Stochastic Structural Dynamics

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Lecture No. # 24

Markov Vector Approach

We have been studying, how the Markov property of response vector of dynamical systems, driven by white noise excitations can be formulated, in terms of Fokker Planck equation; we continue with this discussion.

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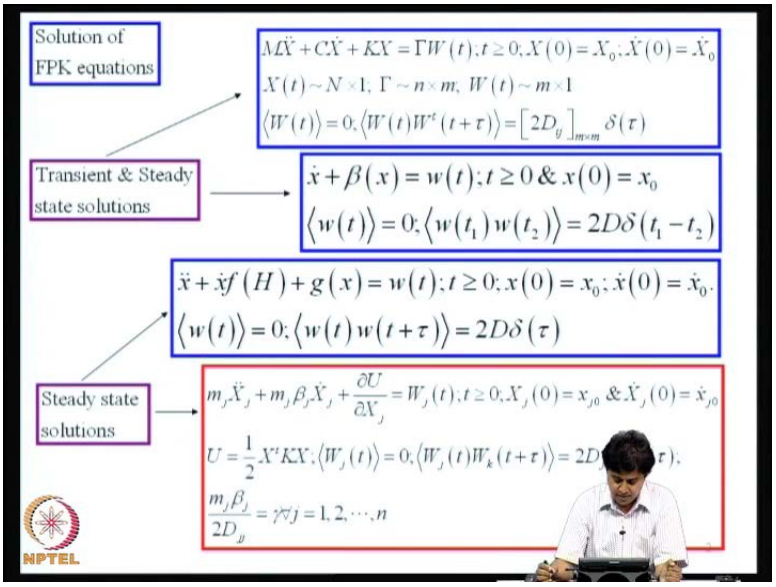

So, what we did was, we formulated the governing equation for the time evolution of the transition probability density functions. And we showed that, for linear multi degree freedom systems under white noise excitations, we can obtain complete solution, that involves both steady state and transient regions; and the result that we obtain, agrees with what we got using ample integral approach earlier.

Similar solutions are also possible for first order non-linear and differential equations of this kind driven by white noise and driven by white noise. Here, steady state solutions

can be obtained quiet easily, but the transient solutions need to be obtained using series expansion methods, by seeking a solution in every variables separable form.

But for more generate class of problems, even single degree freedom systems or multi degree freedom systems, it is often not possible to exactly solve the governing Fokker Planck equation. Nevertheless, for certain class of problems, the steady state solutions can be obtained; and I showed that, for this class of systems, that is, where non-linearity is in damping as well as stiffness, but the non-linearity in damping is a function of the energy, system kinematic energy and potential energy. For this problem, we showed that, the steady state solution can be obtained exactly.

And similarly, for a class of multi degree freedom systems, I demonstrated that, such exact solution for steady state solutions can be obtained. There are broader class of dynamical systems, where exact solution to the governing Fokker Planck equations are possible, but I will not get in to that.

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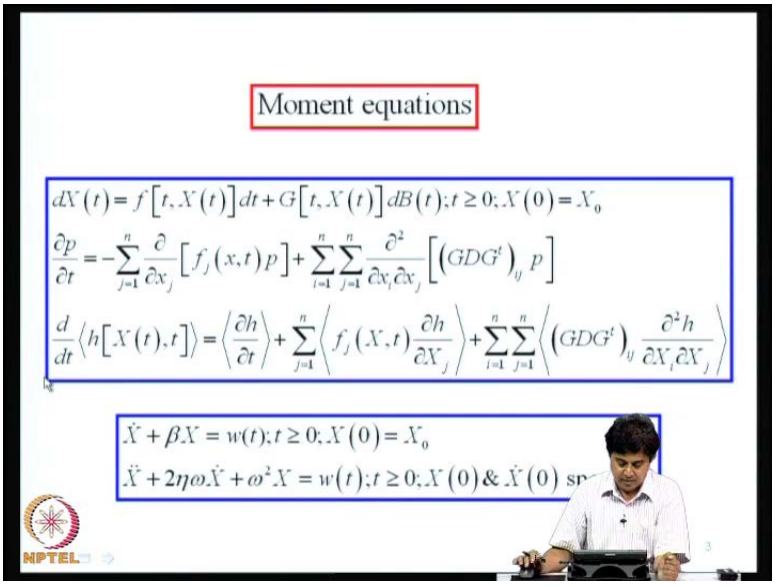

So, we will move on now with the next topic. We also considered the formulation of moment equations. So, for systems govern by these general stochastic differential equation, this is the Fokker Planck equation; and based on this, we can derive the time evolution of moments, where h is a function of X of t, and we derive this equation in terms of the left hand diffusion quotients. And I formulated these equations specifically

for a linear first order system and single degree freedom linear system driven by white noise.

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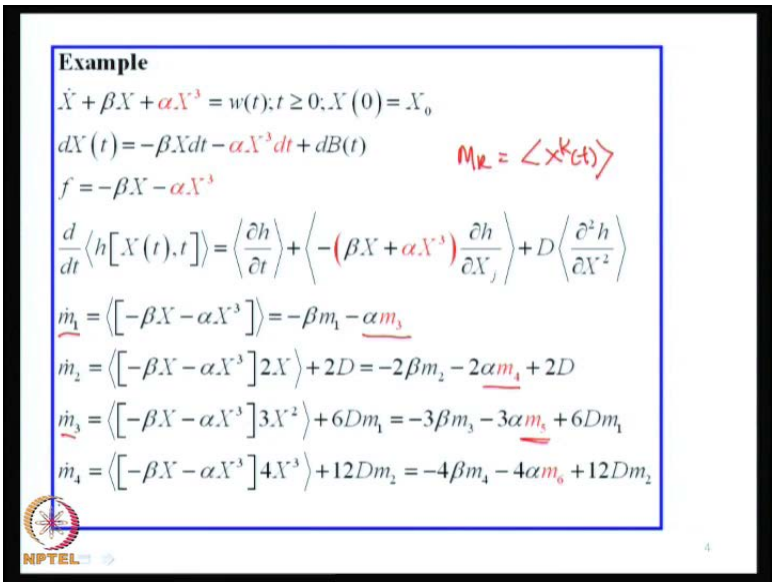

Now, let us continue, we consider now a first order non-linear system, X dot plus beta X plus alpha X cube is equal to w of t, where w of t as before as a white noise; and we write this in terms of stochastic differential equation as shown here, dX of t is minus beta X dt minus alpha X cube dt plus dB t.

So, this drift term will be minus beta X minus alpha X cube, and g d g transpose will be d. So, the moment equation has this general form, d by dt of expected value of h is given by this. And suppose if we now focus on expectations of the kind, m k is X to the power of k of t, so the equation, for m 1 dot is shown here. You carefully look at the right hand side, the equation for m 1 now has the term m 3; this m 3 is nothing but expected value of X cube.

Now, we look at the expressions for m 2, that is expected value of X square, the mean square value. m 2 dot if look at right hand side, I have m 4 which is expected value of X to the power of 4. Similarly, you look at the equation for m 3, which is expected value of X cube; it has expected value of X to the power of 5 on it is right hand side. So, this moment equations of this time here form an infinite hierarchy of equations, such that, at no stage, we get sufficient number of equations, so that we can solve the problem numerically. Suppose, you want to find m 1, you need to know m 3. The moment you

write equation for m 3, you end up reading m 5; you write equation m 5, you need equation for m 7, and so on and so forth.

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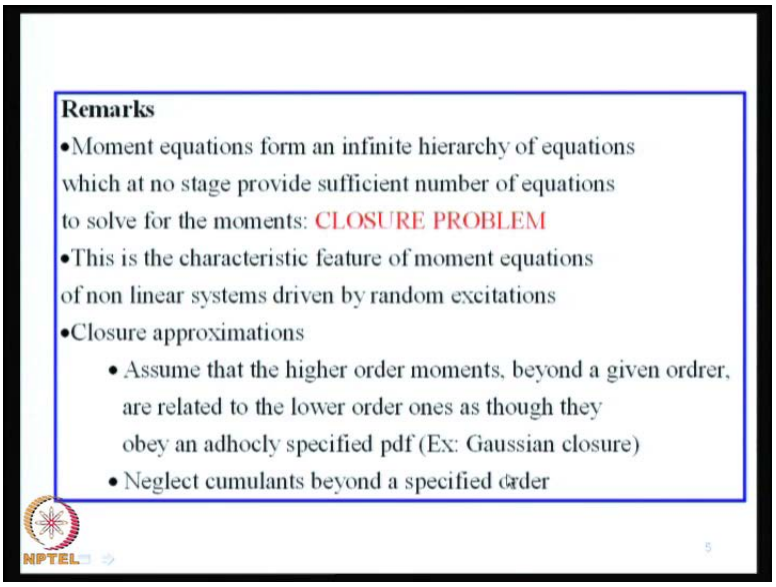

So, this problem is known as closure problem. So, moment equations form an infinite hierarchy of equations, which at no stage provides sufficient number of equations to solve for the moments; this is known as closure problem. This is the characteristic future of moment equations of non-linear systems driven by random excitation.

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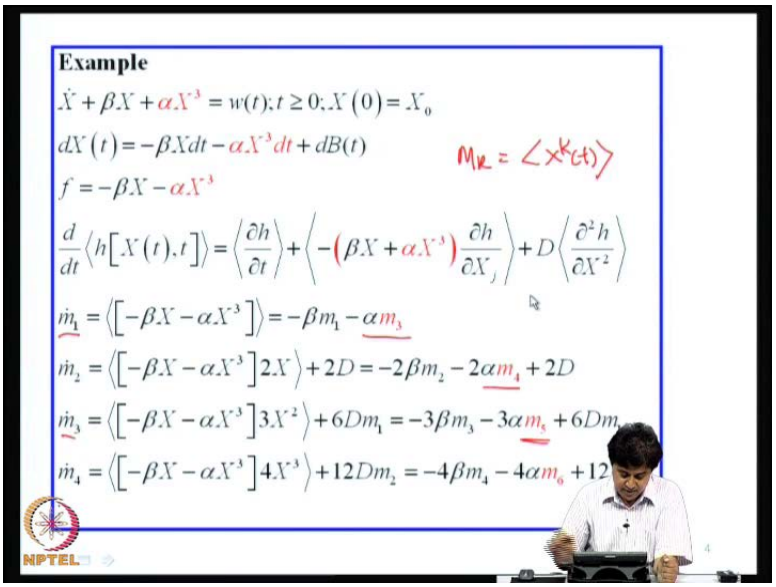

Now, we can develop approximate methods to solve this infinite hierarchy of equations, if we adaquely close this hierarchy by invoking some assumption. For example, after deriving an equation for m 1 and m 2, I can assume that, all other higher order moments m 3, m 4, m 5, etcetera are related to lower order moments, as if the X of t is a Gaussian random process. If you do that, it is known as Gaussian closure approximation. So, it is in some sense equivalent to linearizing the governing differential equation, because Gaussian responses are characteristics of linear systems under Gaussian random excitations. Otherwise of doing it could be to… you know, assume general form of probability density function of the response and we can close it an arbitrary level; we can go up to m 4 following this hierarchy equations, and equation for m 5, m 6, m 7, etcetera can be assumed to be related to the first four order moments, again following certain positive related probability density functions.

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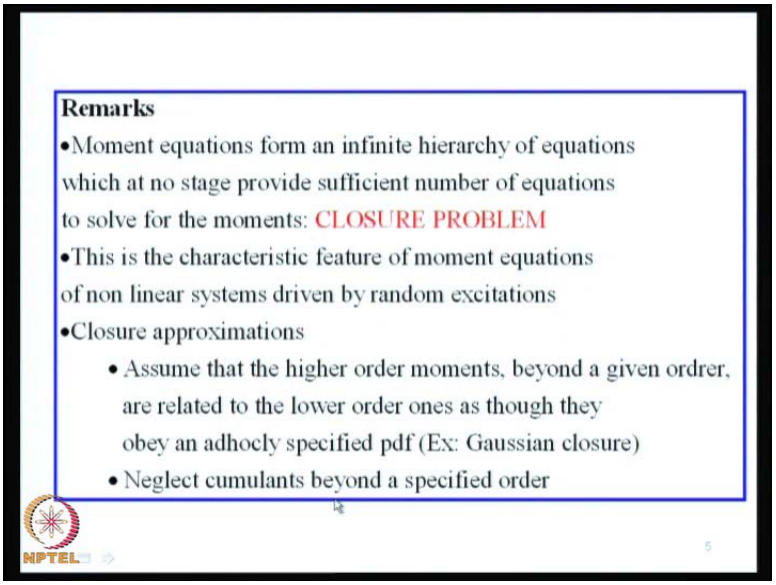

Other choice would be to neglect cumulants beyond a specific order; we can simply say that, fourth cumulants beyond third order are all 0, fourth order are all 0, that helps us to arise the hierarchy of equations and get sufficient number of equations to solve the low order moments. These are Adohc procedures, but nevertheless, there widely discussed in the existing literature.

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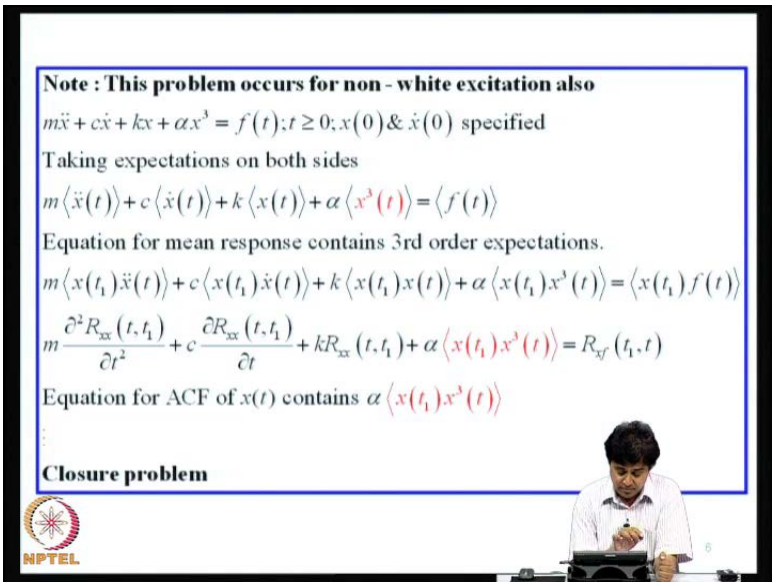

The closure problem can occur, I mean, indeed occurs for non-white excitation also. We have been talking about white noise excitation, but suppose if you consider a doffing oscillator under a random excitation, this can be a band limited noise, stationary 0 mean random excitation. Suppose I am interested in mean of the response, I can take expectations on this equation; so, I get m into expected value of x double dot plus c into expected value of x dot, so on, so forth.

Here, I get alpha into expected value of x cube is equal to expected value of f. now, when you are writing equation for expected value of x, you are getting now expected value of x cube in your equation. Now, similarly, if you want to write now an equation for auto covariance or auto correlation between x of t and x of t 1, I can multiply this equation by x of t 1 and take expectation. So, I get m into x of t 1 x double dot of t expected value plus all these; and here I get, alpha into expected value of x of t 1 x cube of t.

So, in terms of auto covariance auto correlation of x, I can write this dou square R xx dou t square c dou R xx dou t plus k R xx t comma t 1 plus the higher order moment, which at this stage I do not know; this actually you can call this a fourth order moment, which is not known. Suppose you want to now overcome this difficulty and multiply this equation by x cube of t, and try to get an equation for expected value of x of t and x cube of t 1. When you multiply this with x cube of t ,we get x cube of t 1 and x cube of t; that means, a sixth order of moment will appear in your equation. So, again this forms an

infinite hierarchy, and this is a well-known problem known as closure problem. So, moment equations are difficult to deal with, in non-linear random vibration.

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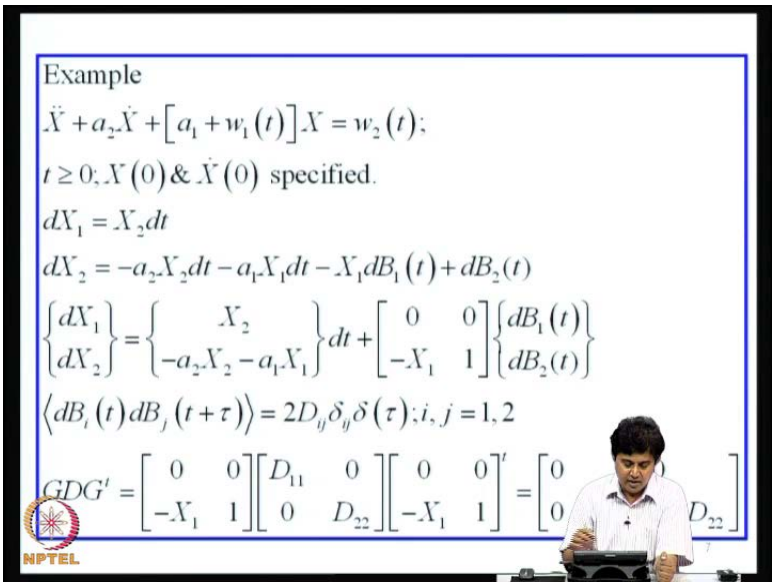

Now, what happens if there is a parametric excitation? So, here, I consider X double dot plus a 2 X dot plus a 1 into white noise into X plus another white noise w 2 of t. Now, here, we can assume initial conditions are specified and I can recast this into the stochastic differential equation form as shown here; and this matrix now, the so call g matrix has the term now minus X 1, so that means, one of the system states multiply the noise terms.

Now, we can write the moment equations for this. Now, it could be an interest to know, whether there will be a closure problem for this case. In fact, if you try to solve the governing Fokker Planck equation here, you will not able to solve this problem. There are no exact solutions to the evolution of probability density function for this class of problems. So, there is indeed a difficulty; but how about moment equations?

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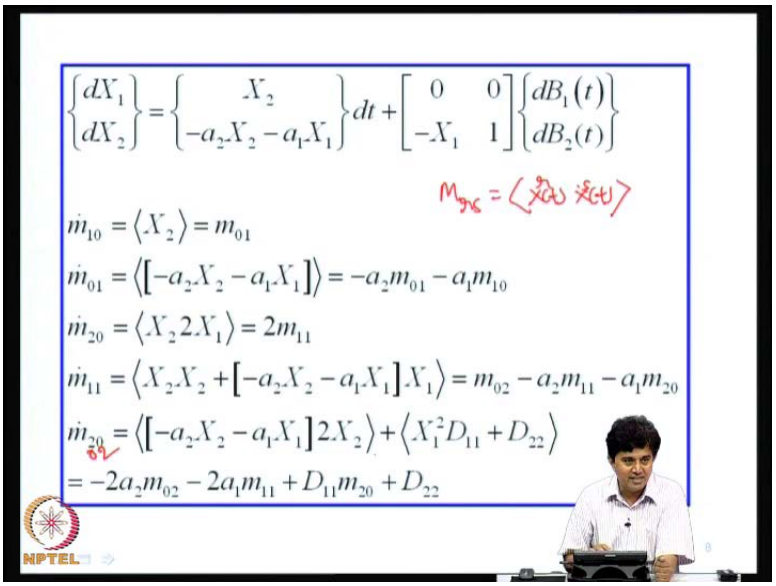

Well, if you now write an equation for m 10, so the notation that I am following is, m r s is expected value of X of t to the power of R X dot of t to the power of s. So, m 10 nothing but expected value of X of t; so, m 10, I get on the right hand side m 01.

So, similarly, if I write m 01, I get on the right hand side m 01 as well as m 10. So, if you look at now the equation at the mean level, so this is the expected value of X, this is the expected value of X dot, we get a close set of equations; this is no problem here, because in the right hand side, I have nothing other than expected value of X and expected value of X dot. So, I can solve this problem and I can get exact solutions.

Although the governing Fokker Planck equation for the evolution of probability density function, I am unable to solve, but for the moments, I get exact equations which are solved. How about higher order moments? You now look at m 20, which is expected value of X square of t, you get on the right hand side 2 m 11, which is auto correlation the correlation between X of t and X dot of t. Similarly, m 11 dot if you write, on the other right hand side I get, you know, m 02, m 20, m 11, etcetera. How about m 20? This is a m 02. I get here an equation which is again contains terms only up to second order; so, that could mean in this case, there is no closure problem.

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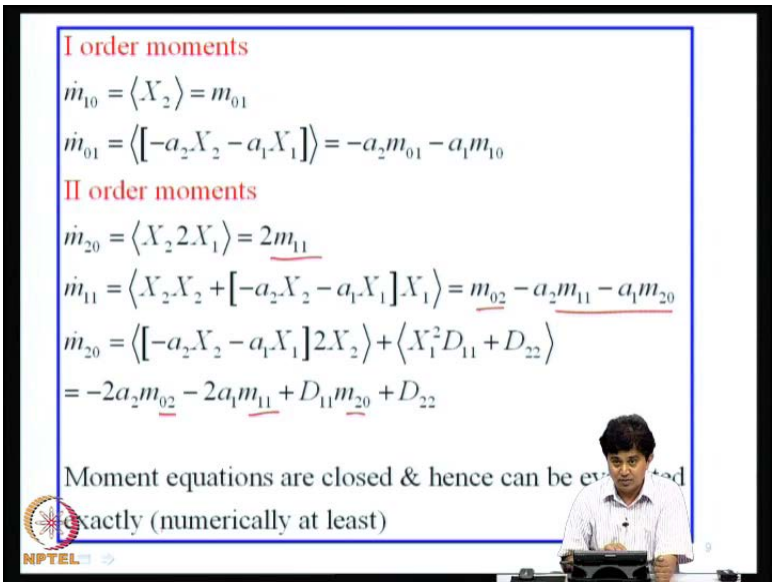

So, you look at first order moments on the right hand side, I have only terms which are first order nature. You look at second order moments on the right hand side, I have terms which are basically the second order. See here, m 02 m 11 m 20; so, that means moment equations are closed and hence can be evaluated exactly at least numerically. So, in this particular case, we get the approach that we are discussing forms source of an exact solution.

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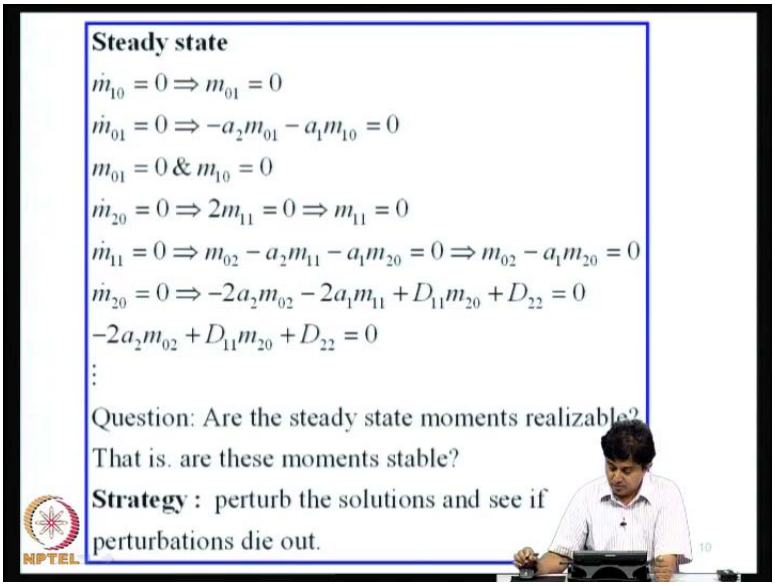

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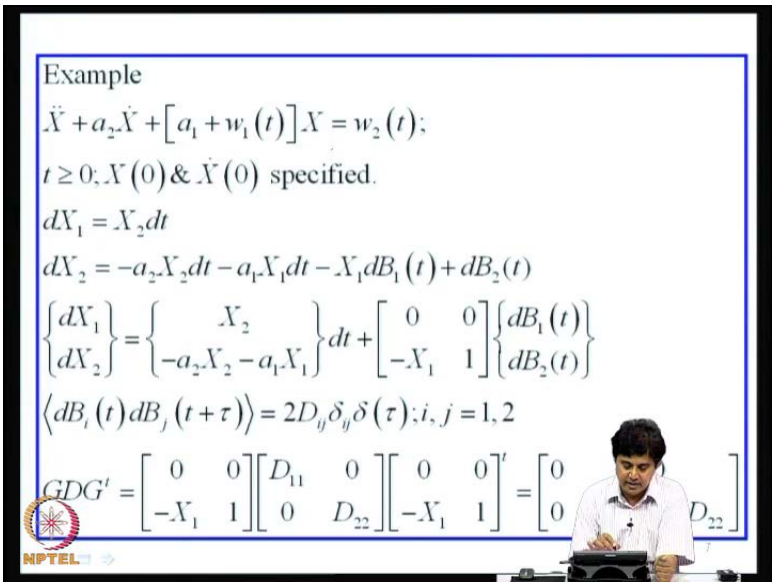

Now, how about steady state here? So, for system driven by parametric excitations, as we know there is no guarantee that system is behavior could be stable, so assume vertically time becomes large; we do not know there is steady state is reached or not.

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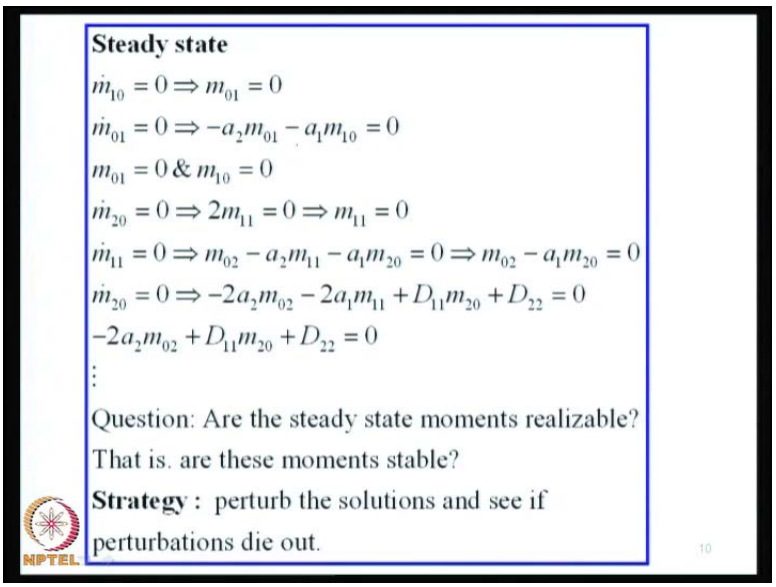

But nevertheless, if we assume that a steady state indeed exist, we can get the solutions for m 01, m 10, etcetera. So, m 01, m 10 are all 0; m 20, I get certain expressions. Now, the question are arises, whether these steady state solutions are realizable? They may not be realize, they may be on stable. So, what we do? We give a small perturbation to this

moments and see whether the perturbations decay in time or not; that means, I study now the stability of moment equations, which has set out the deterministic equations.

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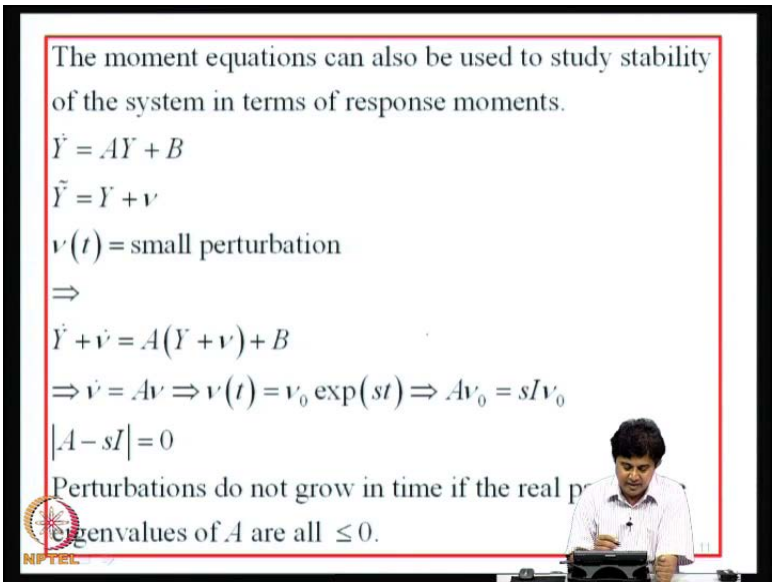

So, that can easily be done; for example, if you have a set of equations, which is Y dot is equal to AY plus B, and you give a perturbation Y plus nu, where nu is a small perturbation, I can get an equation for time evolution of nu, which is nu dot is equal to A nu. So, if you now seek for the this set of equations and solution of this form, nu naught exponential st, again Eigenvalue problem governing nu naught and s, and we can show that the solutions are stable, provided the real parts of the Eigenvalues of a matrix are non-negative. Now, for stability, there non positive; if it is 0, the solution do not grow; but if it is negative, the solution decays.

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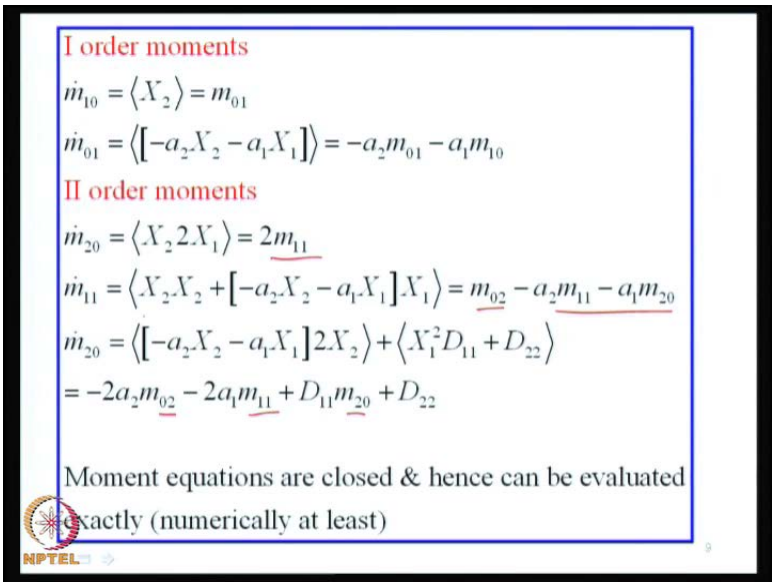

So, in any case, we can form this kind of equation Y dot equal to AY plus B from the governing moment equations. And we are able to study now the stability of response moments; the stability of response moments need not guarantee the stability of sample process of the system. It only means, mean, standard deviation, variance, etcetera, co-variance whatever we are discussing are stable; it need not mean the samples of X of t need to be stable; so, that has to be core in mind. If you want to study stability of sample, there is an altogether different you know approach that we are not discussing in this course.

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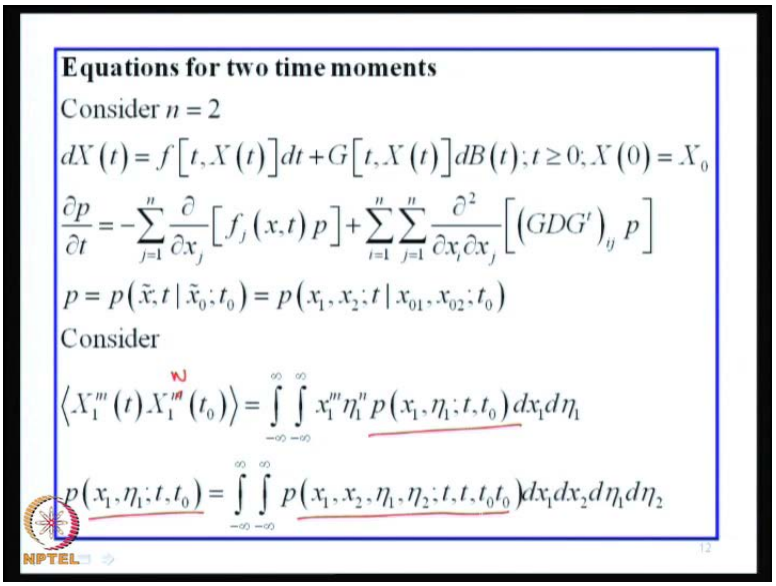

I have discussed till now equations for time evolution of moments, where I am considering only one time instant. Now, how about second order moments or equations for two time moments? So, for example, let us consider n equal to 2, it can be 3, 4, etcetera, but for the purpose of discussion will consider n equal to 2.

Now, this is an equation dX of t f of t X of t plus G dB t etcetera. Now, this is a governing Fokker Planck equation and this is a transition probability density function. Now, there are two time instance, t and t naught in the problem already; suppose, if I now consider expected value of X 1 to the power of m of t and X 1 to the power of n at t naught, so this can be written as X 1 to the power n n 1 to the power of n X 1 n 1 t comma t naught dx 1 d eta 1; this is a definition of expected value. For this join density function, I will now write it as a marginal density of four-dimensional join density function; that is, this density function I am writing it as a marginal of a fourth order density function.

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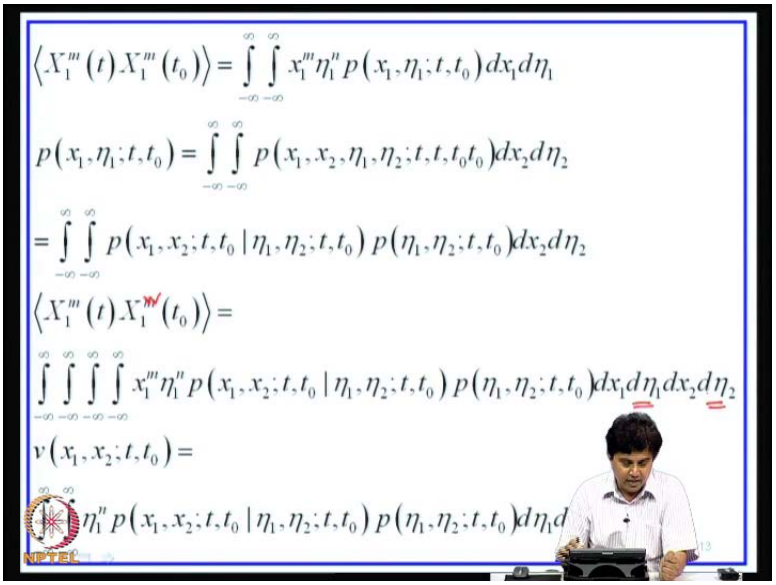

Now, we can therefore look at this four-dimensional probability density function and use Markov property, and write that in terms of the transition probability density function and a second order density; actually, this is first order; eta 1 and eta 2 are elements of vector random process which is Markov. So, X 1 to the power of m X 1 to the power of n t naught is given by this now. And if we now call the part of this integrant as a nu of X 1 comma X 2 t comma t naught, eta 1 to the power n and the only this term is retained; that

means, I retain look at only terms involving integration with respect to eta 1 and eta 2 and that integrant I will call it as v or nu.

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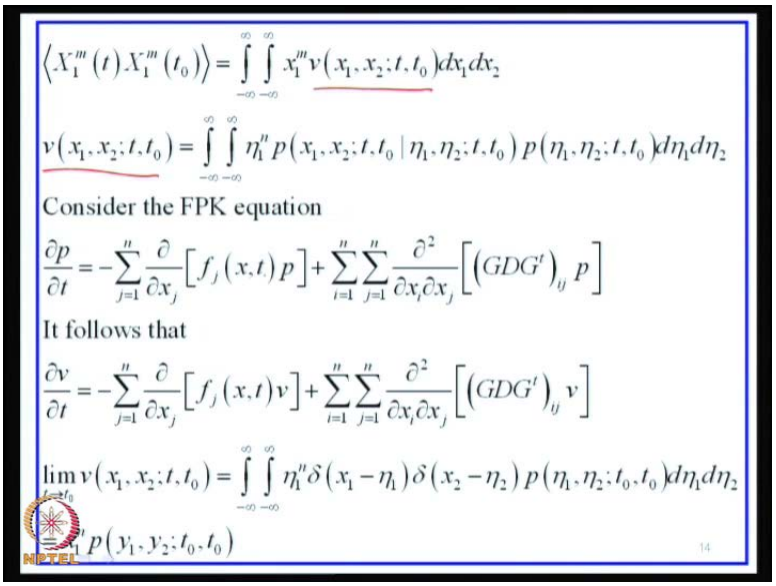

Now, the requisite moment can be written as, x 1 to the power of m nu dx 1 dx 2, now, where nu is given by this. Now, in the governing FPK equation, we can see that, the FPK equation is also satisfied by this function nu, because this involves x 1 x 2 t and t naught; and if I substitute, if I carry out this integration with respective eta 1 and eta 2, this equation is not affected, because eta 1 and eta 2 are not the independent variables here. So, therefore, it is immediately follows that, the function nu is also satisfied by the same operator.

Now, how about initial conditions and bounded conditions? So, this now I have a Fokker Planck like, FPK like equation, where dependent variable is this function nu. I can establish the initial condition by considering the definition of nu at t equal to, as t goes to t naught, and I get this as, x 1 to the power of n t y 1 y 2 t naught comma t naught; so, this is a fairly smooth function. So, it is good way to specify the initial condition. Then, I can actually solve this problem in principle; so, that would mean, using FPK equation, not only I can formulate a strategy to solve one time moments but also two time moments; and by the same token, higher order moments can also be, equations for higher order moments also can be set up.

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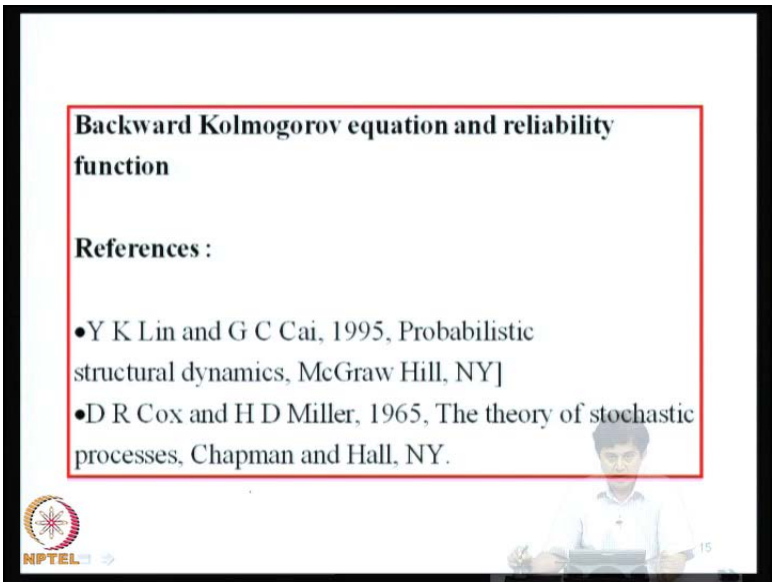

I will not get into the details of this solutions etcetera, but we now move onto as related topic. What is known as backward Kolmogorov equation and reliability function? So far we have been looking at Fokker Planck equation, where the independent variable is t and X. And we assume the systems start from t equal to t naught, and t is greater than t naught; so, the independent variable is greater than t naught, and in that sense, it is a forward equation. Now, if we treat t naught itself as an independent variable along with associated X naught, then we get another equation known as backward Kolmogorov equation. In terms of that, backward Kolmogorov equation is useful for studying first passage times.

So far what we discussed was, the probability density function of the response and its moments. But this description does not helps us to solve the more important problem of first passage time, and extreme, and so on, so fourth, but the backward Kolmogorov equation assist us to you know tackle that problem. So, the two references: this one is a book by professors Lin and Cai, and other one is the book by professors Cox and Miller, which has certain useful background to this discussion.

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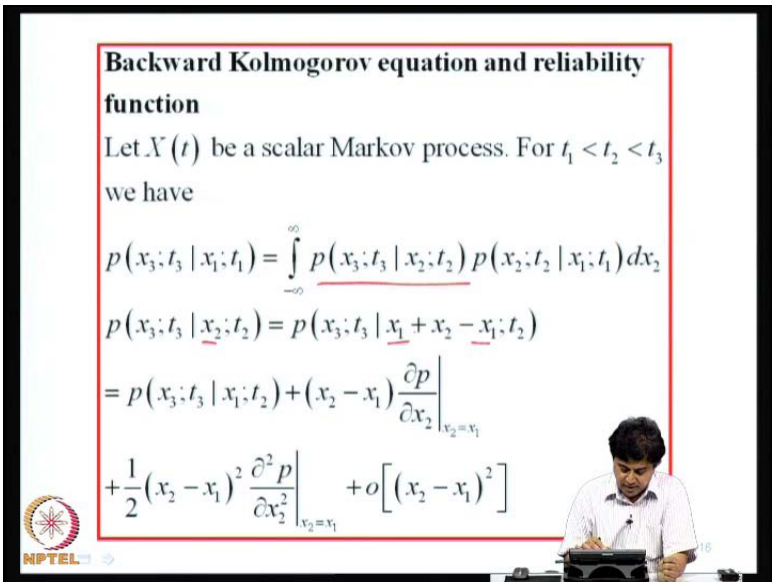

Now, I will start the discussion on a scalar random process. Let X of t be a scalar Marcov process, and I take three time instance, t 1, t 2, t 3, and I have this equation which is the Jeff man Kolmogorov equation, which that transition probability density function must satisfied. So, what I will do now? I will consider this transition from t 2 to t 3, and I will rewrite the argument x 2 as x 1 plus x 2 minus x 1; that is, I adding and subtracting x 1, and I will now perform a Taylor's expansion of this function around x 1. So, if I do Taylor's expansion around this and retain terms up to second order, I get this equation; that means, this is a function of x 2, x 3, t 2, t 3, I am performing Taylor's expansion only around x 2; x 2 written as x 1 plus x 2 minus x 1. So, I am expanding this around x 1.

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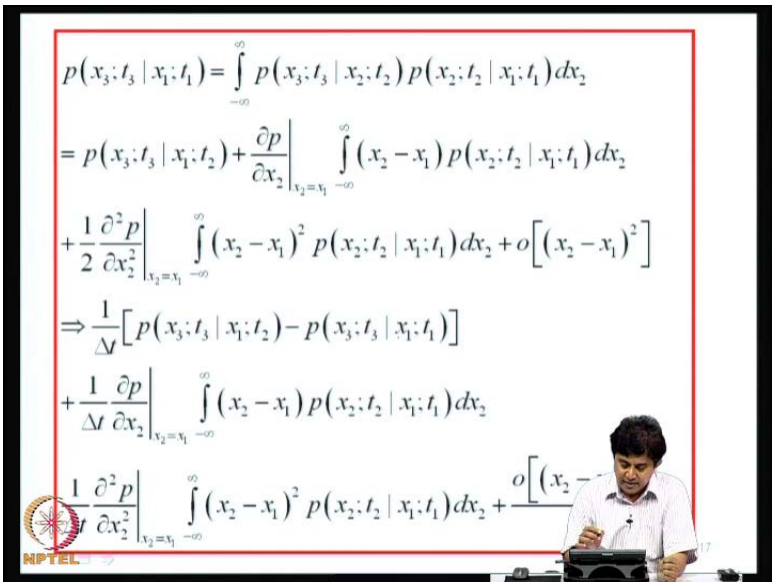

After doing that, I will substitute that back into the Chapman-Kolmogorov equation and carry out the integration by parts. And if you do that, this is straightforward exercise, we can recast the equation as 1 by delta t; I can transpose some of the terms to the left hand side and divide both sides by delta t, and I can write this as the kind of 1 by delta t into transition from, I mean, x 3 t 3 x 1 t 2, x 3 t 3 x 1 t 1 plus these terms, which have displayed here.

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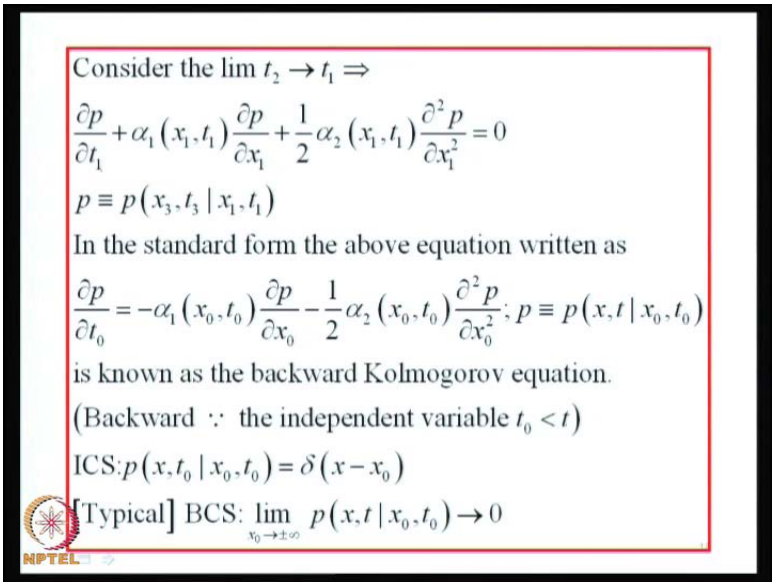

Now, if you allow delta t equal to 0, we get now the time derivative with respect t 1; that is dou p by dou t 1 plus some, this is the incremental moments alpha x 1 t 1 dou p by dou x 1 plus half alpha 2 x 1 t 1 dou square p by dou x 1 square, where p is now the transition from t 1 to t 3. So, this is known as backward Kolmogorov equation. In the standard form, the above equation is written as dou p by dou t naught as, minus alpha 1 x naught t naught dou p by dou x naught minus half alpha 2 dou square p by dou x naught square, where p is probability density function x comma t x naught t naught. This is called backward, as I said already, because the independent variable t naught is less than t. Initial conditions at, as t naught t goes to t naught, it is a direct delta function, and typical boundary conditions, as x naught goes to plus minus infinity, this goes to 0.

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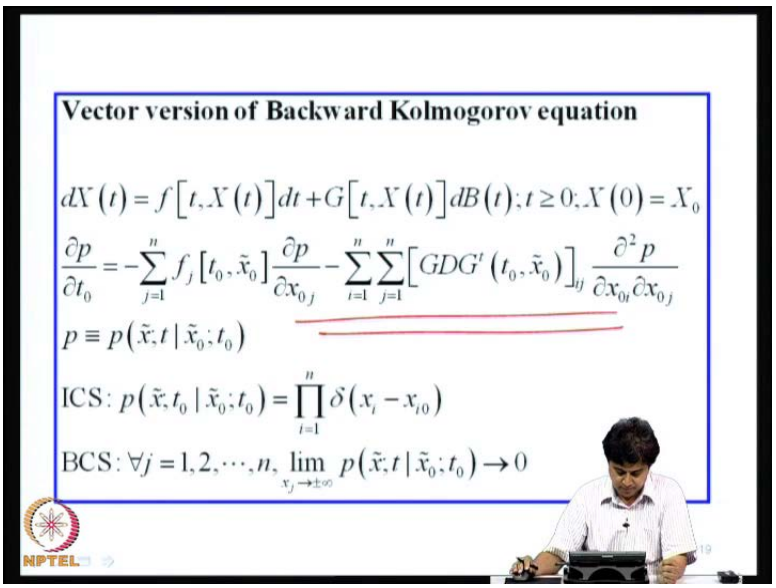

Now, a vector version of this equation can also be set of… this is straightforward and we logic is same for this derivation and this is displayed here. So, you should notice that, now the independent variables are t naught, x naught and t naught are the independent variables.

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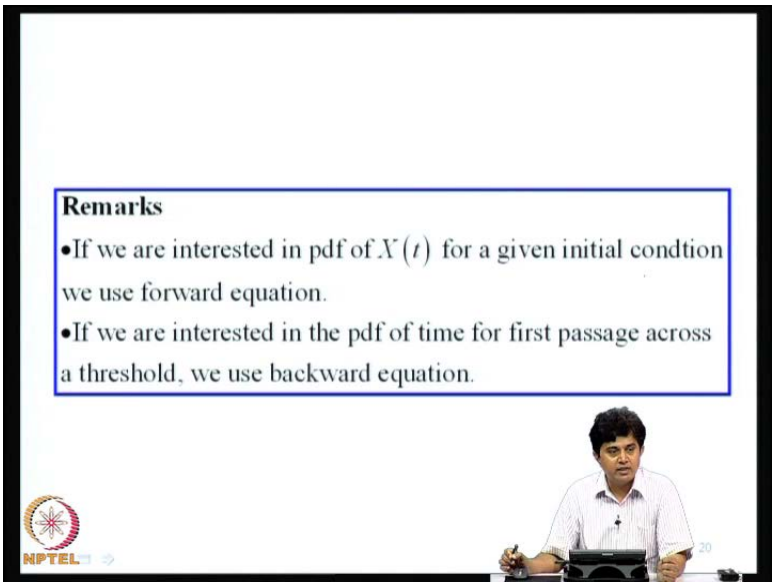

Now, when to use forward equation and when to use backward equation? The same transition density function satisfied both the equations. So, when do we solve forward equation and when do we solve backward equation? If we are interested in probability density function of X of t for a given initial condition, we use forward equation; that means, start t equal to t naught to the given initial conditions, and you want to know what is known as probability density function after some time t, you use forward equation. But if you are interested in knowing if the trajectories originate, if you are interested in PDF of time purpose for passage across a threshold, we consider the backward equation; I will explain this further.

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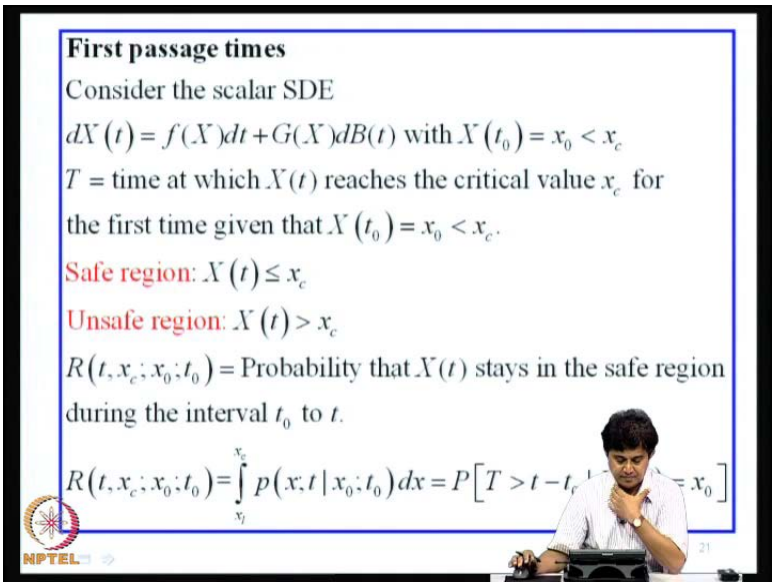

So, what that means? So, we talk about now first passage times. Now, let us consider the scalar SDE; now, I am considering a scalar function, dX t f of X dt G of X dB t, which is specified initial condition. And I defined a barrier x c, which a critical barrier, and if X of t reaches that barrier, we says the system has failed; and what I am interested in is, if system start selecting from safe region, what is the time required for the first crossing of the critical barrier? That can be used as a descriptor of life time of the this structure.

So, t I defined as time at which X of t reaches the critical value of x c for the first time, given that X of t naught is a X naught, and X naught itself less than a critical barrier; that means, t equal to t naught, where starting from the safe region. So, safe region is X of t less than or equal to x c and unsafe region X of t greater than x c.

So, I define R which is function of X naught, t naught and t, and x c and X naught; that mean, trajectory originated t naught from X naught and reach a level x c at time t. So, this is probability that X of t state in the safe region, during the interval t naught to t. So this is nothing but integral of x l to x c probability density of X of t condition X naught t naught dX, which should be interpreted as probability of t greater than t minus t naught; that means, first passage time is greater than t, if t naught is 0 condition on the fact that X of t naught is X naught.

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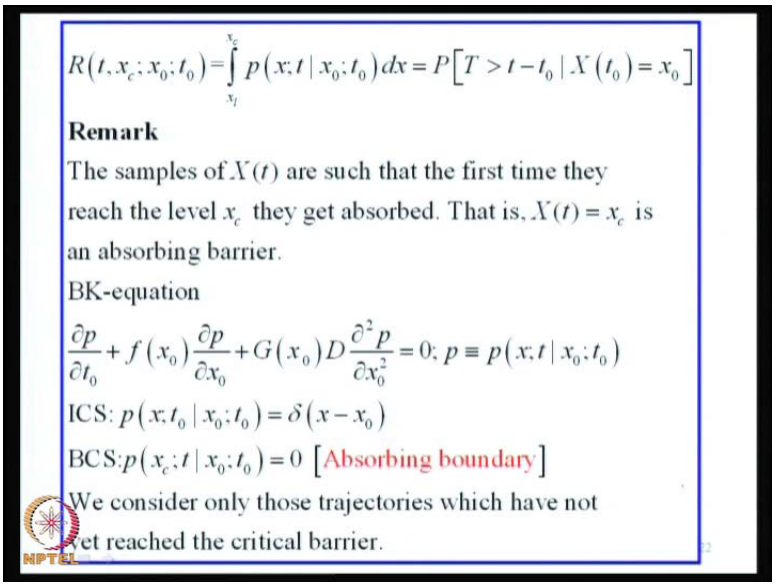

Now, the samples of X of t are such that, the first time they reach the level x c, they get absorbed; that is, X of t equal to x c is an absorbing barrier.

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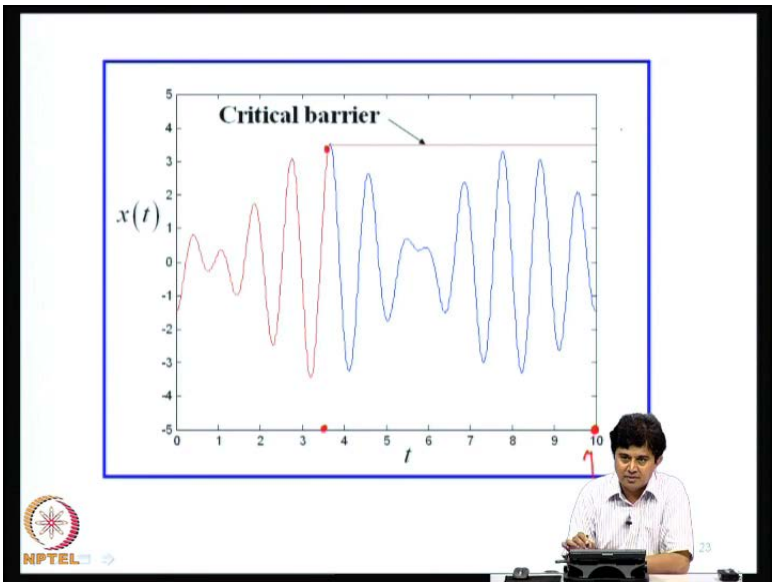

So, what that means is, suppose we are looking at sample of x of t, and imagine that this line as shown here is a sample of x of t, suppose at say x of t equal to is a 3.2 or whatever, I place a barrier. So, you see that, this time instant here, the trajectory reaches this level and it gets absorb. suppose I am interested in knowing what happens at this

time instant, what is the probability that first passage time is greater than 10 seconds, if I am interested?

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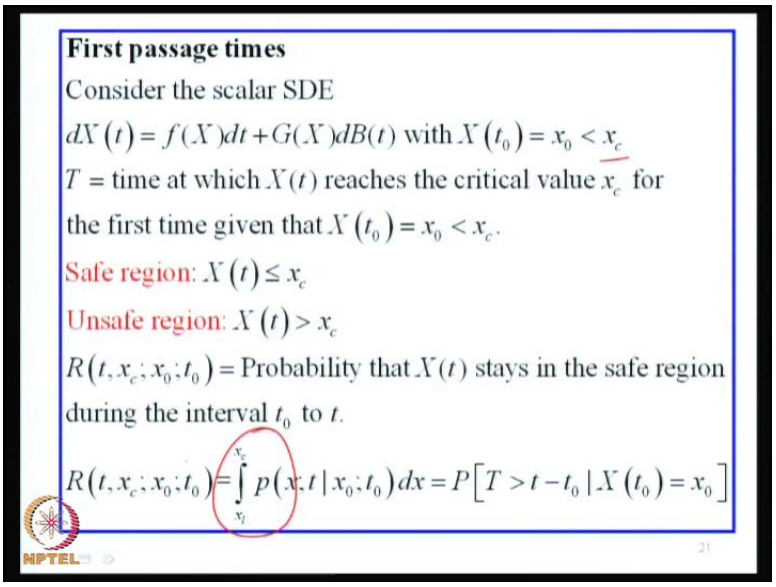

So, I have to look at only those trajectories, which have not yet been absorbed; and this integration that I am showing here is across only those trajectories, which have not yet been absorbed.

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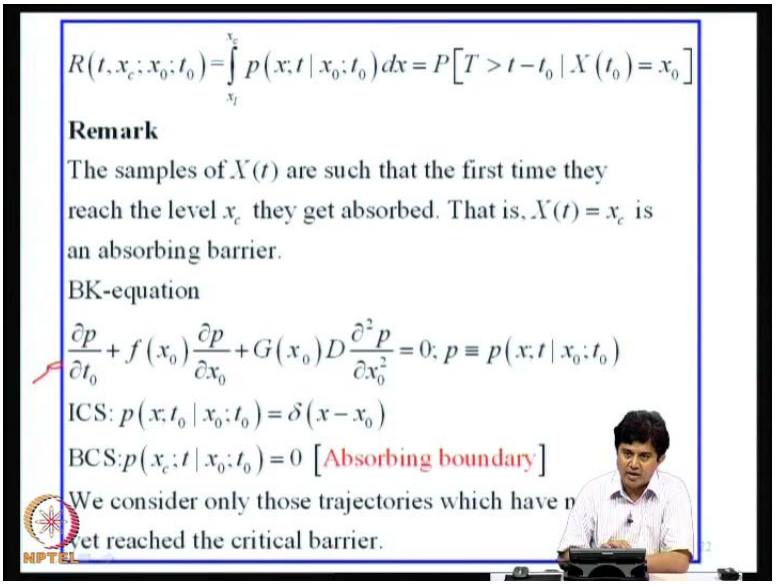

So, we have to now place an absorbing barrier for this equation; so, this is the backward Kolmogorov equation. And initial condition is that, t equal to t naught is a direct delta

function, and let the critical barrier we say that, probability density function is 0; that means, we consider only those trajectories, which have not yet reach the critical barrier.

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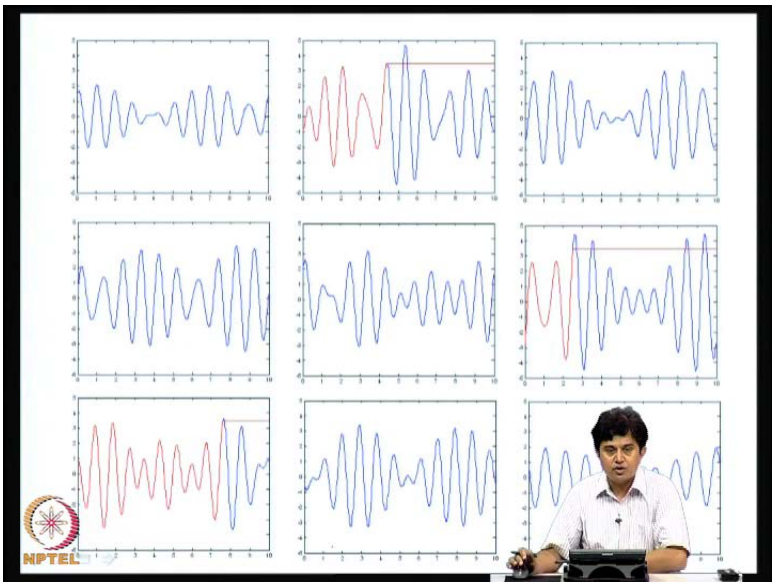

So, I have shown here a few trajectories of a same random process. This is the trajectory, at equal to 10 second, it has not yet been absorbed. So, this stage in our reckoning, but whereas this goes out, because I am interested in what happens in 10 seconds, at the end of the time period. So, this trajectory is already been absorbed, this survives, this survives, this survive, this is absorbed, this is absorbed, this survives, this survives. So, among those trajectory which has survived, I will do the integration and find out what is a probability the first passage time is greater than 10 seconds.

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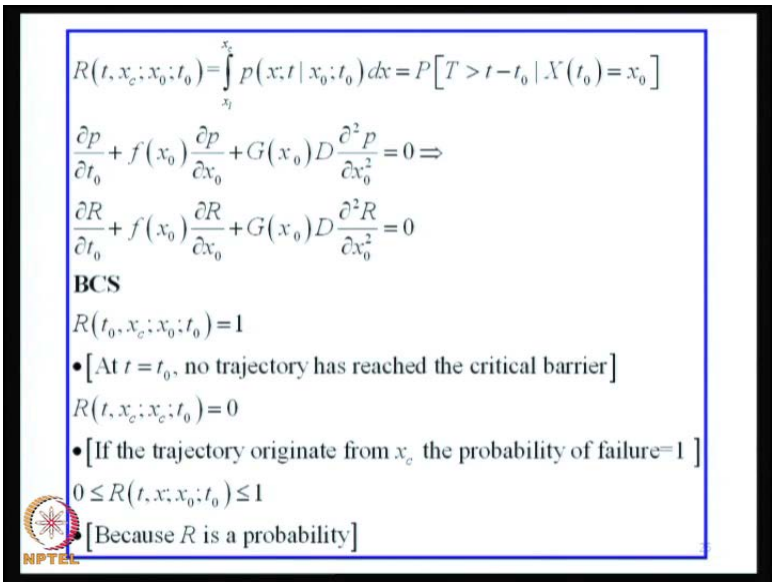

So, how do we write that? Now, I have now got the backward Kolmogorov equation and I have got this expression for R, which is actually the probability that, first passage time is greater than t minus t naught condition on the factor X of t naught is x naught.

Now, you can see that, the backward Kolmogorov equation is also satisfied by this function R; that means, you perform this integration x l to x c on dx, which is not an dependent variable here; the form of the differential equation is will remain the same, because what you will do? Multiply by dx and integrate over x l to x c, and that first term will be dou r by dou t naught, second term will be f of x naught dou R by dou x naught, and so on so forth. That means, the backward Kolmogorov operator is applicable even for R, which is the reliability function; R can be thought of us safety function or reliability function or survival function, because it means that, we are looking at a first passage time being greater than a specified time t.

So, what are the boundary conditions on it? R of t naught x c x naught t naught is 1; that means, at t equal to t naught, no trajectory has reached the critical barrier, that translates as one of the Markov. Then, R of t comma x c x c comma t naught is 0, that means, if the trajectory originate from x c, the probability of failure is 1; therefore, survival is 0, and R be a probability is bounded between 0 and 1.

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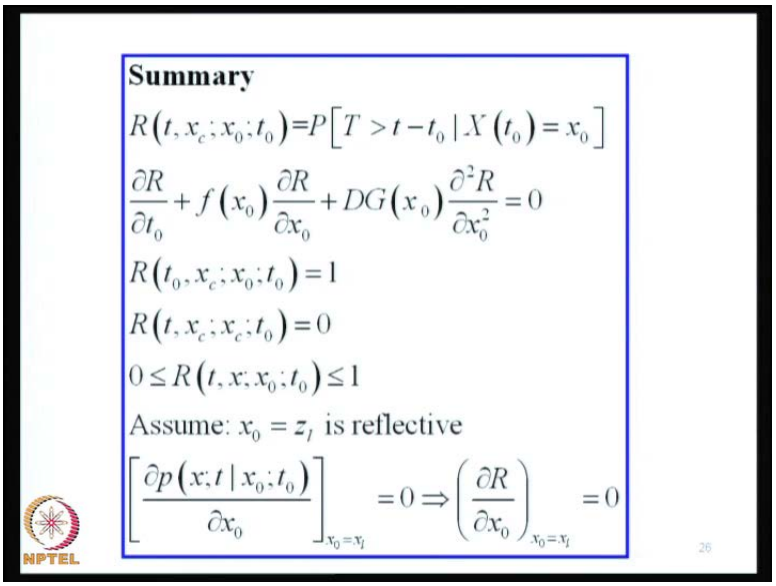

So, this actually constitutes an Eigenvalue problem; so, we to summarize that, this is a function that we are looking at, R is probability of t greater than t minus t naught condition X of t naught equal to x naught. This is the governing partial differential equation, these are the condition that has to be satisfied; and at the lower end, we are placing a non-critical barrier and we call it as a reflective barrier. Here, if trajectory reach x lower barrier, it is not absorbed, but it can return to the safe region subsequently.

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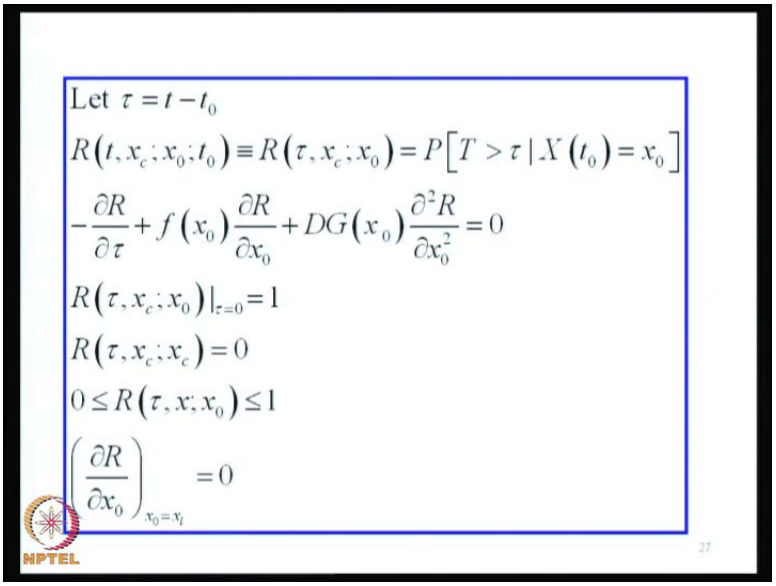

Now, if we put tau as t minus t naught, I can make some simplification; this R can be written as tau of x c semicolon X naught; that means, probability of t greater than tau condition X of t naught is equal to x naught is what I am looking for. And I get the governing equation in terms of R, in terms of independent variables, which are tau and x naught. So, again the boundary condition following the similar logic can be written like this; this is the boundary condition on the absorbing barrier. And if we can solve this problem, this set of partial differential equation, I mean, this partial differential equation along this set of conditions, we have now as forced to derive in a probability of first passage times.

Now, the solution for the entire function, that is the probability distribution of first passage time would require solution of a partial differential equations. Suppose, if you are interested in moments of first passage times, situation be simpler.

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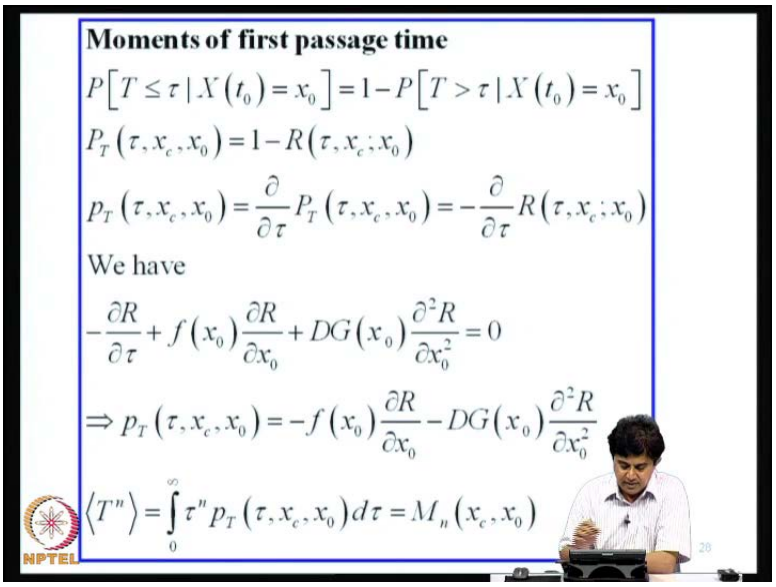

So, if you look at that, now let us consider the probability distribution function of t, which is probability of t less than equal to tau condition on X t naught equal to x naught, which is 1 minus R, which is our survival function.

So, we see that, probability distribution function of first passage time is 1 minus a survival function. Now, the probability density function is derivative of this with respective tau, because tau is a state variable. So, that will be dou by dou tau this, and this is minus dou by dou tau R; that would mean that, we have this equation which

governs R and first term here is nothing but probability density function. So, I can tau as an independent variable now, I mean tau would no longer be an independent variable of this differential equation; this becomes an ordinary differential equation now. So, p T can be written like this. From this, if I now find out the nth order moment - expected value of T n - is will be 0 to infinity, tau n this density function d tau, and I call it as M n, with barrier x c and x naught.

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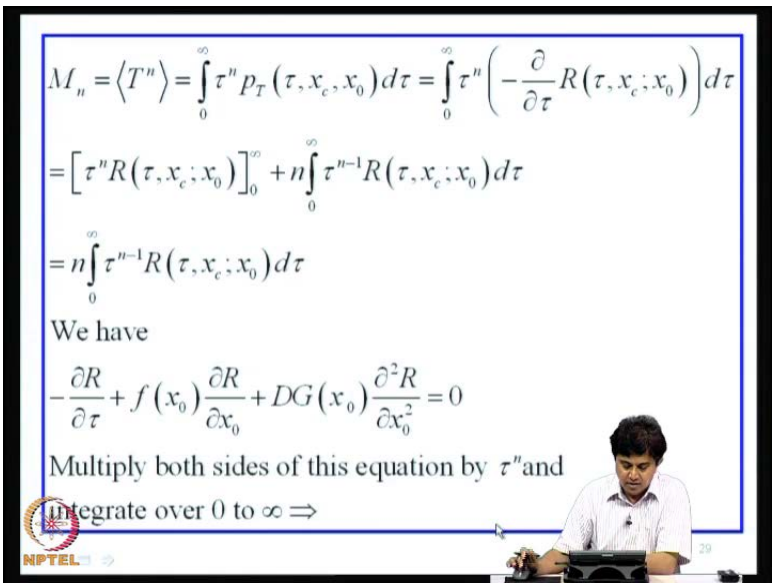

So, if you look at M n, this is the definition as I said just now, and we can now use the governing equation and integrate this by parts, and we can see that, we reach this equation M n is n into tau n minus 1 R d tau. Now, we can simplify by this by reverting back to the governing equation for R, what will do is, in this equation, we multiply both sides by tau to the power n and integrate on 0 to infinity.

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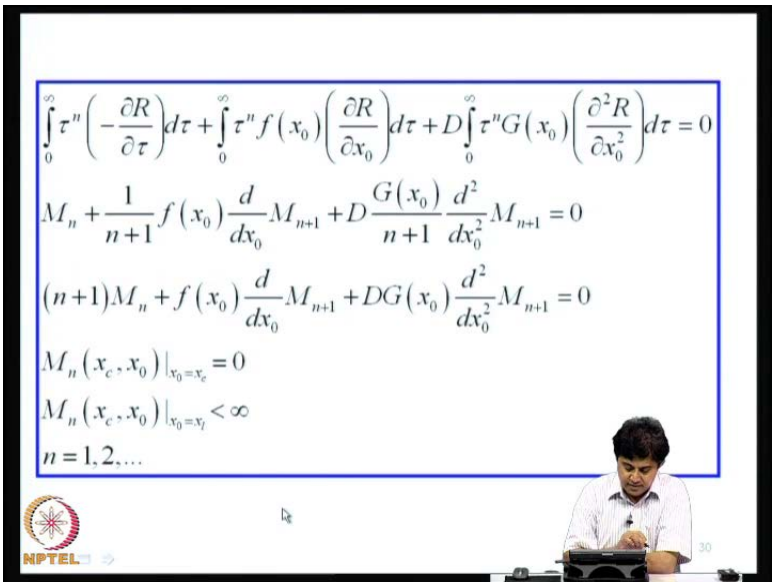

If we do that and rearrange the terms slightly, I get a set of recursive relations for first passage times. So, the equation that I get is recursive relation for moments of first passage times. So, n plus 1 into M n plus f of x naught d by dx naught M n plus 1 plus DG of x naught d square M n plus 1 by dx naught square is equal to 0 and along with this stipulated boundary condition. So, we can solve this now sequentially.

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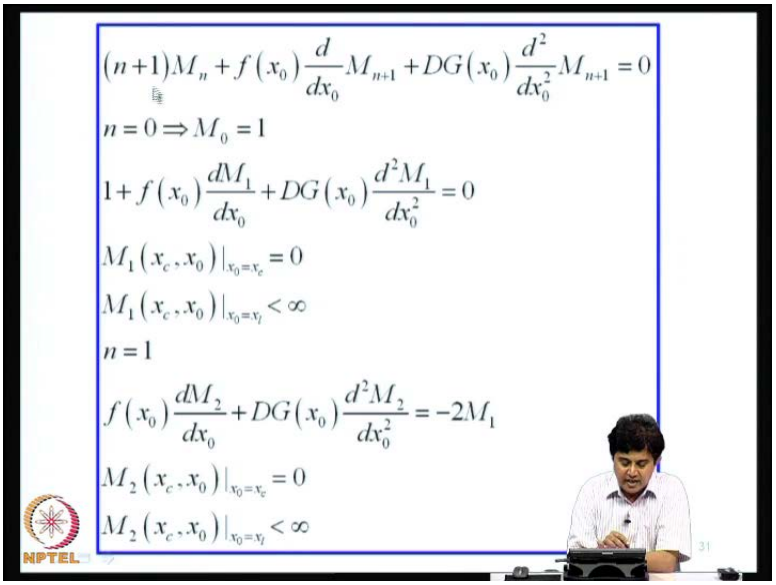

So, if you put n equal to 0, we know that M naught is 1, because that is a normalization condition on a probability density function; area under probability density function is 1,

therefore, M naught is 1. So, I get this simple equation and this can be solved; and if n equal to 1, what happens is, on the right hand side, I get M 1, which have already solved.

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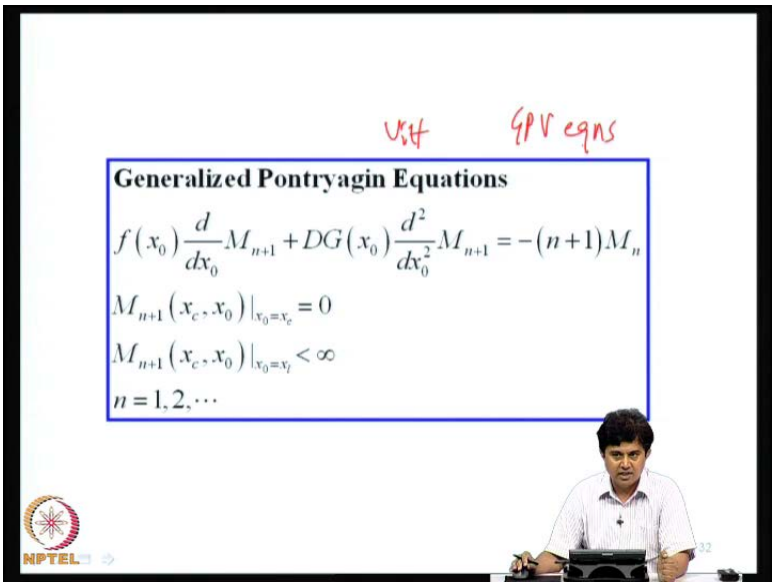

So, I get a sequential set of equations, which I can tackle one by one and this set of equations are known as generalized Pontryagin equations. So, there are called GPV equation. How about This is for first one-dimensional random process - scalar random process. How about multi-dimensional case?

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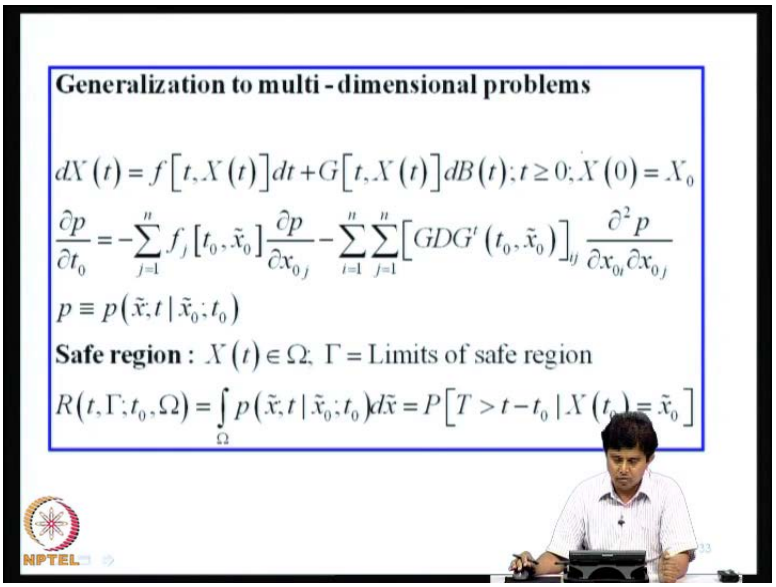

The same logic could work; the algebra is likely tedious. So, we look at now the vector version of the stochastic differential equation and associated backward equations; and we now define a safe region, it is no longer a barrier, now it is a region in safe face. So, X of t belongs to capital omega; we call it as a gamma is a limits of safe region.

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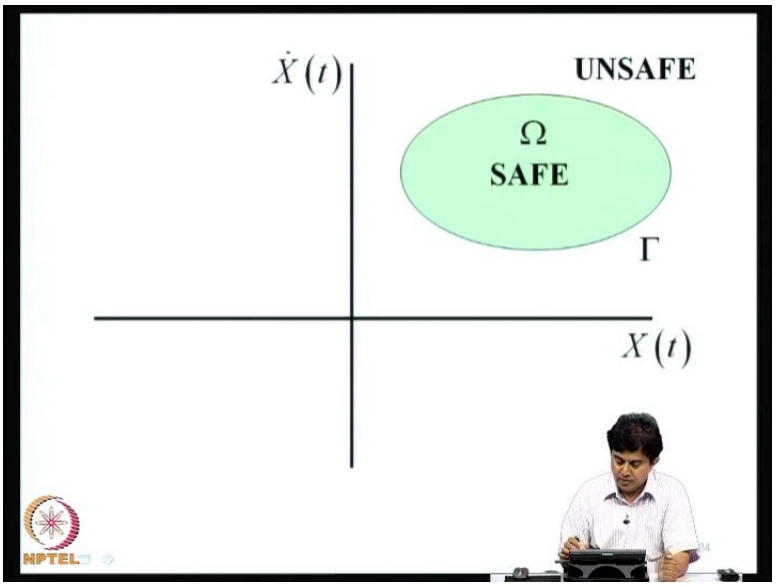

That means, if you in the phase plane, the two-dimensional problems, this can be safe and this boundary is gamma, and this is the answer. But this is the one proposed one possibility.

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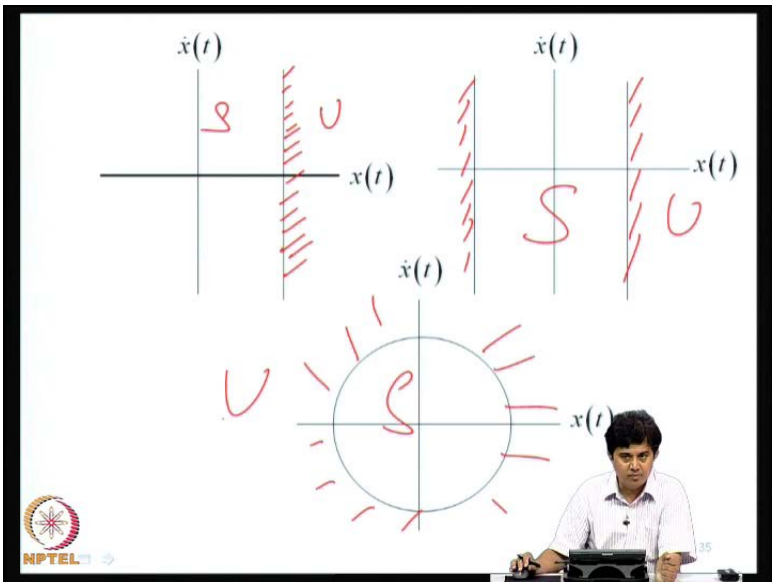

But there could be other situation; for example, we can have a single barrier problem, this is unsafe, this is safe; or two barrier problems, this is safe, this is unsafe; and kind of region, so this is safe, this is unsafe. So, the boundary conditions for the associated governing partial differential equation have to be formulated, based on the physics of the failure criteria, that you are interested in studying; so, this can be fairly involved.

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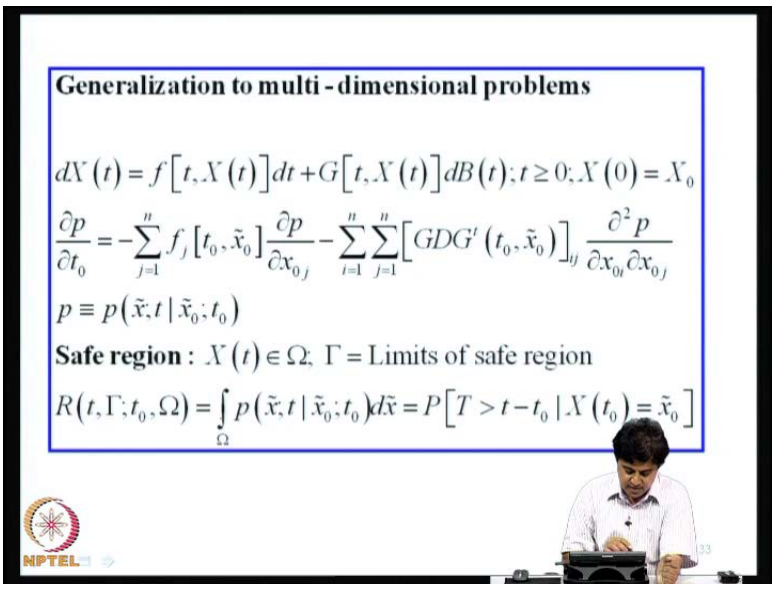

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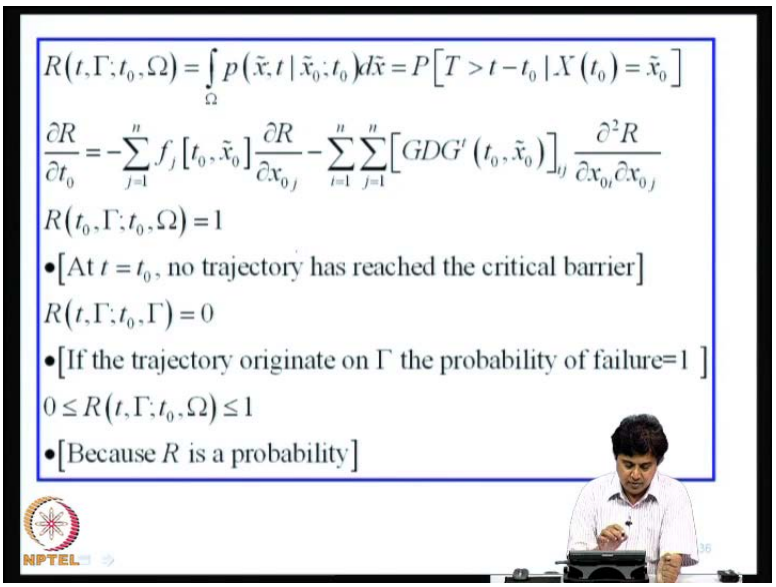

So, in any case, we define R as the probability of x tilde colon t condition on x naught t naught over the safe region. So, again I place a boundary on gamma absorbing boundary

on gamma and this equation has to be solved in conjunction with absorbing boundary. So, if you look at that, again we can consider the governing backward Kolmogorov equation, and verify that the reliability function is again satisfied by the same operator and we have the governing partial differential equation for the survival of the system - probability of survival. And following the same argument that used for scalar random process, we can continue to use similar arguments and derive the associated boundary condition. In principle, therefore, we can formulate the required partial differential equation, which governs the first passage time. But now the first passage time in a region, it does not requires barriers in a as in a scalar problem.

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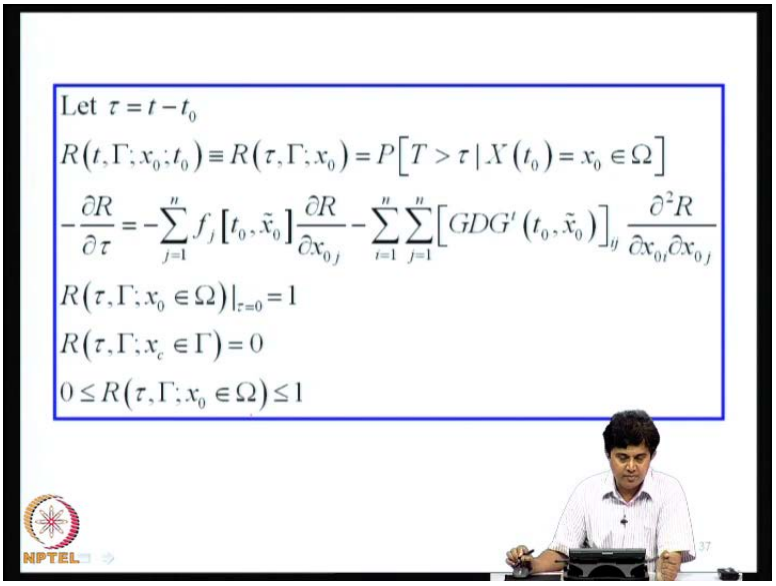

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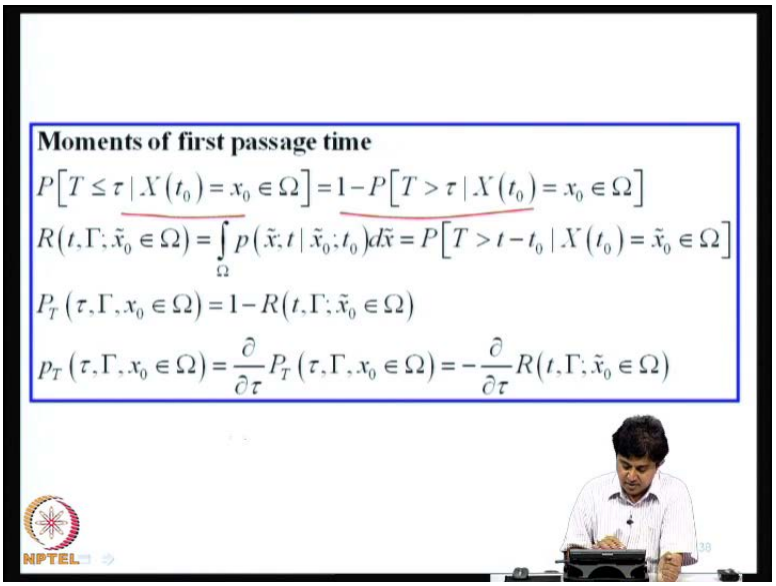

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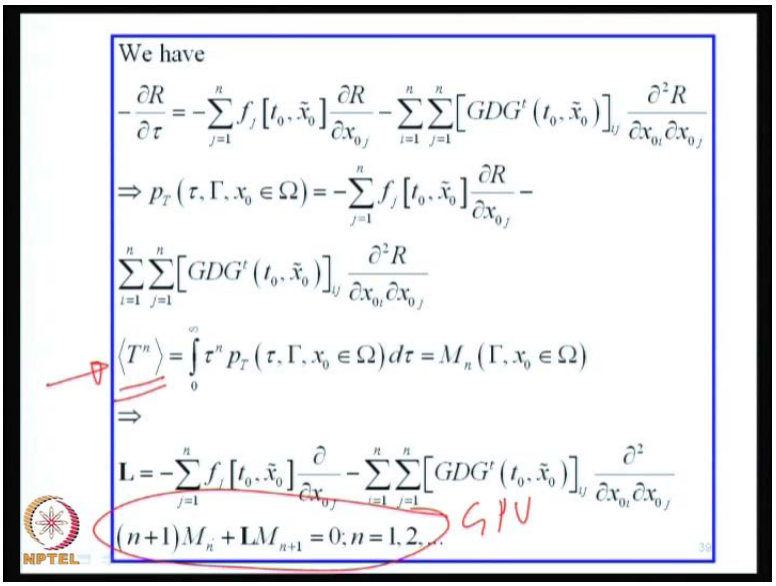

So, some of these can be simplified if you introduce tau is equal to t minus t naught and simplified version of this can be obtained; and from this, we can look at moments of first passage time; the logic is exactly the similar. We start with PDF of first passage time - probability distribution function of first passage time - and express in terms of express theta 1 minus R, and go back to the equation of R and derive the equation for the moments of first passage time as shown here. And we get this set of equation, that is, n plus l M n plus LM plus 1 is 0, for n equal to 1, 2, etcetera. So, these are the GPV equations for the system.

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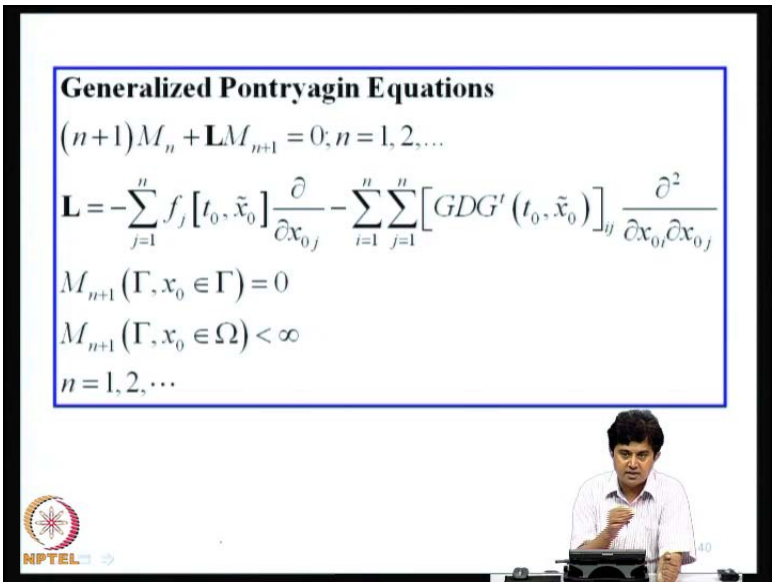

So, these are again recursive relation; you solve for n equal to 1, then you proceed to n equal to 2, and so on, so forth. In the existing literature, people have used various method of weighted residuals, finite element formulations, etcetera, to tackle these set of equations, and there are certain numerical approaches that exist, which can be used to solve these problems.

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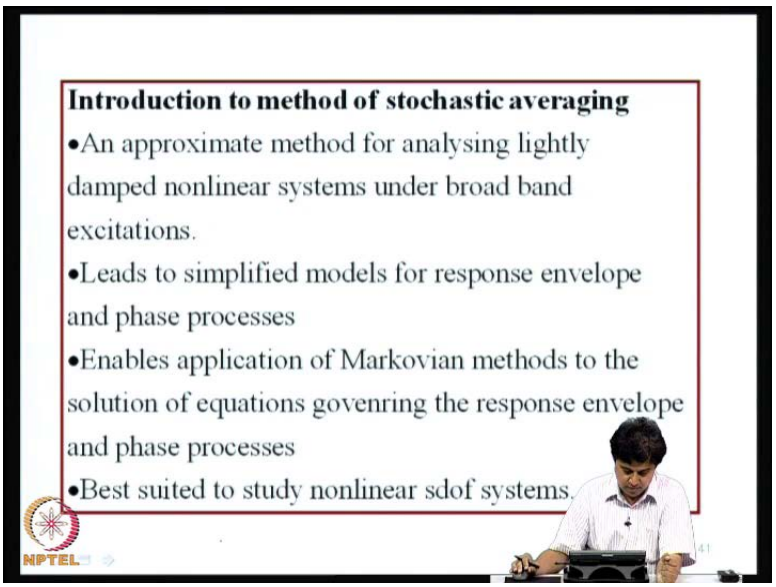

Now, if you consider what we have been doing till now, based on Marcov property of response vector, we derive first the transition probability density function by solving the

governing Fokker Planck equation, then we derive the equation for one time moments, derive the equation for two time moments, and so on, so forth. And using backward equation - backward Kolmogorov equation - we formulated the problem for first passage times.

If we recall the discussion that we had, when we study random vibration of a single degree freedom systems, we consider properties of envelope source - envelope and phase processes. So, how to study envelope and phase processes, when response process Markovian properties? So, that is a question that I would like to discuss now; and that is based on an approximate method of response analysis known as method of stochastic averaging.

So, what is method of stochastic averaging? It is an approximate method for analyzing lightly damped non-linear systems, under broadband excitations. So, it is applicable to weakly non-linear systems. Under broadband excitations, the excitations are not necessarily white to start with; they need to be broadband.

If use this method, you get simplified equation for the envelope and phase processes. So, it means simplified model for response envelope and phase processes. And once we get this simplified models, we can show that envelope and phase together constituted Marcov vector; it can be shown that I will briefly demonstrate. And if moment you get the Markovian property we can applicable to envelope and phase, you can write the associated Fokker Planck equation, and get moment if you want or write the backward equation and solve first passage problems associated with envelopes. So, you can do lots of analysis. These methods are best suited for studying single degree freedom systems; they become unwieldy, if we look at multi degree freedom systems, but within the frame work of their own limitations, it still provide very powerful means to study envelope and phase of non-Gaussian random processes. So, that is the main advantage; so, what is this method of averaging?

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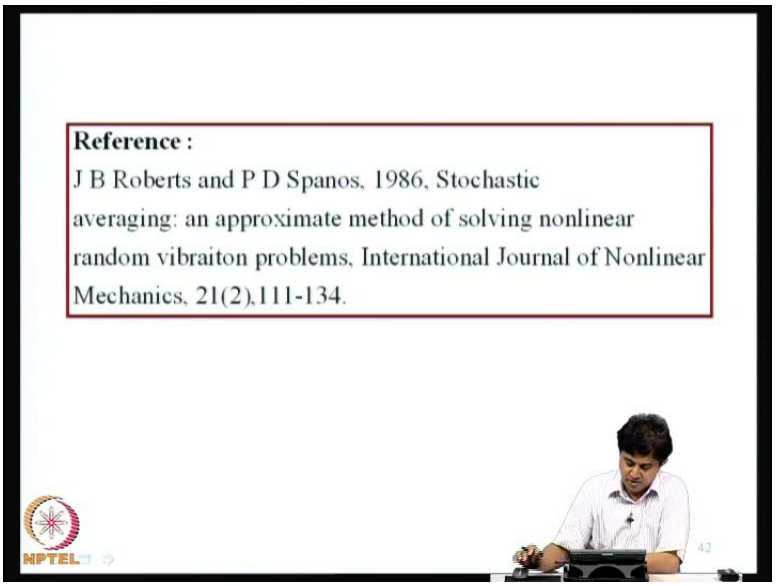

Before I get into that, I would like to cite a reference; this is a review paper written by Roberts and Spanos in 1986. It appeared in international journal of non-linear mechanics. So, this paper reviews various application of stochastic averaging, for solving a class of non-linear random vibration problems.

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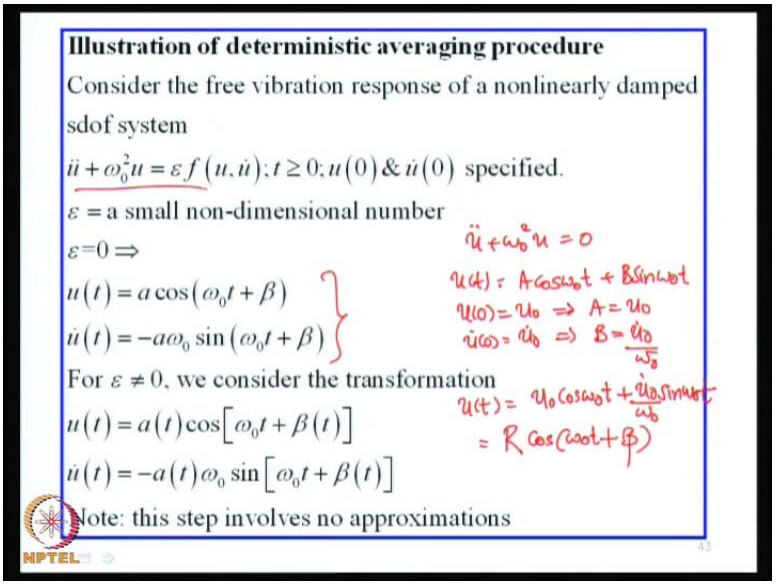

Now, as a prelude to discussion of stochastic averaging, we will briefly look at, what is averaging in a deterministic context. Now, let us consider the free vibration response of a nonlinearly damped system, the system will u double dot plus omega naught square u is

equal to epsilon into f of u comma u dot, and time recon from 0 and initial conditions are specified. The parameter epsilon is a small non-dimensional number; if epsilon is 0, I have u double dot plus omega naught square u equal to 0. So, u of t can be written as, a cos omega naught t plus B sine omega naught t, and you can use initial conditions u of 0 is naught could imply a equal to naught, u dot of 0 is naught dot implies B is equal to naught dot by omega naught; and I can write u of t therefore as, u naught cos omega naught t plus naught dot by omega naught sine omega naught t and this can be written as some R cos omega naught t plus beta, where R in beta R in terms of naught dot and omega naught.

So, when epsilon is 0, I can always write this solution as a cos omega naught t plus beta, where A and beta are functions of specified initial conditions; and the velocity is u dot of t is the derivative minus a omega naught sine omega naught t plus beta. So, when epsilon is 0, these two are exact, and a and beta are constants; there not functions of time, there are only functions of initial conditions.

Now, what will do is, for epsilon not equal to 0, we will consider a transformation, which is fashioned after this exact solution. And we will write that, even when epsilon is not 0, I take u of t to be a cos omega naught t plus beta as before, except now that a and beta are functions of time. And u dot of t, I again make the assume the same function of for minus a omega naught sine omega naught t plus beta of t, but a and beta as I said functions of time. So, if I can derive consistent set of equations for a and beta, this satisfy these two equations and the governing equation; this will be an exact transformation; at this stage, there is no approximation in our discussion.

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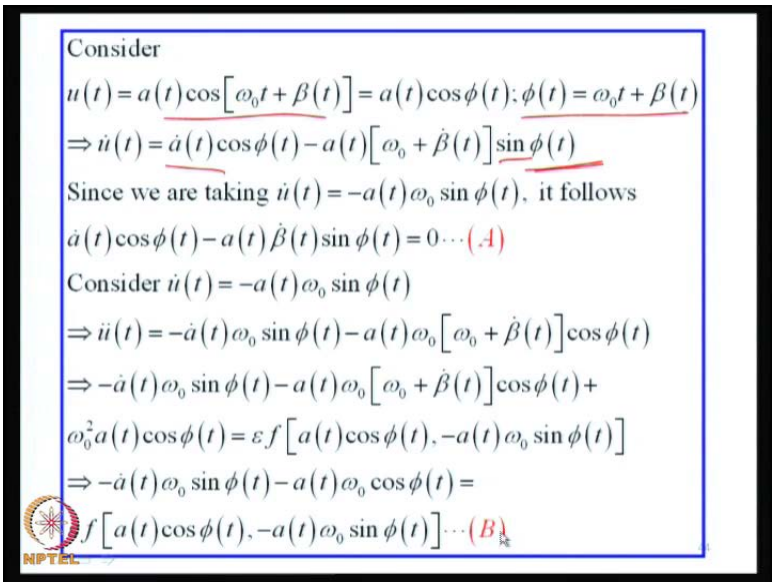

Now, I have u of t is a cos omega naught t plus beta of t and I will write phi of t as omega naught t plus beta of t, and write u of t as a of t cos phi of t. Now, if this is correct, then a time derivative of u of t would be a dot cos phi t minus a as it is, and cos becomes sine, and derivative of phi is omega naught plus beta dot. But we are taking u dot of t as minus a omega naught sine phi, therefore, the remaining terms in this equation must be equal to 0.

So, what are the remaining terms in these equation? a dot cos phi minus a beta dot sine phi, this must be equal to 0, fine. Now, you consider now, u dot is equal to minus a omega naught sine phi; that is, what would happen if this condition A is satisfied? From this, now you evaluate u double dot; u double dot is what? Minus a dot omega naught sine phi minus a omega naught, sine becomes cos, and phi dot becomes omega naught beta dot. So, I get an expression for u double dot and I have an expression for u, and I will substitute that into the governing differential equation and get this equation. A slight reordering of the terms, because two of the terms are going to get cancelled; I get an equation which is B. Now, A and B are the two equations which govern a and beta, which are not known. So, I can solve for a dot and beta dot from these equations.

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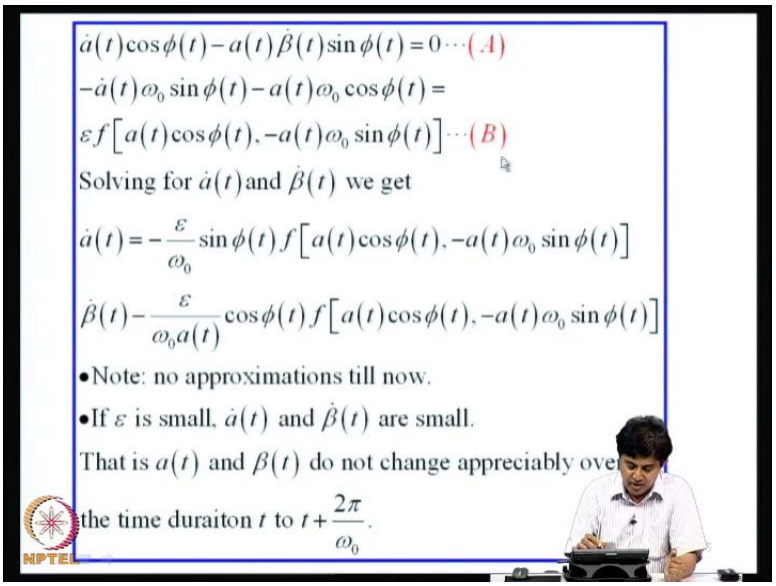

So, the two equations are, this is A and this is B; and solving for a dot and beta dot, we get a dot as minus epsilon omega naught sine phi into some function of a cos phi and a omega naught sine phi; and beta dot is similarly shown here, minus epsilon omega naught a cos phi; and this function again let me emphasis, till now we have not made any approximation, this is exact.

Now, we can bring in some arguments based on epsilon being small. If epsilon is 0, a and beta are constant; if epsilon is small, you can expected a and beta will be functions of time, alright, but it could be slowly varying functions of time. So, they vary much slowly; a of t varying much slowly than sine phi of t itself, because when sine this function complete 1 cycle of oscillation, the change in amplitude could be quiet small. Now, that is a and beta do not change appreciably over the time duration t to t plus 2 pi by omega naught, where t changes from t to t plus 2 pi by omega naught; this trigonometric functions undergo dramatic change; whereas the change in amplitude and phase will be quiet small, if epsilon is small. If epsilon is 0, there would not be any change.

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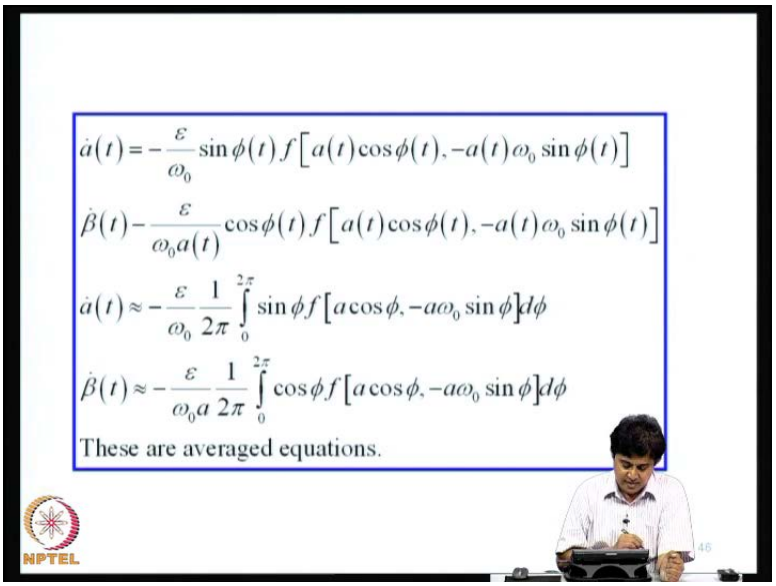

Based on that what we do is, we average over a cycle the right hand side of these equations by treating the functions a and beta in the procedural averaging as constants; over a cycle, they do not change as much as the function themselves. So, if I integrate from 0 to 2 pi and take the average 1 by 2 pi integrate pi, this integration can be performed by treating a and beta as constants. If I do that, I get here of approximate equations a dot and beta dot, and these who called as an averaged equations. At many times, we can solve for this a and beta or study the fixed point and their stability, and so on and so forth.

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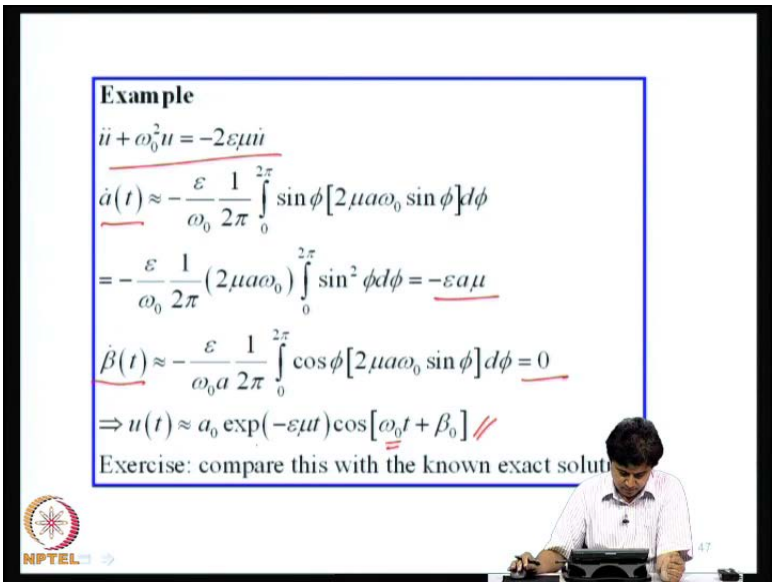

So, I will consider a few examples, so that this idea is established in your mind. So, you consider now the first example, u double dot plus omega naught square u is equal to minus 2 epsilon mu u dot; this is a damped free vibration of linear single freedom systems. So, here a dot, now this function is specified, so I can plug that function here and actually calculate a dot. And if I do that, I get a dot as, a dot I get as minus epsilon a mu and b dot as 0; and if I now solve this equations, I get u of t as a naught exponential minus epsilon mu t cos omega naught t plus beta naught. This is quite similar to the exact equation that you would get, if solve these equation by writing the correct complementary function and exposing the initial conditions. The changes would be in terms of frequency and slight changes in an amplitude, but by in law, this could be a reasonably good approximation to the given problem.

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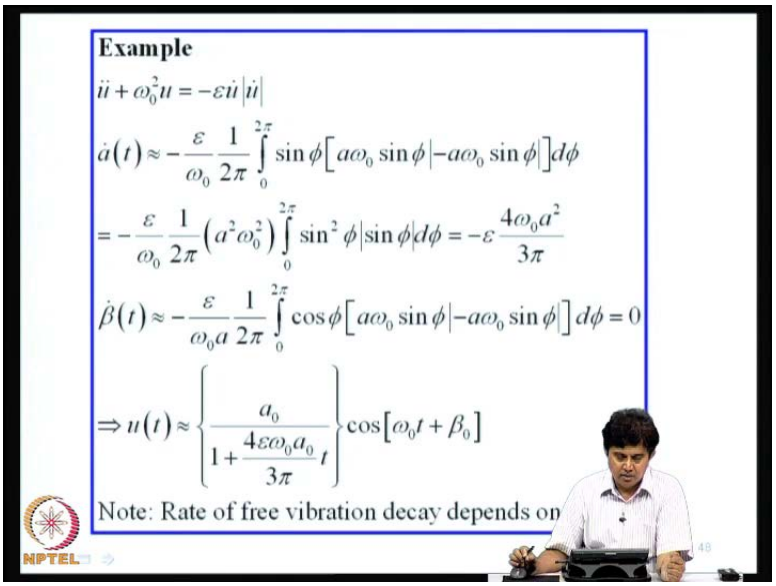

A slightly more complicated problem, u double dot plus omega naught square u is minus epsilon u dot into modulus of u dot.

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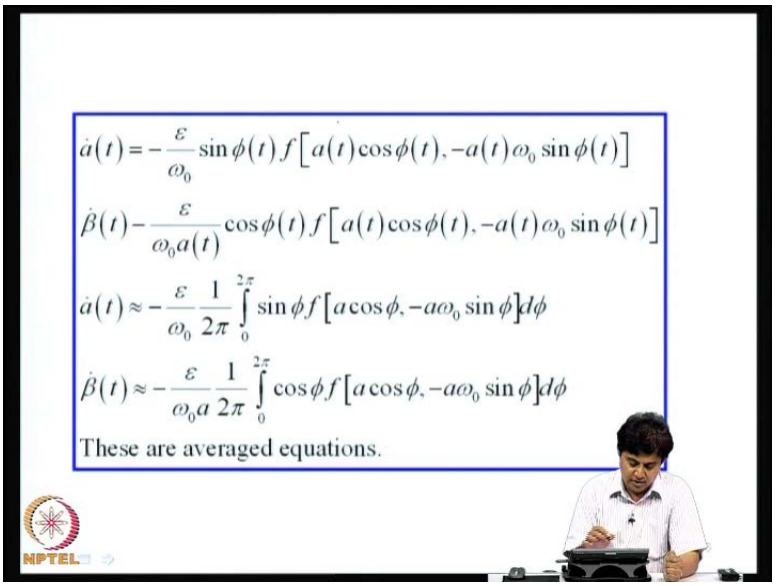

So, again, I go back now, look at these equations; I have now the simplified equation for a dot and beta dot, and f is given now.

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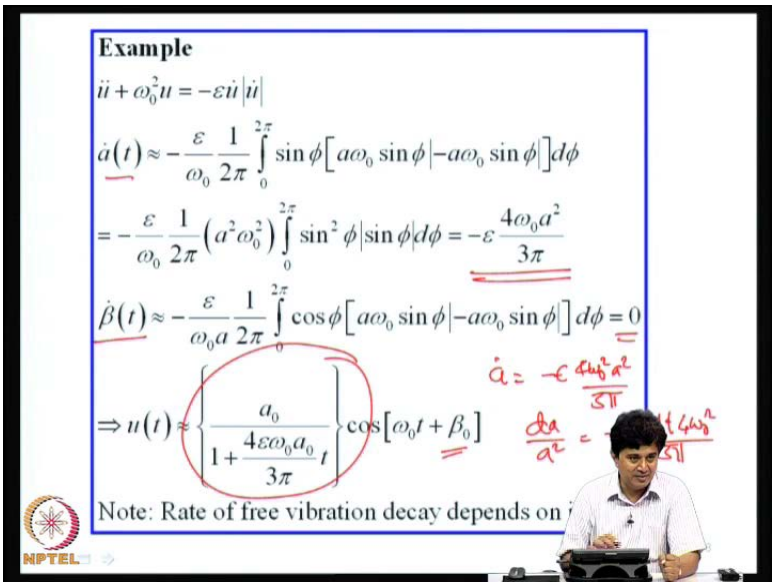

So, I substitute that, I get the equation for a dot and I have to integrate this; as I said, while integrating this, a and beta are written as constants; so, that means integration is done only with respective phi and I get this equation. Similarly, beta dot if I do, in this case also, it turns out that beta dot is 0. Now, this equation a dot is equal to minus epsilon 4 omega naught square a square by 3 pi; we can solve this. This can be written as

da by a square is equal to minus epsilon dt 4 omega naught square by 3 pi etcetera, and I can integrate both side and solve for d of t.

If I do this, I get this as the expression for the amplitude a of t; and beta is beta dot is 0, therefore, beta naught will be constant. Now, you look at now the nature of a of t; as t times to be infinity, you can see that the denominator here becomes increasingly large and the amplitude goes to 0. But the rate at which this amplitude goes to 0 depends on the multiplier here, which is a function of initial conditions; that means, for the same system starting with different initial conditions, you get different rates of decay.

So, if you work to compute the logarithmic decrement for this, it would mean the logarithmic decrement is a function of initial conditions; whereas in a linear system, where the concept of logarithmic decrement is basically applicable, logarithmic decrement is independent of initial conditions. The rate at which free vibration decays, in linear system, it is independent of initial conditions, but here, that the capability here, that principle no longer holes. So, the rate at which the oscillation decay depends how you start the system.

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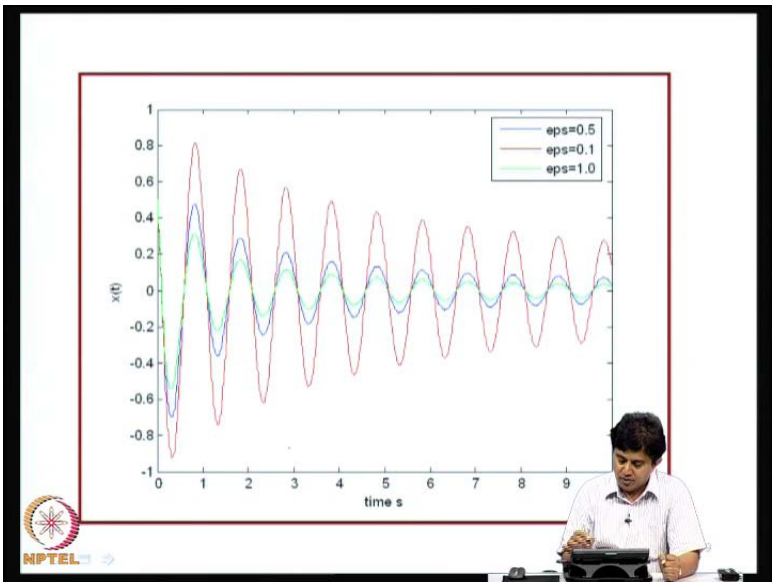

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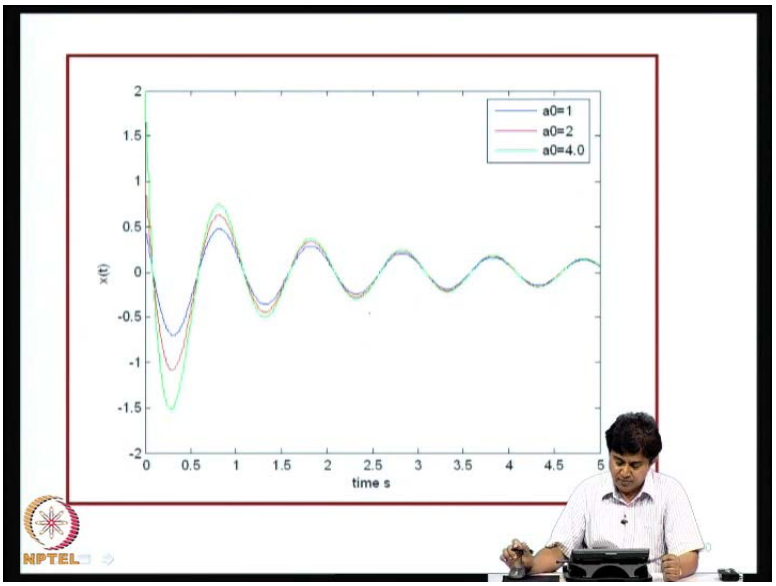

So, here, I have shown different trajectories of the response using the approximate solution obtained using averaging, for different values of epsilon; and you can see that, as epsilon increases, the decay is rapid; and here I have shown for same epsilon, but with different initial conditions. So, if you compute the rate at decay is occurring, you can see that rate this actually dependent on the initial conditions, especially for small times. So, this is one of the complex feature associated with non-linear system behavior.

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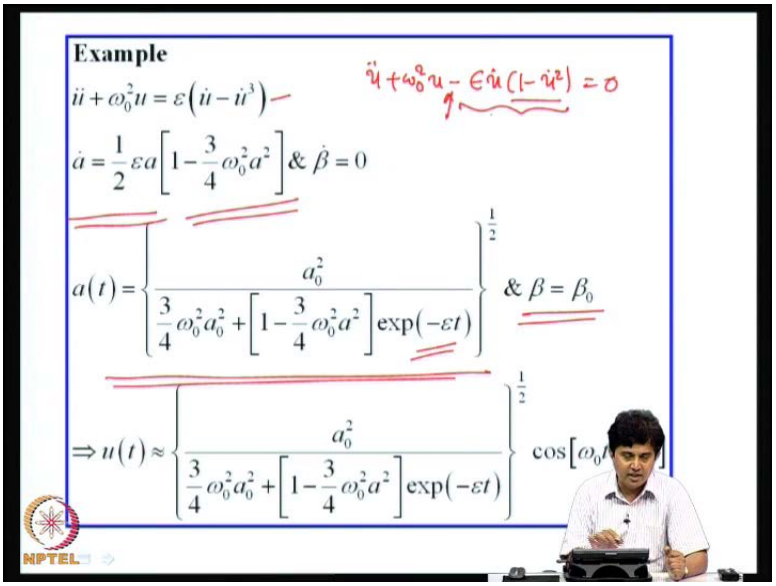

One more example, this is a system with limit cycles. So, here we are considering this equation; so, to understand this, we can write it this as omega naught square u minus u into 1 minus u dot square equal to 0. So, this is our dissipation term. Now, you can see here, there is a negative sign here and we have here 1 minus u dot square. So, if u dot is small, 1 minus u dot will be greater than 0, and we will be dealing with a negative sign for the damping term; that means, motions in the neighborhood of origin tend to grow in few operations.

Whereas if we start with initial conditions, where 1 minus u dot square is large, then the next sign of this term would be positive and the large oscillation is not decay. So, small oscillations grow and large oscillations decay, that could mean, in between them, there is a stable periodic solution - isolated periodic solution. And that is known as a limit cycle. So, we can go through the method of an averaging and we can get this equation; and I skip this details, you fill up the details, it is an exercise. So, I get a dot is this and we can solve this equation. And if you do that, we get beta as beta naught and a as this; the time is sitting here. So, as t becomes very large, this function becomes 0, and I am left with 4 by 3 omega naught square, square root of that as amplitude of a of t. So, that captures the qualitative feature of the response, that we expect even before solving the problem; so, this is a solution.

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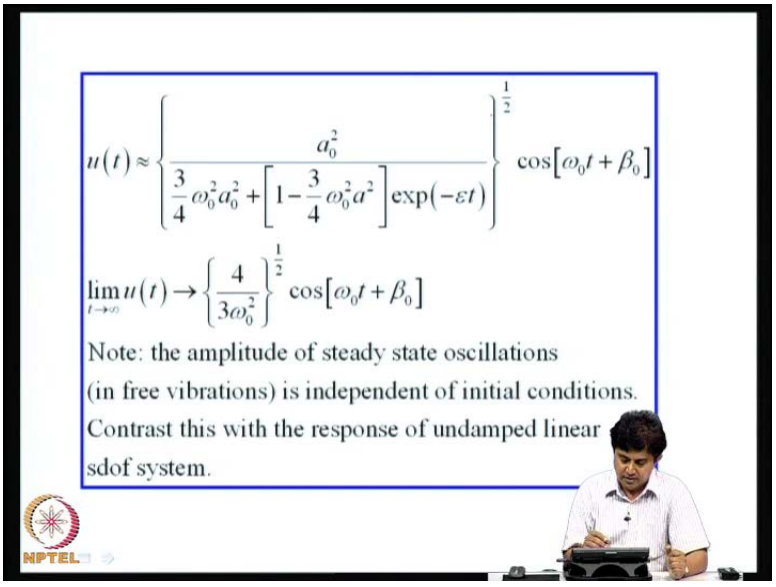

And as t tends to infinity, we get 4 by 3 omega naught square, square root of that into cos omega naught t plus beta naught. This is the amplitude of steady state oscillation in free vibration, is independent of initial conditions. Contrast this with the response of un-damped linear single degree freedom systems. You take un-damped linear single degree freedom system, the solution of that system is periodic, but amplitude is function of the initial conditions. So, here, there is no forcing, solution is periodic; so, in that sense, it is similar to un-damped free vibration, but the amplitude of free vibration in periodic state, it is independent of initial conditions; so, that is where the non-linearity comes in.

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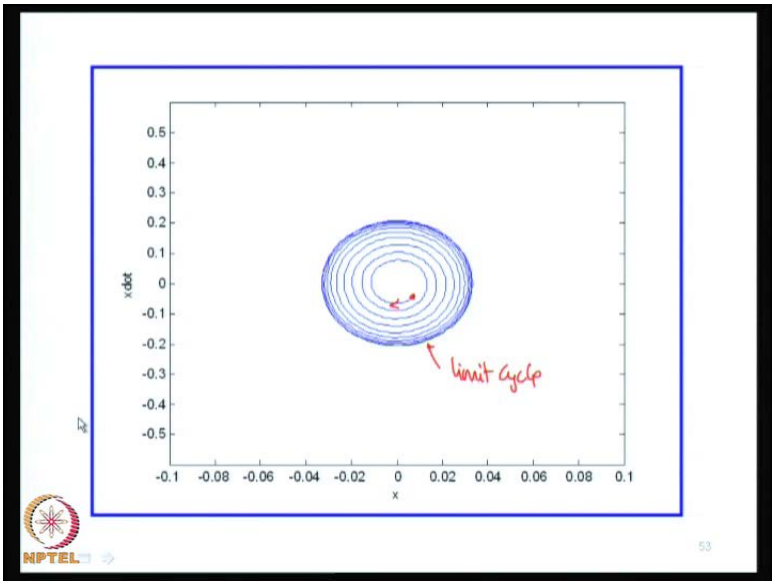

So, this is one trajectory, where the solutions have originated here, and they are growing and this is my limit cycle. So, this is phase plane plot, a velocity versus displacement; a small initial condition, the solution grows and reaches this outer circle.

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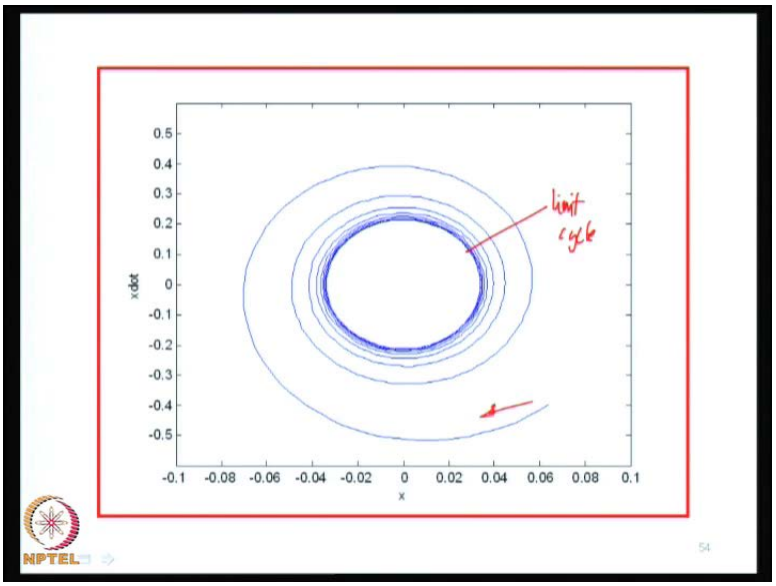

Here, the solution originates from this point, a large initial condition and decays; it moves in this direction, and it spirals down and I get the limit cycle; this is the limit cycle. So, that means, limit cycle here is approached from outside, large oscillations decays to that stable limit cycle.

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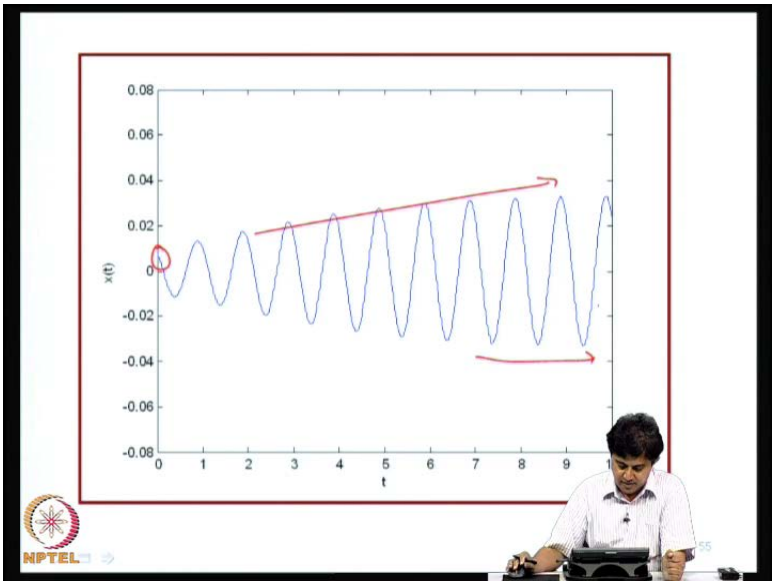

Now, this is time history of x of t, where in the first case, where motions started with a small value and it is now growing to reach the amplitude of limit cycle oscillations.

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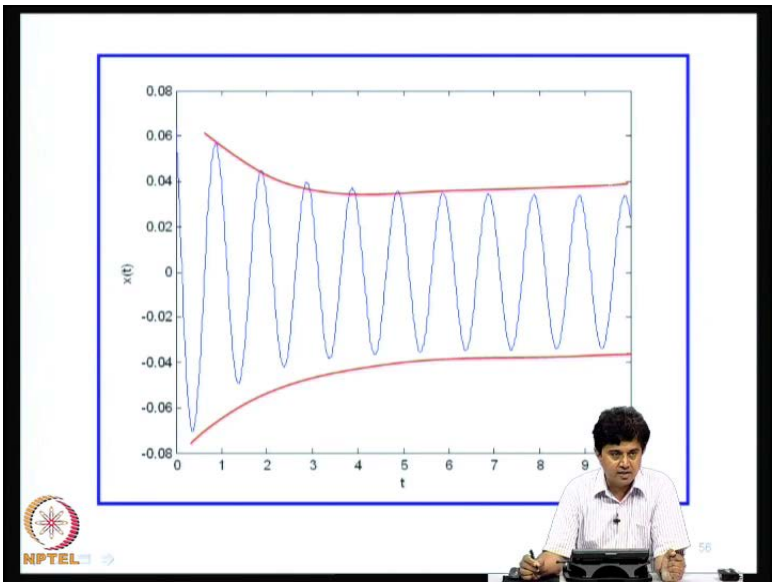

In this case, the solution is starting in a large initial condition, it is now decaying to reach the limit cycle oscillations. So, it is moving towards the limit cycle from outside, whereas in the previous case, it was from inside. So, this is the behavior of non-linear systems without any random excitations, where we have used the concept of averaging.

So, in the next part of the lecture, we will consider how to implement this averaging principle, if there is noise present or the system is driven by random excitations. So, we will consider that in next lecture and we will close this lecture at this juncture.