Stochastic Structural Dynamics

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Lecture No. # 22

Markov Vector Approach-2

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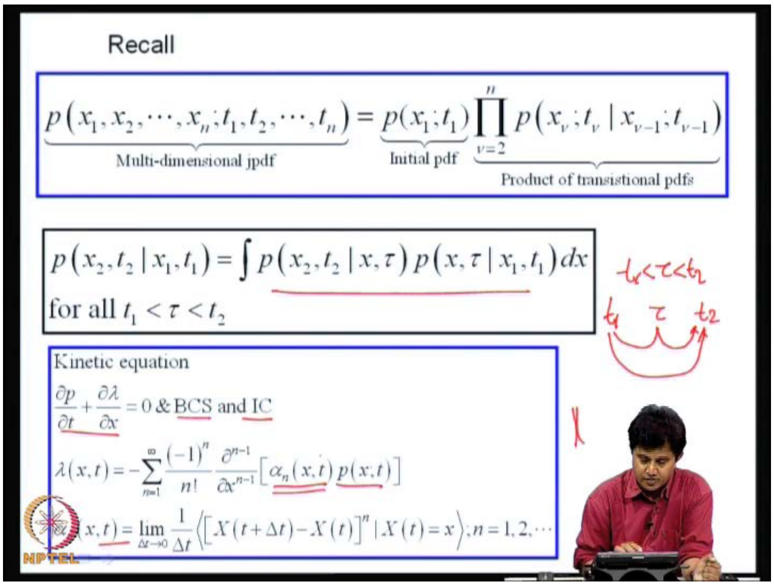

So, in this lecture, we will continue with our discussion on properties of Markov processes and how they can be used to analyze probabilistic characteristics of response of randomly driven dynamical systems.

A quick recall of what we did in the previous lecture; we showed, that for a Markov random process, the nth order joint probability density function can be expressed in terms of the initial probability density function, and products of, what are known as transitional probability density function.

So, complete specification of a Markov process, therefore, could be in terms of the initial probability distribution function and this transition - transitional - probability density functions for different choices of t 1, t 2, t 3, t n and for different choices of n.

We also showed that, the transitional probability density function need to satisfy a compatibility condition known as Chapman-Kolmogorov Smoluchowski equation, where if we consider three time instants x 1, x 2 and tau, so that, t 1, t 2 and tau, so that tau lies between t 1 and t 2; if we consider the transition from t 1 to t 2, and look at transition from t 1 to tau 1, tau to t 2, you should get the same answer. So, for that to happen, the transition PDF should satisfy this condition and this is known as the Chapman-Kolmogorov Smoluchowski equation.

We also went through a sumo Tds derivation of the governing equation, for evaluation of probability density function of a scalar random process. We showed, that this equation known as kinetic equation has a found, dou p by dou t plus dou lambda by dou x equal to 0 and we discuss the associated the boundary condition and initial condition. This lambda, lambda for x comma t is expressed in terms of the probability density function, and quantities denoted by alpha n, which were known as derivative moments.

So, if you have to utilize this kinetic equation to solve any given problem, we should first evolve a method to determine these derivative moments, because they are the parameters in this governing differential equation.

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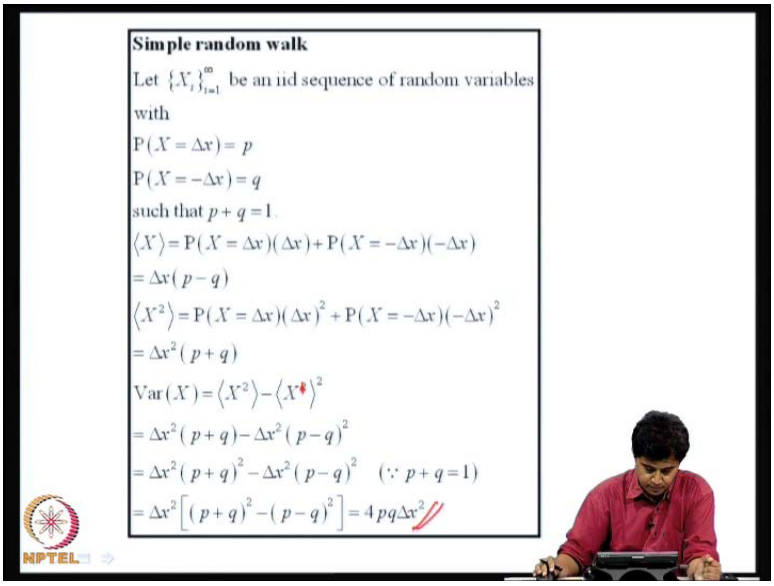

So, in this lecture, we will consider some of this questions; and we will begin by considering some properties of simple random walk, and how they behave as delta x and delta t goes to 0; this I had discussed in earlier, one of the earlier lectures; we will proceed beyond what we did in this previous lectures.

So, let us consider, X i i 1 running from 1 to infinity to be on sequence of identically distributer and independent random variables, with three states; probability of X equal to delta x is p, probabilities of X equal to minus delta X is q, so the p plus q is 1. So, the expected value of the X would be delta x into p minus q, and mean square value would be delta x square into p plus q; p plus q is 1, but I still choose to return this as p plus q; variance should be the mean square value minus square of the mean; and we write this in terms of the mean square value and the square of the mean, and manipulate these expressions and we get the variance to be 4pq delta x square.

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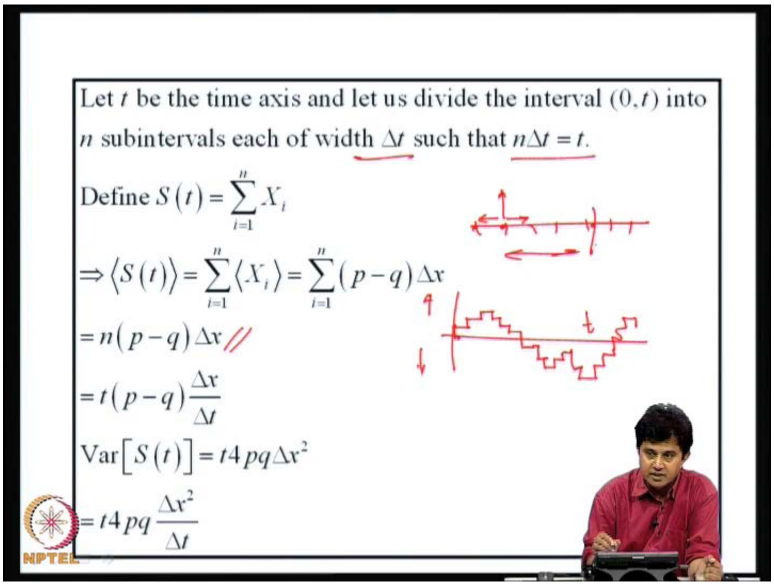

Now, let us consider t to be the time axis and let us divide the interval 0 to t into n sub intervals, each of width delta t, such that, n delta t is t. Now, I define S of t, a sum of X i from i equal to 1 to n. Expected value of S of t is expectation of the right hand side, which I show, this to be n into p minus q into delta x; now, substituting for n in terms of t, I can write it as, t into p minus q delta x by delta t. Variance of S of t is 4pq delta x square into t, and that I can write it as, the 4pq t into delta x square by delta t. Now, this is simple random walk; so, what we are doing is, we are considering n time instants and a every time instants and I am tossing a coin, and if I get head, I will move this way; and if I get, I mean actually one-dimensional, I will move backward or forward, depending on whether I get tail or head; and S of t tells me, after n such tosses, where I am in time axis. So, we are executing random walk on a line, and that we show, for example, forward moments if I show on this direction and negative one in this side, suppose at this moment I get a head, I move up, I stay put till the next toss, and if I get a head, I move up, stay put, move up, stay put, come down, come down, so on and so far; so, this is one realization of x sequence of toss. So, at any time t, S of t tells me in my position on these line.

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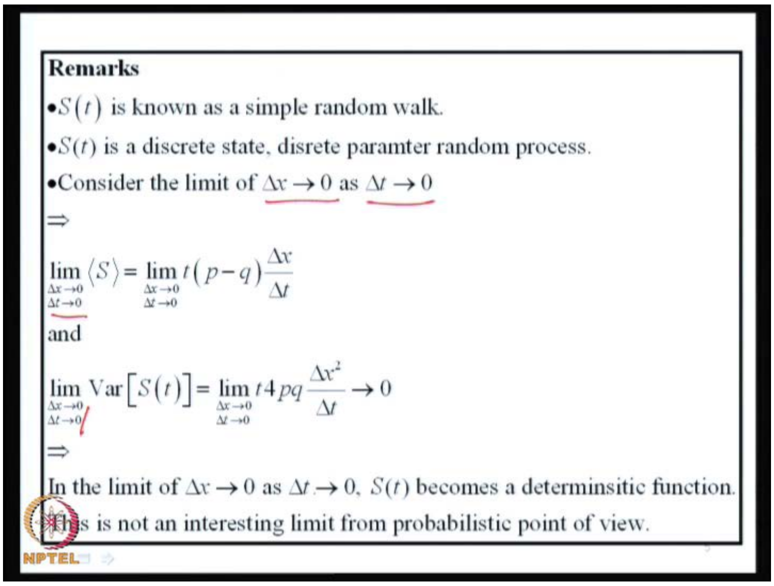

So, S of t is known as simple random walk. Now, the S of t is the discrete state, discrete parameter random process; it is a Markov chain. Now, what I am interested here is, what happens to this random walk, as we take limits of delta x going to 0 and delta t going to 0. If I do that, limit delta x is going to 0, delta t going to 0 of expected value of S, I get t into p minus cube delta x by delta t, which is the meaningful limit, because both delta x delta t are going to 0; so, it can be a meaningful limit.

But on the other hand, if we look at a similar limit on variance, that is, limit delta x is going to 0, delta t is going to 0, you will look at variance, where to look at 4pq t delta x square by delta t; as delta S goes to 0, the numerator goes to 0. So, this can be written as delta x into delta x divided by delta t; so, we can see that the variance goes to 0.

That means, if variance is 0, that corresponding quantity is essentially deterministic; so, in the limited delta x going to 0 and delta t going to 0, S of t becomes a deterministic functions. This is not an interesting limit from the point of view of probabilistic analysis, right; S of t is a random walk, which is a discrete state, discrete parameter random process, but under this limit of delta x going to 0 and delta t going to 0, we get a deterministic process; so, this is not interesting; what can we do now?

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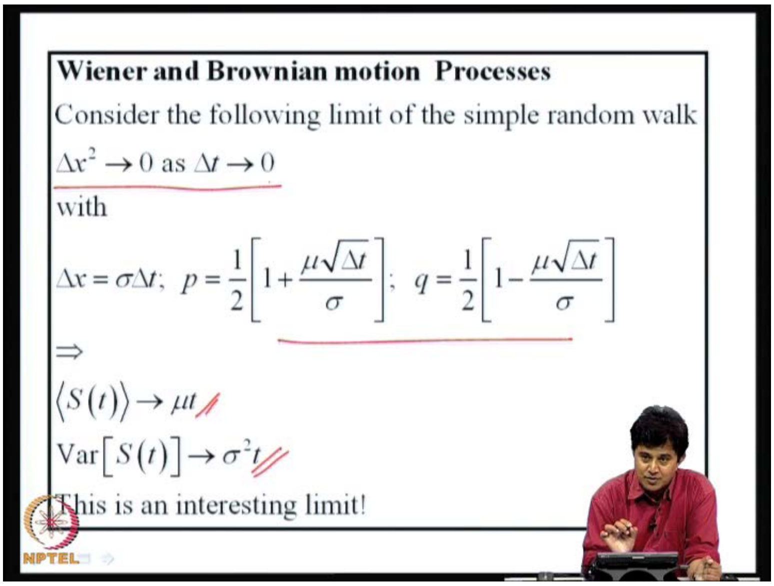

Let us do the following. Now, let us consider the limit of simple random walk in which delta x square goes to 0, as delta t goes to 0; in that case, what happens? I will rearrange this terms slightly, I will write the delta x is a sigma delta t, and p and q are write in the particular form, I introduce the parameter in mu and sigma; and I can show that, under these assumptions, the expected value of S of t goes to mu t, and variance goes to sigma square t. Mind you the limiting operation is quite peculiar, it is delta x square goes to 0, delta t goes to 0; that means, delta x goes to 0, square root of delta t goes to 0.

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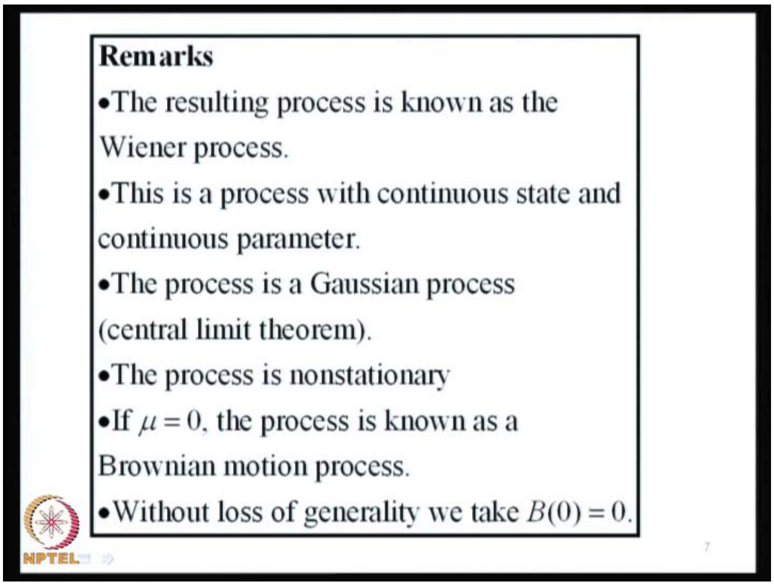

So, the resulting process is known as wiener process. A simple random walk, in which delta x square goes to 0, as delta t goes to 0 leads to a random process, which was continuous time in state and that process is known as Wiener process.

The process is Gaussian, because S of t obtained by adding independent random variables, and we can invoke central limit theorem and prove that the process is actually Gaussian. The process is non-stationary, because mean and variance, function or functions of time; and if mean is 0, we see that the process is a Brownian motion process.

Now, without loss of generality, we take B of 0 to be 0. So, the Brownian motion process is a non-stationary Gaussian random process, which is obtained as the limit of a simple random walk, in which delta x square goes to 0 and delta t goes to 0. Because of this peculiar property, the sample behavior of the Brownian motion will have many pathologies, and that we need to understand, if you want to proceed with these models.

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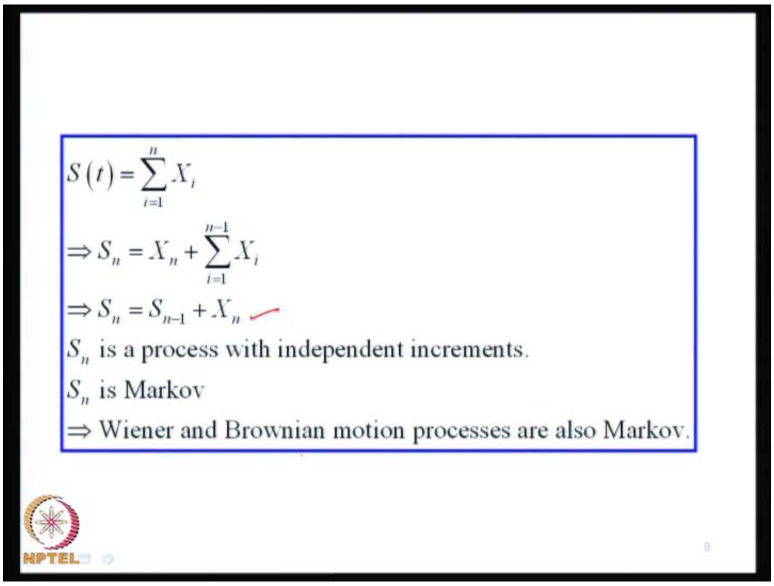

Now, let us look at this random walk a bit more closely, S of t is sum of i equal to 1 to n X I; therefore, S can be written as, X n plus S n minus 1 as shown here. So, S n is the process with the independent increments, S n is Markov. Since wiener and Brownian motion process is obtained as limit of random walk, the Wiener and Brownian motion processes are also Markov in nature.

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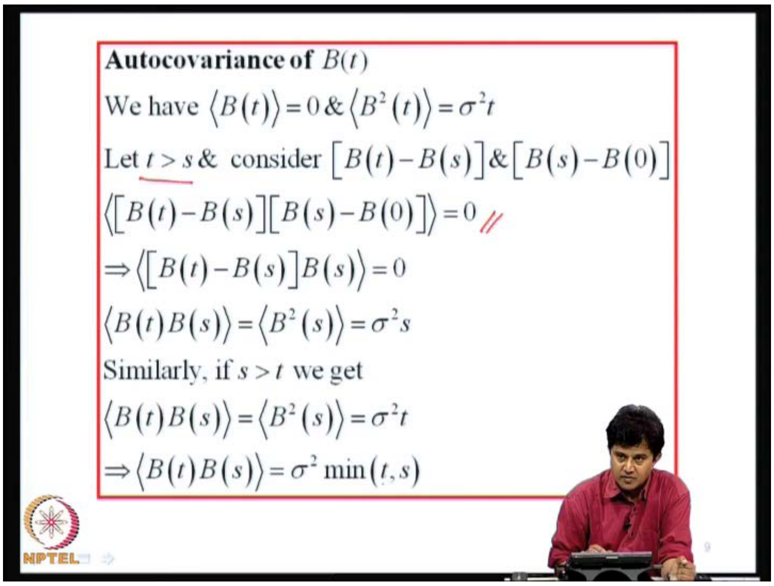

Now, let us look at some of the properties of Brownian motion process. I would derive the mean, the mean is 0, and variance we have derived. Now, let us look at the mean is 0 and the variance is sigma square t, that is what we have to done. Now, let us look at auto covariance of B of t. Let us consider two time instant, t and s, so t is greater than s; and we consider the increments, B of t minus B of s and B of s minus B of 0. So, I am considering the 0 s and t, three times instants and looking at the increments.

Now, since the process as independent increments, the expected value of B of t minus B of s into B of s into B of 0 is 0, because processes independent increments. Now, B of 0 have to taken into be 0; therefore, B of s into B of t minus B of s 0. So, that would mean, expected value of B of t into B of s is nothing but B square of s, which is sigma square s. So, here, I am assuming t is greater than s; if you assume s to be greater than t, we get this expectation to be sigma square into t; or in other words, I can write auto covariance of B of t as expected value B of t into B of s is, sigma square and min(t,s).

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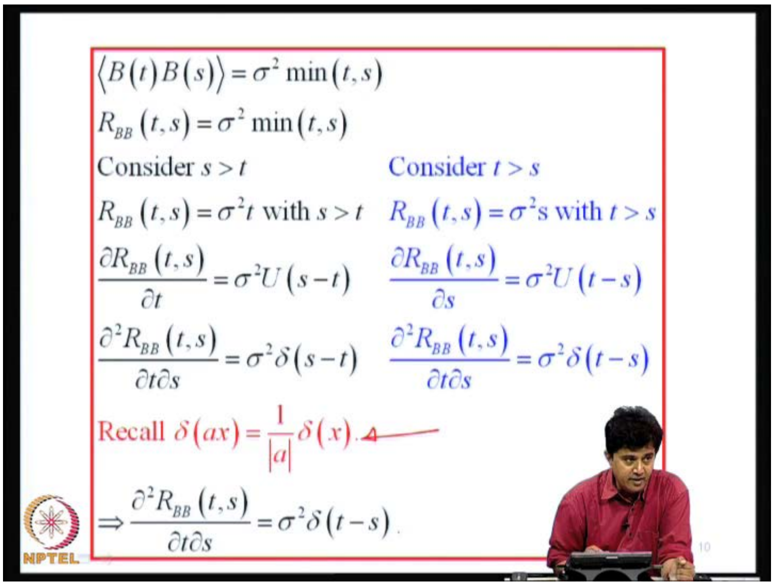

So, I will write R B B t comma s is, sigma square minimum of t comma s. Now, let us consider s to be greater than t; I am interested in derivatives of Brownian motion process. So, if I now differentiate the auto covariance with respect to t, I will get, if I assume s to be greater than t, R BB sigma square t is greater than t; therefore, this derivative can be written as, sigma square into unit step function s minus t.

If I now differentiate the next time, the step functions becomes a Direct delta function and I get sigma square s minus t. We can also consider situation t greater than s separately, and using similar logic, we can also show that, dou square R BB t comma s dou t dou s is sigma square Direct delta of t minus s. But if you recall, Direct delta of a x is nothing but 1 divided by modules of a into Direct delta of x. So, using that property, we can show that, the second derivative of R BB with respect to t and s is, it can be written as, sigma square delta of t minus s; this is, as we know is auto covariance of white noise process.

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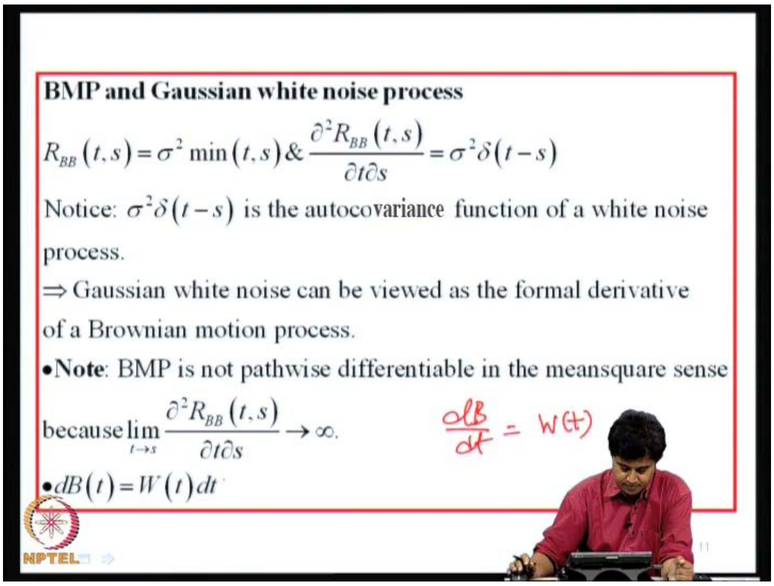

So, we have R BB of t comma s is, sigma square minimum t comma s and its second derivatives sigma square delta of t minus s. Sigma square delta t minus s is auto covariance function of a white noise process. Therefore, a Gaussian white noise process can be viewed as the formal derivative of a Brownian motion process.

Now, you should notice that BMP is not path wise differentiable in the mean square sense, because as limit t tends to s, we are already getting this in Direct delta of function, that shows that it is not differentiable in the mean square sense, right; path wise it is not differentiable. But if you differentiate the auto covariance of the Brownian motion process and allow Direct delta representation in that, in that generalize sense, we get the second derivative of auto covariance Brownian motion process, leads to the auto covariance of white noise processes.

Therefore, we say that, white noise - Gaussian white noise - can be interpreted as a formal derivative of Brownian motion process. The word formal emphasizes, that I am not talking about samples; a sample of white noise do not exist, because the variance of white noise is infinity and therefore it is not physically realizable. On the other hand, I can write dB by dt, formally can be written as, you know, the formal derivative of Brownian motion is white noise; I can write the increment of the Brownian motion process as W of t into d t.

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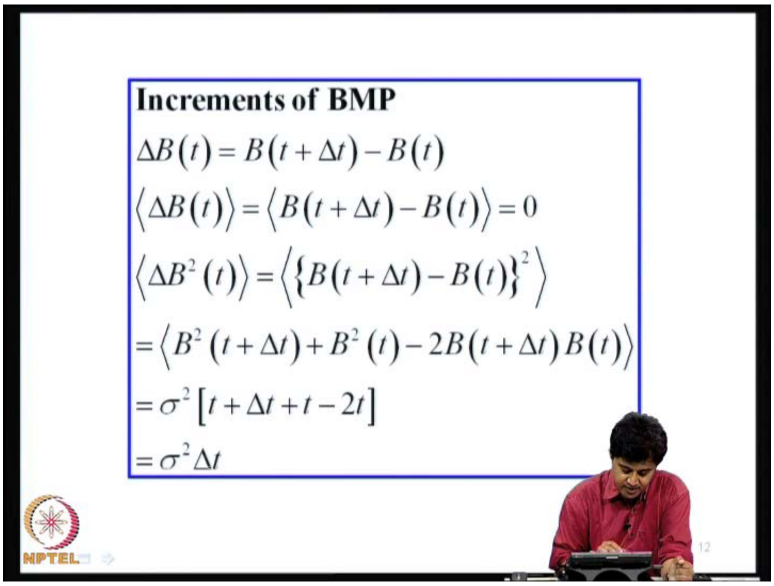

So, the increments of Brownian motion process delta B t is B of t plus delta t minus B of t. What is mean of this? Mean is 0. What is it variance? The B of t plus delta t minus B of t whole square; you expand this and we use the, we need auto covariance of B of t plus delta t and B of t, and we know that the sigma square minimum of t comma s if you use and simplify this, we can show that the variance of the increment is sigma square delta t. We should notice that the increment is on delta t, but the variance which is square is again linear function of delta t.

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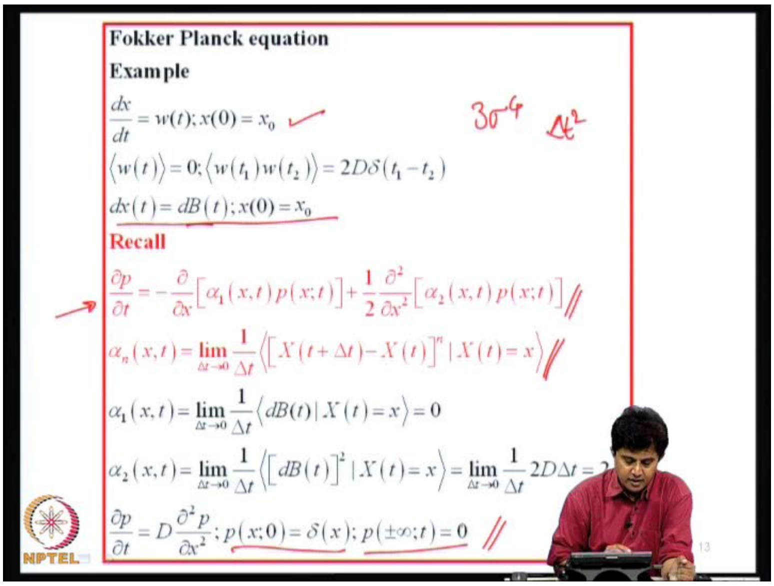

Now, let us look at simple problem, dx by dt is white noise, with initial condition x of 0 is x naught, and the mean of this is 0 and auto covariance is delta function. Now, this is the governing equation, I will write this as, dx of t as dB t, with x of 0 as 0. Now, we have this Fokker-Planck equation, dou p by dou t, which is minus dou by dou x alpha 1 x comma t p of x comma t etcetera etcetera.

What is alpha n? alpha n are this derivative moments, which involves nth moment of the increment. Now, let us start with alpha 1, alpha 1 will be limit delta t to 0 1 by delta t dB of t, x of t plus delta t minus x of t , d x of t, which is dB of t; so dB of t condition x of t equal to x is 0, because expected value of dB of t 0; alpha 2 is expected value of dB of t whole square condition x of t equal to x and this is 2D delta t, and delta t delta t gets cancelled, and I have 2D.

Now, if you consider alpha 3, it will be dB cube of t; expected value of t dB cube is 0, because B of t is a Gaussian random process with 0 mean, so expected value of cube of random Gaussian random variable is 0. You consider the fourth moment, fourth moment will be 3 into 4th power of the variance, and this will involve delta t square and delta t square in the numerator divided by delta t will be delta t; and as delta t goes to 0, alpha 4 goes to 0. So, all higher alphas go to 0, for based on one similar logic, and therefore, the kinetic equation terminates at only two terms; so, this is a kinetic equation that we have to write.

Now, we talking about derivative - formal derivative - Brownian motion process; therefore, the boundaries are x equal to plus minus infinity, because we are talking about Gaussian random processes. So, the initial condition will be this and boundary condition would be this. So, we have to solve this equation, under this prescribed initial conditions and boundary conditions.

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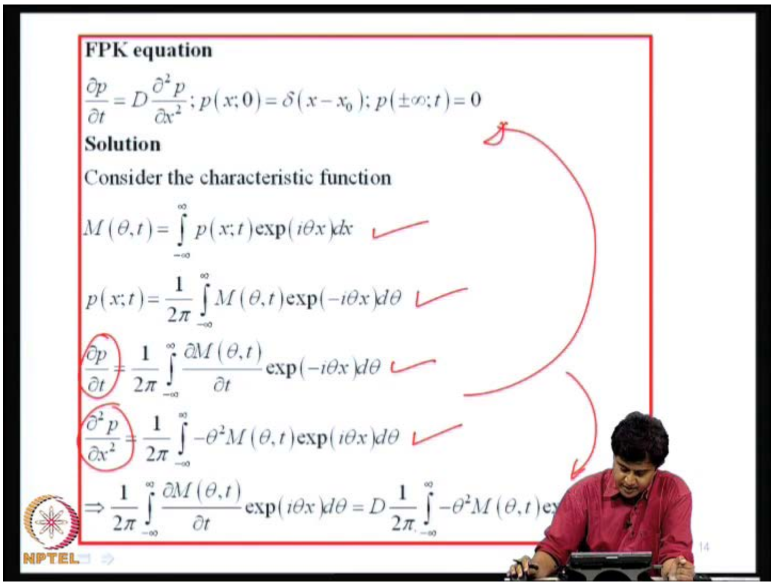

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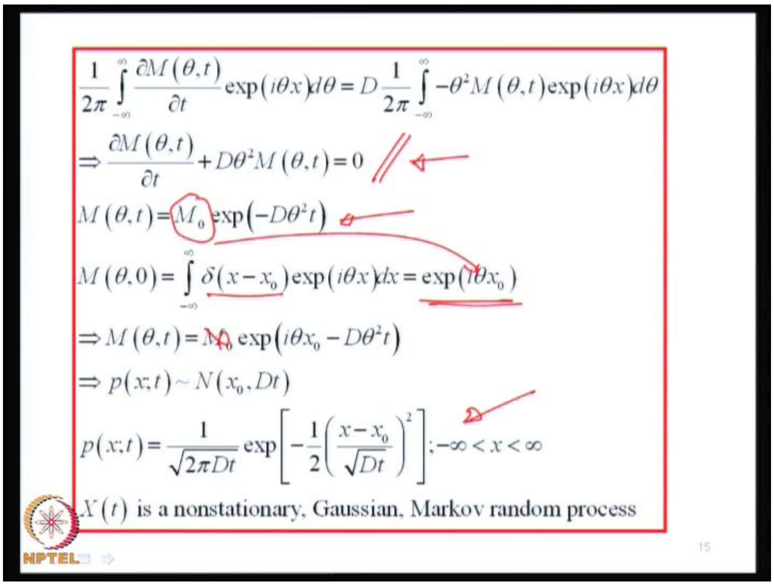

Now, in this particular example, it is easy to solve this, we can consider the characteristic function, I will call it as m of theta comma t, which is the Fourier transform of the density function. And I will express now in density function in terms of the characteristic function, and find out, now, dou p by dou t and dou square dou p by dou x square, this is like using Fourier transform technique to solve partial differential equation.

So, dou p by dou t would be in terms of this and dou square p by dou x square will be in terms of this. Now, we substitute these two into the original equation, I get this; little bit of simplification would lead to ordinary differential equation, for the characteristic equation the characteristic function, and I can get m of theta comma t M naught exponential minus D theta square t. Now, what is M theta comma 0? Which is actually the Fourier transform of the density function - probability density function - at t equal to 0, which is Direct delta function and I get this i theta x naught.

So, this M naught which is arbitrary constant now turns out to be this. So, if I now substitute into this, this M naught already written, this i theta x naught minus D theta square t; this is nothing but the characteristic function of a Gaussian noise random variable. So, it would mean that, probability density of x comma t is normal function n, with mean x naught and variance Dt, and this is written here. So, from this, you can see that, x of t is non-stationary Gaussian and Markov.

So, I am talking about the formal derivative of Brownian, you know, Brownian is actually, this is we are talking about Brownian motion process, dB by Dt equal to w of t is equation I am solving. So, these are the property of b of t; this is a Brownian motion process that am talking about. So, this tells as, briefly illustrate how the kinetic equation can be used to analyze a relatively simple problem. We would be interested in this approach, if this approach can be generalized to more complicated systems; so, how do we do that?

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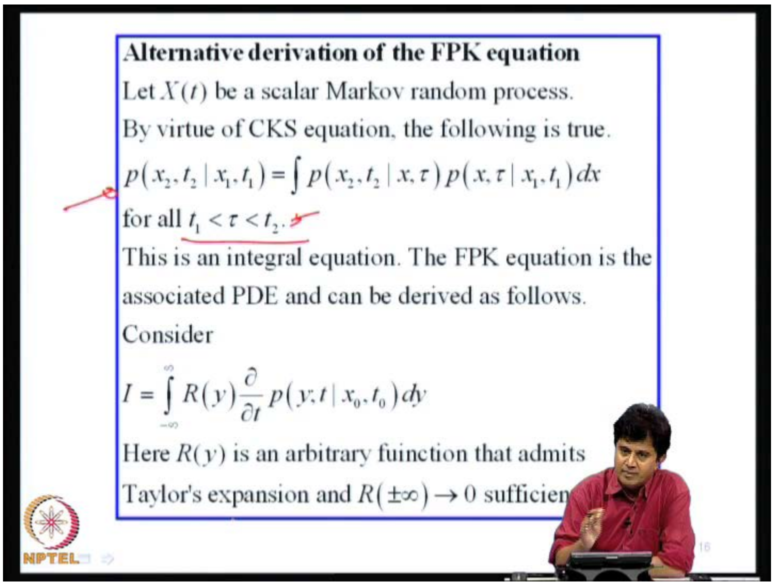

Before we get into that, there is an alternative derivation of the Fokker-Planck equation, in terms of the Chapman-Kolmogorov Smoluchowski equation and that is instructive; so, let us go through that. So, let X of t to be a scalar Markov random process and the virtue of Chapman-Kolmogorov Smoluchowski equation, the following will be true; this integral equation is true, where t 1 is t 1 tau 2 are ordered as shown here.

Now, this is an integral equation. Now, for an given integral equation, we can always derive an associated partial differential equation; and indeed, such that partial differential equation would be the Fokker-Planck equation. We can show that; that would mean Fokker-Planck represents the consistency condition for the process to be Markov, just as this CKS equation represents the consistency conditions for process to be Markov. So, how do we do that? We do, we start by considering an integral minus infinity into plus infinity R of y dou by dou p by dou t d y. This R of y presently arbitrary function, that admits Taylor’s expansion that immediate smooth and well behaved, and also it approaches plus minus infinity sufficiently fast. What exactly mean by this, it will become clear, as we go along; it is a well-behaved function.

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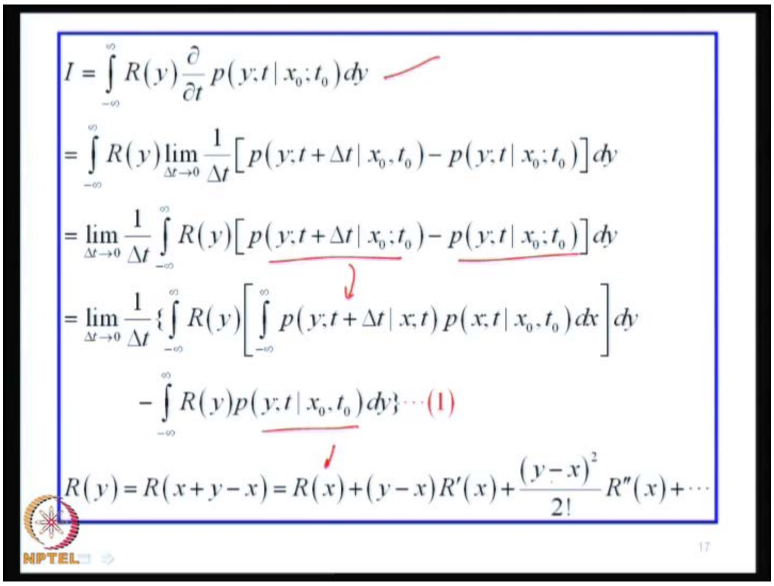

Now, let us look at this integral i, and I will write this as, this derivative in terms of this limit, dou p by dou t is nothing but 1 by delta t p of t plus delta t minus p of t divided by delta t is delta t goes to 0. So, I will look at this, now, p of y t plus delta t condition x naught t naught, and I will use the CKS equation to represent that in the first term, this is this; in the second term remains as it is. So, I have utilize the factor x of t is Markov. This R of y, I will now expand in Taylor’s expansion around x, I will consider R of y and expand around x as shown here, and this expansion is permissible, because I assume R of x is sufficiently smooth; so, I go back and substitute this into the first integral.

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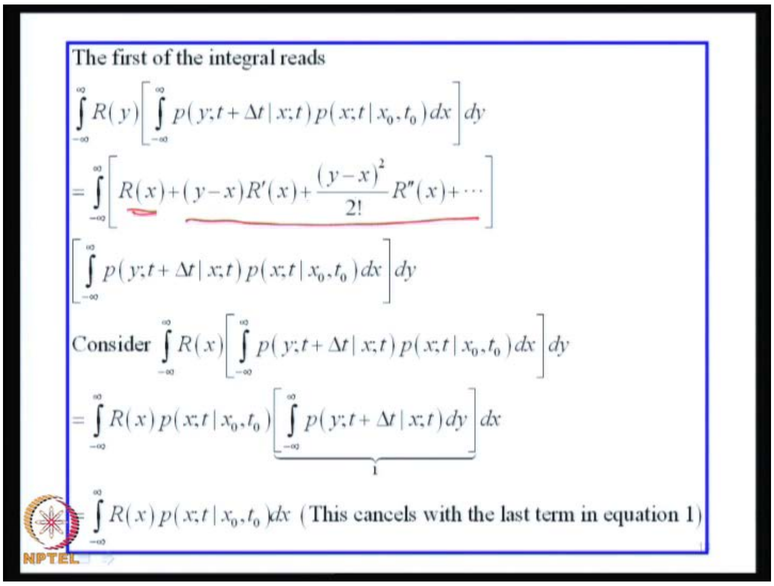

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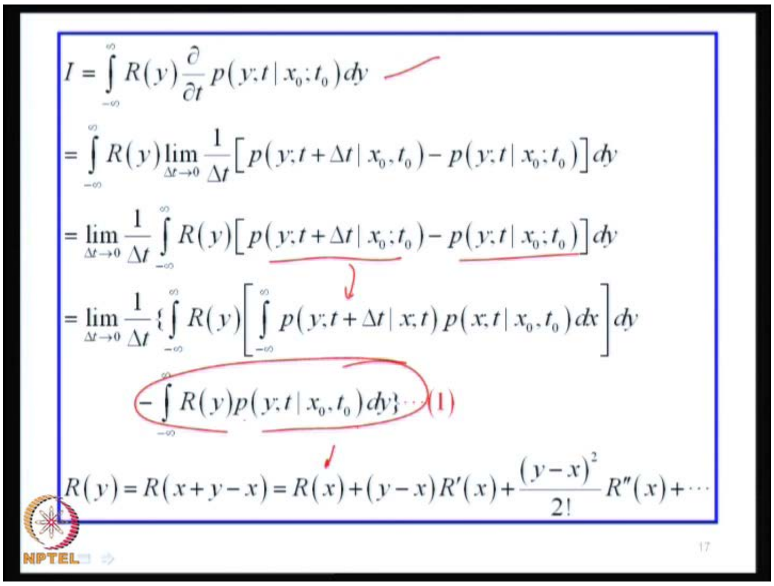

So, R of y into the terms involving the in the CKS equation, I will write like this, and R of x R of y, I am writing in terms of Taylor's expansion and this remains as it is. And I am now considering the first term here, R of x into this here; and we can interchange order of integration, and one of the integration with respect to y, the time actually 1, because the area under the probability density function; and I get R of x p of x x at t condition x naught t naught, this term, and this term just cancel with the second term, that is present here; so, this system now forget; so, what we are left with now? Terms I contributing from the second term onwards.

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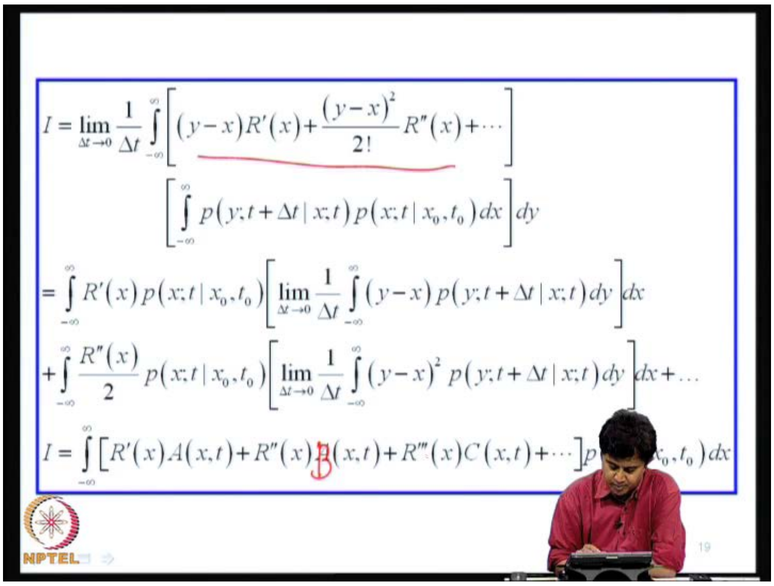

If we do that, first one is gone; so, I am writing in this in terms of this. And a little bit of manipulations of rearranging this terms will show as, that this i can be written as R prime of x into A, R double prime of x into A, this is B or triple prime into C, etcetera into dx.

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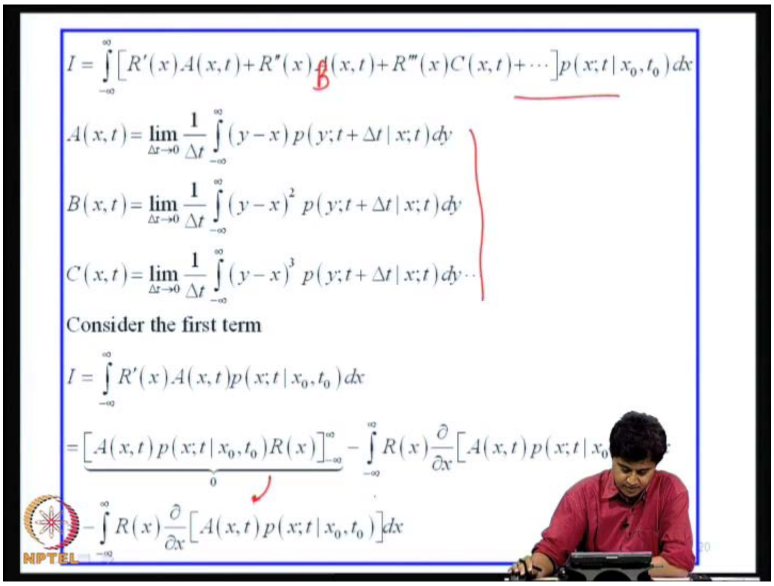

Now, this is what it is, where is the A, B, C, etcetera are the you know nothing but the incremental moments in the delta x that I was writing, that alpha n writing in terms of moments of delta x, same things here. If you consider a first term in this R prime of x A x comma t p x colon t dx, I can do an integration by parts; and the first term in the bracket goes to 0, because I am assuming that R x of goes to 0 faster than the other terms; that is what I meant, when I said R of x goes to 0, as x goes to infinity sufficiently fast; that is, it should faster than this, whether this term is 0; I restrict my R of x to only that; so, I get this expression, that is, this expression (Refer Slide Time: 25.00).

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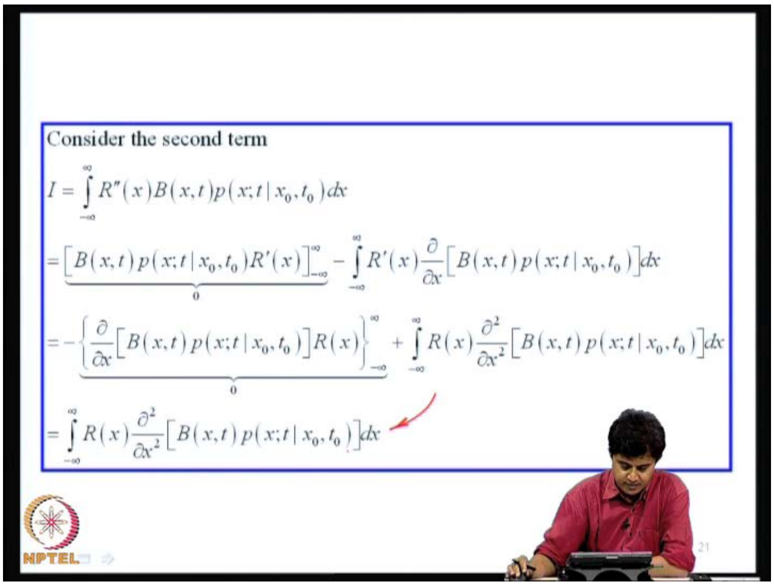

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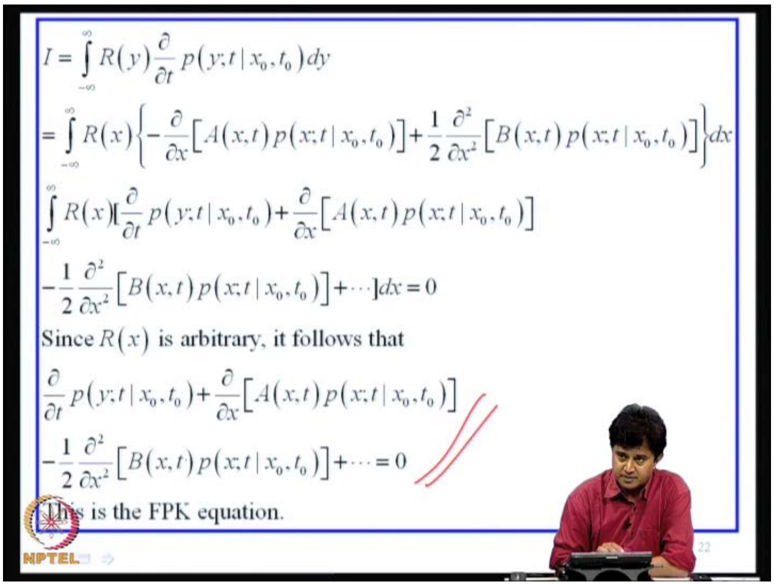

Now, the second term which involve double derivative, will again integrated by parts twice. And here, again I put this as 0, with an assumption the R prime of x goes to 0, as x goes to plus minus goes to infinity, that is what again I meant what is meant by sufficiently first. So, I get here these terms has R of x dou square x by dou x square into this.

Now, let us go back to the definition of i and write in this terms of this, and use this simplified versions, I get an equation R of x into a set of terms inside the bracket into dx equal to 0. Since R of x is arbitrary, it follows that the term inside the bracket should be 0, because this is 2 for all R of x. And we get that, this is nothing but the Fokker-Planck equation, that we derived by alternate argument. Here, again, there are the infinite the number of term shown here; that if we are dealing with systems driven by Gaussian white noise Gaussian, where white noise is obtained as formal derivate in the browning motion process, we are seen that all the other terms goes to 0 and we are left with only three terms in this, and that becomes, you know, amenable for possible solutions.

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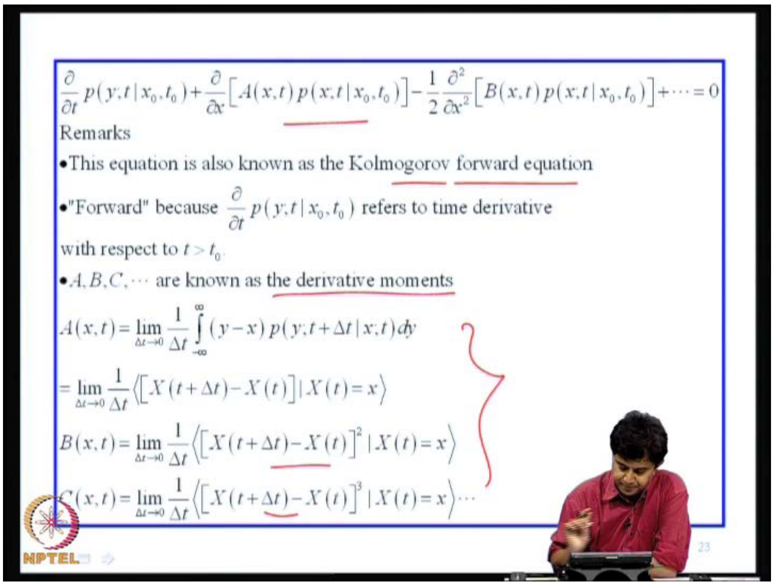

Now, let us make a few remarks; this equation displayed here is also known as Kolmogorov forward equation. The word forward here is because it contains derivative dou by dou t of the density function, which refers to time derivative with respect to t, which is greater than t naught; so, that is what, why we call it as forward equation. So, this connotation also automatically implies, that there is something known as backward equation; so, that will come to later.

And we will show that, the backward equations are useful for solving first passage times and problems of first passage probability, and so on and so forth. Right now, the forward equation; the parameter A, B, C are known as the derivative moments, which are nothing but the alpha discuss in a kinetic equation; there exactly the same. So, they are actually, moments of alpha A is actually the first order moment, B is the second order moment of the increment, C is the third order moment of the increment, and so and so forth.

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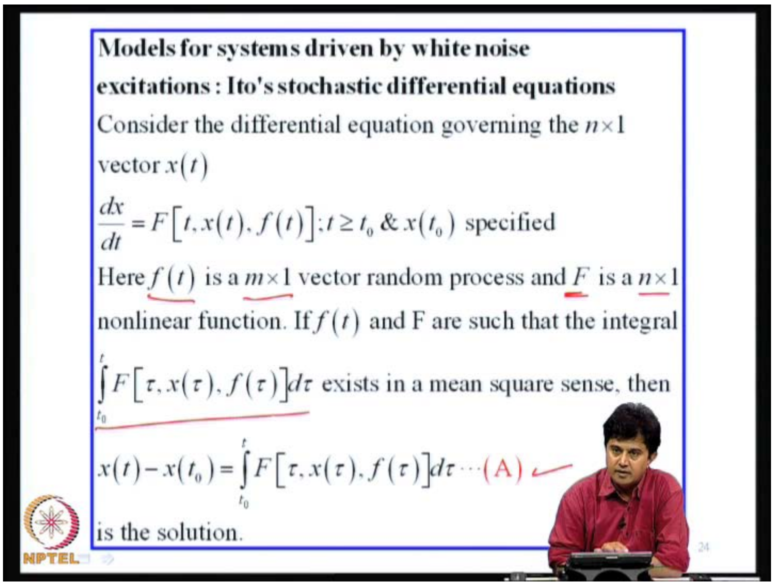

Now, we have already discussed properties of Brownian motion process. And we will now restrict the admission to models of system driven by white noise excitations and this will lead to what are known as Ito's stochastic differential equation. So, here, we restrict our attention to excitation, which are formal derivative the Brownian motion processes. In deriving the incremental moment, we will use the properties of increments of Brownian motion process.

Now, consider the differential equation governing the n cross one vector x of t, dx by dt some function of t comma x of t comma f of t, where t greater than t naught and x of t naught is specified. Let f of t be a m cross 1 vector random process and capital F is n cross 1 non-linear function. If f of t and capital F are such that, this integral t naught to t, suppose, if we integrate the both sides with respect to t, I will left with t naught to t f of tau x of tau f of tau d tau. If this integral exist in a mean square sense, then the solution can be written as x of t minus x of t naught t naught to t F of tau comma x of tau f of tau d tau etcetera; this is a solution provided; these integrals can be interpreted in a mean square sense.

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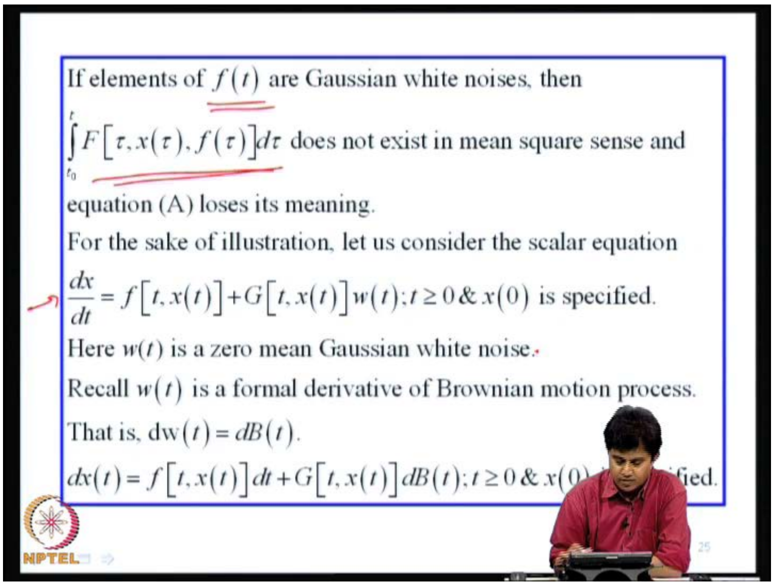

If on the way, if f of t if elements of f of t Gaussian white noises, then these kind of integrals, this integral does not exist in the mean square sense and equation a loses its meaning as a solution, because integrals itself is not define in the mean square sense. So, for sake of illustration, let us consider scalar equation, dx by dt f of t comma x of t plus G of t x of t into w of t, where w of t is 0 mean Gaussian white noise. Now, we require the w of t is the formal derivative of Brownian motion process; that is, d w of t can be written as dB of t. Now, I can rewrite this equation dx of t f of t comma x t dt plus G of t comma x of t into dB t, where t greater than or equal to 0, and x of 0 is specify, in terms of increments I am writing.

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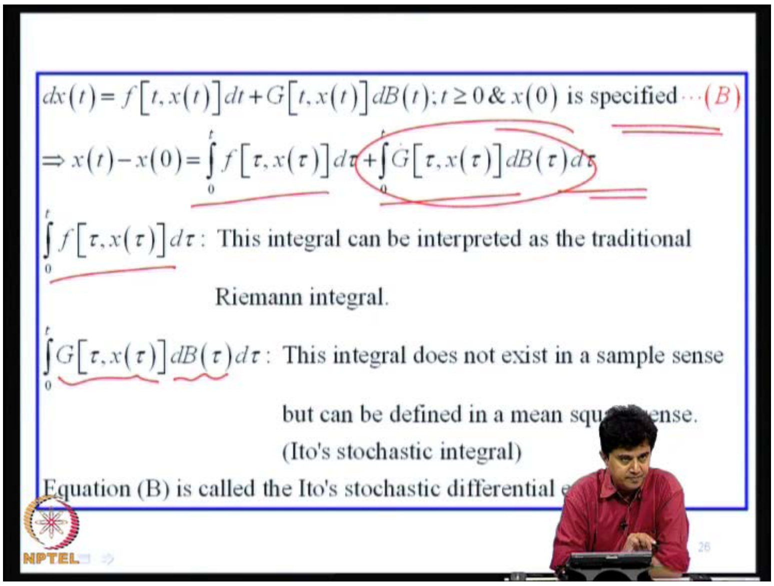

Now, I can therefore now write this as, x of t minus x of 0, as 0 to t f of tau comma x of tau d tau plus 0 to t G of tau comma x of tau dB of tau d tau. Let us look like this two integrals. First of these integral, you know, is integration with respect to tau; so, this integral can be interpreted in the traditional Riemann sense, in terms of area under the curve, and so on and so forth; whereas second integral, if we look at, this is random, this is also random; this integral does not exist in a sample sense, but can be defined in a mean square sense. So, this, that means, you cannot Riemann interpretation for this, but it can be interpreted in a mean square sense and that is what Ito’s stochastic integral theory enables us. So, this equation B, we call it as Ito’s stochastic differential equation; it is a differential equation to the extended; this is an integral equation, where this integral is an interpreted suitably in a mean square sense.

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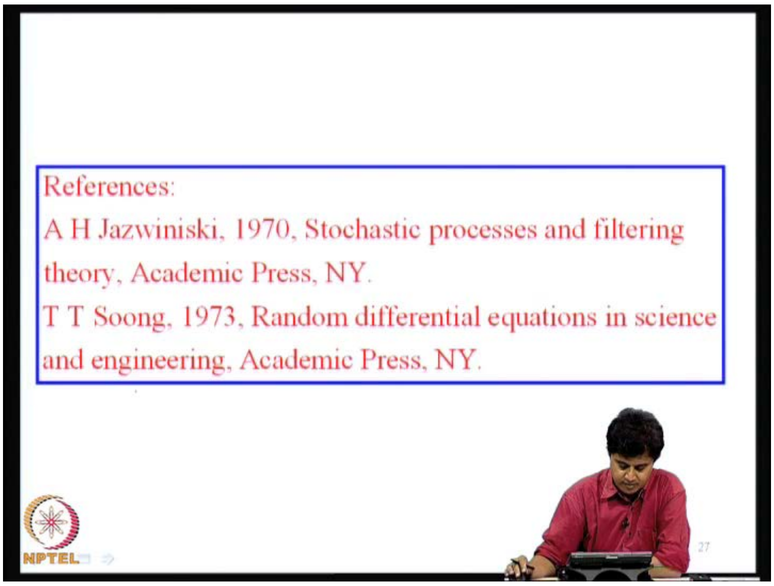

So, there are two reference that you may find useful; to study this further, in these lecturer, I will not be getting to greater details of Ito's s d is, but these two references is, one by Jazwiniski Stochastic processes and filtering theory and other one is Soong Random differential equations in science and engineering, which has descriptions which perhaps are accessible to engineers.

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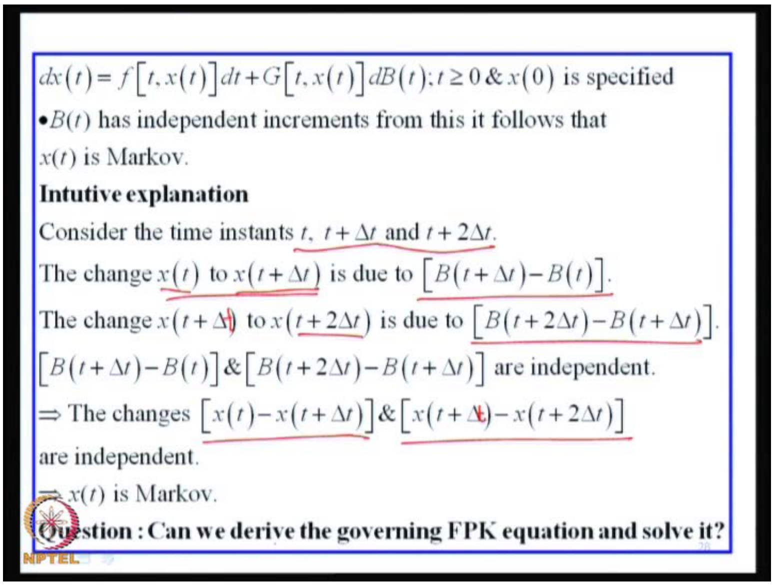

Now, let us look at this equation, again, dx of t f of t comma x of t dt plus G of t comma x of t dB t. We know that B of t has an independent increments, and from this, it follows that x of t Markov.

How do we explain that? The claim I am making is solutions of Ito's differential equations - stochastic differential equations - have Markov property. Therefore, I can write the associate Fokker-Planck equation and solve them; that is the line of thinking that I am trying to develop. Now, how do show that the solutions here is Markov; so, let us consider the time instants t, t plus delta t and t plus 2 delta t. The change from x of t to x of t plus delta t is due to change in B of t from t plus delta t minus B of t; this is the changing in the Brownian motion process, so this produces this change.

Similarly, change from t plus delta 2 t plus 2 delta t is due to this increment, B of t plus 2 delta t minus B of t plus delta t. These increments are independent; that means, a increment B of t plus delta t minus B of t and B of t plus 2 delta t minus B of t plus delta t are independent. Consequently, the incremental change in x of t, which is this and this, from t to t plus delta t and t plus delta t to t plus 2 delta t are also independent. That would mean, x of t has independent increments, and therefore, it has Markov probability. The question therefore, now, that we interested further would be, can we derive the governing FPK equation and solve it? How do it proceed?

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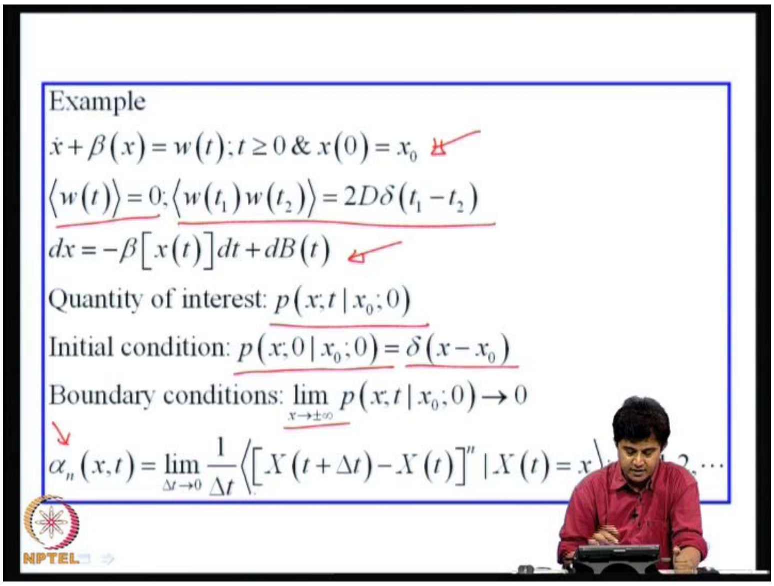

So, let us consider a simple example, x naught plus beta of x equal to w of t, where t is greater than is equal to 0 and x of 0 is x naught. w of t is a Gaussian white noise 0 mean and auto covariance given in terms of Direct delta function. So, I can derive this equation in terms of Ito’s s d, as dx equal to minus beta of x of t into dt plus dB t.

What is the quantity of interest? Probability density function of x colon t condition on x naught equal to 0. So, therefore, initial condition would be, p of x semicolon 0 x naught semicolon 0 is their delta functions of x minus x naught; boundary conditions as x goes to plus minus infinity, this probability density function goes to 0. These are the incremental moment, that I would be leading to formulate the Fokker-Planck equation - alpha 1 and alpha 2.

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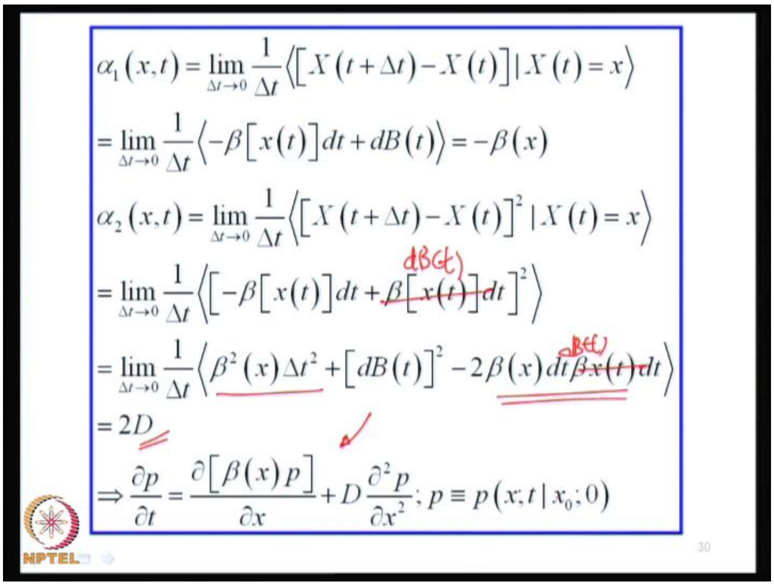

So, how do we derive them? alpha 1 is limit of delta t going to 0 1 by delta t, the first order moment of the increment x of t plus delta t minus x of t, conditioned on x of t equal to x. So, what is dx of t? dx of t is minus beta of x of t dt plus dB t; so, minus beta x of t dt plus dB t.

What is the expected value of this? Condition on x of t equal to x, it is nothing but beta of x; expected value of dB of t is 0. What is the second incremental moment? The square of this increment, condition x of t equal to x; square of this is, minus beta dt plus dB t whole square, so you expand this, it is beta square delta t square dB square minus 2 beta into dB of t. Now, this term, it will be beta square into delta t square divided by delta t could be delta t, as limit delta t goes to 0, that goes to 0.

This second term will be 2D and this is mean of dB t is 0, therefore, this term goes to 0; so, I am left with 2D. So, therefore, what is the Fokker-Plank equation? dou p by dou t is equal to dou by dou x of beta f x into p plus D into dou square p by dx square, where p is this transition probability density function. So, from governing equation of motion, have been able to derive now the FPK equation, which again can be viewed as equation of motion for evolution of probability density function.

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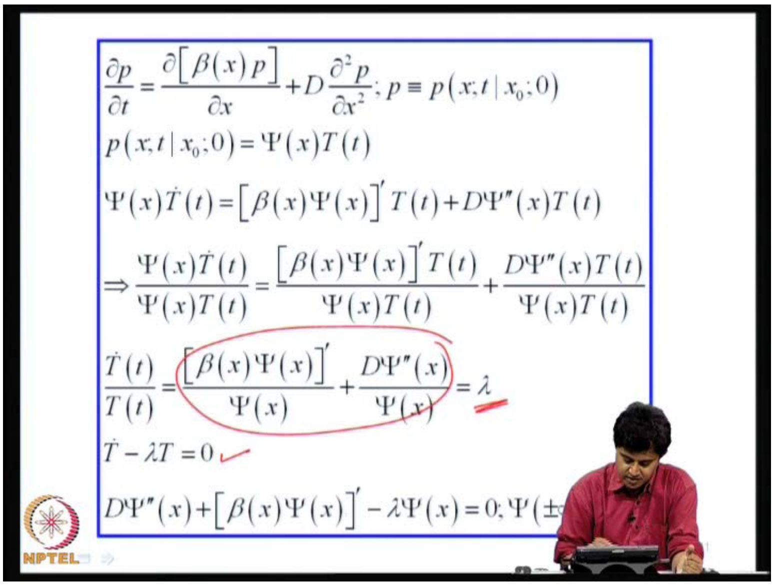

Now, how do we solve this? A standard method for solving partial differential equation is using method of variable separation. So, we can seek the solution in the form, psi of x into T of t. So, we substituted that into this, this becomes psi of x T dot plus beta of x psi of x prime into T of t plus d psi double prime into t; it is divided by psi T and simplify, I get T dot by t is equal to beta psi prime divided by psi plus p psi double prime by psi; and if we look at this term, is a functions time alone, this term is the function of x alone. If you change time, and this term, second term cannot change; if you change x, the first term cannot change. So, only way they can be equalize, we have to be equal to constant. So, I get T dot minus lambda t equal to 0 and I get another equation for psi, and second order equation with boundaries at plus minus infinity.

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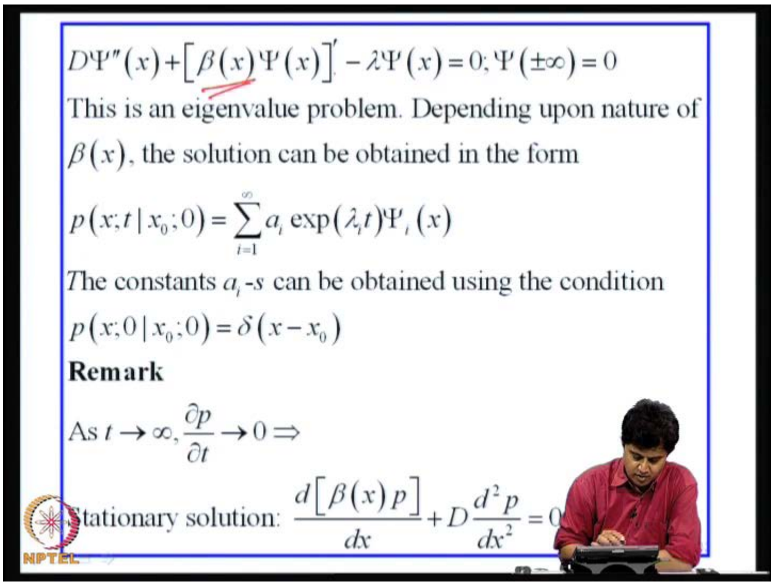

So, this is the equation, this is the eigenvalue problem; lambda is eigenvalue, psi of x will be the eigenfunctions. And depending upon the nature of beta of x, this solution can be obtained in the form, i equal to 1 to infinity a i exponential lambda i t psi i of x. The constant a i's can be obtained using the initial conditions, that t equal to 0, this function should get Direct delta function and we will get a i.

Now, the task of finding psi of x depends on choice of beta; so that typically obtained in terms of hyper geometric functions like, hermite polynomials, laguerre polynomials, and so on and so for. Some point the next few lectures, I will touch upon some of the illustrations. Right now, I proceed with how to formulate the governing equation, I am not so much focusing on the exact methods for solution.

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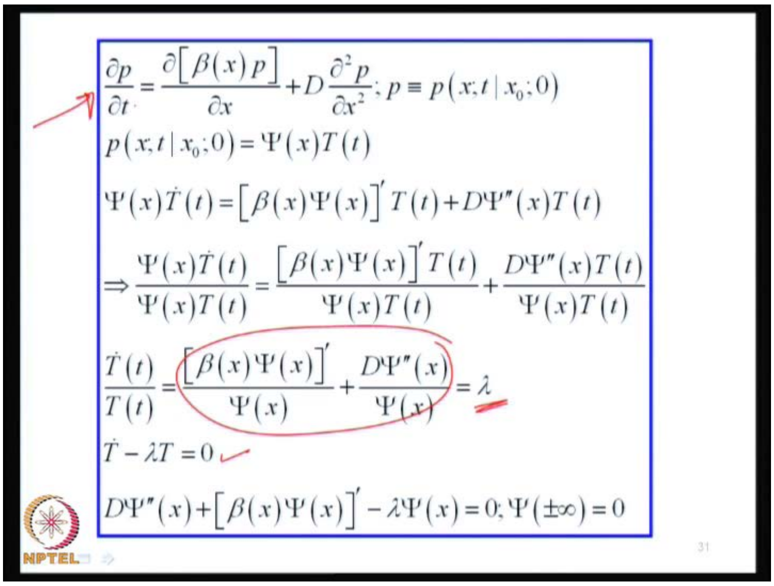

Now, before will leave this, we have to see here, if we focus on this equation, as t tends to infinity, if system admits the steady state, then the probability density function will become independent of time and I get dou p by dou t equal to 0. So, if that happens, the governing equation for stationary solution will be an ordinary equation - ordinary differential equation - d by dx of beta of x into p plus D into d square p by dx square equal to 0. This can again be solved; this is easy to solve; I will illustrate the solution later, when I consider specific examples.

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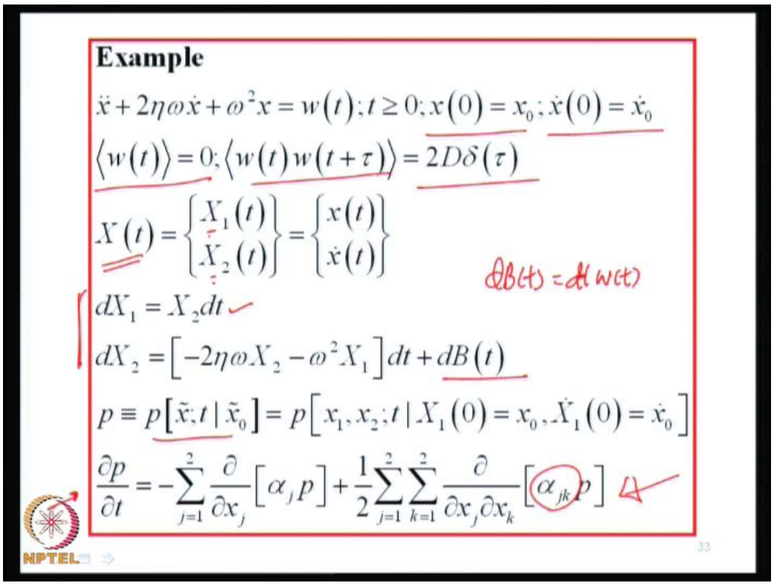

Now, we will come to one of the simplest problem in vibration analysis, namely, a single degree freedom system under Gaussian white noise. So, let us the system initial condition be, x naught and x naught dot, and mean of w t is 0 and auto covariance of w t is 2D into Direct delta of tau. This is second order differential equation. First, what I will do is, I introduce state vector x of t consisting of x 1 and x 2, where x 1 is x and x 2 is x dot; this is displacement and this is velocity.

So, what will be the x 1 dot is actually x dot; therefore, dX 1 is equal to X 2 into dt, that would be the first equation; dX 2 by dt would be X double dot, which is equal to 2 eta omega x dot omega square, omega square x is taken in the other side, plus w of t; so, that would be, dX 2 be therefore minus 2 eta omega X 2 minus omega square X 1 into dt plus dB of t, because, w of t dB of t is what? dt into w of t, because Brownian motion where interpreting as white noise - Gaussian white noise - where interrupted as formulate derivative of Brownian motion; so, I get this equation.

So, now, I am ready to launch my calculation for finding alpha 1 and alpha 2. So, the probability density function that I am looking for, is now actually for the vector X of t; it is scalar, because these vectors has two elements. The Markov property is satisfied by X of t, it is not satisfied by X 1, it is not satisfied by X 2, it is satisfied by X of t; that what the initial remark I made, that elements of vector Markov process need not to be a Markov themselves.

And also, you can see the displacement and velocity will have to different levels of differentiability, because X double dot is proportional to a white noise. So, therefore, X dot is not differentiable, but X is differentiable, in the mean square sense. So, that is another point that I have made earlier. Now, the evaluation equation for the probability density function the d p t f is shown here, and now to proceed further, I have to find now this alphas here and alphas here; how to do we that?

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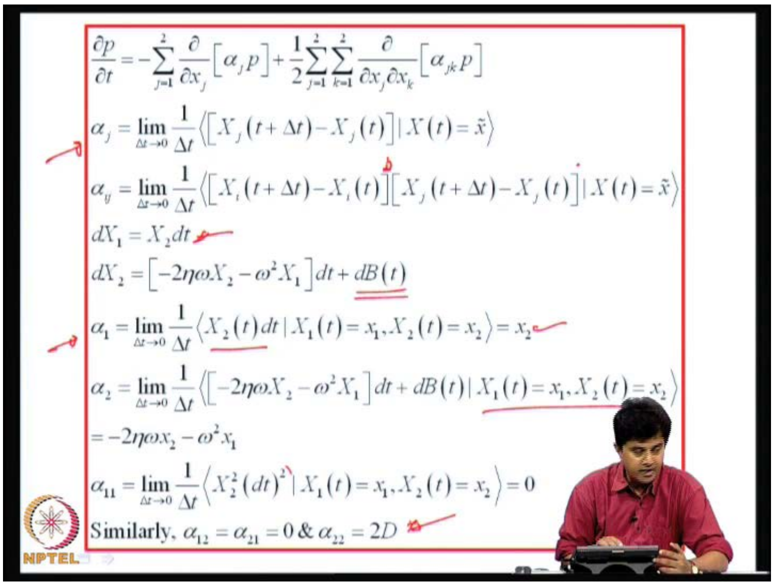

Alpha j is limit of delta t going to 0 1 by delta t increment condition X of t equal to x delta, and alpha i j is actually X i of t plus delta t minus X i of t into X j of t plus delta t minus x of t condition X of t x delta. Alpha 1, let us look at, limit delta t goes to 0 1 by delta t d X 1 is X 2 dt, therefore X 2 dt condition X 1 is equal to X 1 of t X 2 of t equal to X 2, this is nothing but X 2, because this is X 2 here; it is condition at X 2. Alpha 2 would be, minus 2 eta omega X 2 minus omega square x 1 plus dB of t condition 1 x 1 of t x 1 x 2 of t is x 2; so, this will be minus 2 eta omega x 2 minus omega square x 1; the third time will 0, because the expected value of dB of t 0.

Alpha 1 1 will be what? Square of X 2 square into dt square condition in X 1 t equal to x 1 X 2 t equal to x 2; now, under these limiting operation, this will be 0, therefore alpha 1 1 will be 0; this will be X 2 square dt square 1 by delta t, as the delta t goes to 0, that will be 0. So, I get alpha 1 2 equal to alpha 2 1 equal to 0; and alpha 2 2, we have to square this and take expectations by conditioning this way, and the first two terms will vanish and cross term will vanish, and dB square of t will lead to 2D; that we are done previously, therefore I am not repeating here.

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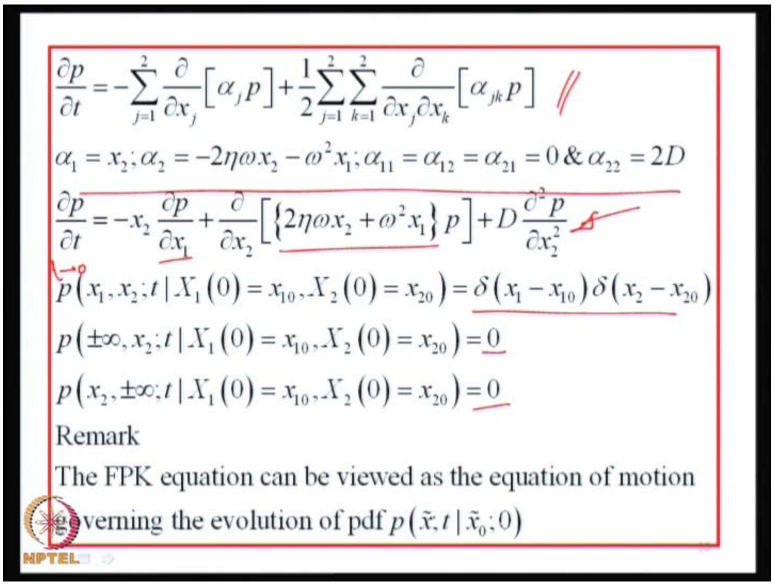

So, I am ready with now the governing equation; this is the general form and we have derived now this alphas. And if I substitute into this, I get dou p by dou t is minus x 2 dou p by dou x 1 dou by dou x 2 into alpha 2 p plus p into dou square by dou x 2 square. The initial conditions, at t equal to 0, displacement is x naught, velocity x naught dot and both are Direct delta functions. So, as t tends to 0, this condition would be satisfied; and similarly, I have at boundary plus minus infinity, the density function is 0; so, x 1 equal to plus minus infinity this is 0; x 2 equal to plus minus infinity equal to 0.

So, the problem now on hand now reduces to solution of a partial differential equation, under this initial condition and boundary conditions. So, this is the different exercise in mathematics that can be done. We will see how it can be done in the next lectures, but right now, we are focusing on how to formulate the problem. So, again, let me emphasize that, the FPK equation can be viewed as a equation on motion governing in the illusions of the probability density function.

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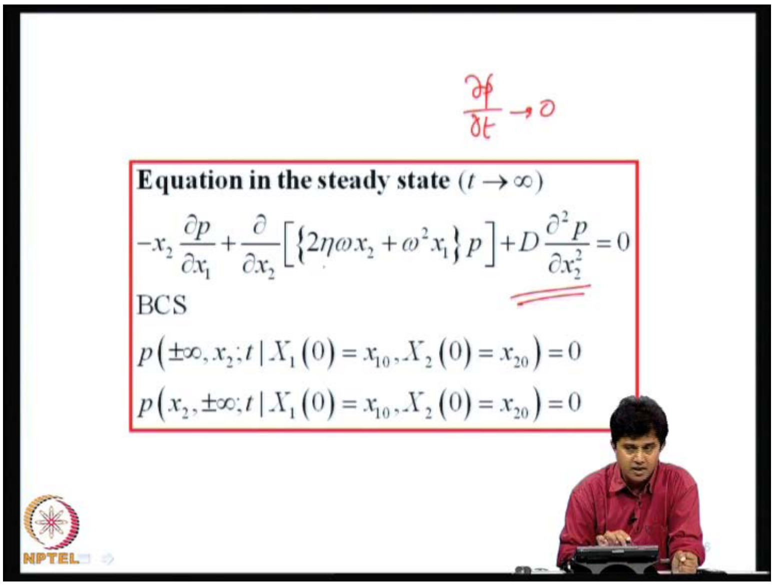

Now, if a steady state is possible, as t tends to infinity, the probability density function would be become independent of time; therefore, dou p by dou t would go to 0, and I get a simplified version of this equations, which is still a partial differential equation, because now I have two independent variables x 1 and x 2. And if you are interested only in stationary solutions, we can directly tackle this equation.

You should of course independently verify, whether the mechanics of the problem admits the steady state solutions; if this, for example, if system is un damped, there would not be any steady state; although you can write such solution, try to see what you get, it would not be correct. And in this boundary conditions, of course, again, at plus minus infinity on x 1 and x 2, the density functions are 0. So, these are the associated boundary condition, under which we have to solve this reduced partial differential equation.

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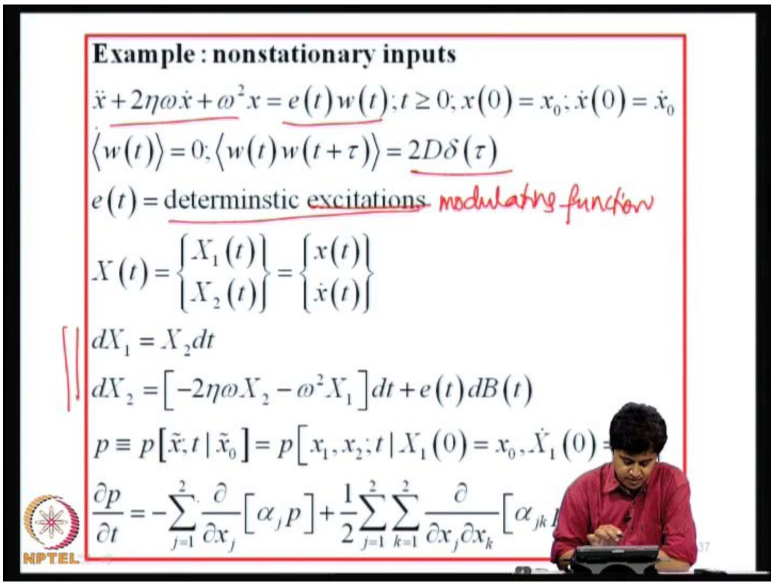

We considered stationary inputs; so, I want to now going in some sequence, I will consider non-stationary excitations, non-linear systems, non-Gaussian excitations and multi degree system; and I will show that, for each of this cases, we can formulate the governing Fokker-Planck equation. So, let us consider now non-stationary inputs; so, this same single degree freedom system, now it is driven by e of t into w of t, where e of t is the deterministic modulating function; w of t is as before white noise is 0 mean and auto covariance, in terms Direct delta functions. As before I define the vector x of t as x 1 x 2 and I get this equation state in space representation, and again look at the governing Fokker-Planck equation.

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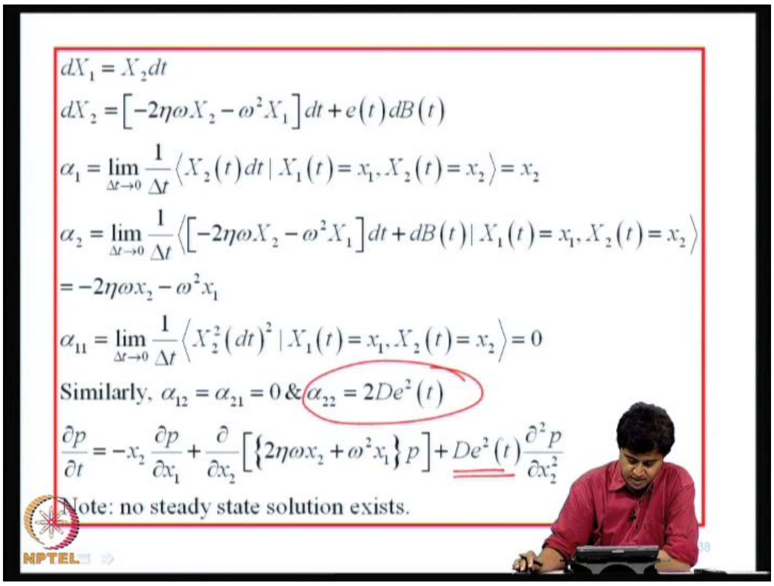

Here, nothing really changes, we get the same values for alpha 1 and alpha 2; except that, for alpha 2 2, I get the effect of the envelope, which is 2 De square of t. So, the governing equation will be quite identical to what we got earlier, except that now, I have in this term, d into e square of t dou square p by dou x 2 square. Clearly, in this case, we cannot talk about a steady state solution, because on the right hand side, there is t; that means, if the incremental moment themselves are functions of time, then there is no basics on which we can talk about steady state solutions; p will continue to be function of time, even for large times.

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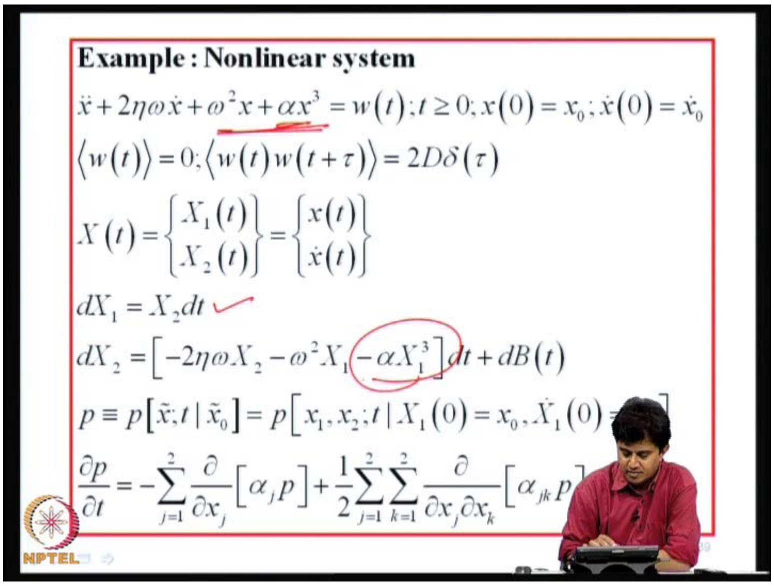

How about nonlinearity? So, I introduce a slight change, I add a cubic non-linear term; so, this system now become doffing oscillator with cubic non-linear term. The problem formulation is essentially similar, except that one of the incremental moment would now change, to allow for this additional influence of non-linear terms. So, I have dX 1 as X 2 dt, dX 2 as 2 eta omega X 2 omega square X 1; this is the new term, minus alpha X 1 cube this.

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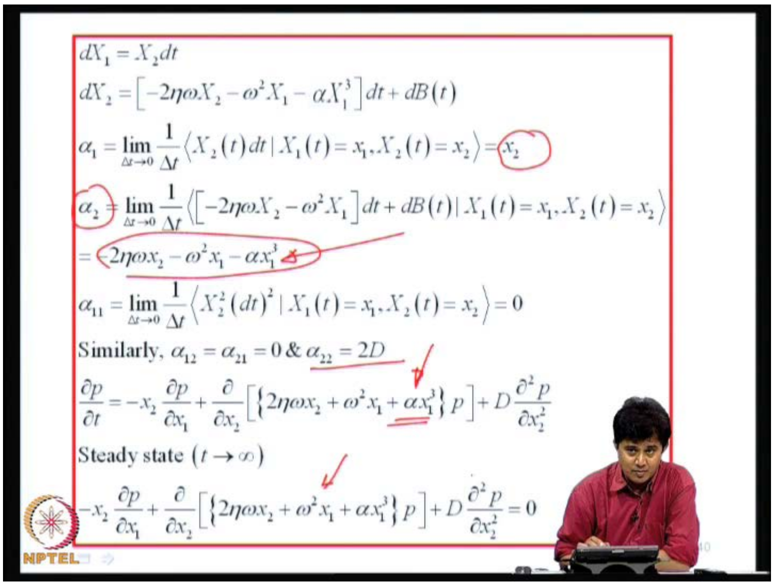

Now, alpha 1 would remain as x 2, but this alpha 2 would now contain the influence of non-linear term, and alpha 1 1, alpha 1 2 and alpha 2 1 will be 0, and alpha 2 2 will be 2D. Now, the governing equation will have the influence of nonlinearity through an additional term in the co efficient here. So, here again, since the incremental moments are independent of time, one can thing of steady state solutions and we can look at the reduce partial differential equation, governing the steady state solution.

So, I will be showing the subsequent lectures, that exact solution to this problem is obtainable; that means, for a doffing oscillator undr white noise, the steady state response is exactly determinable. It should be appreciated, because if you considering doffing oscillator under harmonic excitation, under steady state, no exact solution exist for deterministic problem. When excitation is white noise, the problem somehow becomes simple enough to allow for an exact solution.

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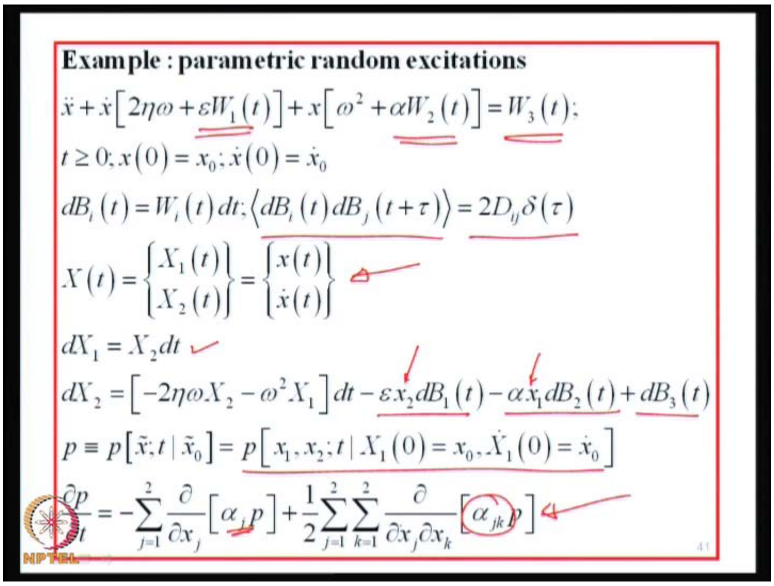

Now, we can make the problem slightly more complicated. We can have external excitation and also parametric excitation; so, I have now linear single degree freedom system with three white noise excitations - two of them are parametric and one is external. So, again I consider system starting from x naught and x naught dot, and the three random white noise processes, I will assume that, there are all correlated, and this is the metrics of cross covariance functions; so, dB i t dB j t plus delta tau is 2D ij Direct delta of tau.

Now, again I define the state vector X 1 X 2, which is x x dot and get the equation in this form. Now, there will be addition terms here, containing increments of three Brownian motion processes and they are multiplied by systems states for the two terms, and of course, third term is an excitations. Again, we are interested in finding the evaluations of the transitional probability density function, which is now function of x 1 x 2 t condition on x naught and x naught dot. So, that FPK equation will have these forms and need not to determine alpha j and alpha jk.

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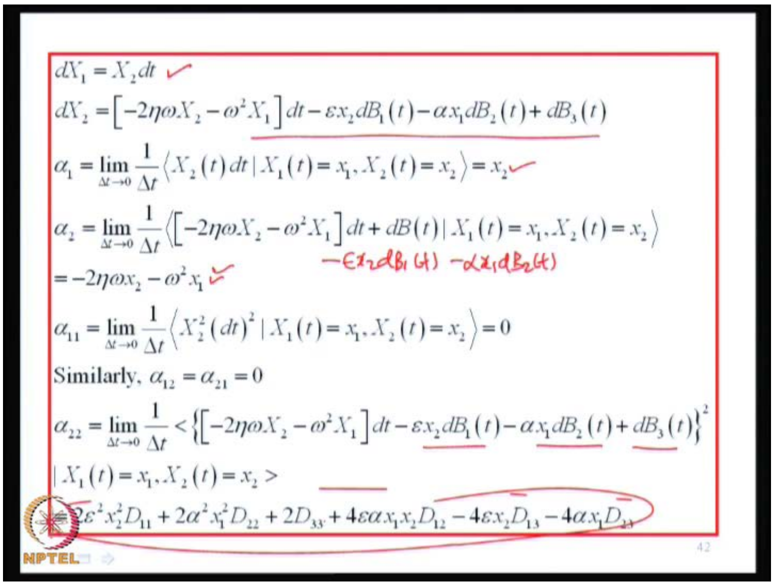

This is to dX 1 X 2 dt and dX 2 is having this additional terms. Alpha 1 would be still x 2; alpha 2 could not change, because the expected value of dB of t and other these terms are stills 0; therefore, this would not change. So, here, we have to add dB 1 t minus alpha x 1 dB 2 t and their mean values are 0, therefore, it would still be this. Alpha 1 1 would again be 0; similarly, alpha 1 2, alpha 2 1 will be 0, but alpha 2 2 will now be expected value of square of this condition on X 1 of t is x 1 X 2 of t is x 2; so, this involves some calculations; the only term that will remain, after we take expectation and apply the limits, will be associated with variances of these three terms and later cross covariance’s; and if you do that, I get this alpha 1 2 to be this.

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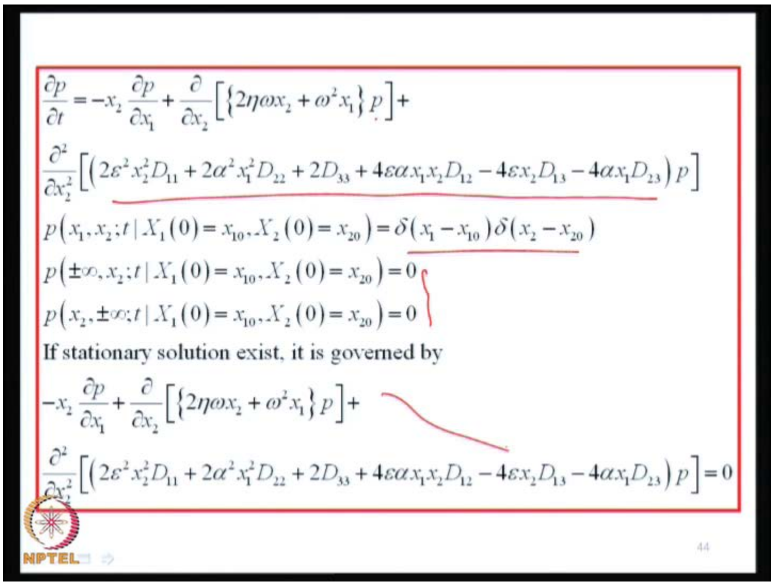

Now, I have governing FPK equation will be, dou p by dou t minus x 2 by dou p by dou x 1; this term also could not change, but now the terms involved the second derivative will be more complicated; they will have these terms. The initial conditions would be remain the same, the boundary condition would also be the same. And if a stationary solution exists, we are not sure, when there are parametric excitations, we know we are never sure if there is a steady state. The system became a unstable, there will be a steady state; so, if a steady state exists, it is governing by this reduced equation.

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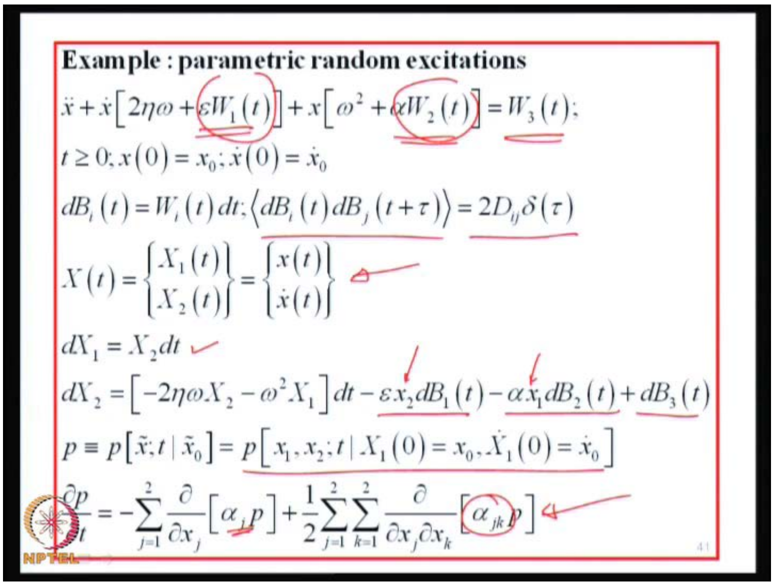

Again, a general class of deterministic problems with this kind of parametric excitation terms is not solvable; there is no exact solution in deterministic case. But it turns out that, for certain cases of problems with parametric excitations, we can get an exact solution using Fokker-Planck equation. Furthermore, I am now right now talking only about probability density functions - evolutions of probability density function; from that evaluation of the probability density function, I can also derive the evaluation equation for moments - response moments. It turns out that, for this class of the problems - linear systems with parametric Gaussian white noise excitations - the moment equations are exactly solvable. So, the Fokker-Planck equations approach which sources of exact solutions for white noise of problems, where the counter parts in deterministic analysis are not solvable.

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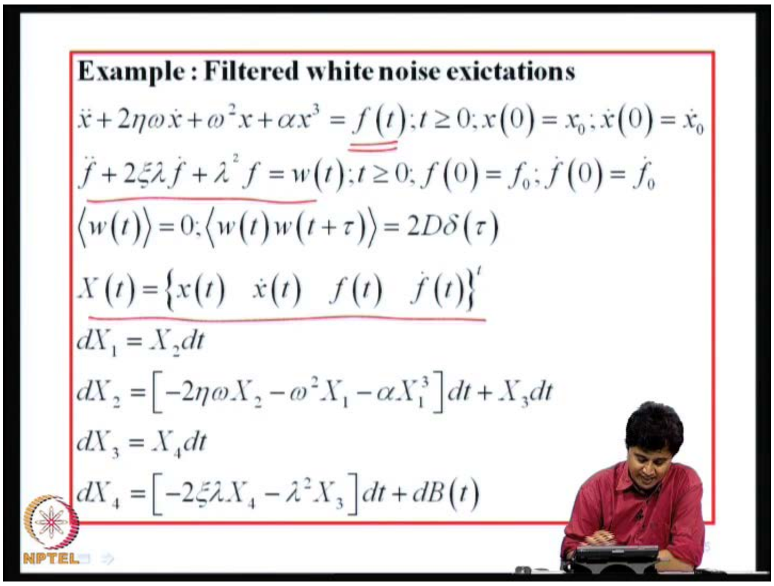

Some more examples, suppose, I have been talking only about white noise excitations; suppose, the filter white noise excitations; the excitation is not white, but it is colored. So, f of t, I modulate as output of a single degree freedom system, which receives as a white noise excitations. Now, handle this problem, I define in a extended vector of responses quantities x x dot f f dot; that means, f of t is a excitations, but excitation itself is modeled using linear system driven by white noise. So, we can show this extended vector will have Markov problem, right; therefore, I can write the governing Markov FPK equation further.

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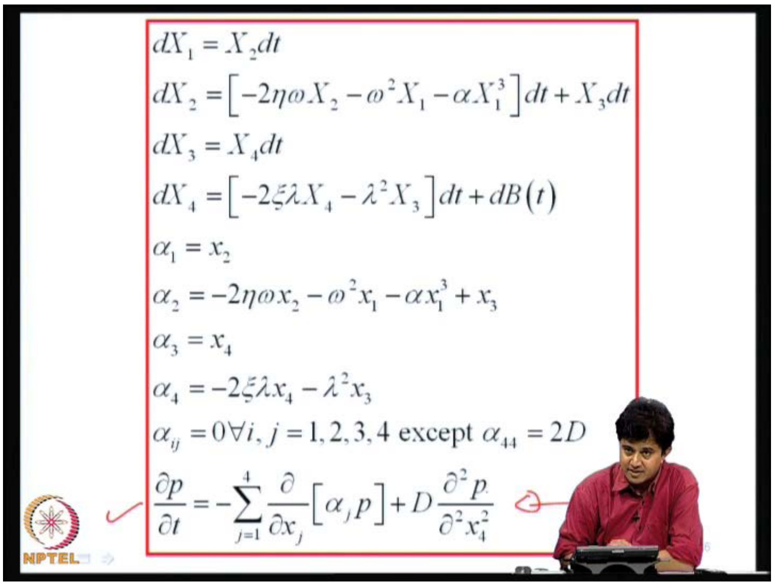

So, I have dX 1, dX 2, dX 3, dX 4 and I can go through this, find out alpha 1, alpha 2, alpha 3, alpha 4 and get the governing Fokker-Planck equation even for this case.

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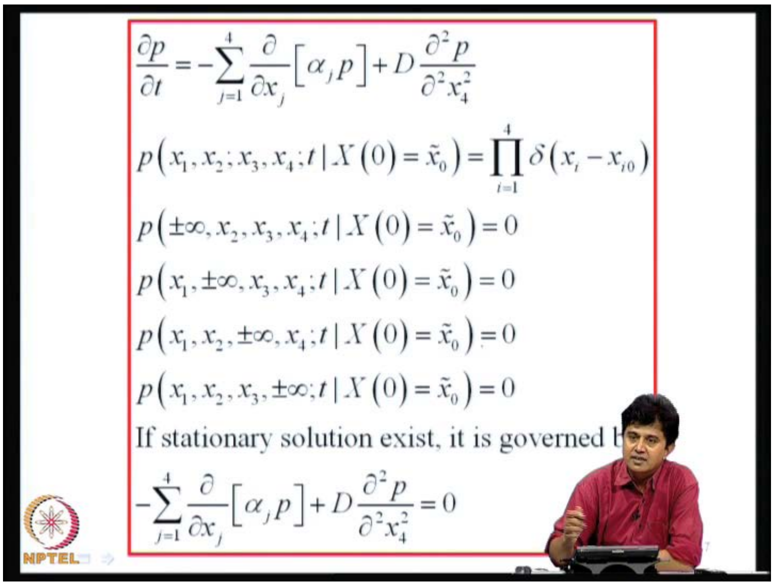

So, I can write these equations and write down the boundary conditions. Now, there are four independent spatial variables x 1, x 2, x 3 and x 4, in addition to time; so, but in principle, the problem is formulated.

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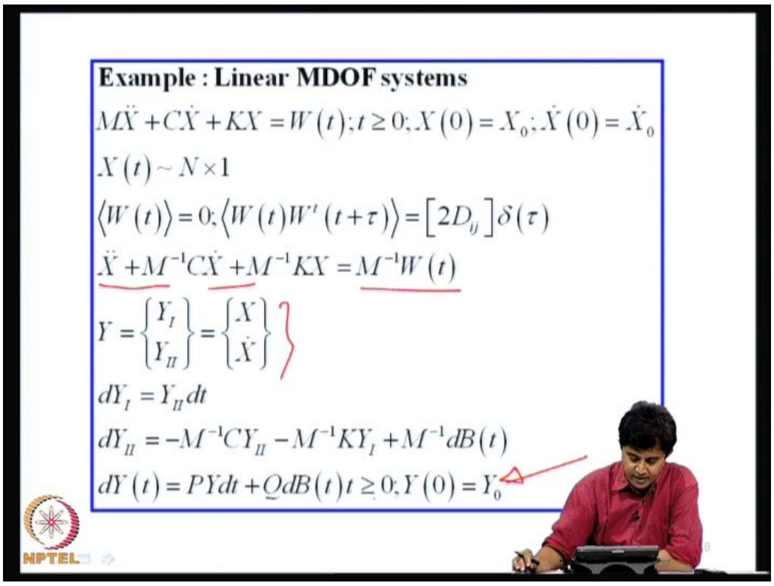

Now, how about linear multi degree freedom systems? I am going to talk about the single degree freedom system. So, I can multiply this governing equation by M inverse and write it as X double dot M inverse CX dot M inverse KX and m inverse this. So, I can introduce now a two-dimensional state space Y i by Y double I, and write an equation for Y i and Y double I, and cross it in the form of the Ito's differential equation.

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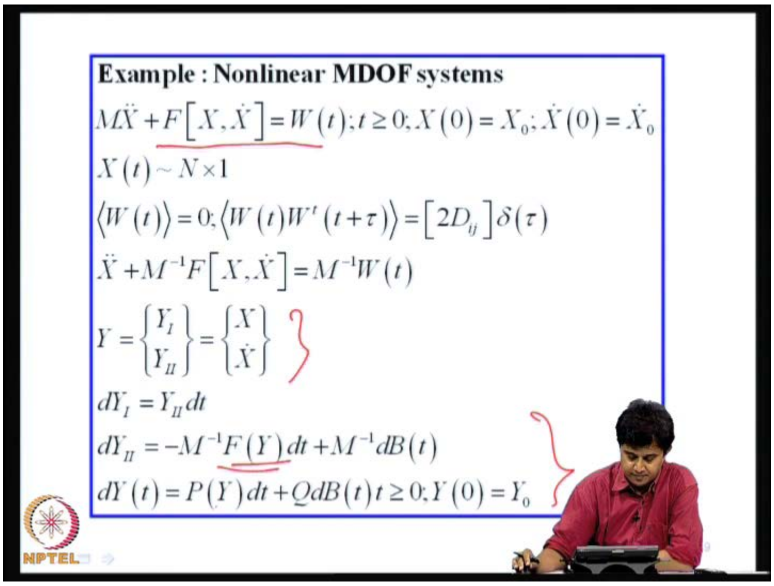

So, it is possible to cause this equation in the Ito's; so, the moment to put in the Ito's form, so I can derive I know the solution is Markov and I can derive the governing

Fokker-Planck equation. Now, what happens if non-linear systems the multi degree and non-linear? So, F of X comma X dot. There again, I can introduce the extended the vectors of Y i Y double i Y double I; and I can still write the Fokker-Planck equation, where this M inverse F of Y is P of Y; so, this is F which is a non-linear function and here still represented in my equation.

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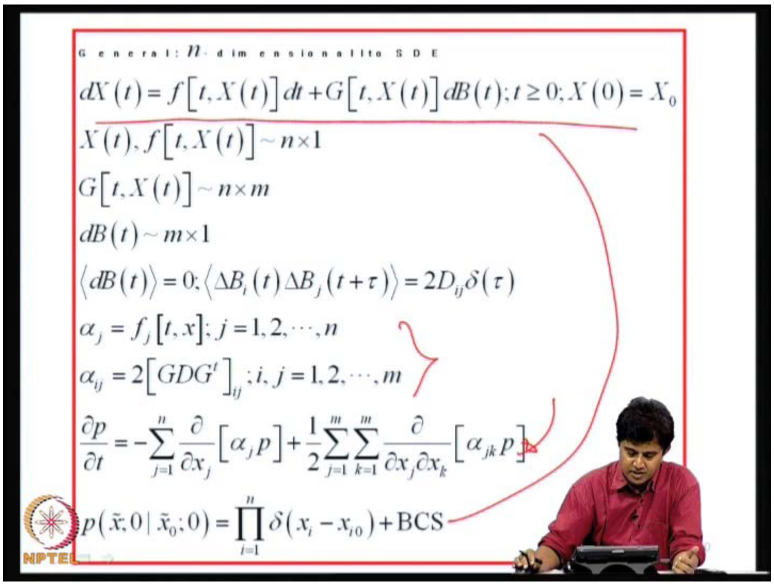

So, now, if we consider the general n dimensional Ito's is stochastic differential equation, I shown the multi degree freedom system can always be cast in this form; so, we can consider a general Ito's multi degree equation, where there are non-linear drift and diffusion terms, this is also known as drift and diffusion terms, I can use this you know state space representation and derive the incremental moments, in terms of this f and this G, and I can still get the associated Fokker-Planck equation. So, the formulation for the Fokker-Planck equation itself is fairly general; it can be parametric excitations, it can handle the non-linear systems, and so on and so forth.

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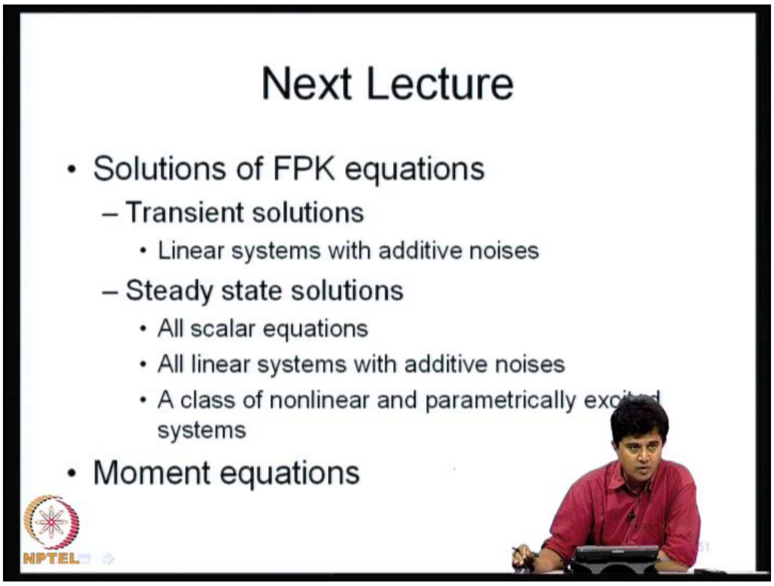

So, in the next lecture, what will do is, we will next think of how to solve the governing Fokker-Planck equation. And how do a transient solutions? How to get a study state solutions? And when we get exact solutions for these problems? And what approximate strategies that we can develop? And how to derive moment equations from the governing equation for evolution of probability density function?

This will consider in the next lecture; we will conclude this at this stage.