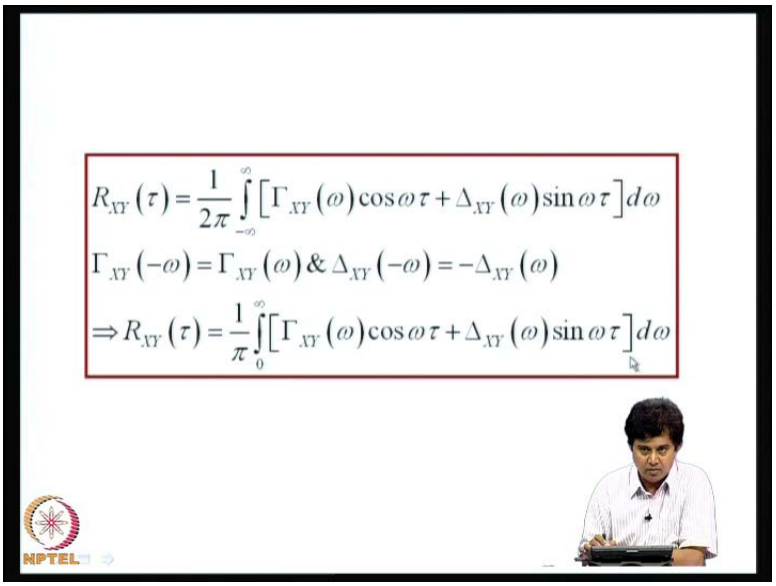

So, this is the summary of the relationship between R xy and gamma and delta. And we can use that property and write R xy of tau as 0 to infinity 1 by pi now, not minus infinity to plus infinity. See, gamma xy is even, cos is even, the so the product is even; this is odd, sin is odd, so odd into odd is even. So, both this integrant here is even; therefore, I can write it as 2 into 1 by pi 0 to infinity.

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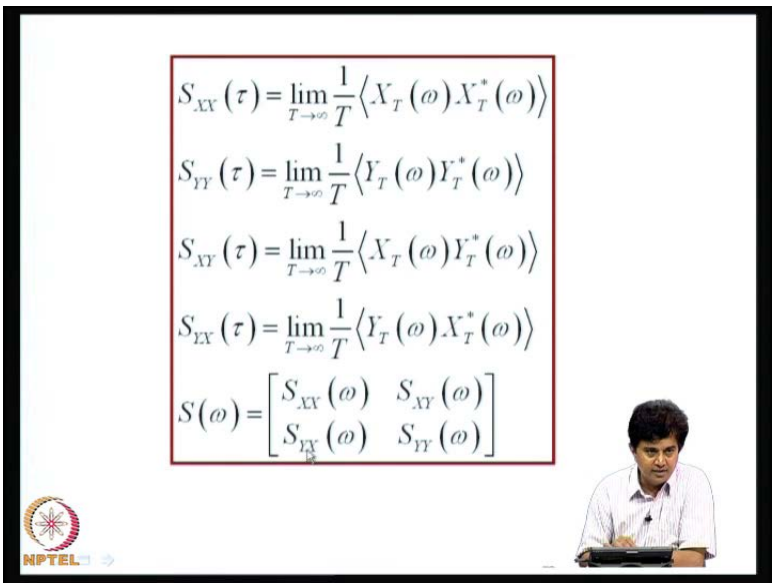

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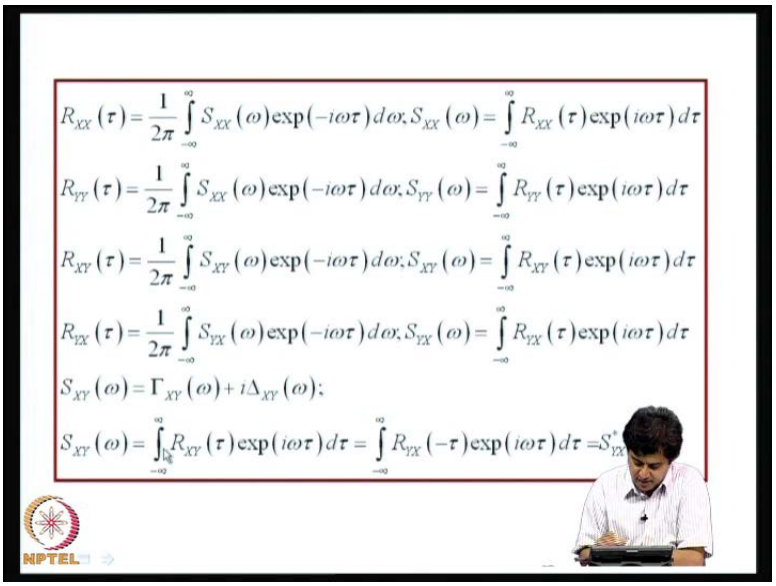

Now, the various definitions of power spectral density function in terms of an expectation of a truncated Fourier transform; this is well known; we have already studied this. And based on this, we can write the matrix of power spectral density functions, and S xy of omega and S yx of omega are not the same, but nevertheless, they are related through this relation, that S xy of omega is S star yx of omega.

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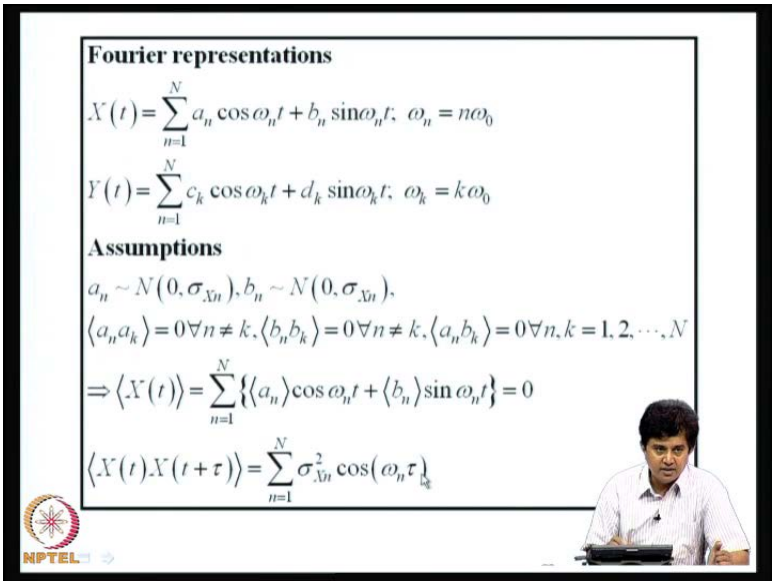

Now, let us do start with Fourier representation for X of t and Y of t. I will introduce for X of t, two families of random variables a n and b n; for Y of t, two families c k and d k.

Now, a n and b n, see, the X of t is stationary, Y of t is stationary, we would as a scalar random process. So, a n whatever conditions we impose on a n and b n, whenever representing X of t will now must continue to apply; that would mean a n is normal 0 mean and sigma xn, b n is similarly 0 mean sigma xn standard deviation, a n a k are independent for n not equal to k, b n and b k are independent for n not equal to k, and a n b k are independent for all n and k.

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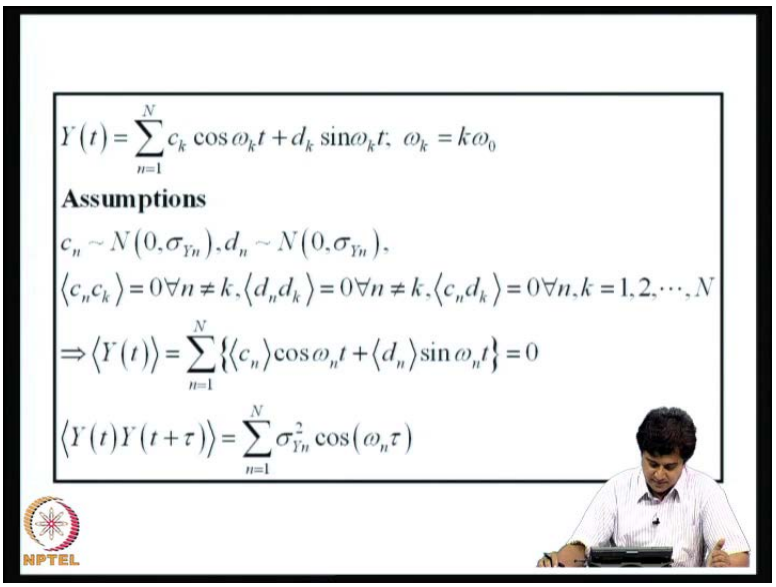

So, X of t as for as a scalar random process is concerned; it has 0 mean and its auto covariance is given by this. So, this is stationary random process, this we have seen. The same logic can be used on Y of t, and we again we can show that, you know, c n is normal 0 mean, sigma y n is standard deviation, and so on and so forth. And expected value of Y of t is 0, covariance of Y of t is n equal to 1 sigma y n square cos omega n tau. So, this sigma xn square and sigma y n square are to be obtained from discretize version of auto power spectral density function of X and Y respectively, so that we have already seen.

Now, what is more what is now remains to be addressed is the properties cross covariance properties and cross power spectral density function properties and capsulated in those functions. So, how do we deal with that?

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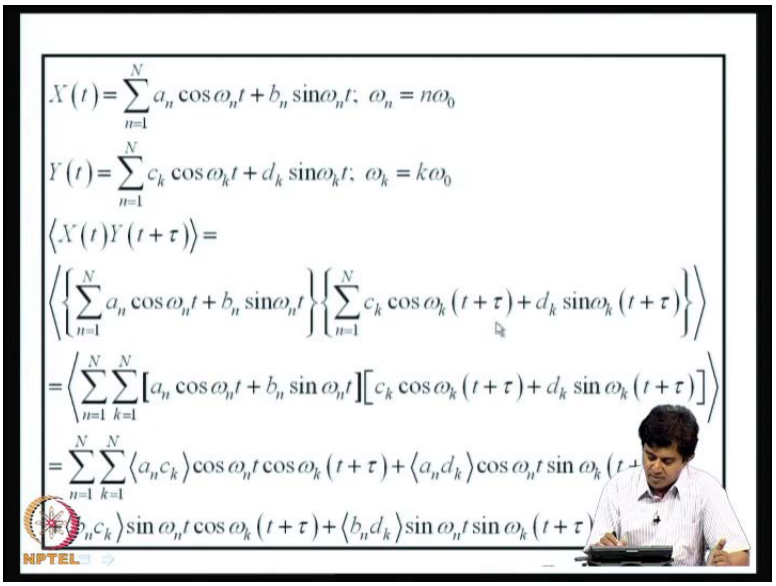

So, let us consider the expectation of X of t into Y of t plus tau. So, for X of t, I have this representation; for Y of t, I have this representation. I will use this and write it as the first term is this first summation, and the second term is the second summation, where t is replaced by t plus tau. So, I run through this calculations, the product of summations become double summation and I will now use the properties that we have encapsulated for a n and b n.

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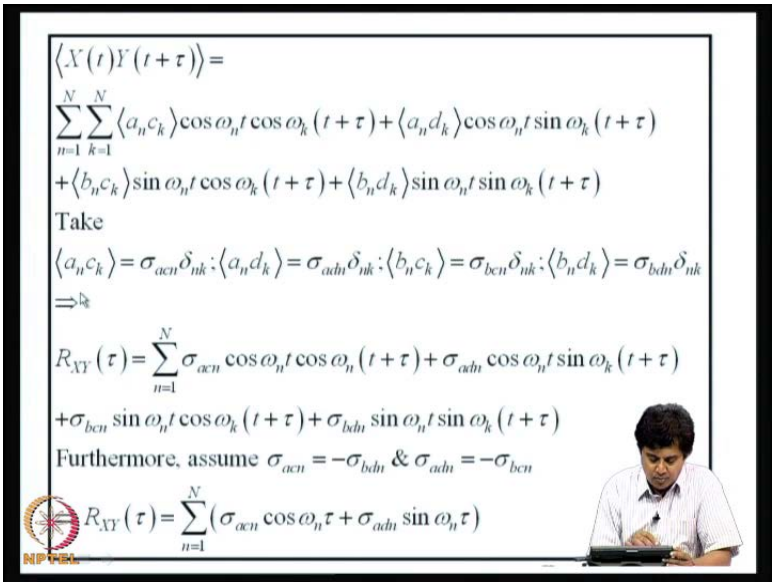

What we will do now is, I have not defined the properties between a n and c k how they are related, a n and d k how they are related, b n and c k how they are related, b n and d k how they are related, that we have not specified. So, what do will do is, we will assume that, a n is independent of c k and d k, if n is not equal to k; similarly, b n is independent of c k and d k, if n not equal to k; that we will be it, but when n equal to k, I will impose certain restrictions on expectations of a n, c n, a n d n, b n c n and b n d n, so that I will be able to simulate the properties of contain properties contained in the cross covariance functions.

So, to do that, what I do here? First of all, I want that, the properties that X and Y are jointly stationary must be honored. So, if that has to be honored, the expected value of X of t in to Y of t plus tau must be function of tau alone. So, for that have to happen, if I take now expected value of a n c k is sigma a c n, delta nk, this is a chronicle delta, which is equal to 1, if n equal to k; otherwise, it is 0. And similarly, if I place restrictions on a n d k, expected value of a n d k, expected value of b n c k, and so on and so forth, I can show that the expression for R xy can be reduced; the double summation collapses into a single summation.

Now, I will make further assumption, that sigma acn is minus sigma bdn and sigma adn is minus sigma bcn. If I do that, I can show that R xy of tau now becomes a function of tau alone. Now, I can select sigma acn and sigma adn to make sure that, this representation actually corresponds to the target value of cross covariance functions.

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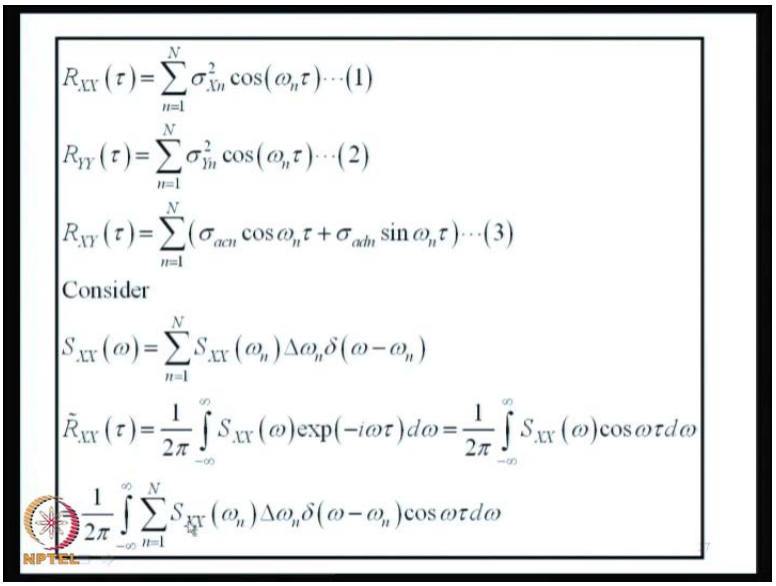

How do we do that? Now, we will consider now R xx, R yy, R xy; this is the 1, 2, 3 are the consequence of are the assume properties of a n, b n, c n, d n, for n from 1 to capital N. S xx omega if I consider S xx omega given in this form, we already seen that the corresponding auto covariance function matches with, can be made to match with function of this form by selecting sigma x n square to be equal to S xx omega n delta omega n divided by 2 pi.

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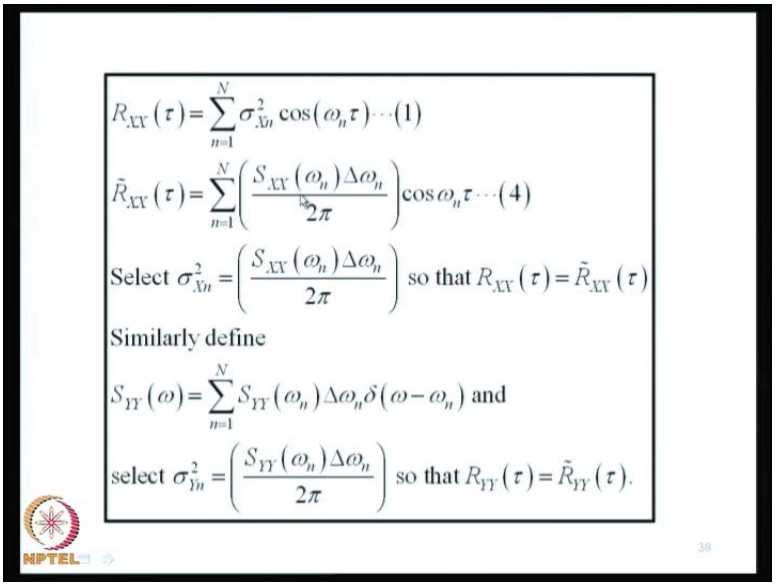

So, that means, this is the target and this is what happens if I use the power spectral density that I mention just now. And these two can be made to agree among themselves, if sigma x n square is taken to be this quantity. So, that R xx tau becomes R tilde of xx of tau. Now, similarly, S y of omega I define in a series like this, the power spectral density functions; and if I select sigma y n square to be given by this, I can show that the Fourier transform of this, which is R tilde yy of tau will match with the target auto covariance of Y; this is similar exactly similar to what we did for X of t.

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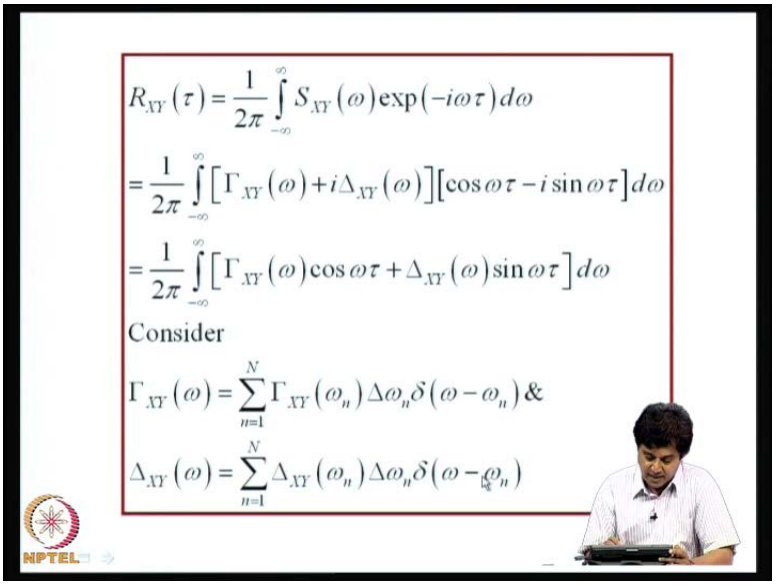

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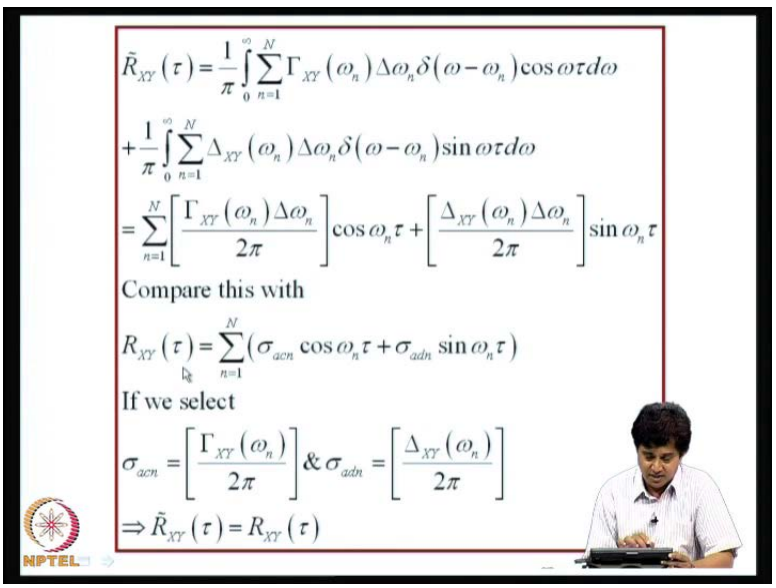

How about R xy? R xy of tau is given by this, and by separating real and imaginary parts, and honoring the fact that R xy of tau is indeed real valued, I get this functions. I have already shown that the imaginary part is 0, and what is the implication of that on properties of gamma and delta. Now, if I now consider gamma to be a sequence like this and delta to be a sequence like this, and select, of course, you corresponding to this sequence, you can take a Fourier transform; and that is quite easy, because we have direct delta functions the integration is very straight forward. If I do that, the expression for the cross covariance function can be shown to be given by this form; it is again of the form some constant of a n multiplied by cos omega n tau constant function of n multiplied by sin omega n tau. Compare this with R xy of tau, and that is suggest that, if I select sigma acn to be this and sigma adn to be this, the cross covariance are tilde xy and R xy of tau match.

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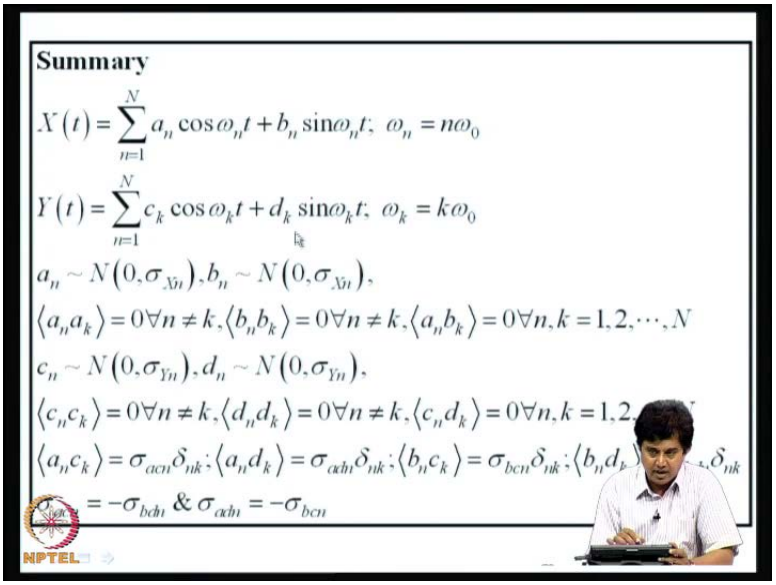

So, I have now the recipe to simulate the vector Gaussian random process. So, summary, I use this Fourier representation for X of t, Fourier representation for Y of t; this sigma x n that appears as standard deviation of a n is obtain from psd of x, sigma y n and that appear for c n and d n is obtained from power spectral density of Y of t, and sigma acn and sigma adn that appear in the description of properties between a n c n, a n d n, b n c n and b n d n is obtained from gamma and delta of your cross power spectral density function. So, the out of power spectral density function of x and y, the gamma and delta functions for correspond to the cross power spectral density between x and y are given.

So, from that, I can always by discretizing them, I can always get sigma x n and sigma y n, sigma can, sigma adn and we are ready to use this series. Once we are able to simulate a n, b n, c n, d n according to this prescription, we will be able to simulate x n y n.

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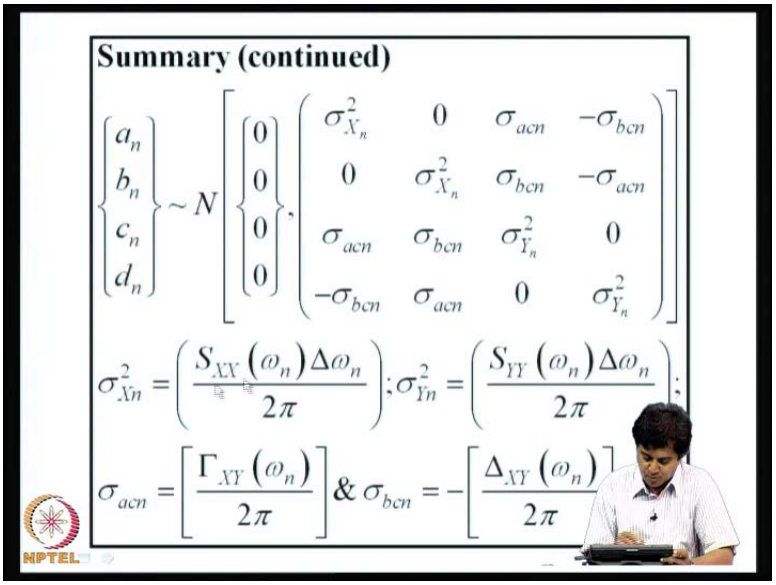

So, summary is a n, b n, c n, d n are four-dimensional normal random variables with 0 mean and covariance given by this. So, to implement the method, we should be able to simulate samples of a n, b n, c n, d n, according to this prescription. So, here, again you need to do the calculation that I mentioned, namely, you have to find the eigenvalues and eigenvectors of this, and do the transformation, and first in the standard normal space and then work backwards and get a n, b n, c n, d n, where all these parameters appearing here are expressed in terms of known properties of X of t and Y of t.

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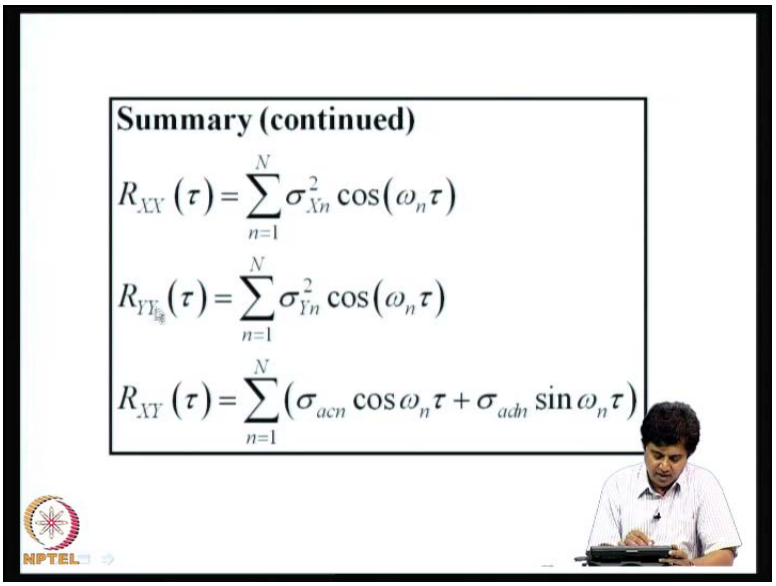

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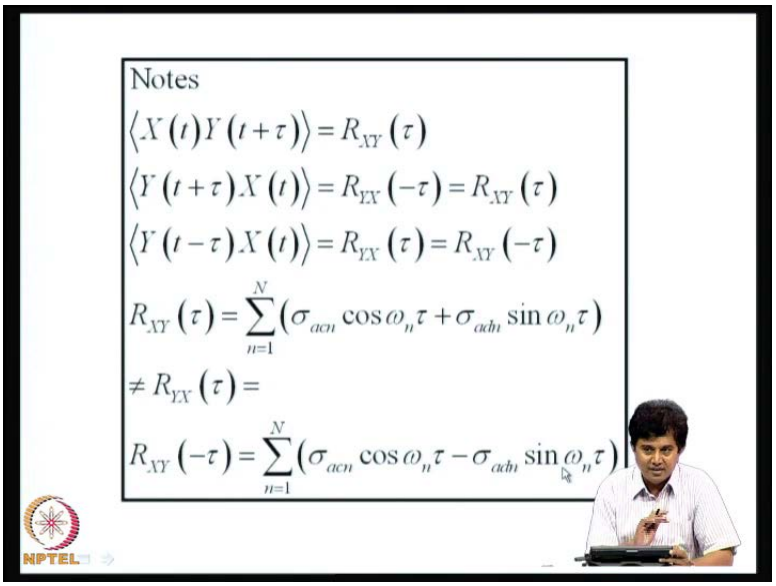

So, after that, we get expression for R xx, R yy, R xy, which is quite satisfactory. Some more notes, expected value of X of t into Y of t plus tau is R xy of tau, and expected value of Y of t plus tau in to X of t is R yx of minus tau, which is same as R xy of tau. Now, if you consider expected value of Y of t minus tau in to X of t, which is R yx of tau, this is same as R xy of minus tau. So, all these you can verify. Now, in our calculation, we got sigma x y of tau a sigma acn cos omega n tau sigma adn sin omega n tau sum from 1 to n, which is not equal to R yx of tau which is R xy of minus tau, because sin omega n of minus tau is minus of sigma adn sin omega n tau. So, things are

ok, these are some simple checks on basic properties, that we should expect from such representations.

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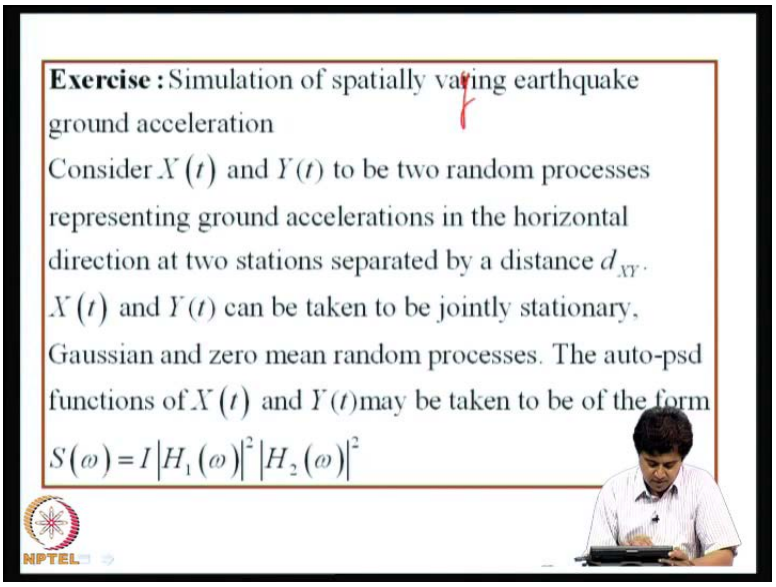

Now, I leave it as an exercise; this is the fairly… this can be a fairly long exercise which can be a accomplished, if you write a computer program. So, as an illustration, we will consider as simulation of spatially varying earthquake ground acceleration. Consider X of t and Y of t to be two random processes, representing ground accelerations in the horizontal direction at two stations separated by a distance d x y; that means, the same event of an earthquake, I take two points on the earth surface and consider the ground acceleration in a given direction; and at one point, I call it as X of t, and another point, it is Y of t.

We can take that X of t and Y of t to be jointly stationary, Gaussian and 0 mean random processes. The auto psd functions of X of t and Y of t may be taken to be of the form, S of omega is I in to H 1 of omega whole square and H 2 of omega whole square; this is typical Kanai-Tajimi power spectral density function model, where H 1 is given by the transfer function of the soil layer and H 2 of omega is an artificial filter function, which eliminates certain anomalies in the low frequency behavior of samples of displacements, that is needed to make the model work well at low frequencies.

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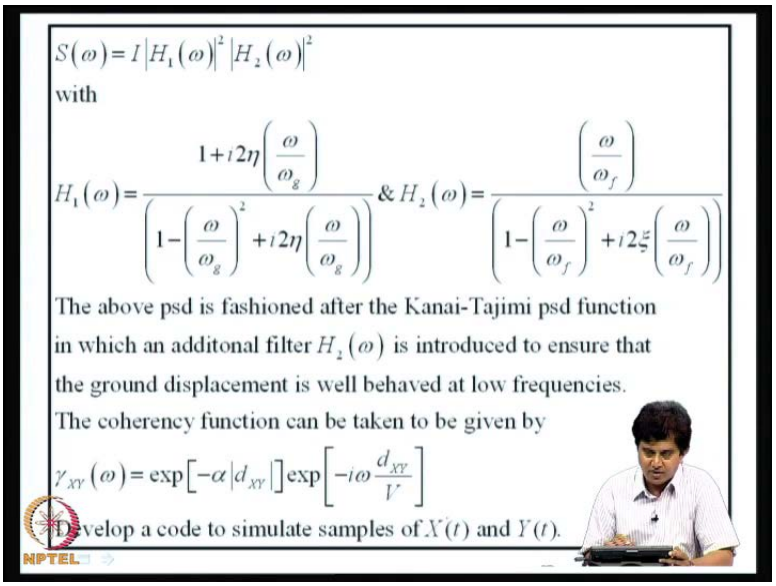

So, the auto psd's are specified; this is the well-known Kanai-Tajimi power spectral density function model. And we introduce a coherency function, that is, S xy by S x square root S xx S yy to be given by this; this is a complex valued coherency function, which depends on the distance between the two stations, and the wave velocity and frequency, and it is a complex valued function. So, from this, you can actually get multiply this by the 2 power spectral density functions, you get the cross psd function; and you can separate them into real and imaginary parts, you will get your gamma and delta; so, you are ready to launch your simulation.

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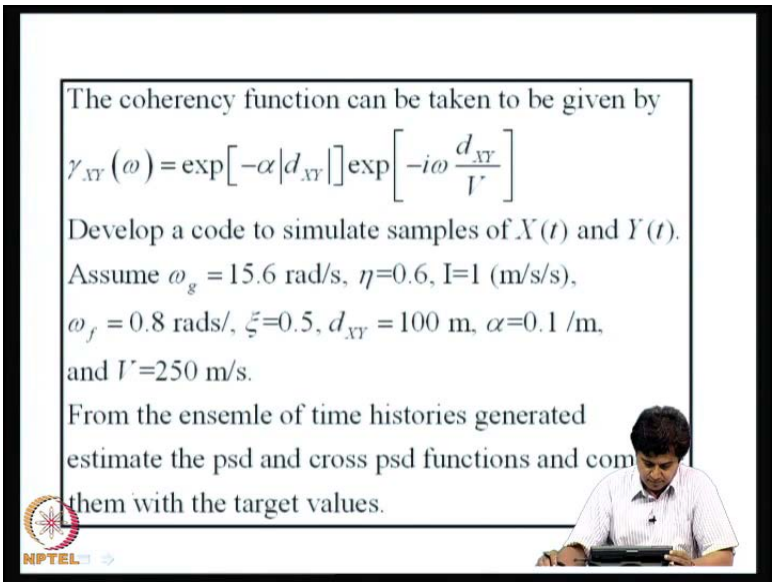

So, problem on hand is to develop a computer code to simulate samples of X of t and Y of t. And to facilitate that exercise, some numerical values I have suggested here; ground natural frequencies 15.6 radian per second, and so on and so forth. So, the exercise consist of simulating, may be say 5000 samples of X of t and Y of t, and then follow it by... From that ensemble of time histories, estimate the power spectral density function matrix and compare it with the target psd matrix. So, this is a quiet an involved exercise. This second part of this exercise, namely, estimating psd matrix from observe data of vector realizations, is something that we need to consider later. At this stage, we do not have I am not discuss this yet, but it is a part of this exercise.

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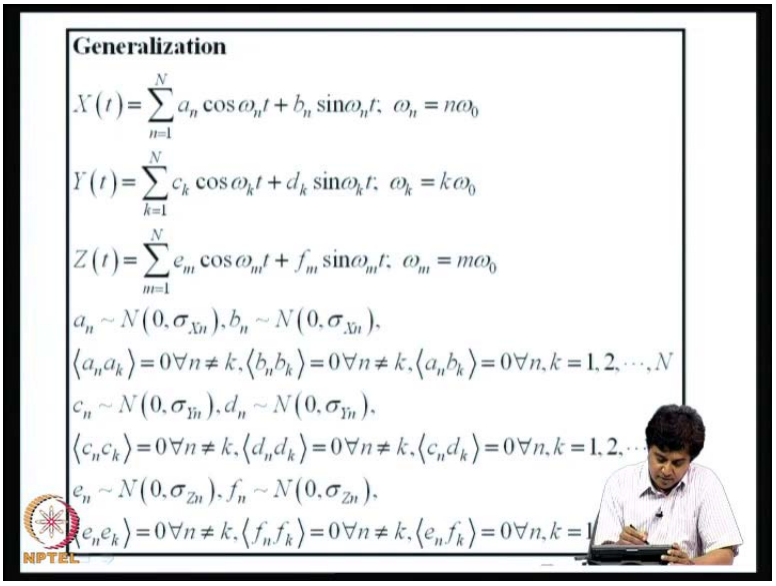

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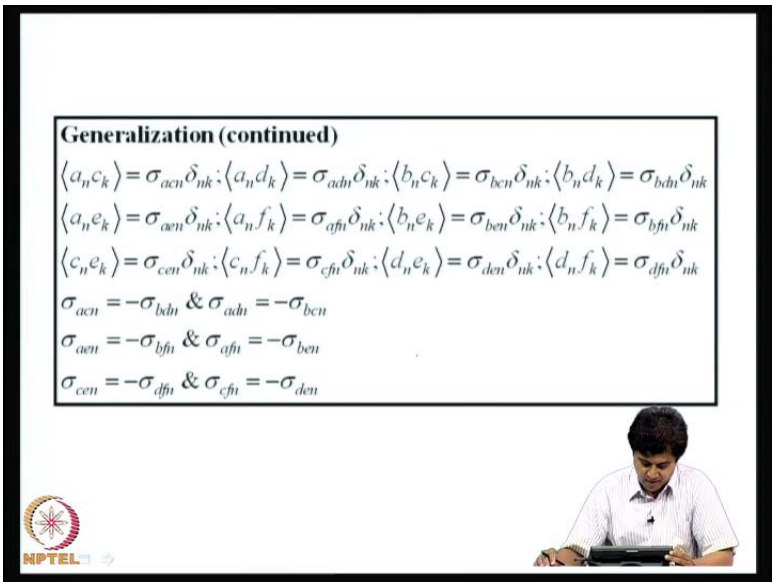

Now, I have talked about pairs of random processes, we can generalize it to say three random processes or n random process. So, this method seems to work out quite well for this. So, we use three Fourier representations, and I have now auto psd for x, auto psd for y, auto psd for z, cross psd between x and y, x and z, and y and z. So, I have to now each one is a stationary random process; so, represent finding properties of a n and b n is purely based on property of psd's of x, y and z, for say, a n b n, c k d k, e m f m, but to find the cross properties between a n c n, a n e n, etcetera. You have to bank on the cross power spectral density functions; so, you can specify all this properties, as for the formulation that have just now outline; and in principle, it should be possible to simulate samples of n dimensional Gaussian vector random processes.

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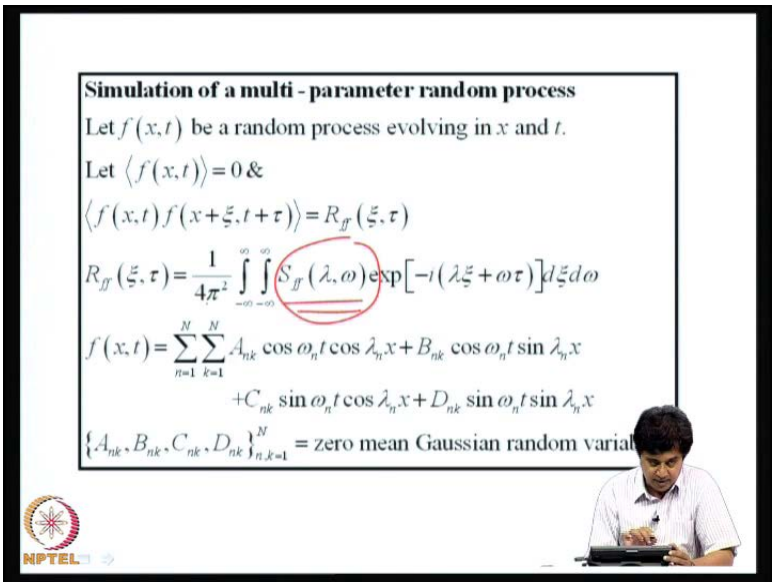

I have detail the various properties that we need to use and I leave that exercise for you to verify. How about multi parameter random processes? For example, if you are modeling wind velocity on a say a chimney, so the wind velocity varies in time, but also varies the space.

So, I have a random process which is a function of x n t. so, in a three-dimensional situation, this x can be x 1, x 2, x y z and t, like ocean waves, and so on and so forth. But here, I am considering a single random processes function of two parameters. This is a scalar situation; so, we can a vector situations also here; instead of just velocity, you can model some other parameter like pressure or something like that. You have two quantities, but we will consider scalar case first. We will assume that, this random field has 0 mean and it is homogenous, homogenous of homogeneity of random field is synonyms are analogous to stationarity of a random process, that would mean expectation of f of x comma t into f of x plus i plus t plus tau is a function of x i and tau, that is the stationarity property.

And corresponding to the two-dimensional auto covariance function, I can define a two-dimensional power spectral density function. One of the parameter will correspond to time, other one to space. This is the wave length, this is the frequency in radian per second, this will be wavelength per unit length. So, I can define the power spectral density function. Now, I can use a Fourier representation involving, A nk, B nk, C nk, D

nk and products of cos and sin functions, evolving in sin and evolving in time and space. Now, these are 0 mean Gaussian random variables; I will adjust properties of these random variables to suit the properties of this specified power spectral density function.

So, the problem on hand is, to simulate samples of f of x comma t, whose power spectral density function in lambda and omega space is given; it is and the process is given to be Gaussian.

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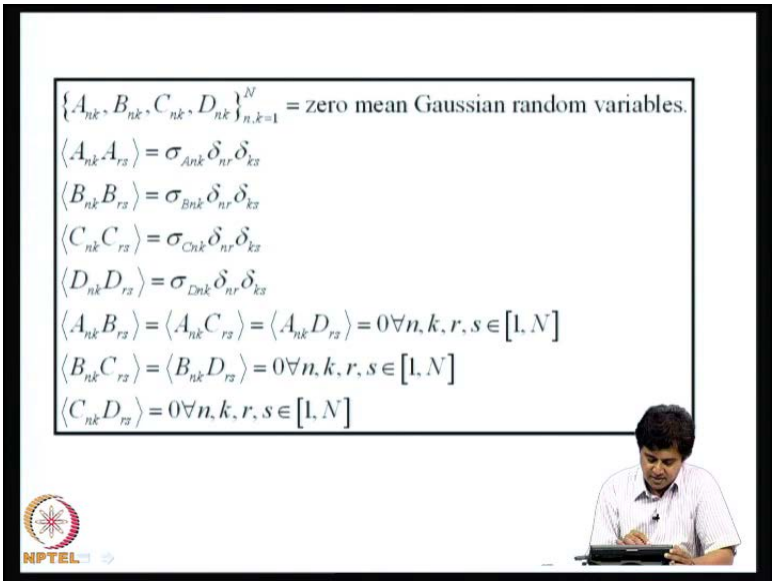

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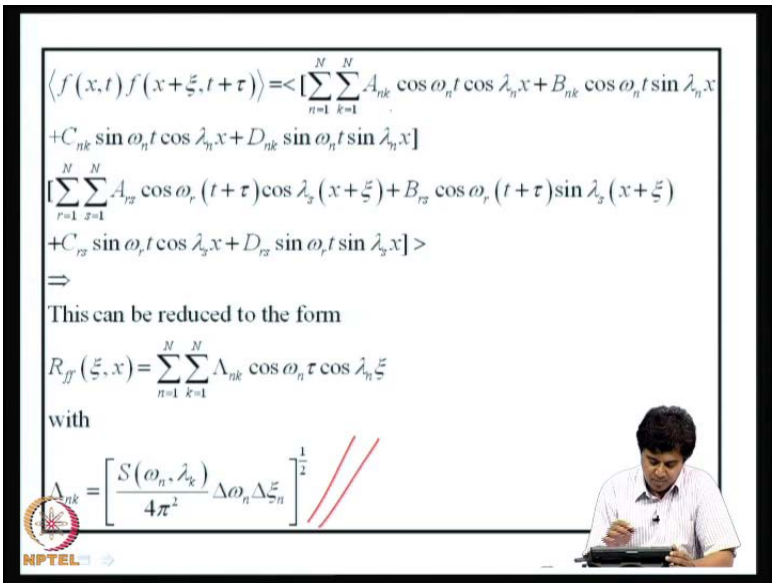

So, you can set up the same logic; we can make all these things for n naught equal to k etcetera to be independent. And when for certain cases, when n equal to R n, k equal to S, etcetera, we will adjust those parameters and take them to correspond to the value of the discretize version of the power spectral density function matrix, and we can show that they auto covariance from such a model. And from a model which is purely based on discretize version of power spectral density function would match, if this A nk B nk are selected in a certain particular manner and you can simulate samples.

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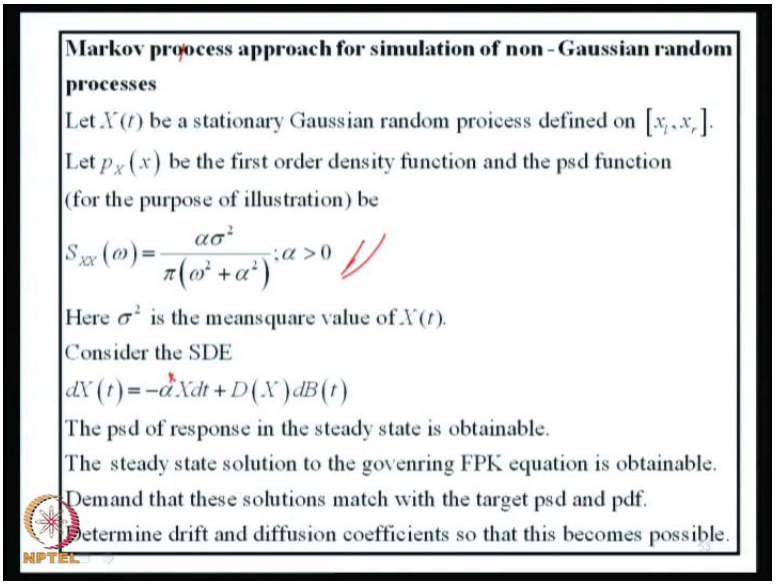

Now, then alternate approach for simulation of non-Gaussian random processes with prescribe power spectral density function. This is based on Markov process theory. So, let us consider this problem; let X of t be a stationary Gaussian random process defined on interval x l to x r, let P x of x be the first order probability density function, and let the psd function for purpose of illustration, I am taking it to be of this form. Now, I want to simulate the samples of X of t; now, what I do is, I consider a stochastic differential equation, dX of t is minus alpha Xdt plus D of x dB t.

Now, this equation is such that, the two time response of X of t can be evaluated and we can in fact compute the auto covariance of the response process. The parameter alpha and this parameter D of X are unknowns; although I have selected alpha to be same as this, it indeed transverse, those two will be the same, but the problem statement is as follows. Suppose this alpha is say alpha star, so the drift and diffusion coefficients are

unknown. For this problem, the steady state solution to the governing of Fokker Planck equation is obtainable; first order equation I showed already, that you can solve the Fokker Planck equation in the steady state.

Now, what I do is, I demand that if I where to obtain the steady state solution of this equation on first order probability density function, you should match with this P x of x. And similarly if I compute the power spectral density function in steady state, it should match with this. So, what is known here, are the solutions of this problem. What is unknown are this alpha star and D of X; so, these are inverse problem. Now, what we do is, we demand that these solution match with the target psd and pdf; that means, determine the drift and diffusion coefficients, so that this becomes possible, that means, we complying with the target psd and pdf, and it becomes possible.

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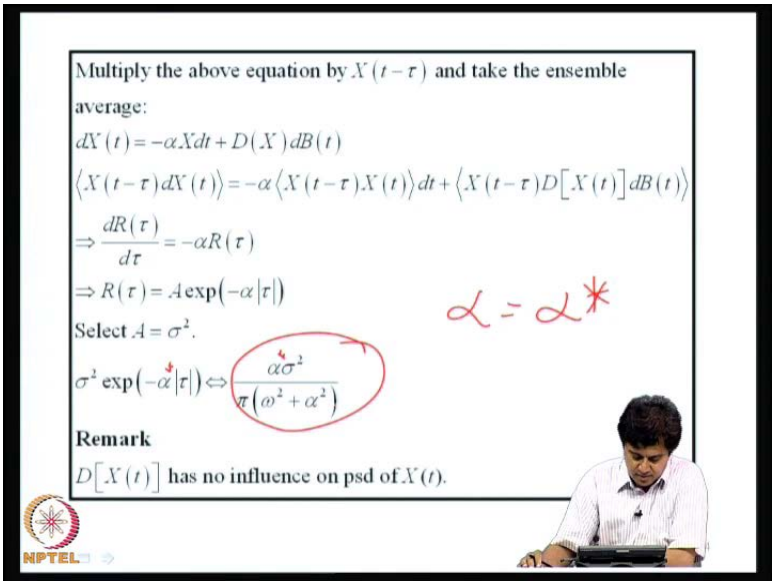

So, how do we do that? So, we can start by multiplying the first of the question by X of t minus tau and take the ensemble average. And from this, we get the equation for covariance function in steady state, and we can show that, the covariance function is of the form a exponential minus alpha mod tau. And corresponding to that, I have this power spectral density function; that means, this alpha I should I have written as alpha star, but so imagine this alpha star, this is alpha star; as for as psd function is concerned, select alpha equal to alpha star. Then, our target is on psd in met; the target on first order

probability density function is yet to be satisfied. Now, what you should notice is, that the diffusion coefficient D of X has no influence on psd of X of t in this equation.

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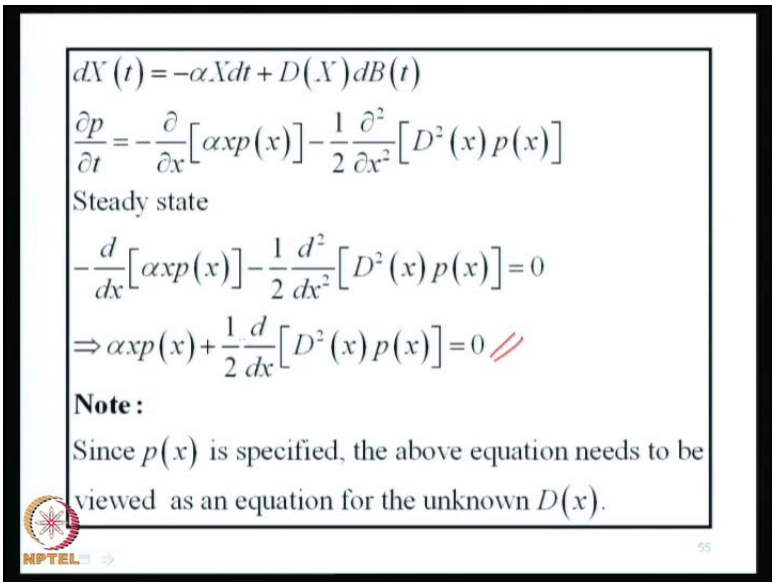

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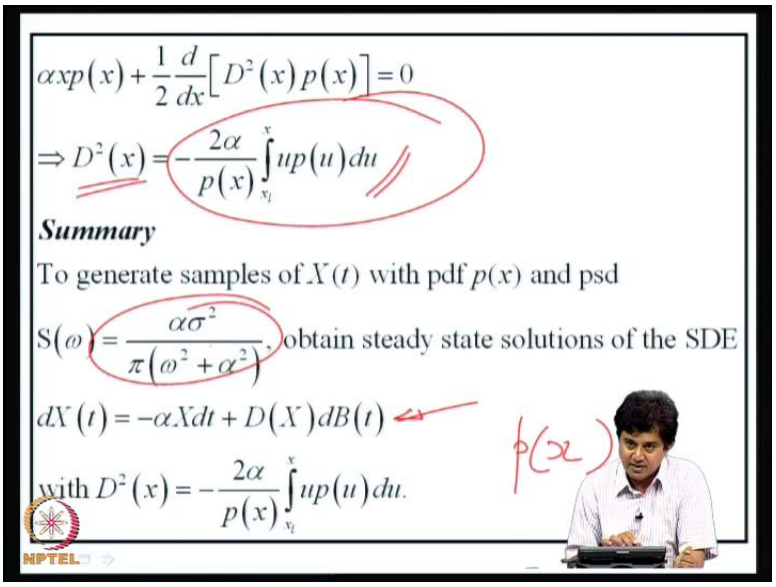

Now, let us consider now the governing Fokker Planck equation. So, dX of t is minus alpha Xdt plus D of X dB t, and associated with that, this is the governing Fokker Planck equation. In the steady state, dou p by dou t would be 0 and I get this reduced equations. So, one of the solutions for this would be of this form, and indeed, we can solve this; this is the first order ordinary linear differential equation and we can solve this. Here, what is

not known? Alpha is known, because I already I solved for that; P of x is given, right; D of x is unknown. So, this has to be viewed as an equation for D of x; that means, for what value of D of x, would P of x would be the solution to this problem? Before so, it is an inverse argument; so, if I do that and solve for D square of x, I get D square of x to be this. That means, if I consider now a stochastic differential equation, whose drift is alpha and diffusion coefficient is D square of x, square root of this is given by this, and simulates samples of X of t according to this rule, the samples in steady state would have this power spectral density and the target P of x as a probability distribution function.

So, that means, where Taylor the SDE is Taylor made, to produce the requisite power spectral density function and first order probability density function; it is an inverse approach. So, he has an exercise, you can consider producing a samples of Rayleigh random process with this power spectral density function or uniform distribution, I leave it as an exercise; I will not be able to, I will not discuss the details.

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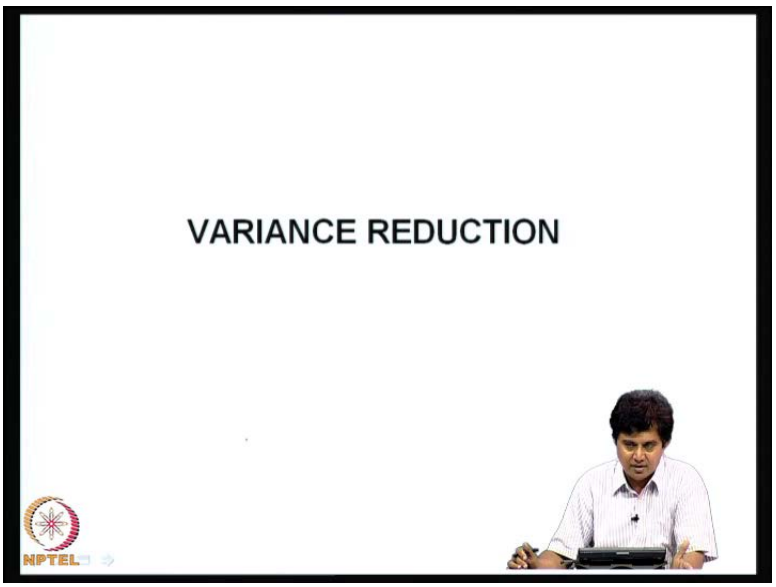

Now, I would like to briefly introduce a new topic, what is known as variance reduction. So, before that, we can summarize what all we have done. We have develop methods for simulation of random variables, scalar, vector, Gaussian, non-Gaussian, completely specified, partially specified. Similarly, methods for simulation of samples of random processes, scalar Gaussian, scalar vector, scalar non-Gaussian, vector non-Gaussian with partial specifications, and also discuss briefly a method based on Markov processes

characteristics of the solutions. So, as for as simulation tools are concerned, we same to be having the requisite a good deal of tools already available with us.

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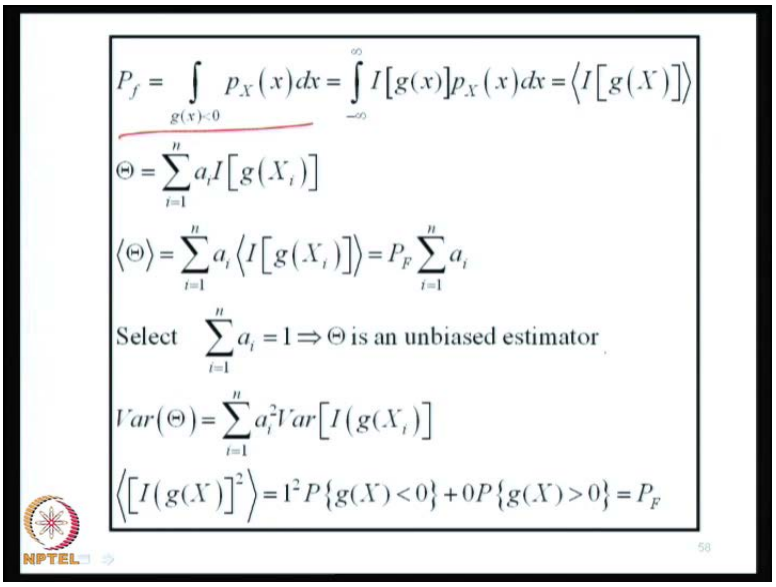

Now, the problem of dynamic response characterization has to be addressed now. So, what we have done is, we have simulated samples of actions. Now, we have to run them through the system dynamics and get the samples of the response, and then be able to solve them and get properties of response characteristics. So, one of the problem that arises there is a is known as variance reduction, and I will briefly explain this in the broader context of Monte Carlo simulations. And I will consider the problem of evaluation of a multi-dimensional integral given by, integral P x of x dx over a region g of X less than or equal to 0.

So, this is the problem that arises in structural reliability analysis. So, the details of this statement of the problem and its relation to problems structural reliability is not of primary concern here. What is of concern here is the computational details of evaluation of this integral using Monte Carlo simulations. So, I can, now, the region over reach integration is being done appears as a limits of this integral, I can first make the limits and minus limit to plus infinity by introducing an indicator function on g of x; this I of g of X is 1, whenever g of X is less than 0; otherwise, it is 0. So, I can write it as I of g of X P x of x dx.

Now, therefore, P f can be written as expected value of indicator of g of x; therefore, this integral formally I am writing it as an expectation. Now, my problem is to estimate this through samples of using samples of x. So, let theta be the estimator, i running from 1 to n a i I g of X i. Now, the mean value of theta is given by, i running to 1 a i expected value of this, and this is known to be P f and this is i equal to 1 to n a i.

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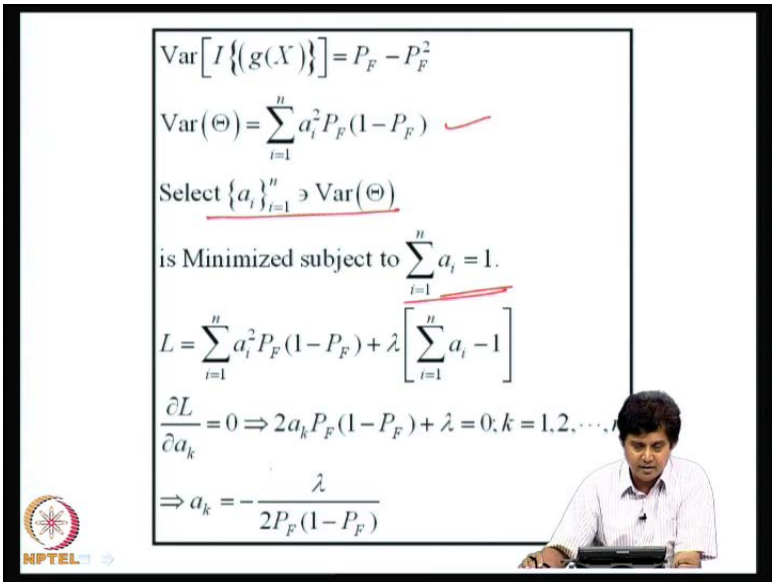

Now, if this summation is equal to 1, we get theta to be unbiased; variance of theta I can do a simple calculation, and I can show that, variance of theta is actually given by this equation, I equal to 1 to n a i square P F into 1 minus P F.

Now, a i is there still unselected. So, what we do? Select a I, i equal to 1 to n, such that, the variance of theta is minimize subject to this constraint; that means, moment this equation this constraint is satisfied, theta become unbiased estimator; and moment this optimization criteria is met, we get an unbiased estimator with minimum variance.

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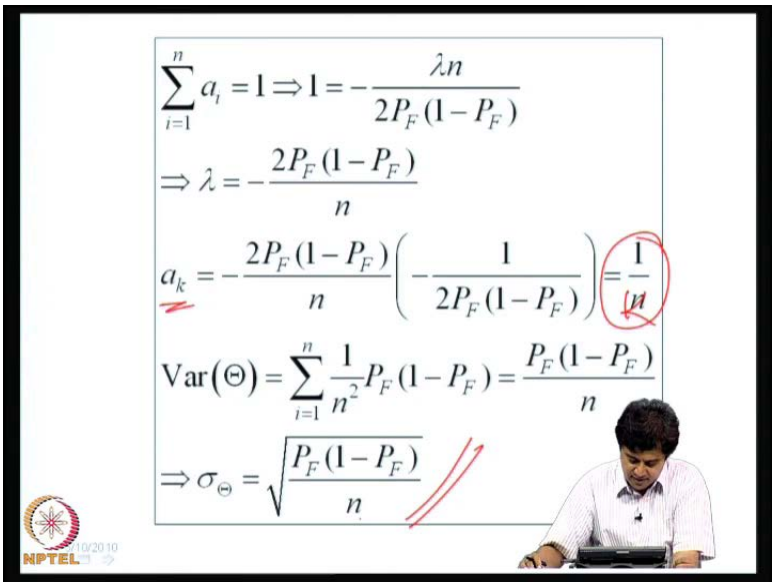

So, we can solve for a i and we have gone through this step in the discussion on estimator for the mean. And we can show that, a k is 1 by k, these are the optimal values of the parameters a 1, a 2, a 3, a n and associated variance is given by square root P F 1 minus P F by n.

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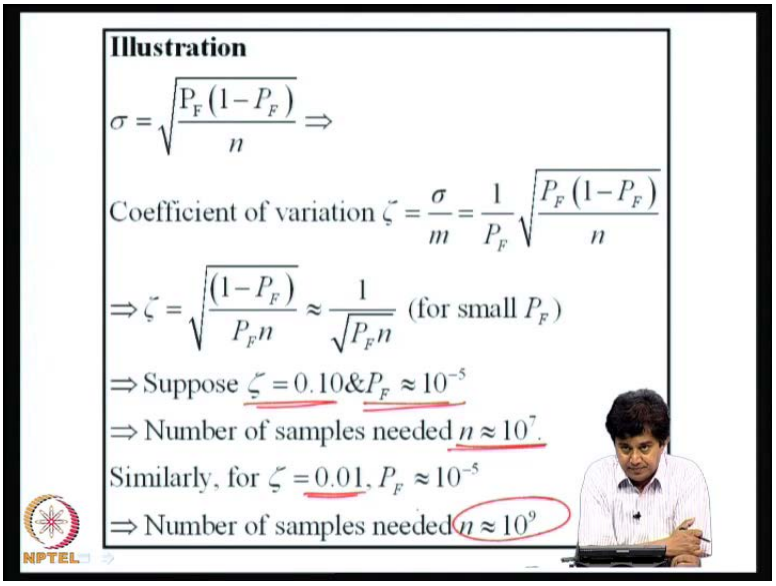

So, this result is quite similar to what we did earlier, so in characterizing the estimator for the mean. Now, let us look at this slightly more carefully. Now, look at the standard deviation which is square root of the variance of the sampling distribution, we get it in

this form; and the coefficient of variation can be defined as sigma by m, that is, 1 by P F into this and a slight rearrangement will show as, that this coefficient of variation can be given as, 1 divided by square root P F n, when P F n is small, which is most of the cases the kind of problem that we are studying.

Suppose the answer is in the range of 10 to the power of minus 5 and you are looking for coefficient of variation of 10 percent; that is, accuracy with which you want compute the value of the integral, then the number of samples needed becomes 10 to the power of 7; that means, you need 10 to the power of 7 samples to evaluate P F, where coefficient of variation is about 10 percent. This coefficient of variation can be viewed as some kind of an error. On the other hand, if coefficient of variation becomes 0.01and probability of this P F, which is probability of failure in structural reliability language is of the order of 10 to the power of minus 5; the number of samples needed becomes 10 to the power of 9.

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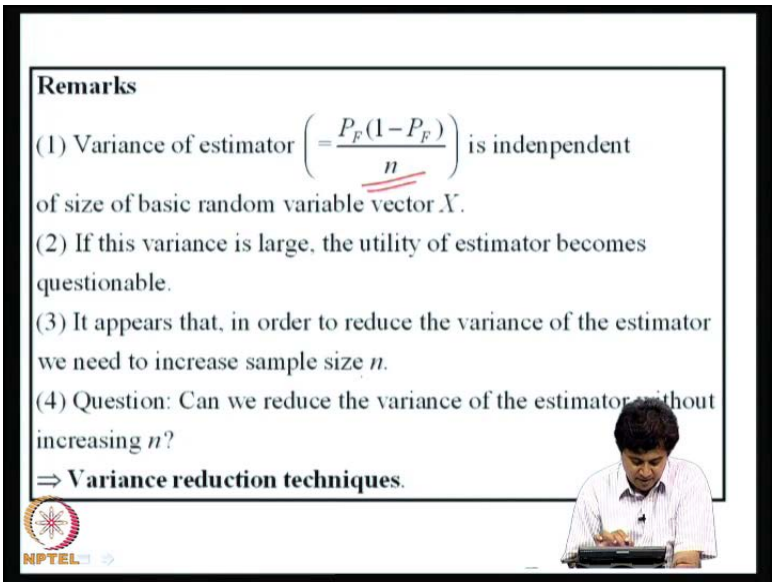

Now, we can make few observations. Now, these variance of estimator, which is P F in to one minus P F by n is independent of size of the basic random variable x; x can be one dimensional, ten-dimensional, hundred-dimensional, this expression is independent of that; this n is samples not a dimensions of x. If this variance is large, the utility of estimator becomes questionable, because our population parameter is deterministic; and we are using a random variable to approximate an deterministic quantity, its variance

should be as slow as low as possible; so, larger variance means the answer is less acceptable.

So, the question of reducing the variance is very important. So, how do you reduce the variance? You look at the expression for the variance, the n is in the denominator here; so, it appears that, in order to reduce the variance of the estimator, we need to increase the sample size n. But, is it the only approach? We can raise this question. Can you reduce the variance of the estimator without increasing n? So, this problem is known as problem of variance reduction. Variance can be increased by increasing sample size, but are there any other ways of reducing the variance without increasing sample size?

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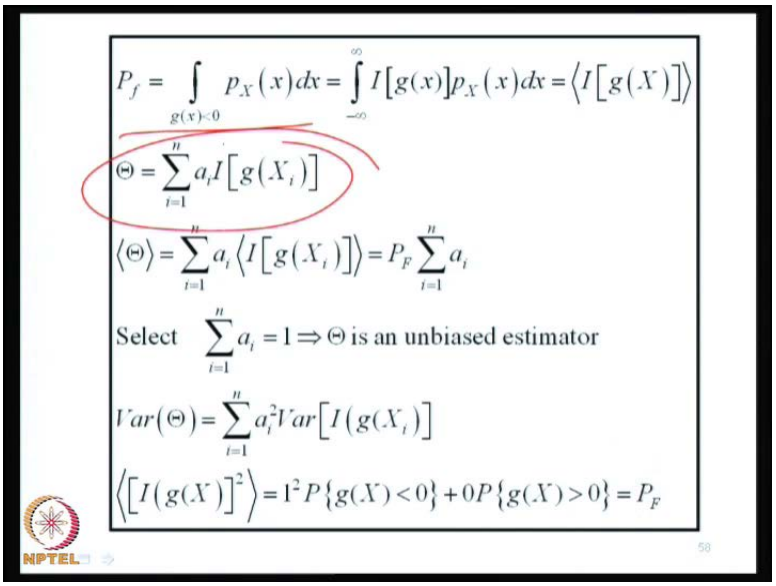

So, for a given value of n, instead of using this as my estimator, can I modify this estimator and reduce the variants? The sample size is fixed, ok. So, this question is of a considerable significants, when we analyze the response, because if you are applying simulation methods for practical structures, if one run of computer simulation of a dynamical behavior of the system takes about, say 5 minutes of C P U, and you are evaluating probability of failure of the order of 10 to the power of minus 5 with coefficient of variation of, say 0.01 as I was mentioning, you need to run the computer code 10 to the power of 9 times. And if 5 minutes in to 10 to the power of 9, it is an hopelessly large amount of computational time and that approach is unlikely to work.

So, we need to develop some intelligent methods to reduce the variance and that issue will consider subsequently. So, at this point, I just want to state, what is the problem of variance reduction. So, at this stage, we will close this lecture. In the next lecture, we will consider questions on simulation of dynamical behavior of systems; we have now completed the problem of simulating samples of random variables and random processes, and this description will be useful in characterizing the inputs and initial conditions to the say the dynamical system. And now, the question of, how does these ensemble of random variables and random processes transmit through the dynamics of the system and produce the response quantities of interest? So, we will address this question in the forthcoming lectures.