Stochastic Structural Dynamics

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Module No. # 06

Lecture No. # 23

Markov Vector Approach-3

In the previous lecture, we have been exploring how to use Markov property of response processes associated with dynamical systems, which are driven by white noise excitations. We will continue that topic.

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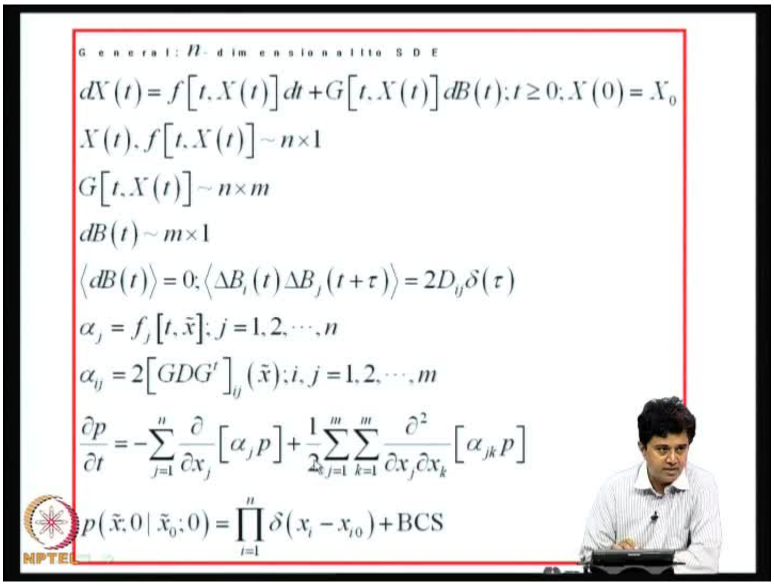

So, what we discussed in the previous lecture was, we considered differential equations governing the dynamical systems and cast them in the form of Ito stochastic differential equations. And, for n-dimensional dynamical system, the equation has this form – f is the drift factor; G is the diffusion matrix; dB is increments of Brownian motion processes. And, we showed that formal derivative of a Brownian motion process can be interpreted as Gaussian white noise. And, the time illusion of the conditional probability density function of the system states conditioned on the initial conditions is given by this partial differential equation. And, this alpha j and the alpha i j are the incremental moments, which are related to the drift and diffusion as shown here. In the previous lecture, I demonstrated how the governing equation, the so-called equation Fokker-Planck equation can be derived for different dynamical systems. And, this equation can be viewed as equation of motion for time evolution of probability density function.

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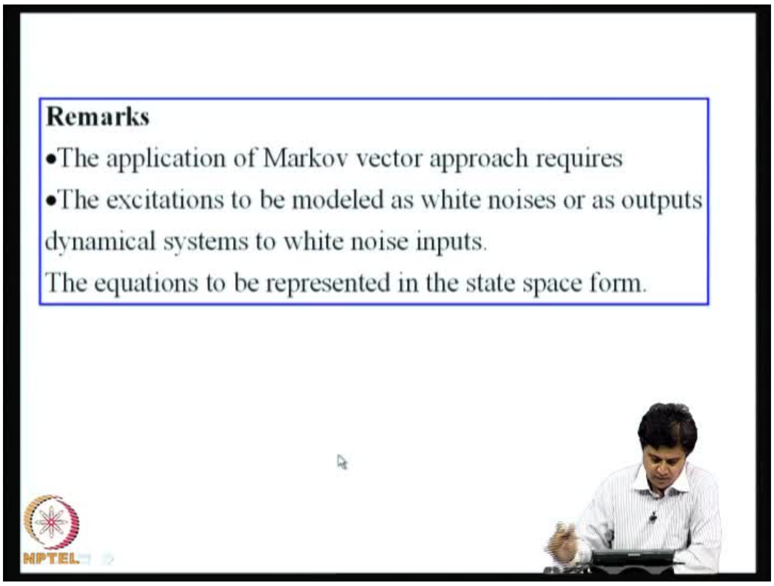

The application of Markov vector approach requires that the excitations have to be modeled as white noises or as outputs of dynamical systems to white noise inputs. And, the equations need to be represented in the state space form. So, these are some of the primary requisites for applying the Markov vector approach.

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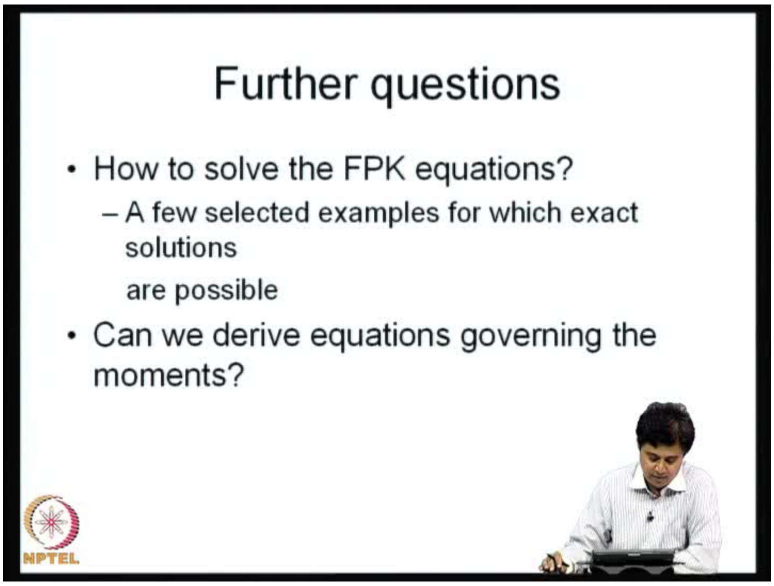

We will consider few questions now in this lecture. How to solve the governing FPK equations? And, I will present a few selected examples for which exact solutions are possible. And then, we will also consider the question, how can we derive equations for evolution of response moments? The equation gives the equation for time evolution of probability density function. But, if you are happy with lesser level of description of response, it is useful to ask what the equations that govern the response moments are. So, we will consider these questions in the present lecture.

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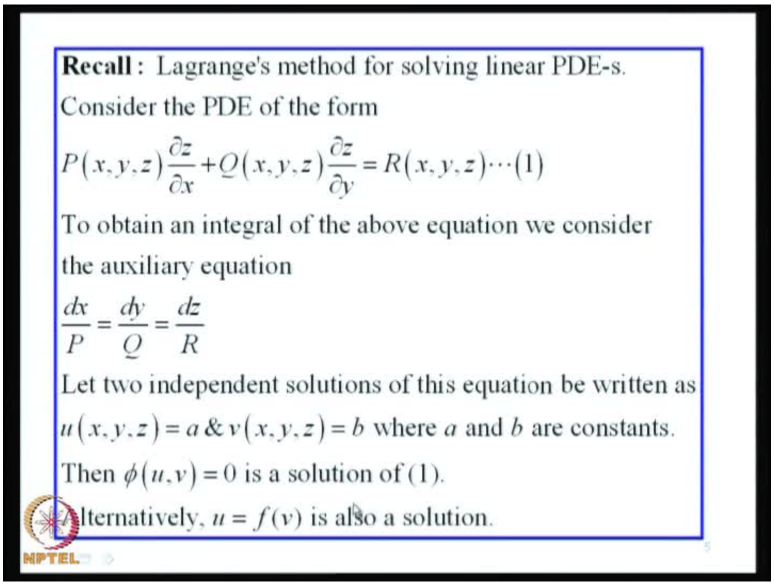

Now, one of the mathematical tools that I will be needing is the so-called Lagrange's method for solving linear partial differential equations. This you must have studied in your under graduate curriculum. So, we consider partial differential difference equations of the form P into dou z by dou x plus Q into dou z by dou y is equal to R of x, y, z. Now, to obtain an integral of the above equation, we consider the auxiliary equation dx by P, dy by Q and dz by R. This is a sense of Lagrange's method. And from this set of equations, we construct two independent solutions to be u of x, y, z equal to a and v of x, y, z equal to b; where a and b are constants. Then, the solution to this partial differential equation can be expressed as some phi (u, v) equal to 0. Alternatively, u as a function of v is also a solution. So, we will be using this in our solution of FPK equations.

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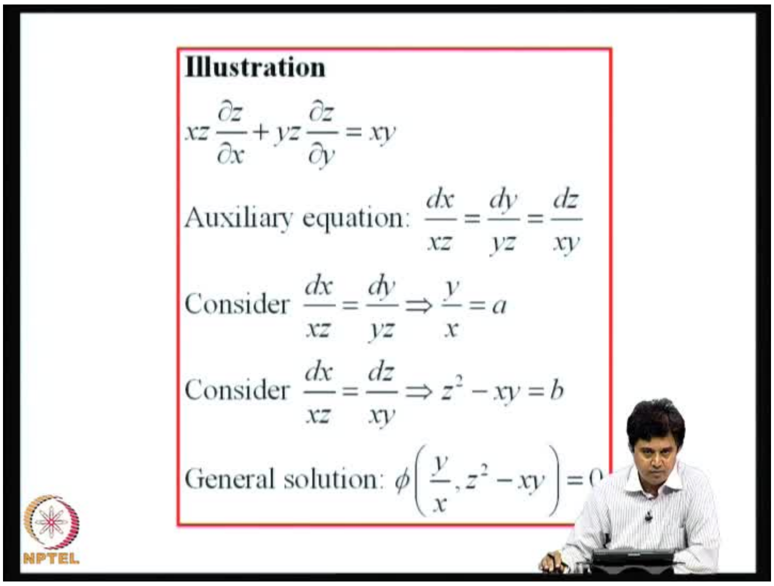

A quick example, a simple example say xz dou z by dou x plus yz dou z by dou y equal to xy. Now, the auxiliary equation here will be dx by xz, dy by yz and dz by xy. Now, we consider these two pairs of equations and the first two of these equations lead to the condition that y by x is a constant. And similarly, the first and the third leads to the equation z square minus xy is b. Therefore, the general solution of this partial differential equation can be written as phi of y by x comma z square minus xy equal to 0. So, this can be verified. I leave that as an exercise. But, you need to recall some of the details associated with this method.

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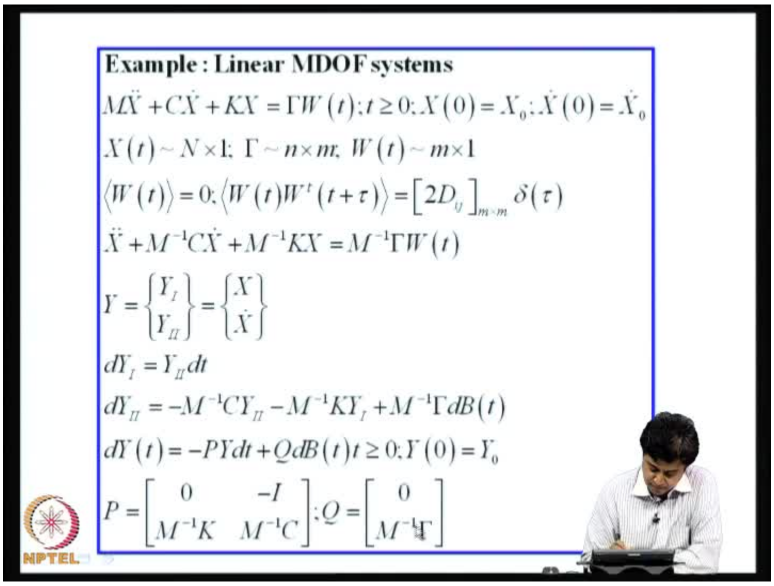

Now, we will start by considering linear dynamical systems under additive white noise. For which we know the exact solution through a normal mode expansion and convolution integral approach. We have already discussed that solution. But, in this part of the discussion, we will consider how we can use the Markov property of the response vector and obtain the solution by solving the governing Fokker-Planck equation. So, we consider n degree of freedom system; X is N cross 1 and it is acted upon by m cross 1 vector of white noise excitations and gamma is a kind of influence matrix, which is n by m. And, system starts from initial condition; X of 0 is X naught and X dot of 0 is X naught dot. W of t is 0 mean white noise; its covariance is given as 2D ij, which is m by m matrix, which is a direct delta function.

Now, first step in implementing the Markov vector approach for solving this problem would be to recast the problem in the state space form. So, to achieve that, what we do is we premultiply this equation by M inverse and I get X double dot plus M inverse CX dot plus M inverse KX is equal to M inverse gamma W. Now, I introduce the state vector Y 1 and Y 2 as X and X dot and recast this equation into a set of 2N first order equations; dY I by dt is actually X dot; X dot is Y double I dt; and, d Y double I dt is obtained from this equation and it is displayed here (Refer Slide Time: 06:33). Now, these two equations can be combined into a single form, which is dY t is minus PY dt plus QdB t with y of 0 be Y naught. The matrix P here is a 2N by 2N matrix, which consists of M inverse K, M inverse C; and, Q is the multiplier for the noise stumps and this is again is 2N cross m matrix.

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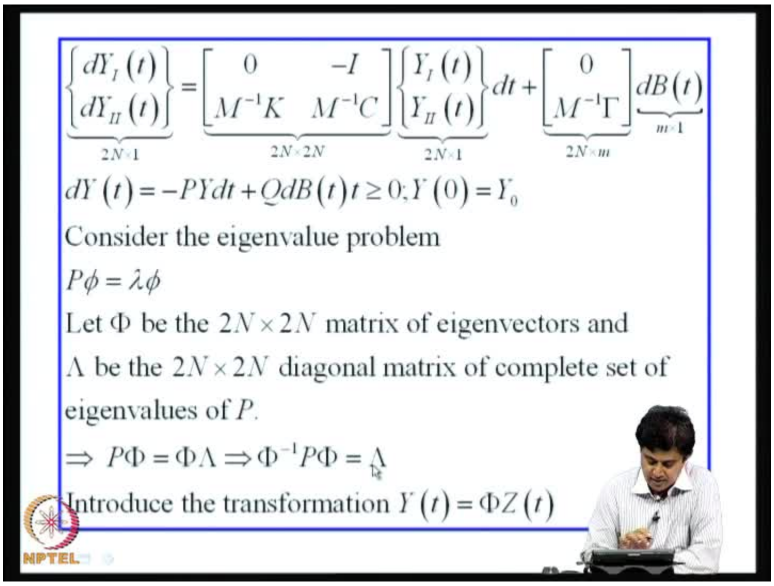

So, the governing equation is displayed in the standard format as shown here and the sizes of various quantities are displayed here. And, this equation dY t is minus PY plus QdB t can be interpreted as an Ito stochastic differential equation of size 2N cross 1. And, response vector Y consisting of Y I and Y double I will have Markov property and we can derive the associated Fokker-Planck equation for the time evolution of the conditional probability density function of Y conditioned on Y of 0. Before we do that, we would like to diagonalize this matrix P that would facilitate certain numerical calculations. So, we consider the eigenvalue problem P phi is lambda phi and we denote by capital phi the 2N cross 2N matrix of eigenvectors. And, capital lambda is the 2N cross 2N diagonal matrix of complete set of eigenvalues of P. Now, P phi is phi lambda. Therefore, phi inverse P phi will be capital lambda; and, capital lambda is a diagonal matrix. So, now, it could be this; I introduce a transformation Y as phi Z.

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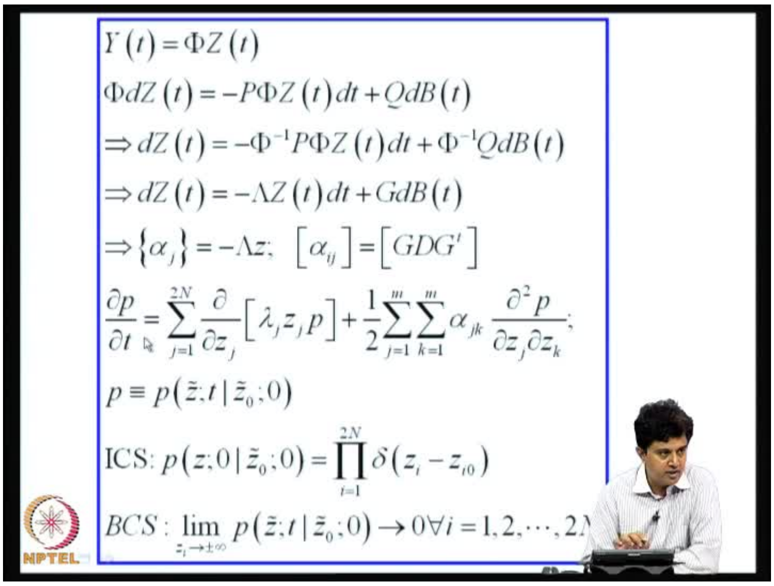

And, use this relation phi inverse P phi is a diagonal matrix and uncouple the equation. So, I substitute that; phi dZ t is minus P phi Z t dt plus QdB t. Now, I premultiply by phi inverse; and, phi inverse phi is I. And, here I get phi inverse P phi Z plus phi inverse QdB as an equation; and, phi inverse P phi is capital lambda, which is a diagonal matrix. And, this phi inverse Q, I call it as G, which is 2N cross m matrix. This alpha j is the drift coefficient, which will be minus lambda into z. And, alpha ij are GDG transpose. These are the coefficients in the Fokker-Planck equations. So, the Fokker-Planck equation itself is given in this form; where p is the conditional probability density function of z evaluated at t conditioned on z at t equal to 0. The initial condition is in terms of products of direct delta functions; and, boundary conditions, we assume that z i tends to plus minus infinity; the function goes to 0.

Now, we need to solve this problem. So, this is a linear partial differential equation with constant coefficients. So, the integral transform methods appear eminently suited to handle this problem and indeed that is the way we look at it. We will use a characteristic function, which is a fourier transform of p and try to solve this problem.

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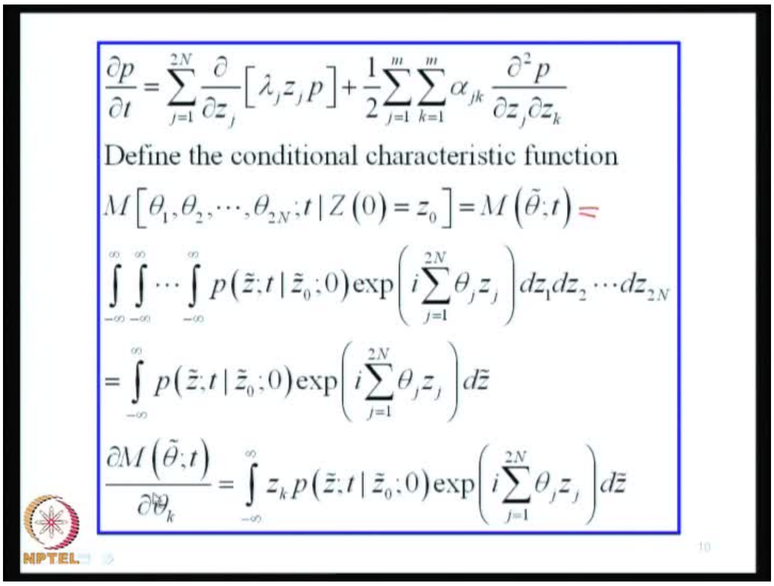

So, to facilitate that, we define the conditional characteristic function M theta 1, theta 2, theta 2N. I denote this as M of theta tilde colon t. Now, this is actually equal to the expectation of exponential i j to 1 to 2N theta 1 z j. This is the definition of the characteristic function. So, this I write compactly in terms of a vector integral; z tilde is a vector. And now, if I differentiate this with respect to theta, I get z k p k – z k p exponential this. See I would need this, because I need in this equation the fourier transform z j into p, is what I would be needing. So, that I have to handle. Therefore, you can see that the fourier transform z k p will be the derivative of the characteristic function with respect to theta k.

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$$M\left(\tilde{\theta}:t\right)=\int_{-\infty}^{\infty}p\left(\tilde{z}:t\mid\tilde{z}_0;0\right)\exp\left(i\sum_{j=1}^{2N}\theta_j z_j\right)d\tilde{z}$$

$$\Rightarrow p\left(\tilde{z}:t\mid\tilde{z}_0;0\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}M\left(\tilde{\theta}:t\right)\exp\left(-i\sum_{j=1}^{2N}\theta_j z_j\right)d\tilde{\theta}$$

$$\frac{\partial M\left(\tilde{\theta}:t\right)}{\partial\theta_k}=\int_{-\infty}^{\infty}iz_k p\left(\tilde{z}:t\mid\tilde{z}_0;0\right)\exp\left(i\sum_{j=1}^{2N}\theta_j z_j\right)d\tilde{z}$$

$$\Rightarrow z_k p\left(\tilde{z}:t\mid\tilde{z}_0;0\right)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\partial M\left(\tilde{\theta}:t\right)}{\partial\theta_k}\exp\left(-i\sum_{j=1}^{2N}\theta_j z_j\right)d\tilde{\theta}$$

$$\frac{\partial}{\partial z_k}\left[z_k p\left(\tilde{z}:t\mid\tilde{z}_0;0\right)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial M\left(\tilde{\theta}:t\right)}{\partial\theta_k}\left(-i\theta_k\right)\exp\left(-i\sum_{j=1}^{2N}\theta_j z_j\right)d\tilde{\theta}$$

So, what that means is M is given in terms of this. Therefore, the probability density function is inverse fourier transform defined with respect to characteristic function. And, since dou M by dou theta k is given by this expression, I get the fourier transform z k into p, will be the inverse transform of this function, which is 1 by 2 phi i, etcetera here. So, I am now ready with using characteristic function to solve the Fokker-Planck equation. There I would need dou by dou z k of this and that is given in terms of characteristic function as shown here.

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$$\frac{\partial p}{\partial t}=\sum_{j=1}^{2N}\frac{\partial}{\partial z_j}\left[\lambda_j z_j p\right]+\frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}\alpha_{jk}\frac{\partial^2 p}{\partial z_j\partial z_k}$$

$$p\left(\tilde{z}:t\mid\tilde{z}_0;0\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}M\left(\tilde{\theta},t\right)\exp\left(-i\sum_{l=1}^{2N}\theta_l z_l\right)d\tilde{\theta}$$

$$\frac{\partial p}{\partial t}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial M\left(\tilde{\theta}:t\right)}{\partial t}\exp\left(-i\sum_{l=1}^{2N}\theta_l z_l\right)d\tilde{\theta}$$

$$\frac{\partial}{\partial z_j}\left[z_j p\right]=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\partial M\left(\tilde{\theta}:t\right)}{\partial\theta_j}\left(-i\theta_j\right)\exp\left(-i\sum_{l=1}^{2N}\theta_l z_l\right)d\tilde{\theta}$$

$$\frac{\partial^2 p}{\partial z_j\partial z_k}=\frac{1}{2\pi}\int_{-\infty}^{\infty}M\left(\tilde{\theta}:t\right)\left(-i\theta_j\right)\left(-i\theta_k\right)\exp\left(-i\sum_{l=1}^{2N}\theta_l z_l\right)d\tilde{\theta}$$

So, now, I will also need a second derivative, which is shown here and derivative with respect to time, which is shown here. And, I substitute all these into the governing Fokker-Planck equation.

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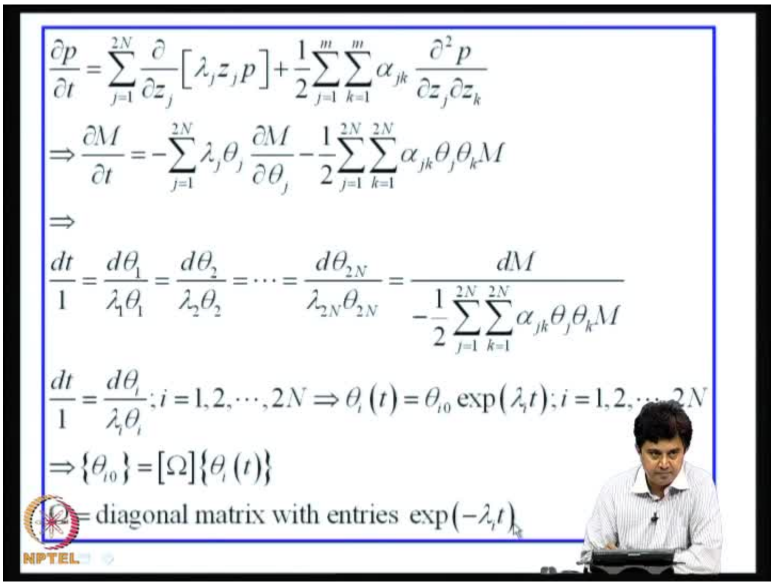

And, I get the equation in terms of characteristic function as shown here. Now, this equation is again a partial differential equation, where independent variables are now thetas and time. Thetas are the parameters appearing in the definition of characteristic function. So, this equation I will now solve using Lagrange's method. So, I will write the auxiliary equation in this form as shown here and I will solve them pairwise; and, construct the solution and substitute into the general solution here, the final expression here and we will be able to tackle this. Some of these details you can understand by studying the accompanying powerpoint slides. So, I will briefly run through this. So, I will write this equation as dt by 1 is d theta i by lambda i theta i. And, from this, I get theta i t is theta i 0 exponential lambda i t for i running for 1 to 2N. This I recast in the matrix form as shown here, where capital omega is a diagonal matrix with entries exponential minus lambda i t.

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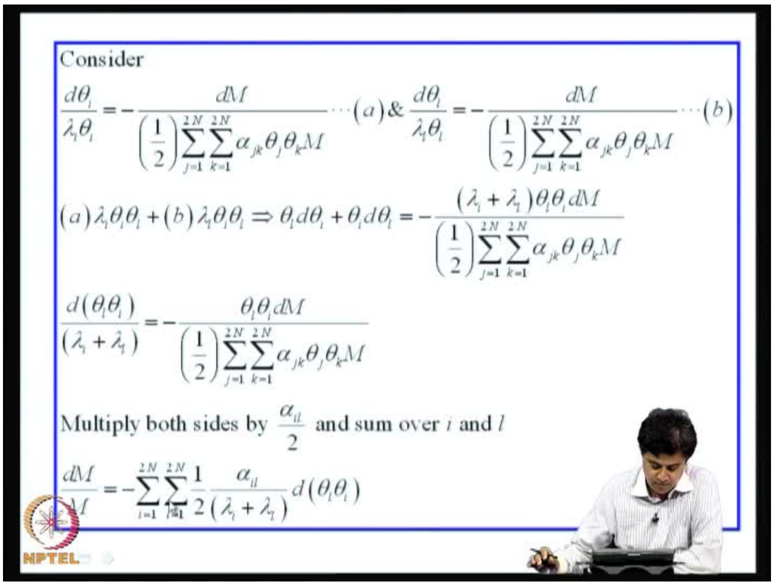

Now, I consider the last term in that auxiliary equation and try to solve for M. So, I consider two terms: d theta i by lambda i theta i and d theta l by lambda l theta l. And, I will manipulate. Suppose if this equation is a and this equation is b. I will multiply a by lambda i theta i theta l; and, equation b by lambda l theta i theta l. And, that leads to left-hand side to be this and right -and side to be this. So, this can be recast as d of theta l theta i divided by lambda i plus lambda l; and, this expression on the right-hand side. Now, I do further manipulations. I multiply both sides by lambda i l by 2 and sum over i and l to get the equation dM by M is equal to this.

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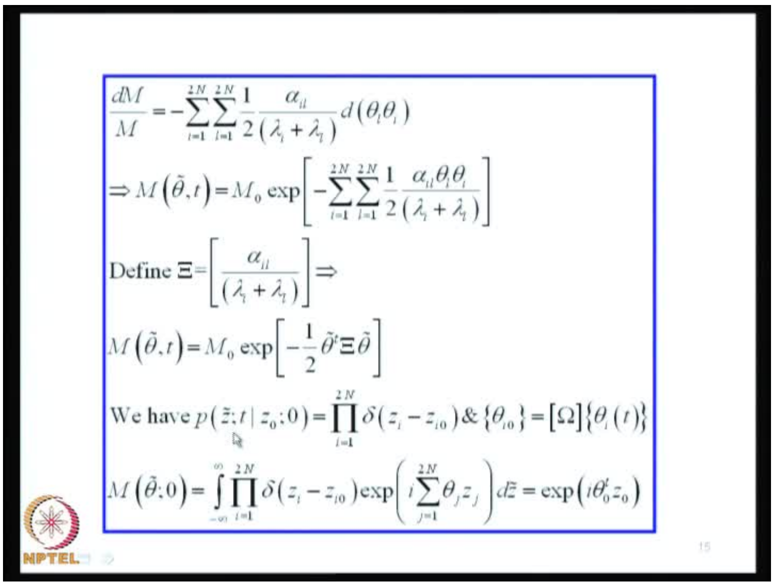

And, this I can solve for M and I get M as in terms of M naught exponent of this. Now, I will introduce capital XI to denote lambda i l by lambda i plus lambda l. And, from this, I get the characteristic function in a compact form as M naught exponential minus half theta tilde transpose XI theta tilde. Now, we have to determine M naught and for that, I use initial conditions. And, if you manipulate this, we will be able to find M naught.

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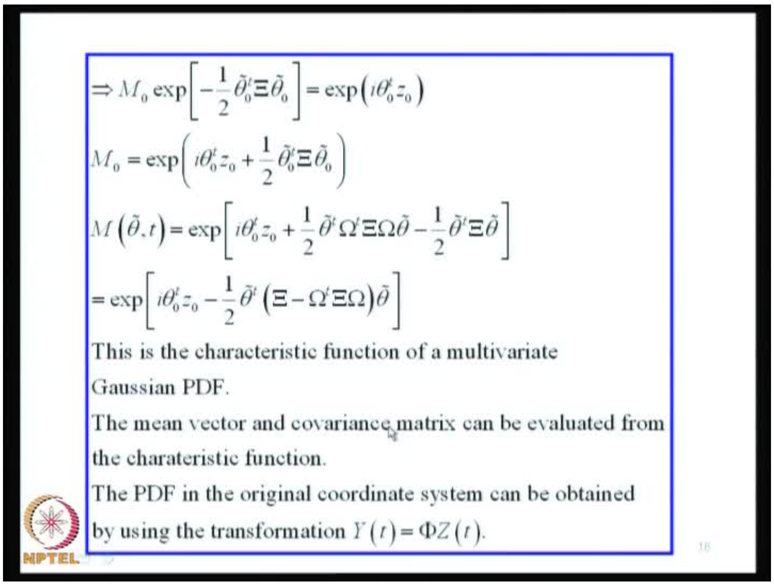

And finally, if you run through all these calculations, you can show that the characteristic function here corresponds to that of a multidimensional Gaussian random vector. This is to be expected, because we have a linear system, which is driven by Gaussian excitation. So, we know by an alternate means, which we have already studied that response is a Gaussian random process. So, we expect beforehand based on that knowledge that a characteristic function corresponds to the double Gaussian random process.

Now, once this is z coordinate, if I take the inverse transform, I will get the Gaussian probability distribution function, density function. And, from this characteristic equation, I will also be able to find mean vector and covariance matrix. And, once I find the mean and covariance matrix, I can use the transformation, which I introduced. Our original variables were Y, but the solution has been obtained in Z; the transformation can be made directly at the level of mean vector and covariance and I will get the requisite multidimensional normal Gaussian density function for the response process. This is valid for all time; it is not just for steady state. So, it is a complete solution. So, this type of complete solution for Fokker-Planck equation is rarely obtainable; linear system under additive Gaussian noise is one instance. It is not always possible to do this.

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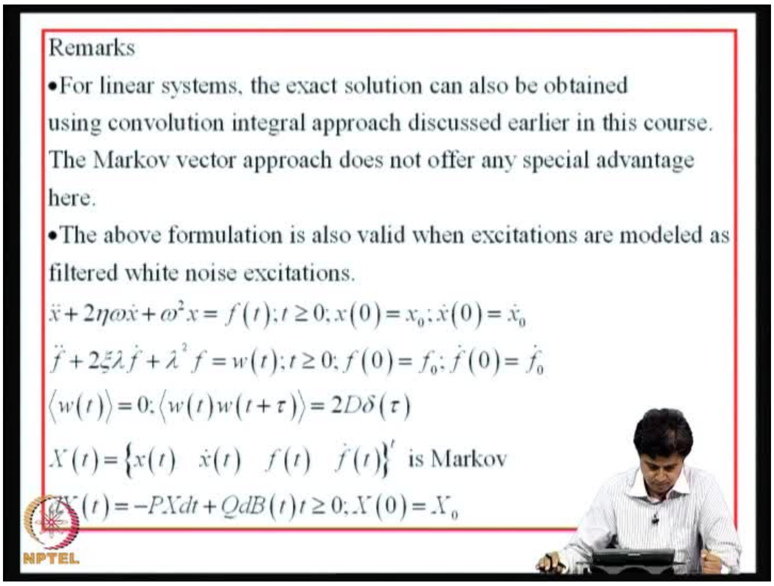

So, we will make some remarks. For linear systems, the exact solution can also be obtained using convolution integral approach discussed earlier in this course. The Markov vector approach does not offer any special advantage here, because we already know how to solve this problem by an independent means. Now, I consider one of the limitation that we could attribute to Markov vector method, is that the excitations need to be Gaussian white noises. But, this is not a very stringent requirement. In the sense, if you do not have a white excitation, you can always make a model for the given random excitation by passing white noise through a filter. For example, if you consider now x double dot plus 2 eta omega x dot plus omega square x equal to f; f is not a white noise process, but it is output of a linear system to white noise excitation. Suppose there is another single degree freedom system with natural frequency lambda and damping xi; receives white noise; and, output of this is f of t. So, f of t can always be modeled as output of linear systems to white noise excitations; under various general conditions it is possible.

Now, what happens is I define now an extended vector that consists of x, x dot, f and f dot. So, at the expense of handling a larger dimensional state space I will be still be able to use Markov vector approach. So, here obviously, the vector consisting of x and x dot will not have Markov property. But, if you consider the extended vector x, x dot, f, f dot, that vector will have Markov property. And therefore, I can cast this pair of equations into a single Ito equation, which is of the form dX of t is equal to minus PX dt plus QdB t. And, I have already discussed how to solve this problem; that means the formulation that I just now presented can also be used if excitation needs not white, but it can be modeled as output of a linear system, which is driven by white noise excitations. And, since the solution is valid for all times, if there is any non-stationarity in terms of a deterministic time envelope multiplying these excitations, the method that we have discussed should be possible in principle to use that for this kind of excitations also.

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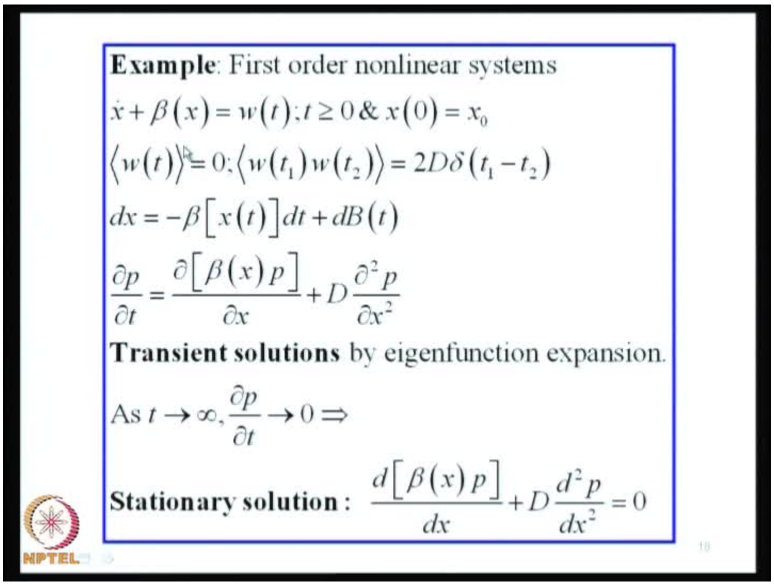

Now, what are the other types of problems that we can solve using Fokker-Planck equation? By solve I mean exactly. Now, I will consider a sequence of problems and just indicate how the problems could be tackled. So, we return to the problem of a first order nonlinear systems – x dot plus beta of x is driven by white noise, w of t. And, this is an Ito’s representation and this is the Fokker-Planck equation. In the previous lecture, I indicated that the transient solution to this can be obtained by using method of variable suppression. And, by using the eigenfunction expansion, we would be able to solve this problem. And, there are problems of this kind, which have been tackled. I will briefly mention that later.

But, if we restrict our attention to only steady state; suppose the system admits steady state. It is not necessary that every system should admit a steady state; it depends on nature of beta of x. Suppose the system admits steady state, then as time becomes large, dou p by dou t will be equal to 0. p is actually the probability density function of x at the time t – single time t. So, if a process is stationary, that probability density function would become independent of time. Therefore, dou p by dou t would be 0. In that case, I get a simplified equation or a reduced equation, which essentially characterizes only the stationary solution. So, if you are not interested in transient, if you interested only in steady state, I can directly tackle this equation. Since in this equation, the two independent variables are x and t, and by bringing in the motion of steady state, I have eliminated one of the independent variables, namely, t, time. I am left with ordinary differential equation of the form shown here (Refer Slide Time: 20:26).

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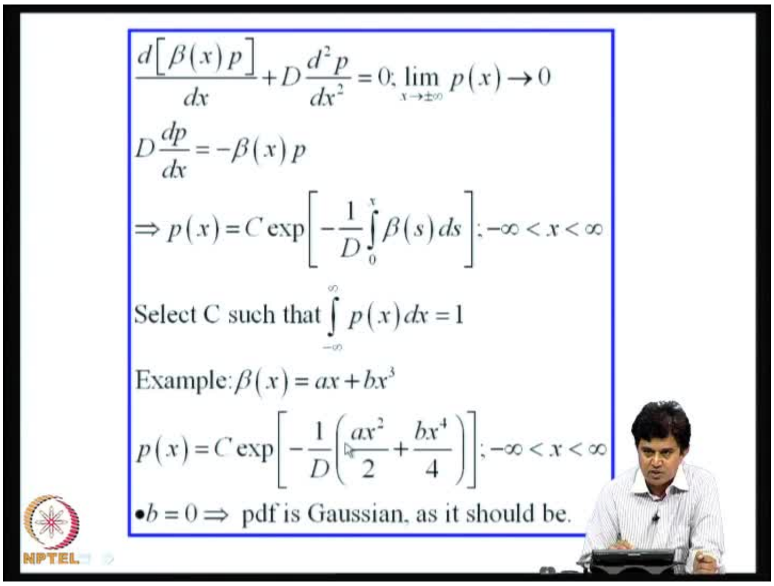

Now, this can be solved under the boundary conditions that x becomes plus infinity and minus infinity; this function goes to 0. And, I can rewrite this equation as D of dp by dx is equal to minus beta of x into p, because there is D by dx that we can put that out and consider this equation. Now, this equation can easily be solved and I get p of x to be C into exponential minus 1 by D 0 to x beta of s d s. Now, we have to select this C, such that the normalization condition is satisfied. We will just give you an example; if beta of x is equal to a of x plus bx cube say, then the probability density function will have this form.

If of course, b equal to 0, we get the Gaussian density function, (Refer Slide Time: 21:15) which is expected because (Refer Slide Time: 21:19) if it is x dot plus alpha x equal to w of t, then the solution is known to be Gaussian. You can find out its variance; you can do power spectral analysis or you can use Duhamel integral analysis and find out the steady state solutions. So, that solutions would essentially match with this solution with b equal to 0. This is as it should be (Refer Slide Time: 21:43). Because system is linear and it is driven by Gaussian white noise, the response is going to be Gaussian. And, in fact, it will exactly match with the solution that you will get by using convolution integral approach.

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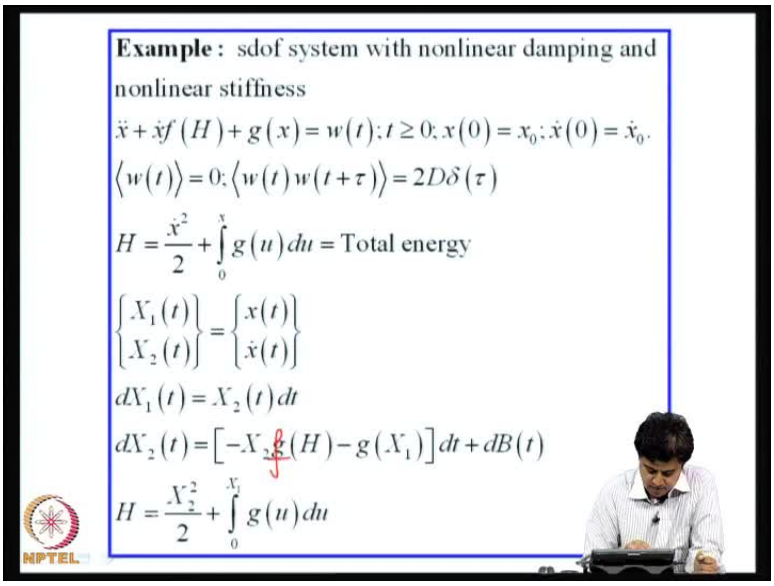

We will consider now single degree freedom systems with nonlinear damping and nonlinear stiffness. The equation of motion that we are considering is x double dot into x dot some function of H – I will explain what this H is – plus g of x is equal to w of t; t is greater than or equal to 0 and initial conditions are specified. w of t as 0 mean and its covariance – it is a delta correlated process. Therefore, expectation of w t into w of t plus tau is 2D delta of tau. H is actually the energy in the system, which is kinetic energy – x dot square by 2 plus the strain energy – 0 to x g of u du. It is the total energy.

Assume that the nonlinear damping is a function of the energy. It is still nonlinear, but it obeys this particular functional form. If all these assumptions of this model are respected, then it is possible to show that the steady state response of this system can be exactly determined by using Fokker-Planck equation approach. Let us see how we can do that. I introduce a vector X 1, X 2 state vector, which is displacement and velocity. I recast the equation into the Ito differential form; dX 1 of t is X 2 dt and dX 2 of t is minus X 2 – this is f of H minus g of X 1 dt plus dB of t. Therefore, H is X 2 square by 2 plus 0 to x 1 g of u du. So, this is the statement of the governing equation of motion.

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$$\frac{\partial p}{\partial t} = -x_2 \frac{\partial p}{\partial x_1} - \frac{\partial}{\partial x_2}\left[\left\{-x_2 F(H) - g(x_1)\right\} p\right] + D \frac{\partial^2 p}{\partial x_2^2}$$

$$p \equiv p(x_1, x_2; t \mid x_0, \dot{x}_0) = tpdf$$

$$p(x_1, x_2; 0 \mid x_0, \dot{x}_0) = \delta(x_1 - x_0)\,\delta(x_2 - \dot{x}_0)$$

$$p(\pm\infty, x_2; t \mid x_0, \dot{x}_0) = p(x_1, \pm\infty; t \mid x_0, \dot{x}_0) = 0$$

Steady state

$$-x_2 \frac{\partial p}{\partial x_1} - \frac{\partial}{\partial x_2}\left[\left\{-x_2 F(H) - g(x_1)\right\} p\right] + D \frac{\partial^2 p}{\partial x_2^2} = 0$$

$$-x_2 \frac{\partial p}{\partial x_1} - \frac{\partial\left[-x_2 F(H) p\right]}{\partial x_2} + g(x_1)\frac{\partial p}{\partial x_2} + D \frac{\partial^2 p}{\partial x_2^2} = 0$$

Now, what is the governing Fokker-Planck equation? It is dou p by dou t equal to minus x 2 dou p by dou x 1 minus dou by dou x 2… This is alpha 1; this is alpha 2; alpha 1 – you can see is actually X 2 (Refer Slide Time: 24:11) and alpha 2 will be minus X 2 f of H and minus g of X 1. These are the drift coefficients. And, we get those first two terms here and the diffusion is capital D, because white noise is multiplied by a constant unity. Therefore, the diffusion term will simply be D. So, p is a transition probability density function as displayed here. And, initial conditions are in terms of product of two direct delta functions and the boundary conditions we assume that at x 1 and x 2 plus minus infinity, the probability density function goes to 0.

Obtaining transient solution for this problem is not straight forward. Currently, no exact solution exists for the complete solution of this problem, exact solution (Refer Slide Time: 25:01). But, an approximate solution can always be generated, for example, by using method of weighted residuals or finite element method. You can directly tackle this partial differential equation and approaches, such as that have been discussed in the existing literature. But, in this discussion, we will restrict our attention to steady state. That would mean dou p by dou t is 0. So, I am interested in now, p would be become the joint density function between displacement and velocity evaluated at the same time. And, in this steady state, that would be independent of time. Therefore, dou p by dou t would be equal to 0. So, I am left with only this right-hand side, which is equal to 0.

This equation is still a partial differential equation, (Refer Slide Time: 25:48) because there are two independent variables here. In the original problem, there are three independent variables: t x 1 and x 2. By assuming steady state, I have reduced one of the dimensions, namely, time. But, still I am left with two independent variables: x 1 and x 2. So, we need to solve this problem.

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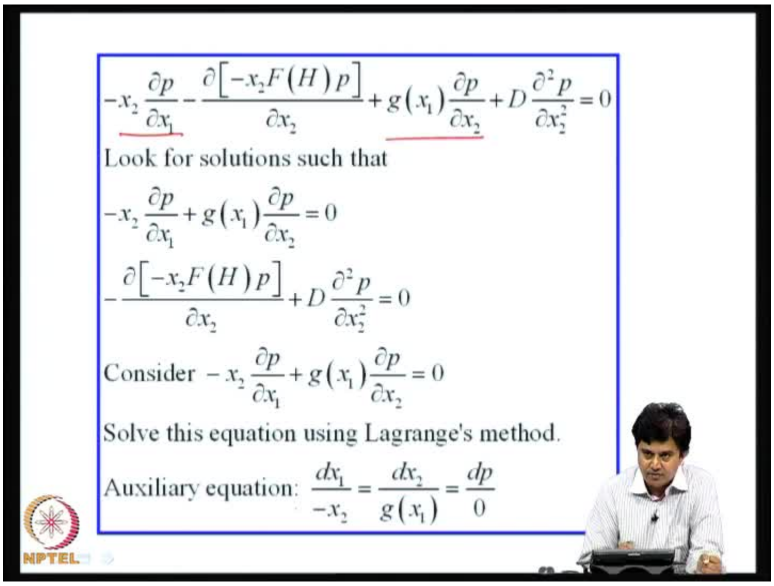

Now, we expand this and slightly rearrange these terms and bring it in the form as shown here – minus x 2 dou p by dou x 1; this is dou x 2; second term – g of x 1 is dou p by dou x 2 plus D into dou square p by dou x 2 square equal to 0. Now, we are looking at stationary solutions. So, x of t as t tends to infinity becomes a stationary random process. For a stationary random process, the process and its derivative are uncorrelated at the same time. So, if process is Gaussian, it automatically implies that x and x dot are independent also. But, here process is not Gaussian, but still x and x dot are uncorrelated. But, we could explore… If x and x dot, we can find out a solution, where a kind of variable separation becomes possible. So, what we will demand is we will demand that the first term and this term; (Refer Slide Time: 27:14) sum of these two is suppose 0; and, the remaining two terms are separately 0. I am looking for a special solution.

Now, we will consider the first part of this equation (Refer Slide Time: 27:28). If these two are independently 0, and I find a p that would automatically satisfy this equation also; the same p, which satisfies these two equations would satisfy this also. Now, we will consider first of this equation – minus x 2 dou p by dou x 1 plus g of x 1 dou p by dou x 2 equal to 0. Now, using Lagrange's equation, Lagrange's approach, we can write this as dx 1 by minus x 2; dx 2 by g of x 1; dp by 0 – this is auxiliary equation for this problem.

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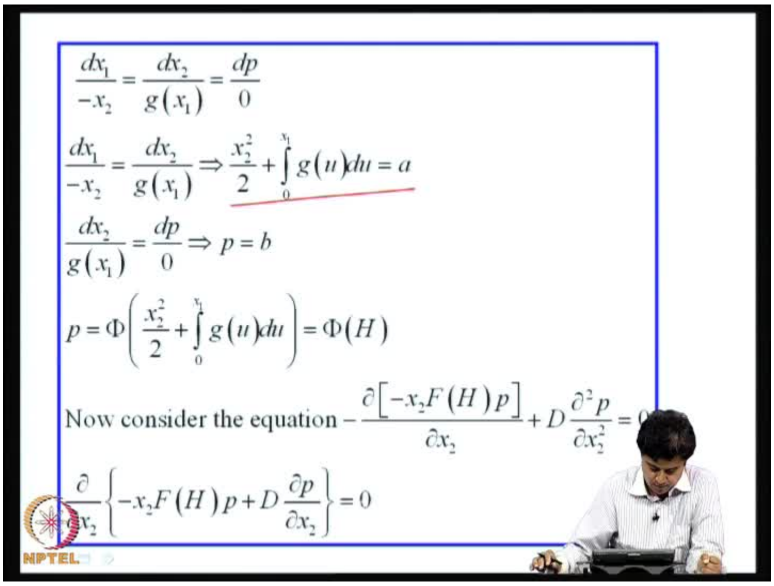

If I consider now the first two of these equations, I get dx 1 by minus x 2 is equal to dx 2 by g x 1. And, from this, I get the condition x 2 square by 2 plus 0 to x 1 g of u du is a constant a. Now, dx 2 by g x 1 is dp by 0. Therefore, p itself should be a constant. So, based on these, I can write a form of a solution, which is p – some arbitrary function capital phi of this function – x 2 square by 2 plus 0 to x 1 g of u du. The argument of this function in fact is H. So, I can write this as phi of H. We will now consider the next equation and write it as dou by dou x 2 of minus x 2 F of H p plus D dou p by dou x 2 square equal to 0.

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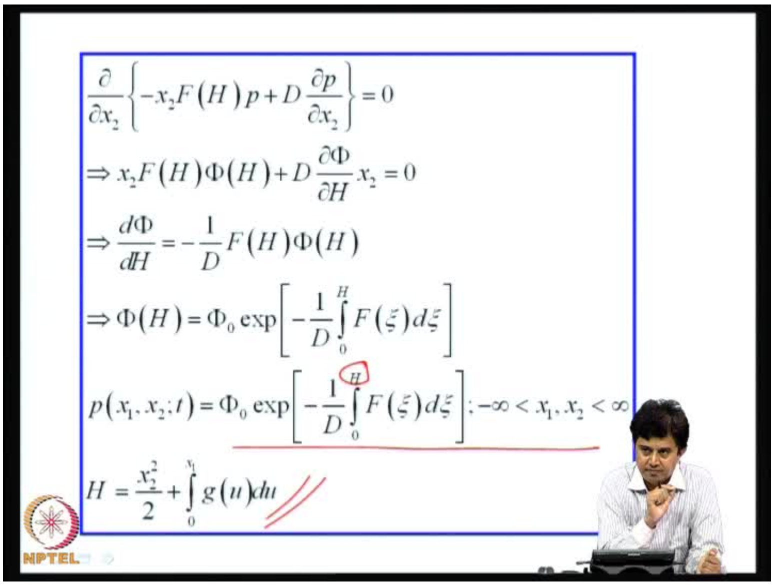

Now, I will demand that the term inside the parenthesis is 0 and I will get… Now, for p, I write phi of H and go through this equation. And, I will now get an equation, which is d phi by dH is equal to minus 1 by D F of H phi of H. This phi of H now can be solved, because this is now independent variable; here is H. So, we can solve this. And therefore, p x 1 comma x 2 will be expressed in this particular form; where this H, which appears as a limit, is this function. So, this is a deliberate effort to consider only that class of dynamical systems for which the Fokker-Planck equation can be solved. So, if you are given a dynamical system and if you set up a Fokker-Planck equation, that equation depending on the nature of nonlinearity etcetera may not be in a form that permits exact solution. So, I am not discussing generic procedures to solve Fokker-Planck equation, but I am doing a kind of an inverse investigation on what could be the nature of nonlinear dynamical systems, which permit exact solutions in steady state to the governing FPK equations. So, that is the theme of our discussion.

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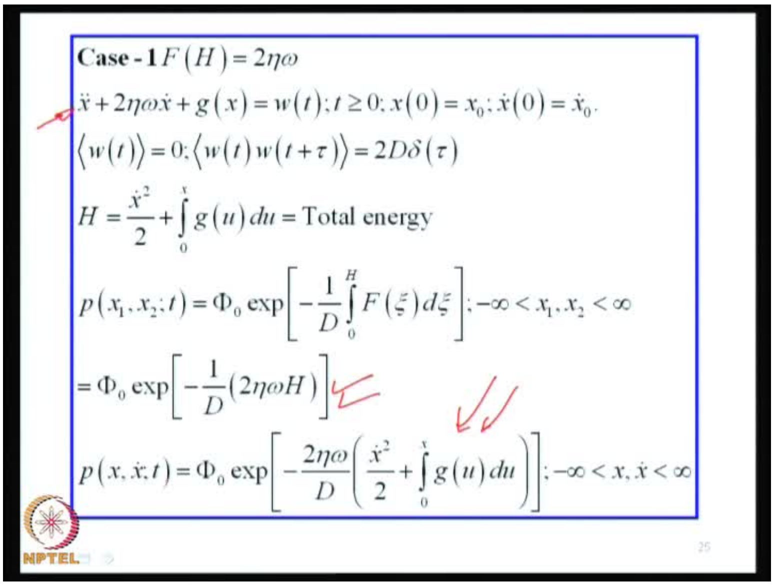

Now, we can start by considering F of H to be 2 eta omega. That means H to the power of 0 some constant into H to the power of 0. So, this is a class of system, where damping is linear, but our spring force or the elastic forces are non-linear. So, this could model, for example, structure with geometric nonlinearity. So, we will show that these types of problems are amenable for exact solutions. So, we can write now… Since we have got the exact solution, I will now write in the exact solution, F of psi to be 2 eta omega and I get this as the solution. And, in terms of x and x dot, this will be the solution.

Now, you can see here that this form of the density function is such that it can be expressed as product of a function of x alone and a function of x dot alone, because this is exponential of a term involving x dot plus an exponential of term involving only x. So, it can be viewed as product of exponentials, of two exponentials: the first one is only function of x dot; the second one is function of x. Or, in other words, the form of this expression (Refer Slide Time: 31:35) suggests that the joint density function, x and x dot are stochastically independent.

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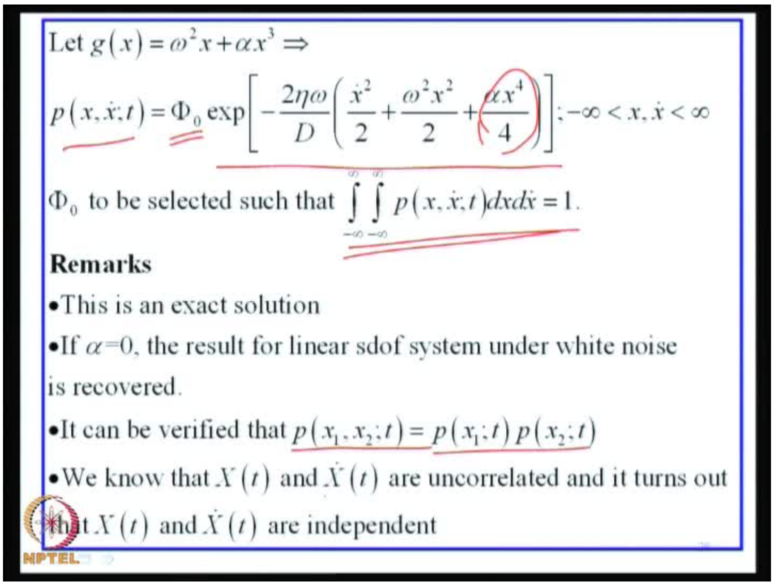

Now, to make these discussions slightly more specific, we can consider the duffing system, where the system has cubic nonlinearity. So, this is the joint density function between displacement and velocity of a duffing's oscillator driven by white noise. The solution is in steady state. This is an exact solution. A duffing system under deterministic excitations admits no exact solutions; whereas, under white noise excitations and if you focus on steady state solutions, it admits an exact solution. So, this is a very interesting feature associated with using Markov method for solving this class of problems. This constant phi naught is still floating around and that we have to select, so that the probability density function is suitably normalized; the volume under the probability density function is unity.

So, we will make quick remarks. This is an exact solution. Now, if alpha equal to 0, if this time goes off, then this corresponds to a 2-dimensional Gaussian density function, which is in fact exact solution for a linear single degree freedom system under white noise excitation. So, this we have derived by using convolution integral approach earlier. So, we can verify that the solution that we are getting is exactly the same as what we got by an alternate approach. Next, we can verify that the joint density function can be written as product of p x 1 and p x 2; that would mean that x and x dot are independent. We know that x of t and x dot of t are uncorrelated, but in this particular case, they are also independent and they are non-Gaussian.

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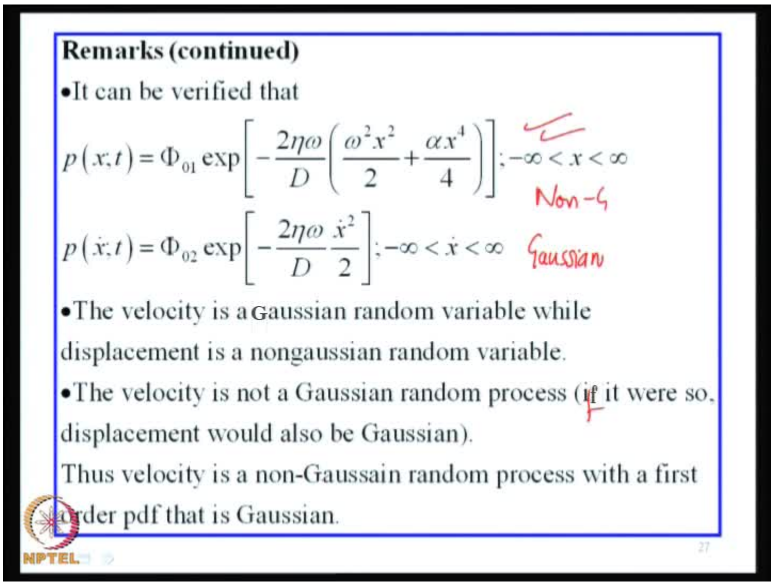

Now, we can verify that the probability density function of displacement has this form. It has in the exponent, a term containing x to the power of 4. That would mean this is non-Gaussian; whereas, velocity, that is, probability density function of velocity at a given time instant t is Gaussian. We should carefully interpret this result; we should not conclude that velocity is a Gaussian random process. If velocity were to be a Gaussian random process, it automatically implies displacement is also a Gaussian random process, because velocity and displacement are related through linear transformation. But, we know that displacement is a non-Gaussian random process. So, we can only say that the velocity is a Gaussian random variable while displacement is a non-Gaussian random variable. It should be noted that velocity is not a Gaussian random process. If it were so, the displacement would also be Gaussian, which is not the case. Therefore, here in this case, (Refer Slide Time: 34:56) velocity is a non-Gaussian random process with a first order PDF, that is, Gaussian. So, it is an interesting feature about the response.

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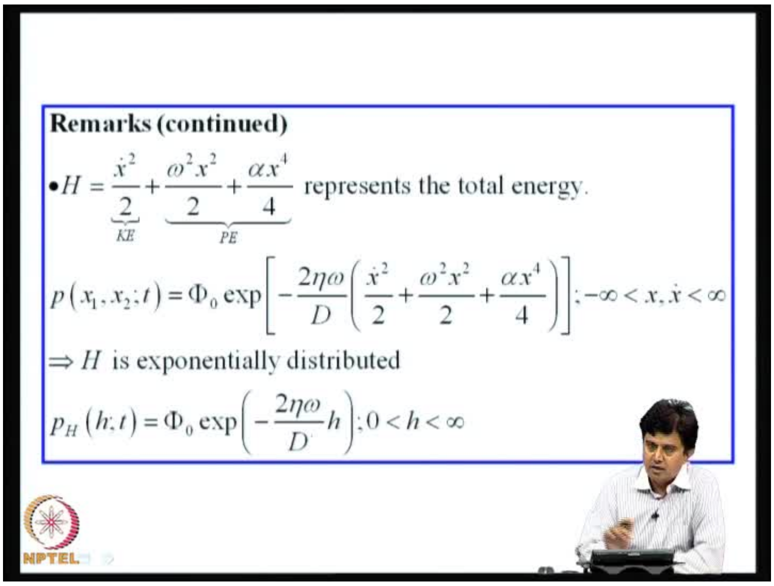

Now, one more remark that we could make is, in the expression for H, the first term is kinetic energy and the second term is potential energy. So, this represents total energy. And, you look at the joint probability density function between displacement and velocity; we get in the exponent, an exponential distribution with respect to energy. That means H is exponentially distributed. So, these types of results are encountered in Maxwell-Boltzmann equations for gas dynamics and so on and so forth. So, it has certain resemblance to those physical laws.

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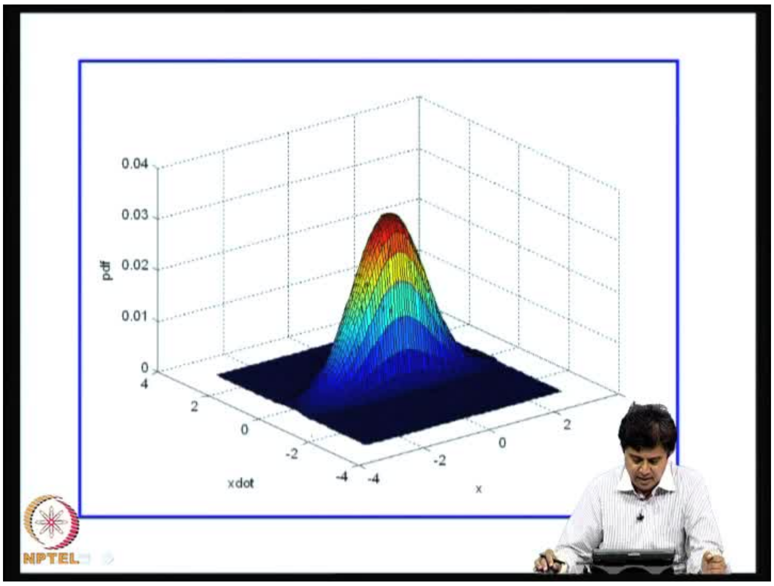

This is a plot of joint density function between displacement and velocity for a nonlinear duffing oscillator. It appears Gaussian-like; it is uni-modal.

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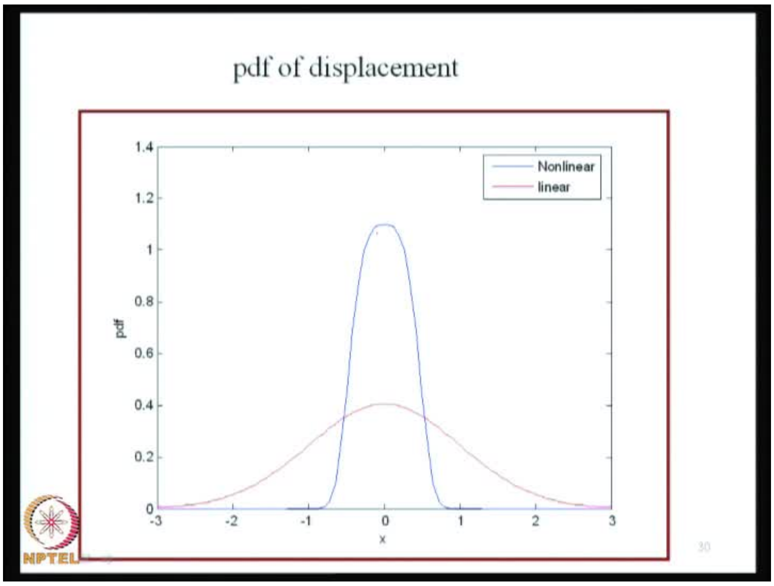

And, if you look at the marginal density function of the displacement, this red line is a Gaussian density function and the blue line is the result for PDF of displacement. So, clearly, we can see that it is non-Gaussian although it has a uni-modal character.

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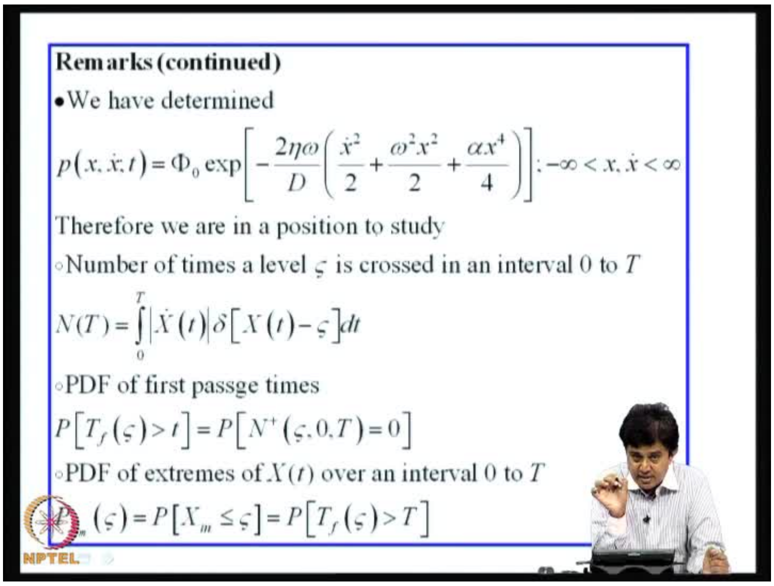

Through this analysis, valid albeit only for the steady state; we are still able to determine the joint density function between displacement and velocity at the same time t. Now, if you recall, much of the discussion that we made concerning level crossing statistics, first passage time and maximum, the basic quantity of interest was the join density function between process and its derivative at the same time. So, it is important to note that the Fokker-Planck equation actually provides that. So, for a duffing oscillator under white noise, I have exactly got the join density function between displacement and velocity at the same time. Therefore, now I can solve for number of times the level alpha is crossed in 0 to t; I can find out its average. And, if levels are high, I can use a Poisson model for that and derive the probability distribution function for the first passage time. And, based on the distribution for PDF of first passage time, I can also get a model for X t. So, that would mean the joint density function between process and its derivative at the same time is the key descriptor if you are interested in reliability related measures of the response.

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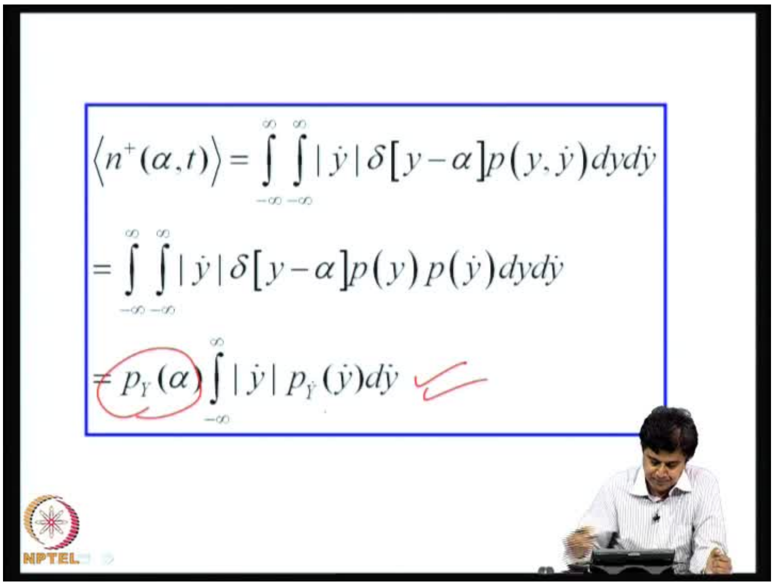

So, for this particular example, we can actually find out the average rate of the crossings of level alpha with positive slope. And, we get for the duffing system in terms of the non-Gaussian density function and an integral on a Gaussian density function. So, this can easily be evaluated and we can go ahead and do the other calculations.

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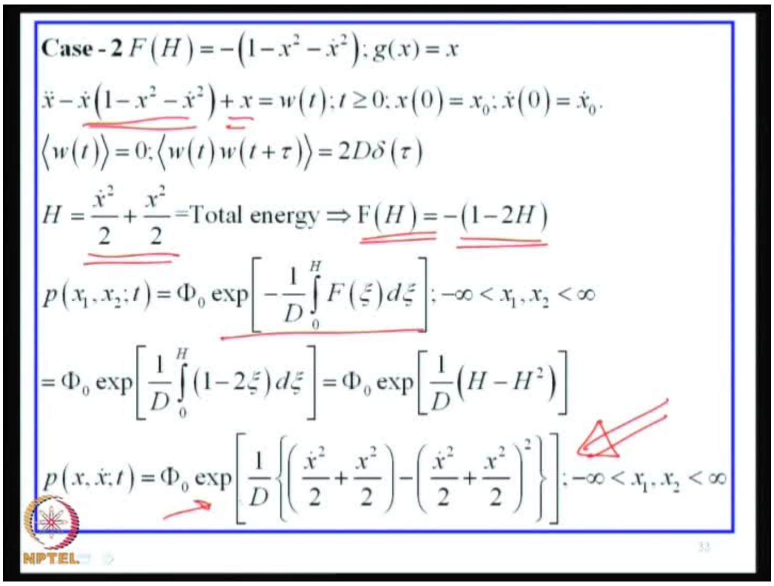

Now, we will consider another example, where F of H now I take it as 1 minus x square and x dot square; and, g of x to be linear. So, this is a nonlinear system with linear stiffness characteristic, but with nonlinear energy dissipation characteristics. Again, it is driven by white noise, delta correlated and all that. And, H is a total energy now, which is x dot square by 2 plus x square by 2. Since the stiffness terms are linear, we get this expression. So, we can go ahead and find out the joint density function between displacement and velocity in the steady state. And, that has this form.

Now, I can substitute for F of H, which is actually minus 1 minus 2H, because this is (Refer Slide Time: 38:54) x square minus x dot square minus of that; it is 1 minus 2H. So, that is what I have written here. And, if you now substitute into this known exact solution, I get this final answer, which is the joint density function between the displacement and velocity in steady state. Now, you carefully look at this (Refer Slide Time: 39:20) equation; you will see that x and x dot are not independent. You can find out marginal densities of x and x dot and multiply them and you will be able to show that the product would not lead to this.

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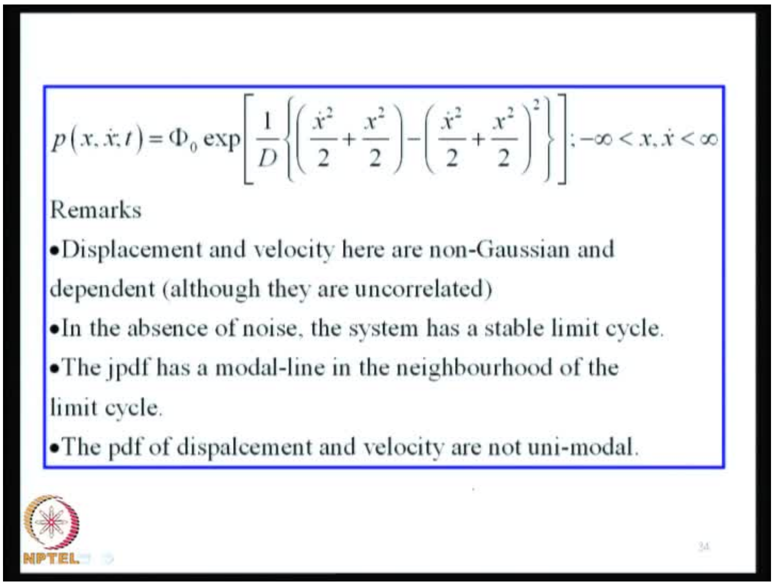

Now, x and x dot are non-Gaussian, but they are dependent although they are uncorrelated. x and x dot need to be uncorrelated, because the process in which steady state… For a stationary random process, a process and its derivative should be uncorrelated at the same time instant. So, this is an example, where process is non-Gaussian, that is, uncorrelated; x and x dot are uncorrelated, but they are jointly non-Gaussian and dependent; mutually dependent. Now, we need to take a look at the system, in the absence of noise, how does the system behave. I will discuss this now and show that the joint probability density function has a model line in the neighborhood of the limit cycle. And, we will see that the PDF of displacement and velocity, which are the marginal density functions, are not uni-modal.

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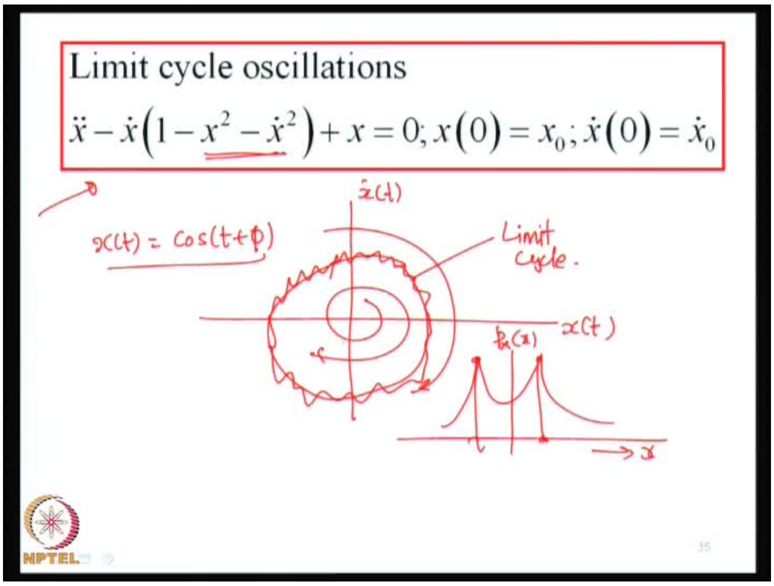

So, to see that, we need to look at this system bit more closely. If I were to plot the phase plane, that is, x of t and x dot of t, I will plot and I will eliminate time. And, I consider free vibration characteristics of these systems here. You look at the second term here. For small oscillations when x and x dot are small, the term inside this parenthesis would be positive, because x and x square plus x dot square will be still less than 1. And, therefore, the net sign of this is negative. Therefore, small motions tend to grow. That means origin is unstable. But, if x and x dot are large enough so that the term inside this parenthesis is positive, the system becomes positively damped and all large oscillations tend to decay. So, small oscillations grow and large oscillations decay. And, what happens is small oscillations grow will like this (Refer Slide Time: 41:38) and large oscillations will decay like this; and in between, there will be a closed trajectory. This is known as limit cycle. That means for this system, the free vibration consist of oscillatory motions.

In fact, you can show that for this system, x of t is cos t plus phi; where phi is an arbitrary phase angle. You substitute into this; you should be able to verify that this is an exact solution, because you will immediately see here 1 minus x square minus x dot square is 0 and x double dot plus x is satisfied by this equation. Therefore, this equation; this is the solution for this equation (Refer Slide Time: 42:15). So, this is actually a circle in the phase plane plot. So, this is the status of the system behavior when there is no noise. Now, upon this if you impose a noise, you could expect that most of the motion will take place around the steady state, will be around the limit cycle, the stable limit cycle. So, if you plot the probability density function of x, you may expect this kind of probability density function. There will be two modes: (Refer Slide Time: 42;47) x… And, this corresponds to the amplitude of the limit cycle system. In the absence of noise, the motion is essentially circular. So, you add a small noise, the motion tends to remain around that circle.

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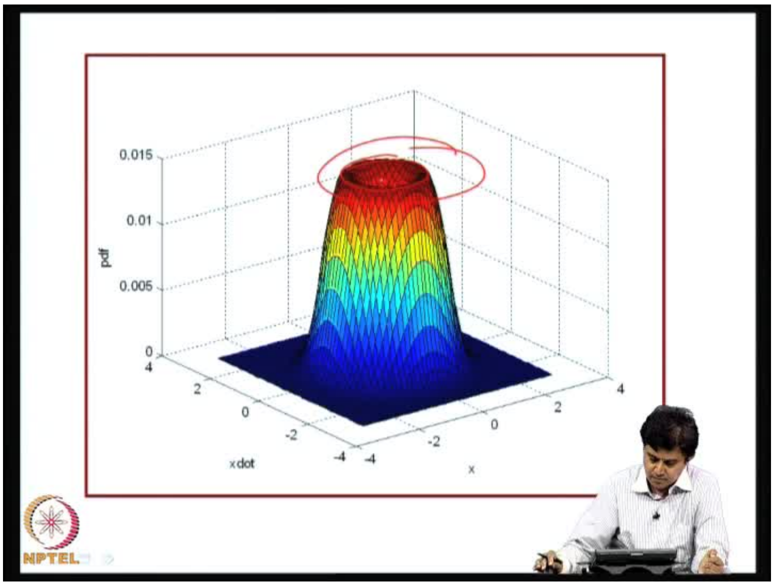

So, if you plot the joint probability density function of displacement and velocity in presence of noise, you see here a kind of a behavior, where there is a depression at x at the origin; and, this circular feature is the reminiscent of the limit cycle in absence of noise. Since for this system origin in absence of noise is unstable, there will be minima in the probability density function at the origin. Unlike when we get the Gaussian models, where origin is stable and a imposed noise, you get a mode in the probability density function, a maxima; whereas for systems where the origin is unstable, we expect a minima in the probability density function and maxima elsewhere, where there are stable fixed points are limit cycles. So, that feature is being displayed here.

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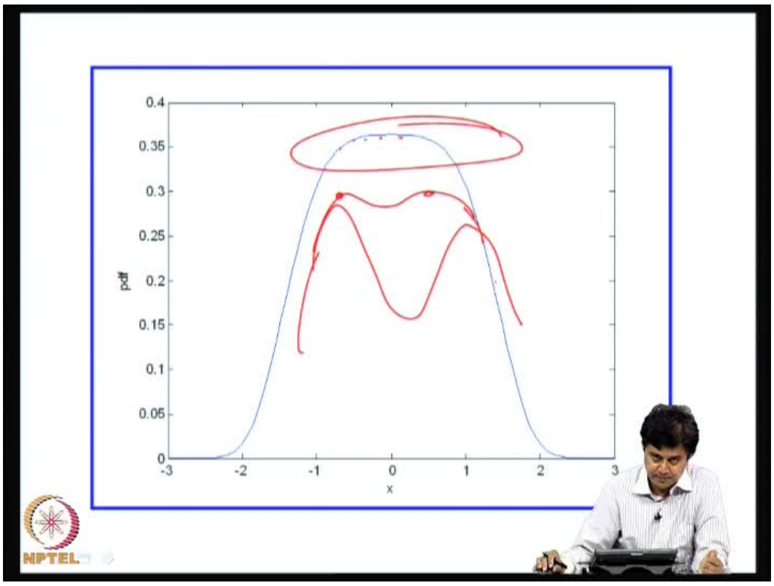

And, this is a probability density function of the displacement process. You can see here that it is essentially non-Gaussian. And, there seems to be several modes here, but if you carefully plot, it may have appearance something like this with two modes. So, as noise becomes small, indeed there will be this kind of behavior in the probability density function. So, this is one of the features of effect of noise on limit cycle systems. And, if you vary some of the parameters of the nonlinear systems and see how the quality of the response changes, in the contest of deterministic analysis, you will see that you will study fixed points and their stability. And, as the parameter of the problem is varied, the number of fixed points and their nature of their stability would change. And, that we call as bifurcation. These bifurcations manifest in the stochastic case in terms of maxima and minima of their associated probability density functions. And, that part of the interpretations can be made from the exact solutions that we have obtained for this system.

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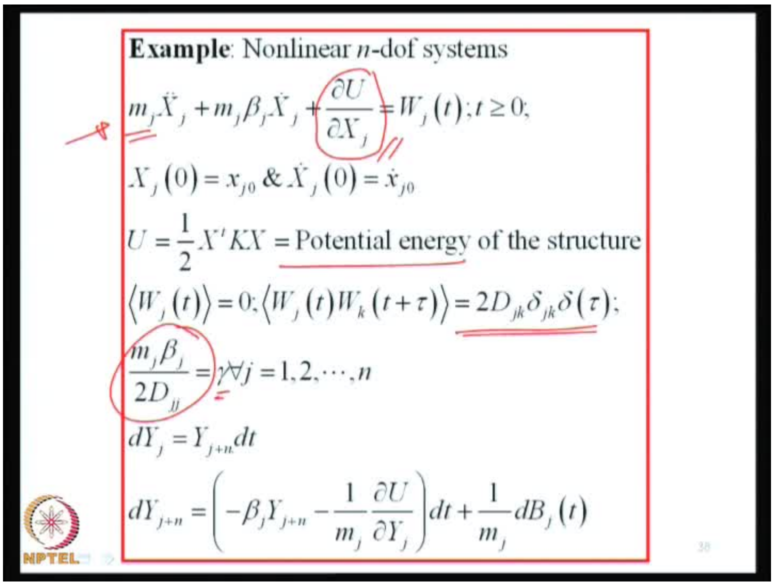

Now, that was a single degree freedom system under white noise excitation, a class of nonlinear models, where nonlinearity could be in damping or in stiffness. How about multi-degree freedom systems? What type of problems are amendable for exact solutions? In fact, the family of problems, which are amendable for exact solutions by their time means the steady state solutions derivable from FPK equations are exact. That is a wide clause; I am just speaking few examples from it. So, here I consider a multi-degree freedom system, where non-linearity in stiffness and that is in terms of derivative of potential energy, dou U by dou X j. So, these equations, the coefficient of accelerations and velocities, the matrices are diagonal. Nevertheless, all those equations are coupled through this nonlinear term.

This W j of t (Refer Slide Time: 46:04) is a vector of white noise processes for j running from 1 to n. And, we assume that they are all independent. W j t is independent of W k t. And also, we impose another condition that the parameter m j, which is the initial parameter and D jj, which is noise parameter; and, beta j, which is the damping parameter. For all j, this ratio is constant, (Refer Slide Time: 46:32) which is independent of j. Now, under only these exceptional situations, we get an exact solution.

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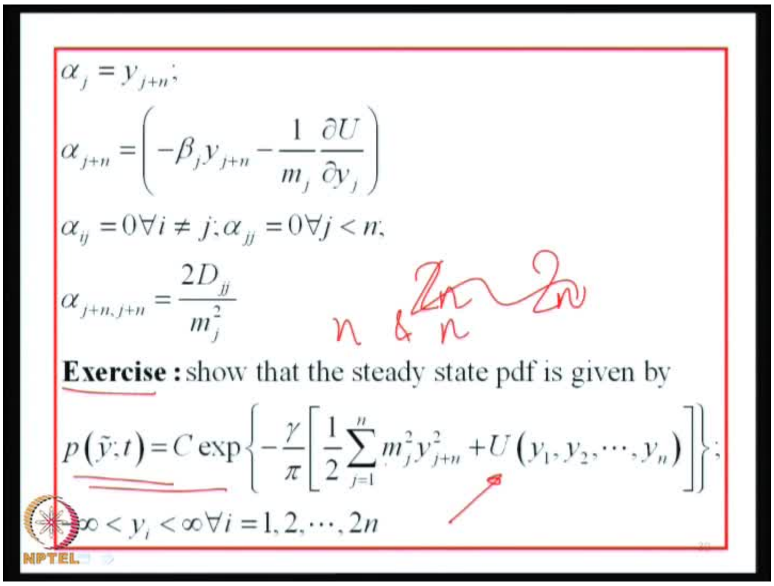

Now, we can rewrite this equation in terms of state space form. I will first use first n terms for displacements and the next n terms for velocities and we get this equation. And, we can derive the drift and diffusion terms in terms of the coefficients of the governing Ito differential equation. And, I will leave it as an exercise for you to show that the join density function between n displacements and n velocity processes is given by this expression in the steady state. This is a fairly comprehensive solution, but valid only for a limited class of systems under certain idealizations. So, this type of exact solutions will serve as benchmarks to test approximate methods. And also, it gives us insights into the influence of noise on system behavior. Because these solutions are exact, the interpretations we make have substantial value.

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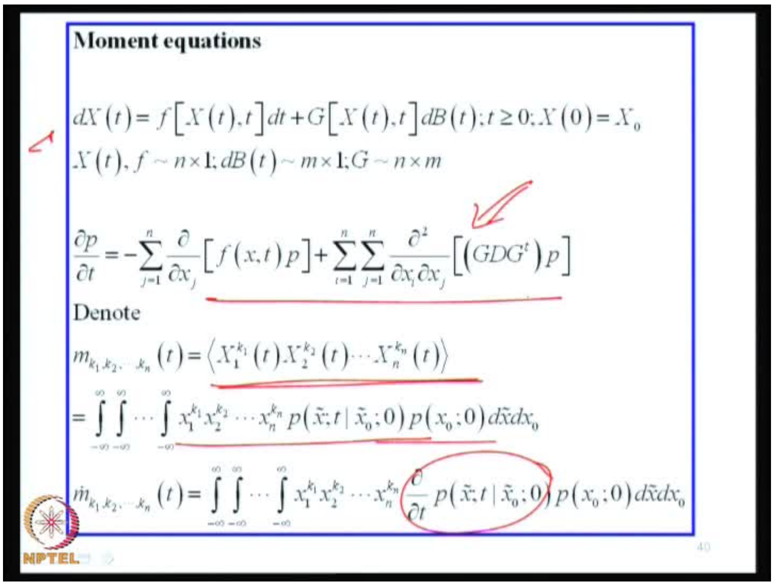

So far, we have been discussing about the evolution of probability density function. Now, how about moments? Can we directly derive the equations for moments. So, that we can do. Conceptually, it is not very difficult. So, we start by considering this, where there is a non-linear drift and diffusion terms and I write the governing FPK equation like this. Now, suppose if I am interested in this expectation, X 1 to the power of k 1 X 2 to the power of k 2, etcetera, I can write in terms of the probability density function and the initial probability… the transitional probability density function and the initial density function as shown here. If I differentiate this with respect to time, I get m dot and this is dou by dou t of p of this transitional probability density function. This is given by the right-hand side of the Fokker-Planck equation. So, I can substitute there.

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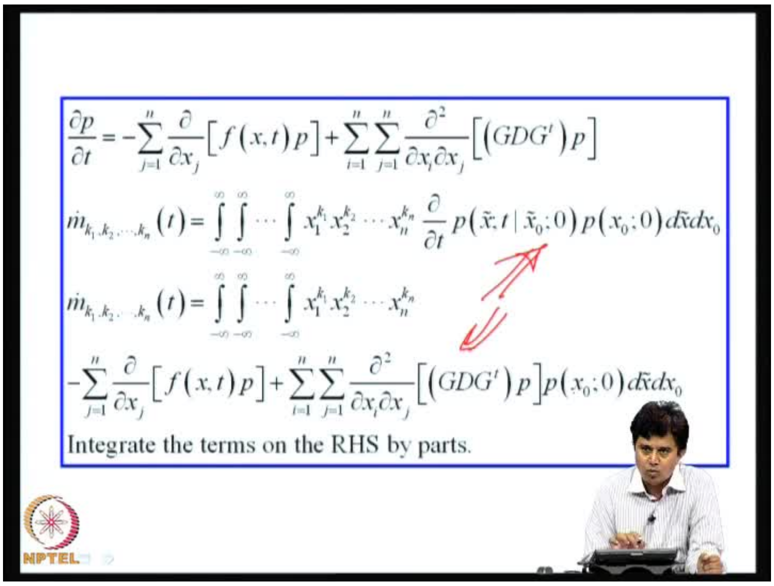

And, in principle, I can get this equation, because this dou by dou t of p is given in terms of the two terms in the FPK equation. And, this can be integrated by parts and we can get the moment equations. Same principle is possible. So, conceptually, there is no difficulty to just derive the time evolution of moments.

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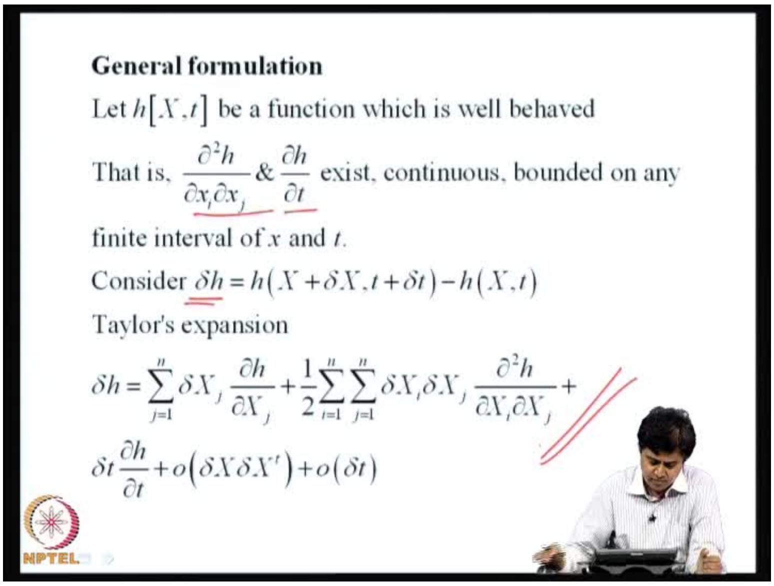

A more general approach would be to consider an non-linear function h of X comma t. We will assume that h is well-behaved; that is, in the sense, its derivatives – second derivative with respect to X i and X j and first derivative with respect to time exist. And, we consider now this increment delta h, which is h of delta plus h of X plus delta X, t plus delta t minus h of X comma t. Based on Taylor's expansion, I derive delta h in terms of h and its gradients as shown here.

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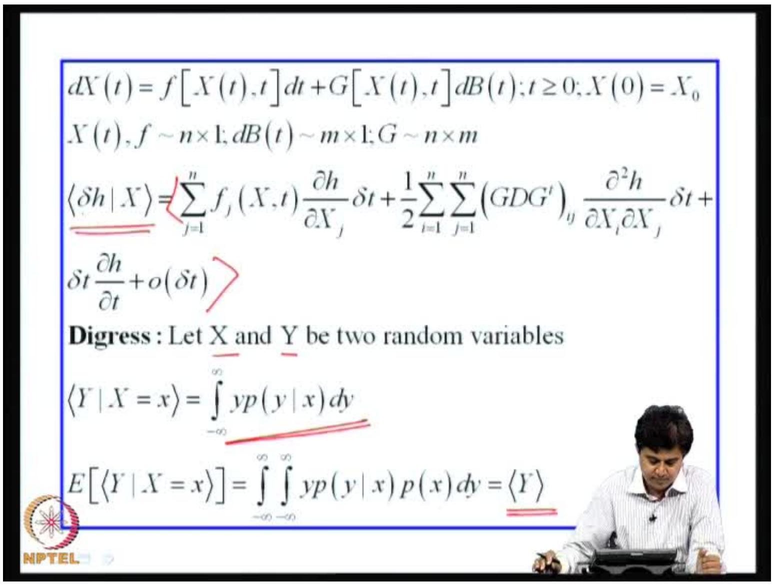

And now, if I consider the conditional expectation of delta h conditioned on X, this is a (( )) kind of terms that I will need in the FPK equation. We can show that this is nothing but expectation of this right-hand side (( )). Now, we will digress briefly. If you consider two random variables x and y and if you find conditional expectation of Y conditioned on X equal to x, it is given by this. This actually is a random variable. And, if you take its expectation, it will be actually expected value of Y. Therefore, this is a conditional expectation. Now, I will take conditional expectation of this.

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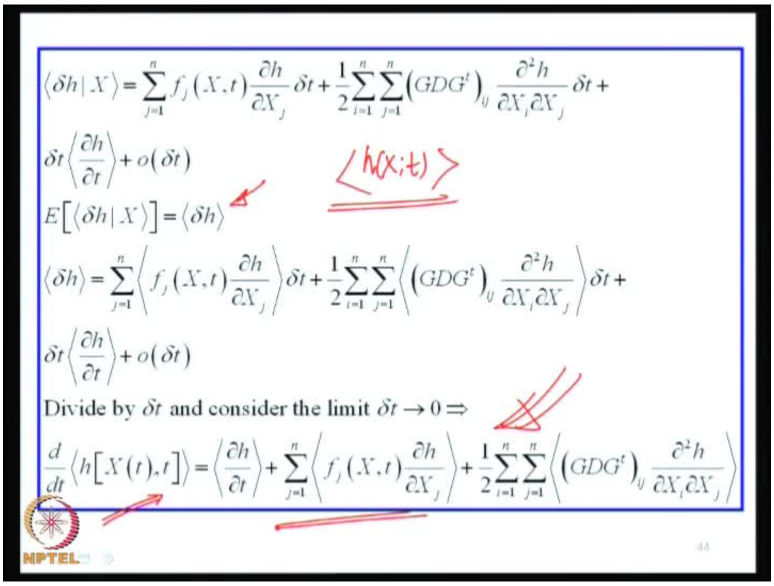

If I consider the expectation of this, my idea is to get expected values of this, the governing equation for this. So, if we use this idea, we can show that the moment equations are given by the equation shown here. This uses the properties of the Brownian motion process and their increments. And, we are also using the drift and diffusion coefficients in the Fokker-Planck equation and we can show that these are the exact equations for time evolution of the moments.

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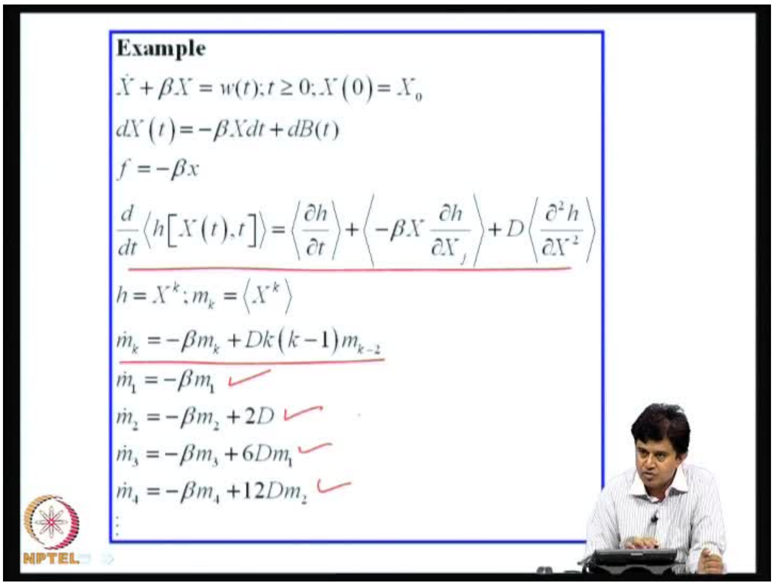

Now, a simple example would throw light on this suppose if I consider X dot plus beta X is equal to w of t. dX t is minus beta X t plus dB t. So, f will be minus beta x. This is the drift term and this is the equation for the evolution of expectation of h of X comma t. Now, suppose h is X to the power of k; m k, I denoted as expected value of X k. So, m k dot is given by this equation. So, you can substitute these terms here and you will see that m k dot is this. Now, k equal to 1. m 1 dot is minus beta m 1; m 2 dot is minus beta m 2 plus 2D; m 3 dot – I have to put here; X k equal to 3 and we will be able to solve this and I get a series of equations, which I can solve. Now, what happens if we look at steady state? So, the time derivative, all these moments vanish and I am left with an algebraic set of equations, which can be solved.

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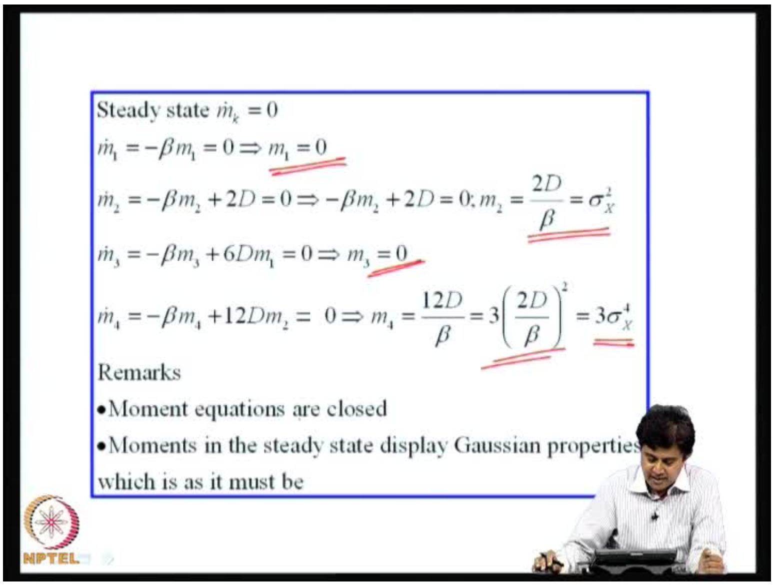

So, I immediately get the mean response, which is 0; the variance, which is this; skewness, third moment, which is 0; and, fourth moment, which is this. And, we show that this is actually three times sigma x to the power of 4, which is property of a Gaussian random variable. So, in this particular case, is very straightforward to write these moment equations (Refer Slide Time: 53:03). And, if you are interested only in steady state, you are only going to solve algebraic equation. So, there is no convolution integral, there is no selection of partial differential equation; you can directly get the steady state solution in a straightforward manner.

Now, one thing that we should notice here (Refer Slide Time: 53:21) is that the moment equations here are closed. In the sense, you want to find the mean, you need not have to know any moment, which is of higher order. And, another thing that we should notice is that we know that the response is Gaussian and it is displaying features; the response moments are displaying the features of Gaussian random variable with 0 mean, for which mean is 0, skewness is 0 and variance, the fourth order moment is three times the 4th power of standard deviation. So, this is skewness, (( )) is 3, what I am getting. So, this matches with known features of Gaussian response.

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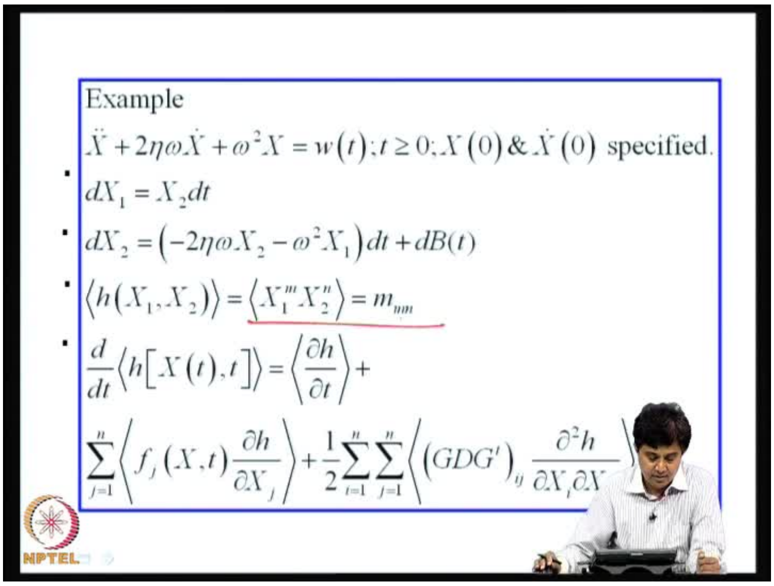

Now, we will consider the familiar single degree freedom system under white noise. So, we can go through this formulation, write the Ito's equation and write these moment equations. You can write equation for…. So, here I am introducing the notation – this notation – X 1 to the power of m X 2 to the power of n; expected value I am writing as m mn.

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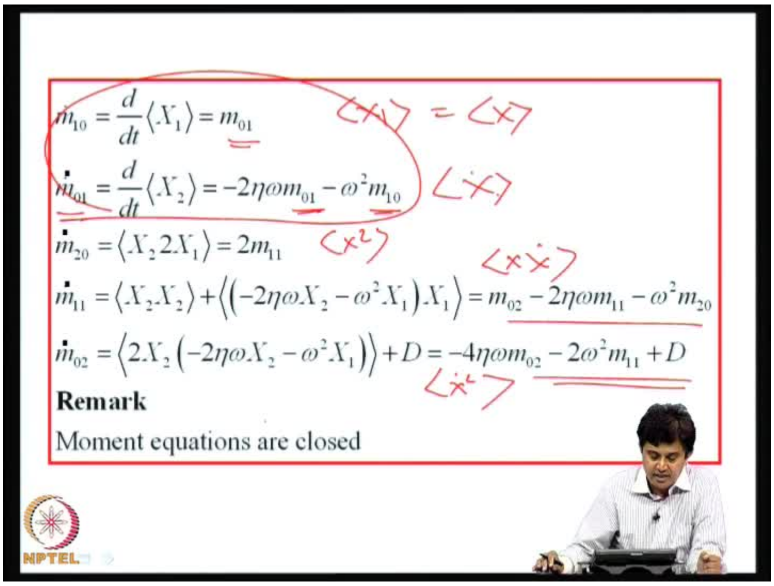

So, m 10 is expected value of X 1; m 01 is expected value of X dot. This is nothing but expected value of X. So, m 20 is expected value of X square. m 11 is XX dot, so on and so forth. So, you can write these equations. This is variance of mean square value of the velocity. So, on the right-hand side, you see that if you are looking at mean, m 10 is coupled to m 01, but equation for m 01 has only m 10 and m 01. So, these two equations can be solved together and I can find out the mean. At the level of characterizing mean, I need not have to know anything more than the mean of the response. Similarly, second order moments, m 20, m 11, m 02 – all these three can be solved using the properties at the second order level. So, these can be solved. So, we say that the moment equations are closed.

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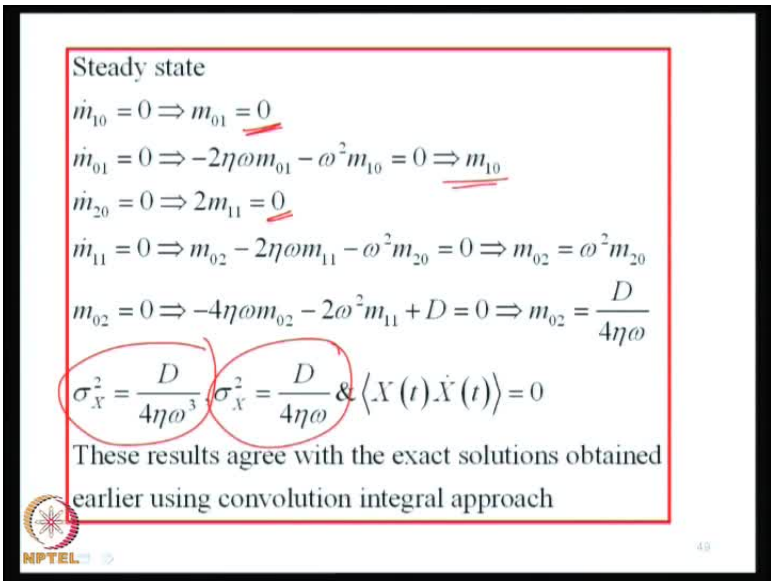

Now, steady state – the time derivative of the moments vanish; mean of displacement is 0; mean of velocity is 0. The correlation coefficient between displacement and velocity is 0, because the response is in steady state. Now, if we analyze this, we get that the variance of the displacement is D by 4 eta omega cube; and, the variance of velocity is D by 4 eta omega, and the process and time derivative are uncorrelated. These are the exact solution that we obtained by two different methods: one by using convolution integral; other by doing spectrum analysis. So, now, this is the third approach, which leads to the same answer. So, these results agree with the exact solutions obtained earlier using convolution integral and spectrum analysis approaches. So, it is quite satisfying.

So, in the next lecture, what I will do is I will consider these moment equations for slightly more general class of problems; and then, we will also consider questions on first passage times – how to use Markov process theory to characterize first passage times. And next, we will also consider questions on how to characterize enveloped processes using Markov property. So, we will take these topics in the next lecture and we will conclude this lecture at this juncture.