Stochastic Structural Dynamics

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Lecture No. # 21

Markov Vector Approach-1

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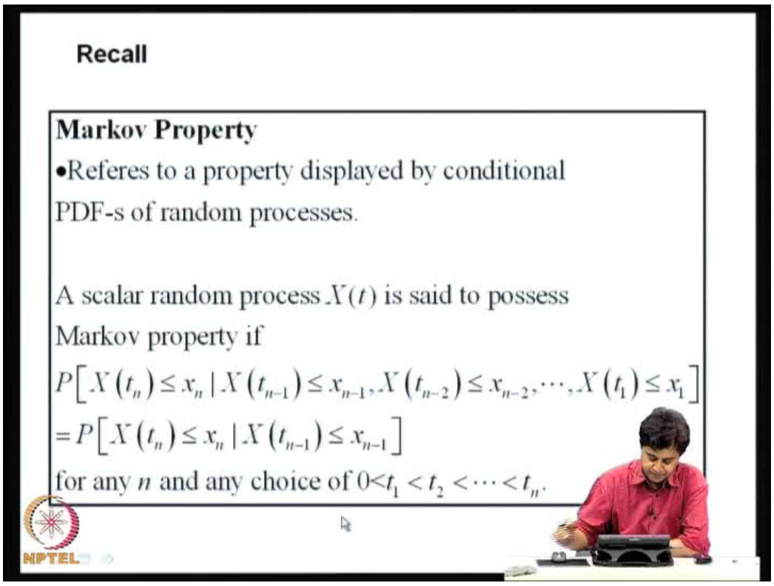

We have been discussing in the previous lecture, a property of random process is known as Markov property and this property would enable us to analyze a randomly vibrating system using a different kind of route, so that is we are developing. So, we will quickly recall what we discussed in the previous lecture. So, Markov property refers to a property displayed by conditional probability distribution functions of a random process. When we say process is Markov, it does not refer to the nature of probability distribution function; for example, when I say a process Gaussian what I mean, is that, the probability density function has a Gaussian character, whereas Markov property is something to do with memory, not so much about the particular form of probability distribution function.

So, the definition of a Markov process for a scalar random process X of t said to possess the Markov property, if the conditional probability distribution function X of t n less than or equal to X n, that is conditioned on X of t n minus 1 less than equal to X n minus 1, so on and so forth, X of t 1 less than or equal to x 1 is the probability distribution of X of t n less than or equal to X n, conditioned only on X of t n minus 1 less than or equal to X n minus 1, for any choice of a n, for any choice of increasing sequence of t 1, t 2, t 3, t n.

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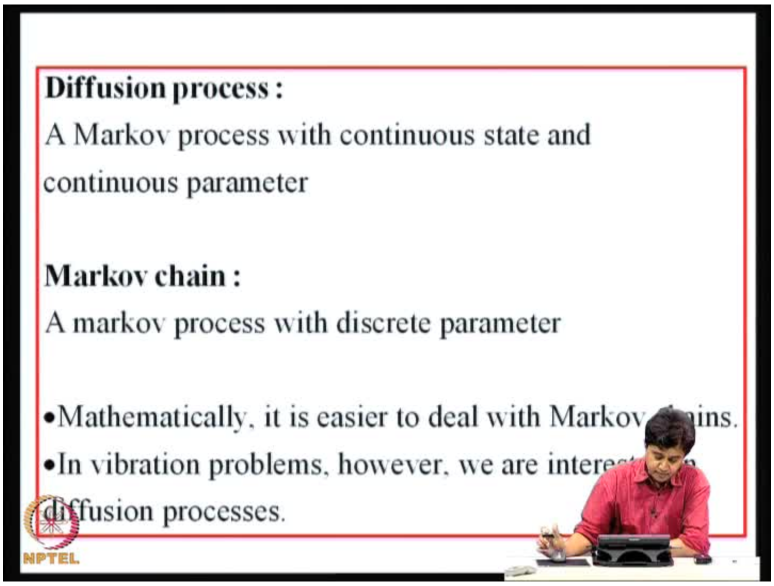

A Markov process with a continuous state and continuous parameter is known as a diffusion process. If the Markov process has discrete parameter, then we call it as Markov chain. Mathematically, it is easier to deal with Markov chain, but however in vibration problems we need to deal with the Markov process, that is the diffusion processes and diffusion processes have several mathematical pathologies and we need to take care of some of that, when we use these tools.

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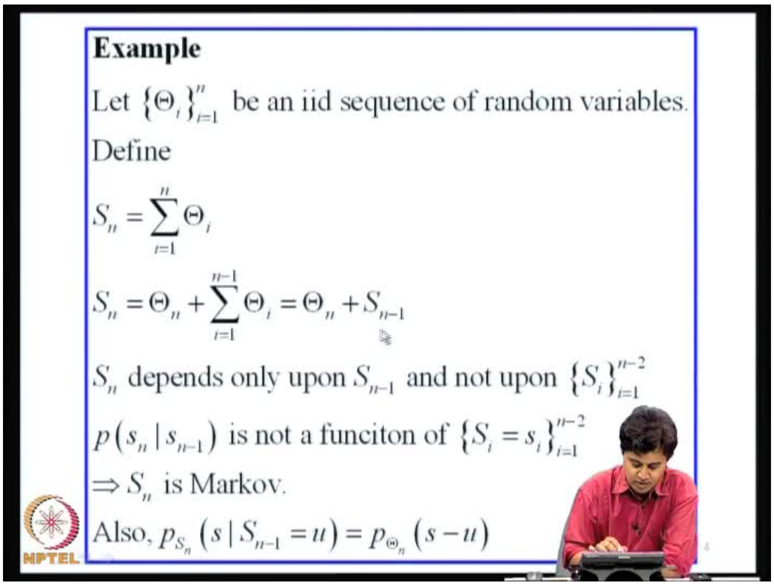

So, we consider few examples, just to fix the idea; let theta i, i equal to 1 to n be an sequence of identical independently distributed random variables and I define S n as sum of this theta i from i equal to 1 to n, I can write the summation from i equal to 1 to n minus 1 and write theta n separately and I can, therefore, write S n as theta n plus S n minus 1.

So, from this we can see that S n depends on S n minus 1 and theta n is a sequence of independent random variables; therefore, the probability density function of S n conditioned on S n minus 1 is not a function of the observed states S n minus 2 equal to s 2, so on and so forth S 1 equal to s 1, that is S n is Markov. Also, in this case, probability density function of S n conditioned on s S n minus 1 u is actually probability density function of theta n evaluated at s minus n; so, this is by simple rules of transformation of random variables, so on this equation.

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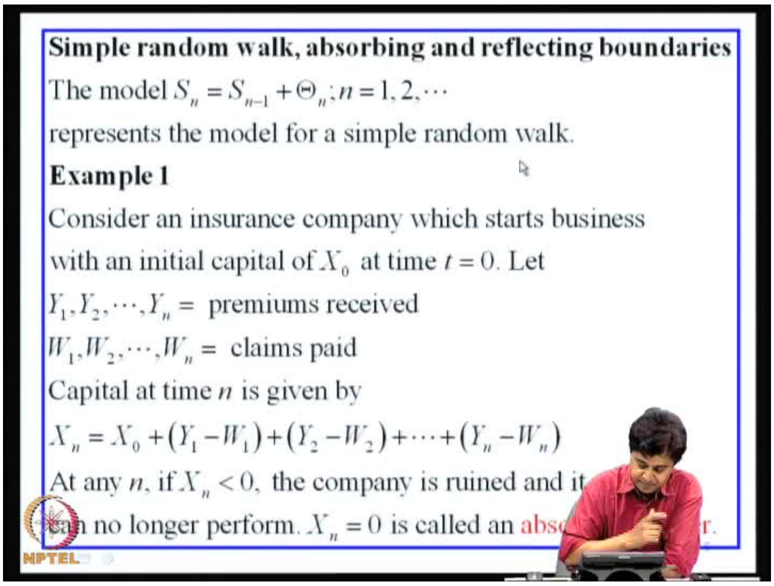

Now, when dealing with Markov processes, we come across what are known as absorbing boundaries and reflecting boundaries, so we could explain what these are through some simple examples. So, we will consider now, S n is equal to S n minus 1 plus theta n, we take that, this represents the model for a simple random walk.

Now, first example that I would like to consider is an example of an insurance company which starts business, with an initial capital of X naught at time t equal to 0. Let Y 1, Y 2, Y n be the premiums received at t 1, t 2, t n; this t 1, t 2 may be days or weeks or months whatever. Then W 1, W 2, W n are the claims, paid at t 1, t 2, t n, so they capital at time n is given by initial capital plus Y 1 minus W 1 plus Y 2 minus W 2 and so on and so forth.

So, at any n if X n becomes negative, that means, capital becomes negative, the company is ruined and it can no longer perform its task and therefore, we call X n equal to 0 as an absorbing barrier. So, that means, random Markov terminates moment, that state is the observing barrier is reached; so, this is a process with one absorbing barrier X n equal to 0.

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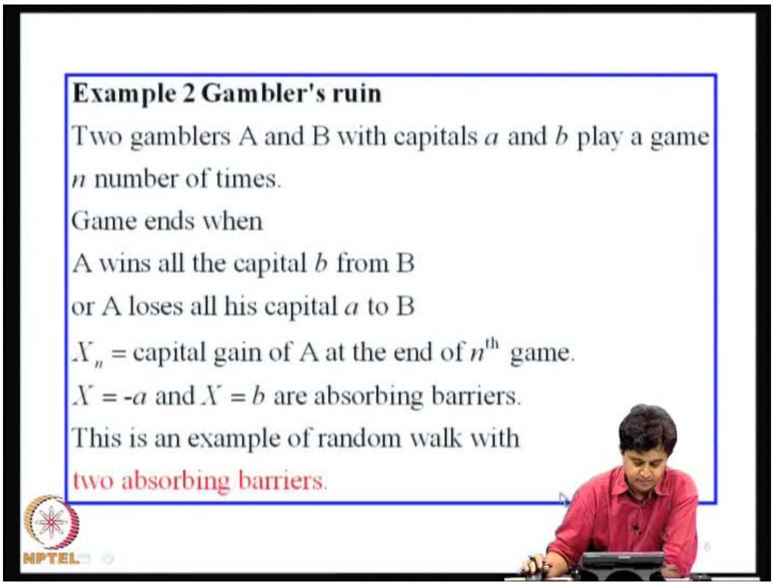

Now, we will consider another example, where 2 gamblers A and B with capitals a and b play a game n number of times. The game ends, when a wins all the capital b from B or a loses all his capital a to B, because gambling cannot continue because one of the person has no longer any capital left with him. So, if X n is capital gain of A, at the end of nth game, then X equal to minus a and X equal to b are the absorbing barriers; here, he loses all is capital, here gains all the capital from player b; so, this is a random walk with two absorbing barriers.

Later on we will see, when we discuss first patches failures, the barriers can be thought of as absorbing barriers; suppose, X of t is the stress time history and if X of t crosses a plus sigma y or minus sigma y, suppose if we say, that is failure for us, then we can think that processes having to absorbing barriers.

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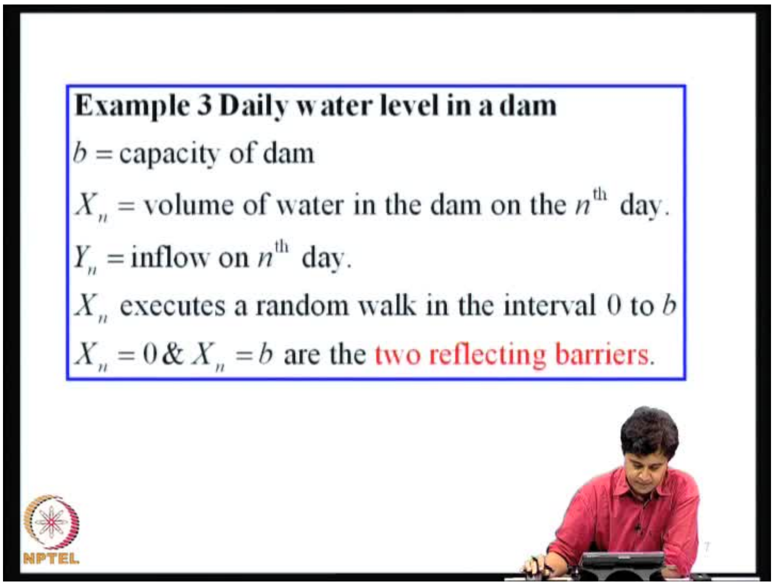

Another example, daily water level in a dam; so, let b be the capacity of dam in some cubic meters or so, then X n is volume of water in the dam on the n th day; let Y n be the inflow on n th day and if X n here, we can say that X n executes a random walk in the interval 0 to b, either the dams becomes empty or dam is full; if it is empty, it has to wait for inflow; if it is full, there will be overflow, whatever is excess overflow on that, I mean, x is water, that gets stored in the dam gets overflow.

If dam becomes empty, it does not mean that the random walk terminates, it waits for the fresh input. Similarly, if dam becomes full, it waits for the input to stop, so that water level reduces; so, these boundaries 0 and b, we call them as reflecting barriers. So, in contrast with absorbing barriers, if the, this state is reached, a reflecting barrier is reached the random walk does not terminate, it can still continue.

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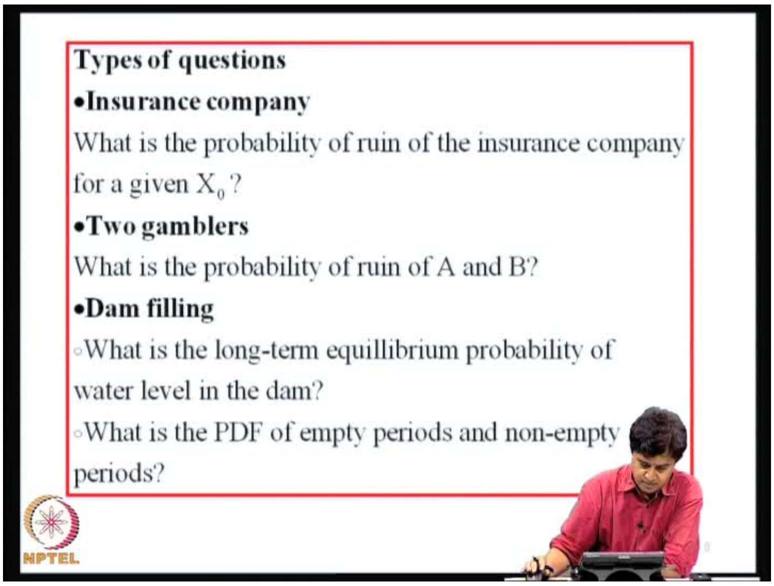

So, these types of questions that we would have ask in this specific context; we can generalize it to problems in structural mechanics in due course, for example, for the problem of insurance company what is the probability of ruin of the insurance company given its initial capital, that is time for ruin.

Then, in the problem of two gamblers what is the probability of ruin of A and B? Then, in the case of dam filling, what is the long term equilibrium probability of water level in the dam or what is the probability distribution function of empty periods and nonempty periods? So, these are type of questions that we can answer using tools of Markov process theory. So, these questions have their counter parts in the contextual structural reliability and that we have to slowly appreciate, as we go along.

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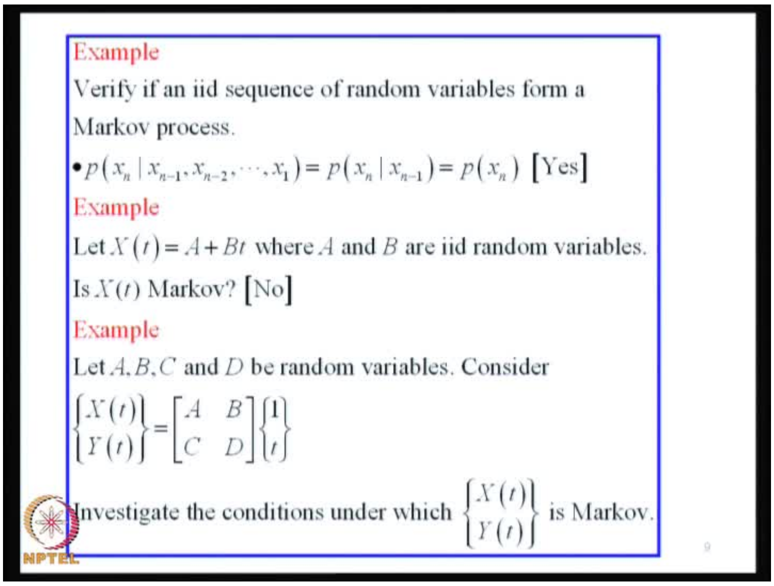

Now, some more examples to appreciate what is meant by the Markov of property; now, first example is, verify if an iid sequence of random variables form a Markov process. So, that is we need to verify probability density function of x n conditioned on x n minus 1 x n minus to x 1, is it this or not, this is, if this statement true or not, indeed it is so, because x i's are iid's and in fact, it is more than that probability density function of x n, condition on x n minus n is simply p of x n, because they are independent sequence; so, we can say that, this process is Markov.

Now, let us consider another random process X of t is A plus B t, where A and B are iid random, variables is X of t Markov, I leave this as an exercise for you to show that, it is not Markov.

Now, an extension of this problem, let us consider A, B, C, D to be four random variables, consider this vector X of t, Y of t, now this can be written as suppose X of t is A plus B t and Y of t is C plus D t, question is under what conditions on A, B, C, D will this vector X of t Y of t be Markov? So, this again is an exercise that would help you to understand the Markov n properties.

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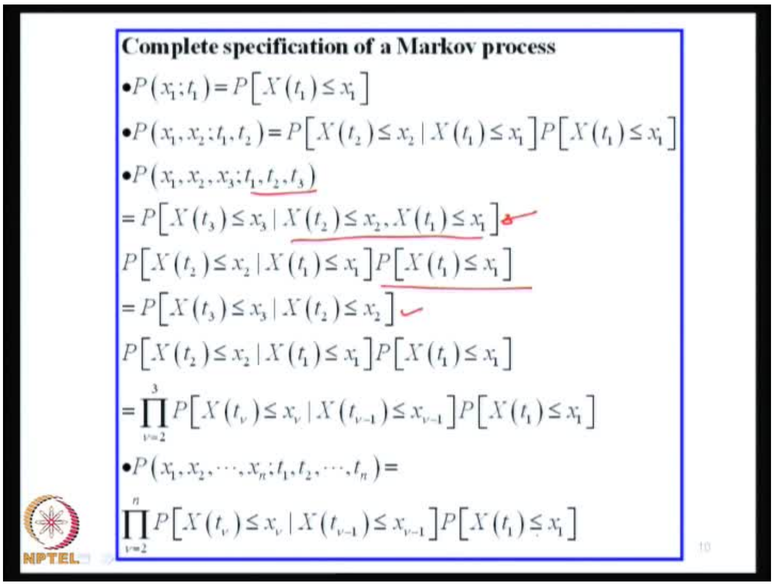

We briefly talked about the complete specification of a Markov process; so, we start with first order distribution function, so probability distribution at t equal to t 1 is probability of X of t 1 less than or equal to x 1. Now, a 2 time instants t 1 and t 2, the probability X of t 2, x 1, x 2: t 1, t 2 would be probability of X of t 2 less than or equal to x 2 intersection X of t 1 less than or equal to x 1 that joint probability t is actually X of t 2 less than or equal to conditioned on X of t 1 less than or equal to x 1 into probability of X of t 1 less than or equal to x 1.

So, this is known, therefore for complete specification I need this also. Suppose, if I take three time instants t 1, t 2, t 3, then I can write the joint density function in terms of a sequence of conditional probability distribution functions, first is X of t 3 less than or equal to x 3 conditioned on X of t 2 less than or equal to x 2 intersection X of t 1 less than or equal to x 1 into X of t 2 less than or equal to x 2, conditioned on X of t 1 less than or equal to x 1 and into probability of X of t 1 less than or equal to x 1, but the since the process is Markov, the first of this terms X of t 3 less than or equal to x 3 is independent of X of t 1 less than or equal to x 1, so I simply write this as this and this is multiplied by the next conditional probability distribution and the first order probability distribution.

So, this I can write it as product of probability density function, probability of X of t nu less than equal to x nu conditioned on X of t nu minus 1 less than or equal to x of nu minus 1 into the first order probability distribution function at t equal to t 1, this nu takes values from 2 to 3. If you now consider n times instants, we can show that using a similar logic, this is equal to product of n transitional probability distribution functions into the first order probability distribution function at t equal t 1.

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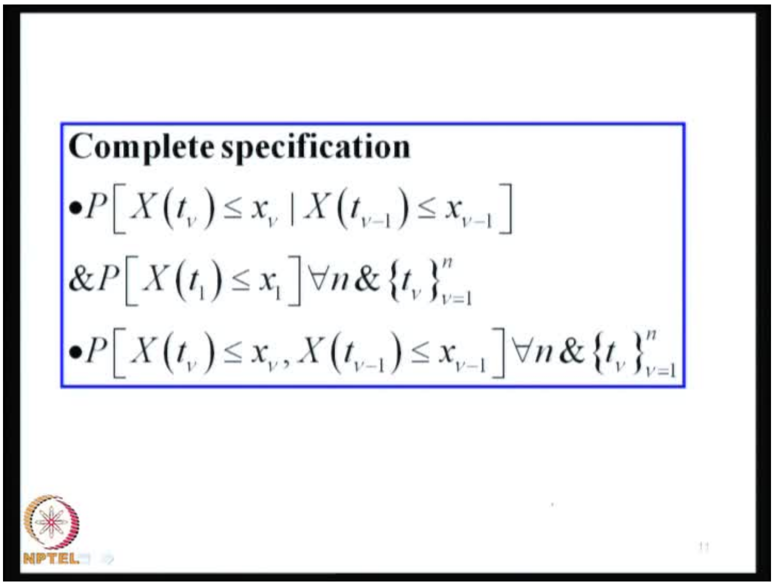

So, the complete specification of a Markov process, therefore, is in terms of the transitional probability distribution function and the first order probability distribution function, for all n and for all choices of t nu, nu running from 1 to n or alternate to this would be, to specify the second order joint probability distribution function for all n and for all choices of nu, running from 1 to n; if you know the second one, you can easily deduce the first one and if you know the first one, that is if you know this transition density function probability distribution function and this probability distribution function, first order probability distribution function, you can deduce the second order probability distribution function, so both are equivalent.

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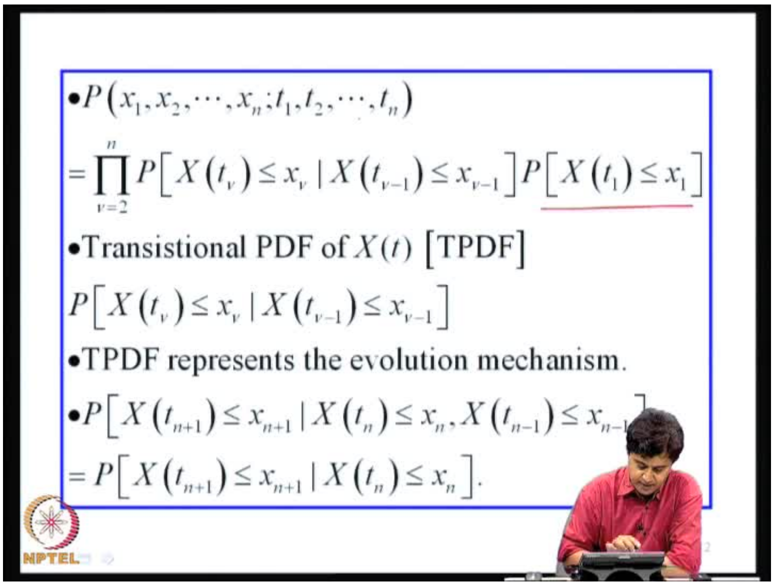

So, these joint densities function as I said, we are writing it as product of this transitional probability distribution function. So, this transitional probability distribution function, we call it as TPDF, I use upper case, let us TPDF following the convention that I am doing; for distribution functions I am using upper case letters and for density functions, I am using lower case letters.

Now, this transitional probability distribution function represents the evolution mechanism of a Markov process. So, you start with first order probability distribution function at t equal to t 1 and go on multiplying this TPDF ‘s to get the joint densities at the time instances that you want; so, this is the evolution mechanism.

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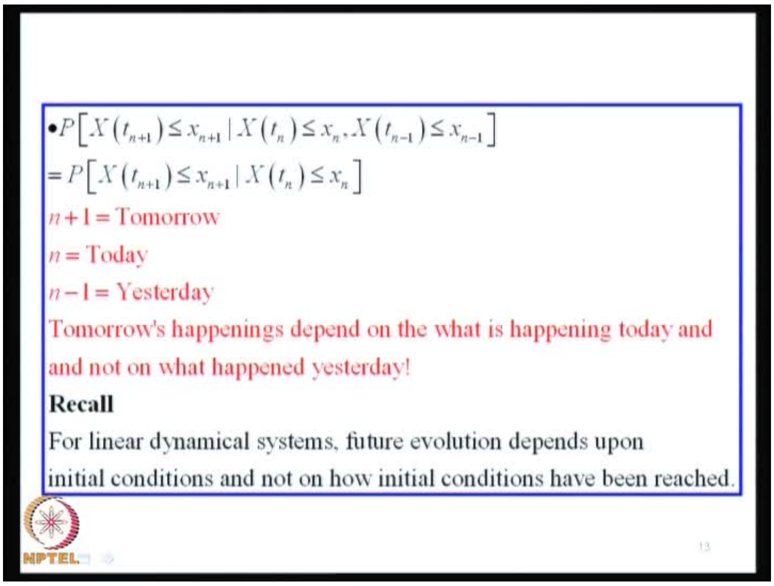

Now, just to give a verbal description, we can consider probability distribution function t n plus 1 t n and t n minus 1, suppose we interpret n plus 1 as tomorrow and n as today, n minus 1 as yesterday, what Markov properties say is, tomorrows happing depend on what is happening today and not on what happened yesterday, that mean, it has one step memory, if time is measured in units of days, this is the Markov property.

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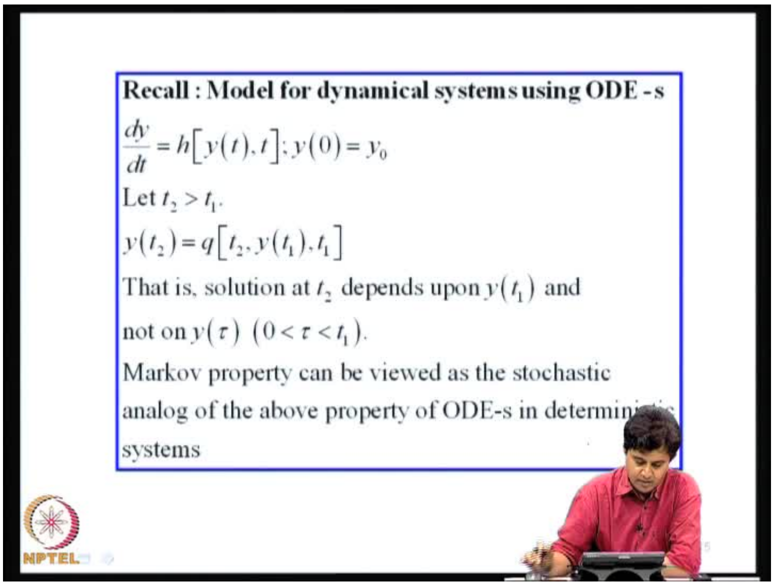

So, you should recall that for linear dynamical system, future evolution depends upon initial conditions and not how initial conditions have been arrived at. So, what that means, if you consider now dy by dt, a deterministic ordinary differential equation, a vector differential equation dy by dt is h of y (t, t), with some initial condition at t equal to 0 provided. Now, if you take two time instants t 2 greater than t 1 y of t 2, if we denote that solution as q of t 2 y (t 1, t 1), this will be the nature of the solution, that means, y of t 2 depends on y of t 1 and t 1 and of course t 2 and not on how y of t 1 has been arrived at.

So, that is solution at d two depends on y of t 1 and not on y of tau, for tau running from 0 to t 1. So, Markov property can be viewed as the stochastic analog of the above property of ordinary differential equations, in the deterministic, in the contrast of deterministic systems.

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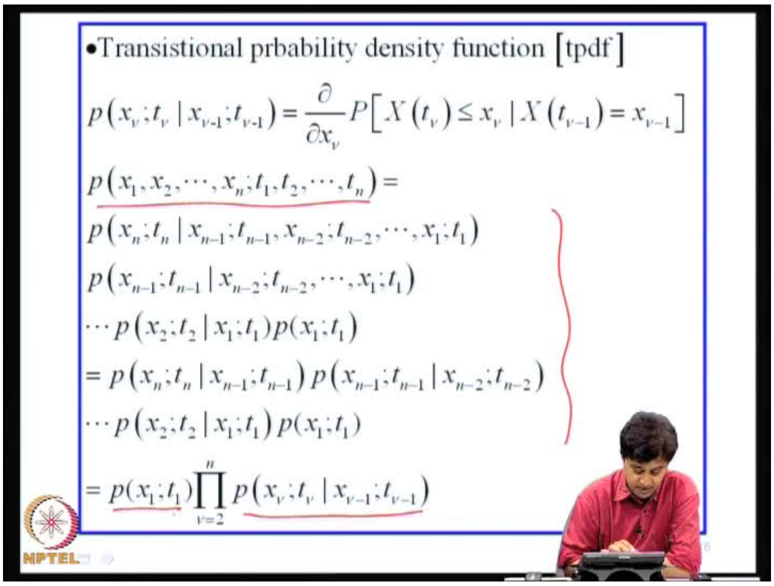

Now, we have defined the transitional probability distribution function, I can define the associated probability density function, by differentiating the probability distribution function; this is straight forward definition. So, lower case p of x nu; t nu, conditioned on x nu minus 1; t nu minus 1, is the partial derivative of the TPDF probability of X of t nu less than or equal to x nu, conditioned on X of t nu minus 1 is equal to x of nu minus 1.

So, the n th order probability density function, you can see through this logic outlined here, we start with the n th order joint density function and write it as a sequence of conditional property density functions and use the Markov property and we can show that the n th order probability density function can be express in terms of the first order probability density function at t equal to t 1 and product of t is TPDF at t nu, t 2, t 3, t 4, t 5, up to t n; so, this is quite similar to what happened with probability distribution function. So, again we can see that complete specification of Markov process, can now, in terms of first order probability density function and the transitional probability density function.

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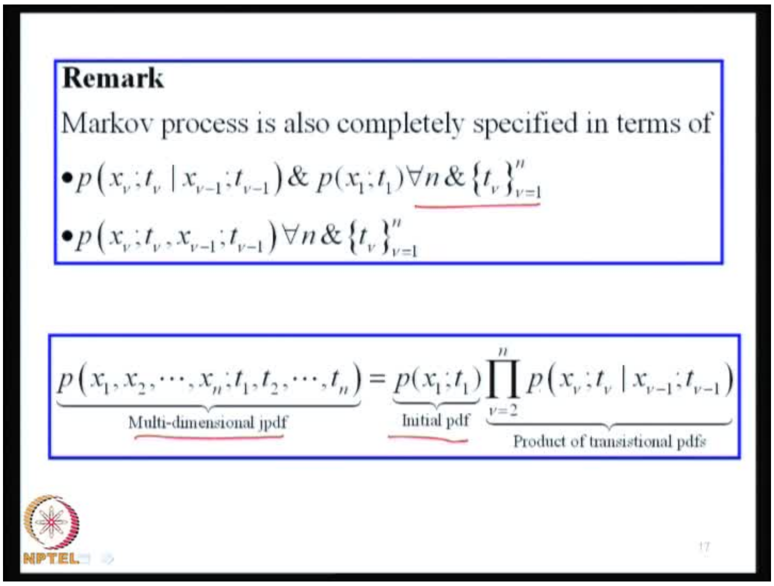

So, that is what Markov process is also completely specified, in terms of transitional probability density function and the first order probability density function, for all n, for all choices of t n; this is very important, that it should be true for all n and all choices of t 1 and t 2, t 3, t n; since the transitional probability density function and first order density function can also be derived from a second order probability density function; the second order probability density function for all n and for all choices of t 1, t 2, t 3, t n is also a complete specification of Markov process.

So, we can again emphasis, the multidimensional probability density function which is needed for complete specification of a random process, is obtain in terms of the initial probability density function and product of transitional probability density functions which represents the transitional mechanism.

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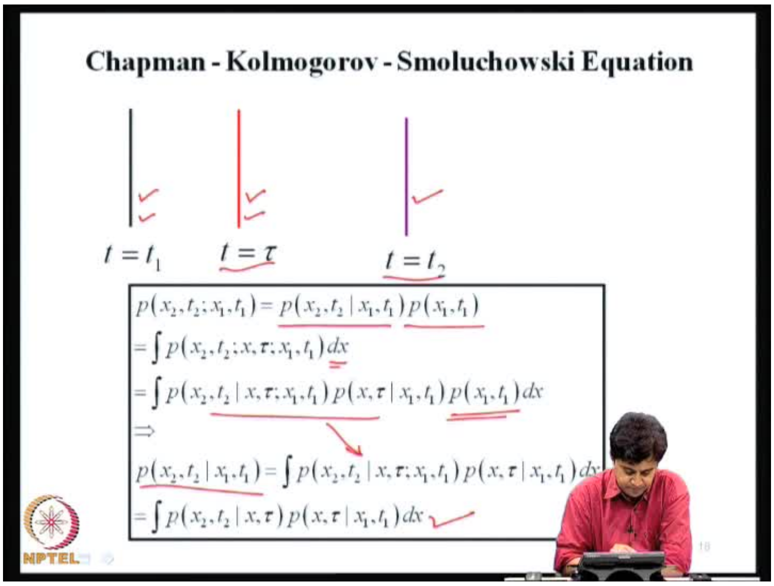

Now, if a process is Markov, the transition probability density function needs to satisfy certain internal consistency conditions. To illustrate that, we consider three time instant t equal to t 1, t equal to t 2 and t equal to t 2, so t 1, tau 1, t 2, so that tau is greater than t 1 and t 2 is greater than tau.

Now, we consider the joint density function between t equal to t 1, t equal to t 2, that is, x 2, t 2; x 1, t 1, this can be written in terms of probability density of x 2, t 2, condition on x 1, t 1 into first order density function. This can also be viewed as the marginal density function of the joint density function between the random variable here, random variable here and variable here, with respect to the states at t equal to tau.

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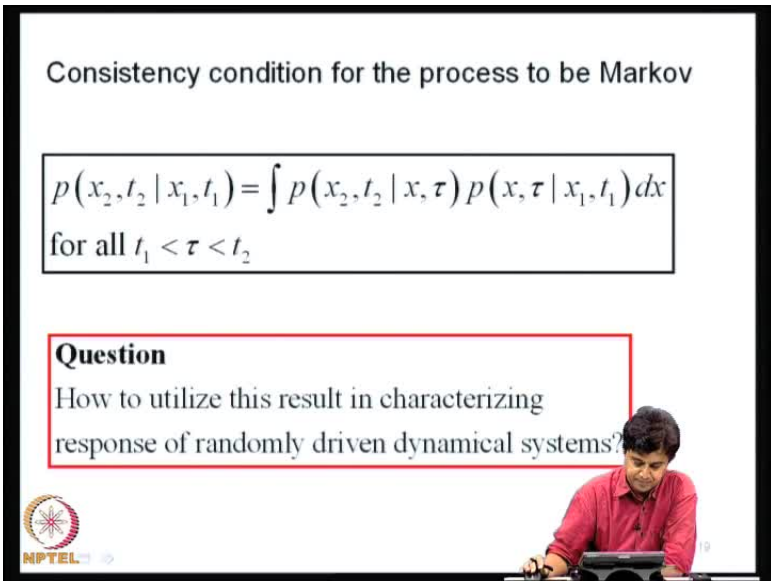

Now, I write this 3-dimensional joint density function, in terms of conditional density functions as shown here and then express this conditional density function of x 2 conditioned on t 2, conditioned on what happens here and what happens here, I write this integral in this form and if I use now the Markov property, this becomes here this p of x 1 t 1 and p of x 1, t 1 here gets canceled and I will do two things, for l h s I will write this and on the right hand side, I will use Markov property and get this relation, that is probability density function of x 2, t 2 condition on x 1, t 1 is given by the integral of x 2, t 2 conditioned on x, tau, which is intermediate time instant into probability density function of x of tau, conditioned on x 1, t 1 dx; this should be true for all t 1, t 2 and tau.

So, this can be viewed as a kind of a compatibility or consistency condition for the process to be Markov. Now, this forms the basis for actually analyzing vibrating systems and deriving the equation of motion for the probability density function; so, the big picture that we should bare in mind at this stage, is to answer this question, that is how do you utilize this result in characterizing response of randomly driven dynamical systems; so, this is our agenda, we should keep that in mind, when we get into some of these details.

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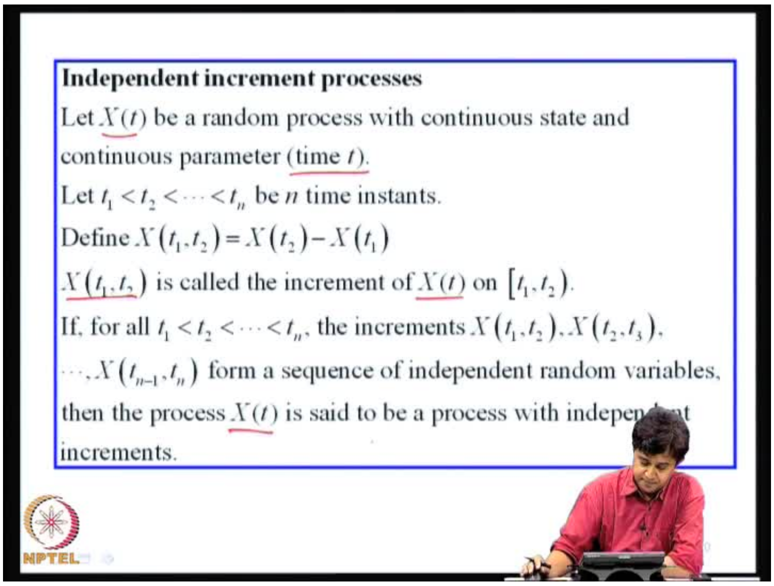

Now, we talk about what are known as random processes with independent increments; now, let X of t be a random process with continuous state and continuous parameter time; I consider increasing ascending sequence of time t 1, t 2, t n, we consider this n time instants and this defines the n random variables X of t 1, X of t 2 and X of t n, now I define another random variable X (t 1, t 2) as X of t 2 minus X of t 1, this random variable is called the increment of X of t on the interval t 1 to t 2.

Now, for given time is instant t 1, t 2, t n, we can also form a sequence of increments, which are again random variables X (t 1, t 2), X (t 2, t 3), so on X (n minus 1, t n), this form a sequence of independent random variables, if this sequence of increments form a sequence of independent random variables, then we say that X of t is the process with independent increments, that is the definition of a process with independent increments.

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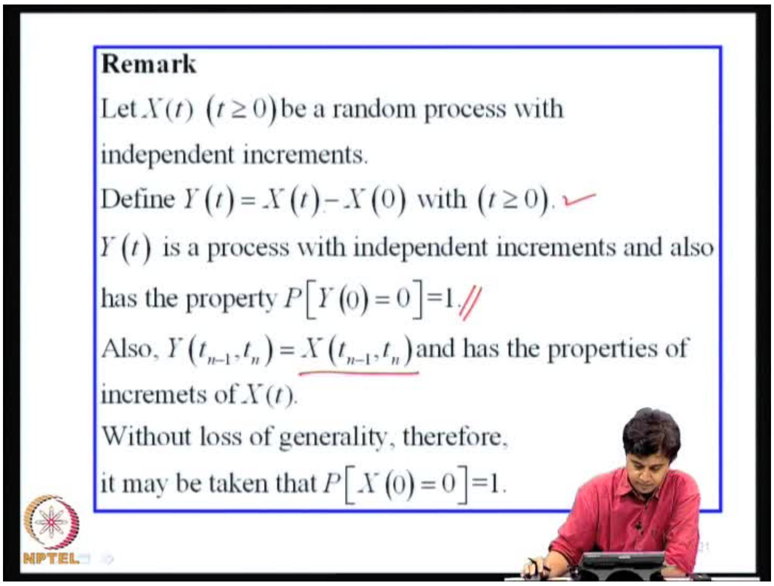

Now, let X of t be a random process with independent increments; now, I define Y of t as X of t minus X of 0 with t greater than or equal to 0. Now, Y of t is a process with independent increments and also has a property that probability of Y of t equal to 0 is 1, because Y of 0 x of 0 minus x of 0 which is 0; so, Y of t equal to 0 has probability of 1.

And also Y of t n minus 1, t n is nothing but X of t n minus 1 t n, if you write say Y (t n minus 1, t n) it is Y of t n minus Y of t n minus 1 Y of t n would be X of t n minus x of 0 and Y of t n minus 1 will be Y of t n minus 1 minus x of 0 and that would get cancelled and it will be same as X (t n minus 1, t n), that means, Y of t has all the properties of this sequence, except that t equal to 0, probability of Y of t equal to 0 is equal to 1.

So, therefore the point is without loss of generality, we can always take P of X of t equal to 0 as 1. So, if it is not true, then we can make this transformation and deal with the process Y of t.

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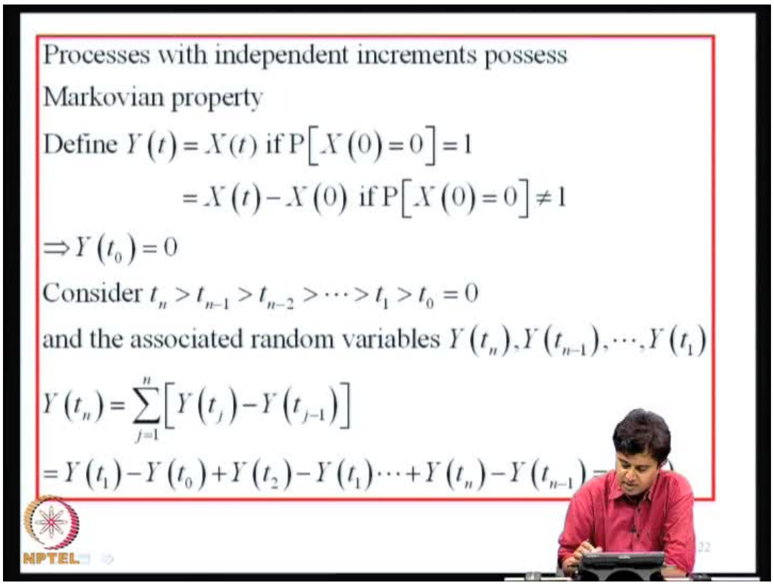

Now, the important feature of process with independent increments is that, such processes have Markov n property. Now, to see that, let us define Y of t equal to X of t, if probability of x of 0 equal to 0 as 1, otherwise X of t minus x of 0, if this is not equal to 1; in any case, Y of t naught will be 0, t naught is 0.

Now, we consider the sequence t n greater than t n minus 1 etcetera, t naught and t naught is 0 and the associated random variables Y of t n, Y of t n minus 1 and Y of t 1. Y of t n can be written as sums of these increments Y of t j minus Y of t j minus 1, we can substitute and verify that, it is indeed true, that means, Y of t 1 minus Y of t 0 plus Y of t 2 minus Y of t 1 etcetera, etcetera, it will be left with only Y of t n.

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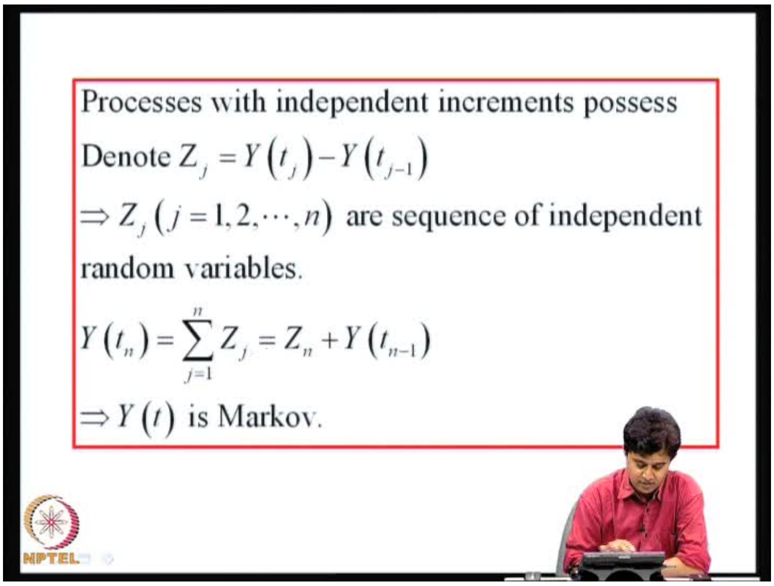

Now, we denote Z j as Y of t j minus Y of t j minus 1, these are sequence of independent random variables. Now, the Y of t n therefore can be written as Z n plus Y of t n minus 1; so, this is in the form of a simple random walk and we already should the simple random walk has Markov n property, therefore Y of t is Markov. You can see here Y of t n depends on Y of t n minus 1 and the change from t n minus 1 to t n is caused due to Z n and Z n, Z 1, Z 2, Z 3, Z n are all independent random variables; therefore, Y of t is Markov.

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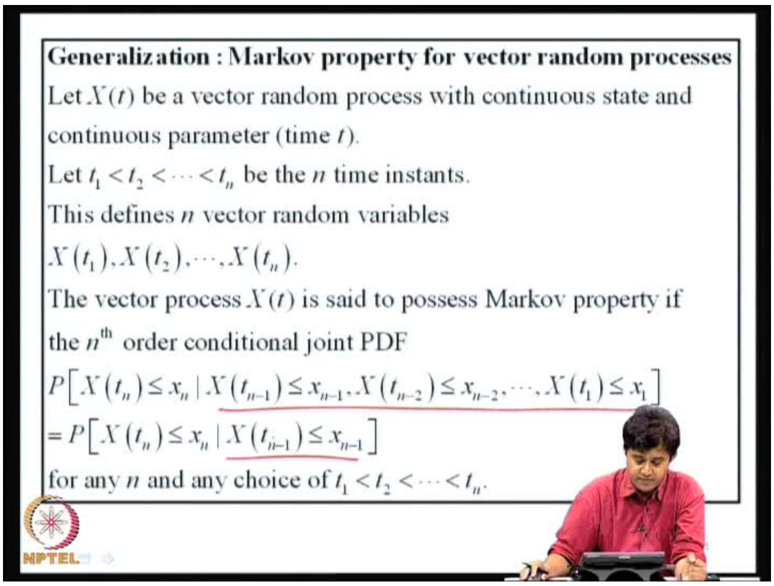

Till now, I have been taking about sequence Markov property of scalar random processes; now, we can generalize this notion to vector random processes. So, if X of t is a vector random process with continuous state and continuous parameter time t, let us consider such a random process, again we consider the time sequence t 1, t 2, t 3, t n, the n time instants and associated random variables X of t 1, X of t 2, to X of t n; X of t 1 is a vector of random variables, X of t 2 a vector of random variable.

So, the vector process X of t is said to possess Markov property, if n th order conditional joint probability distribution function of this vector random variables, X of t n less than or equal to x n conditioned on these events is given by probability of X of t n less than or equal to x n, conditioned only on what happened at t equal to n minus 1 and if this is true for any n, for n choice of t 1, t 2, t 3, t n, we say that the vector random process X of t has Markov property; this is straight forward extension of the definition for a scalar case.

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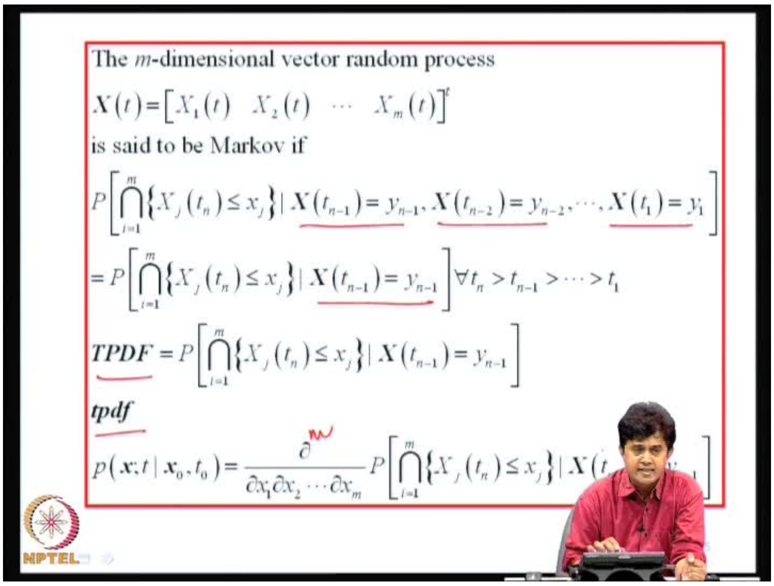

Suppose, if you consider just to elaborate this notion a bit; the m-dimensional vector random process X of t, X of t 1, X of t 2, X of t is said to be Markov, if the probability of intersection of y equal to m x j of t n less than or equal to x j, which is nothing but the probability distribution function, conditioned on this events, where X of t 1, X of t 2 here are all vector random variables and this depends on only the state at t equal to t n minus 1, then we say that X of t is Markov.

So, this quantity, we call it as the transition probability distribution function for the vector random process X of t; associated with this, we can derive the transition probability density function for the vector random process X of t, by carrying out the m time differentiation of this transition probability distribution function; these are straight forward extension of the notions that we are used in scalar case.

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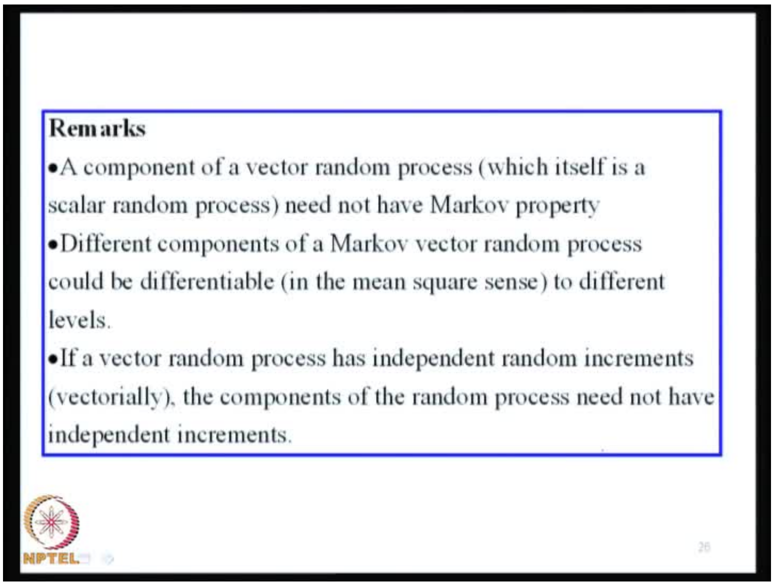

Some remark should be made at this juncture; a component of a vector random process would be a scalar random process, but that need not have Markov property; a vector random process could be Markov, but the components of such a Markov random process need not be Markov; we will see that in due course again.

Now, different components of a Markov vector random process could be differentiable, in the mean square sense to different levels; a Markov process itself is not differentiable. I will I clarify this, what, these things, mean in due course, but we should make notes of this at the outside.

If a vector random process has independent random increments, that is vector ally, the components of the random process, need not have independent increments; so, you need to think about it and verify by actually working out, what exactly this statements mean.

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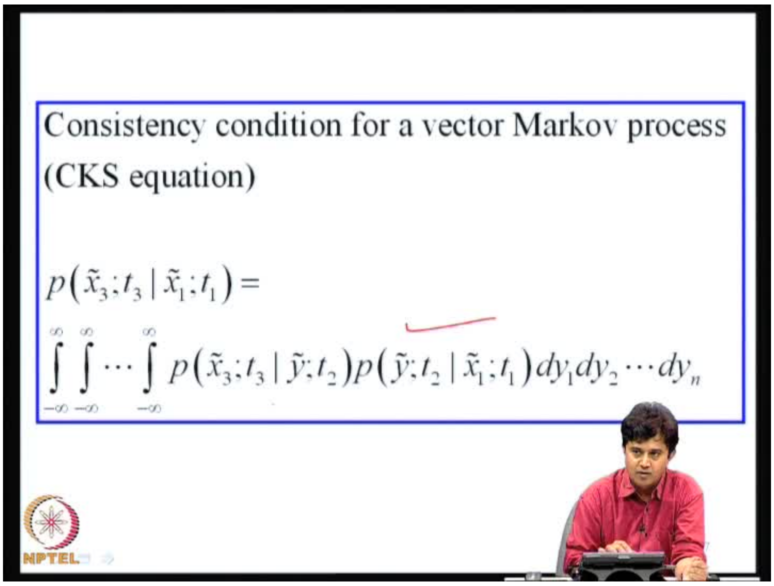

The Chapman Kolmogorov Smoluchowski equation, that is the consistency equation, can be generalized for the vector random process and the mathematical form of that remains identical, except that, now we are writing joint probability density functions instead of scalar probability density function; so, that means, the probability density function of x 3 tilde conditioned on x 1, t equal to t 1 is given by this product of these two TPDF integrated are y 1, y 2, y n; so, this is the condition for internal consistency of a vector Markov random process.

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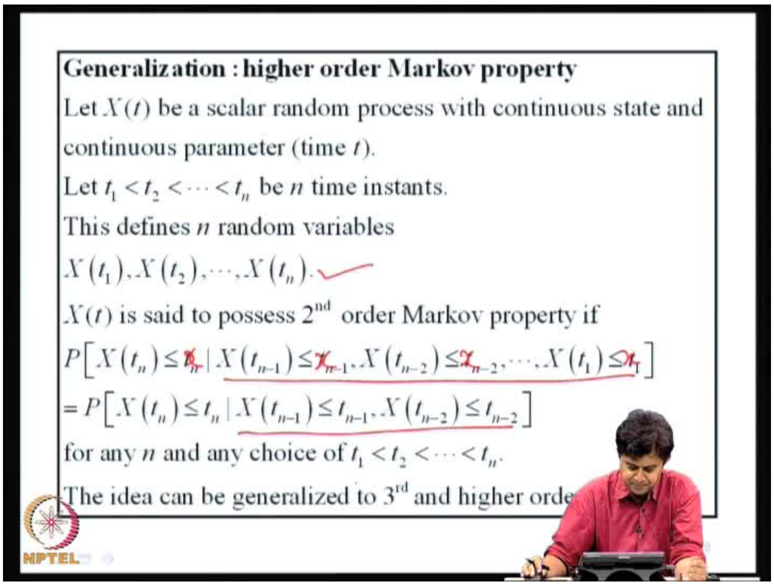

We could generalize the theory of Markov process in a slightly different way; again, let us return to a scalar random process. Let X of t be a scalar random process with continuous state and continuous parameter time t and again, we consider n time instants and associated n random variables. Now, X of t said to possess second order Markov property, if probability of the event X of t n less than or equal to t n, the event the X of t n less than or equal to X n conditioned on these events, depends upon what happen during t equal to t n minus 1 and t equal to t n minus 2, if this is the property, that the probability distribution satisfy, then we say that X of t is Markov with second order. So, this notion can be generalized to higher orders; this, I am mentioning for sake of completeness.

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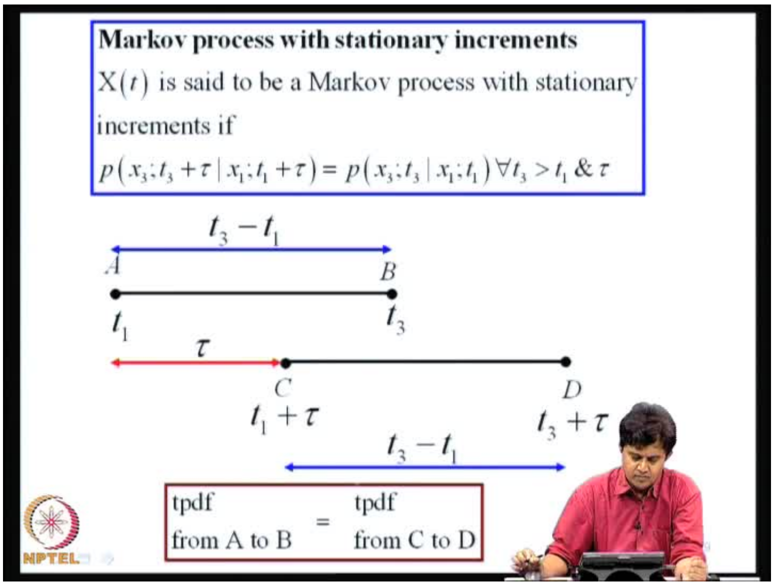

Now, one more thing that we will be talking about is Markov process with stationary increments; now, X of t is said to be Markov process with stationary increments, if you consider the TPDF, suppose you considered two time instant t 1, t 2 and other two time instants t 1 and t 3, that means, I shift this t 1, t 3 by a fixed amount tau, that means, the distance from A to B is t 3 minus t 1 and distance from C to D is again t 3 minus t 1; that means, there are now two random variables, here separated by t 3 minus t 1 and there are two more random variables separated by the same distance t 3 minus t 1.

So, if we now consider the transition probability density function from say A to B and C to D and if it is a function of only tau, then we say that processes stationary increments. So, TPDF from A to B is same as TPDF from C to D.

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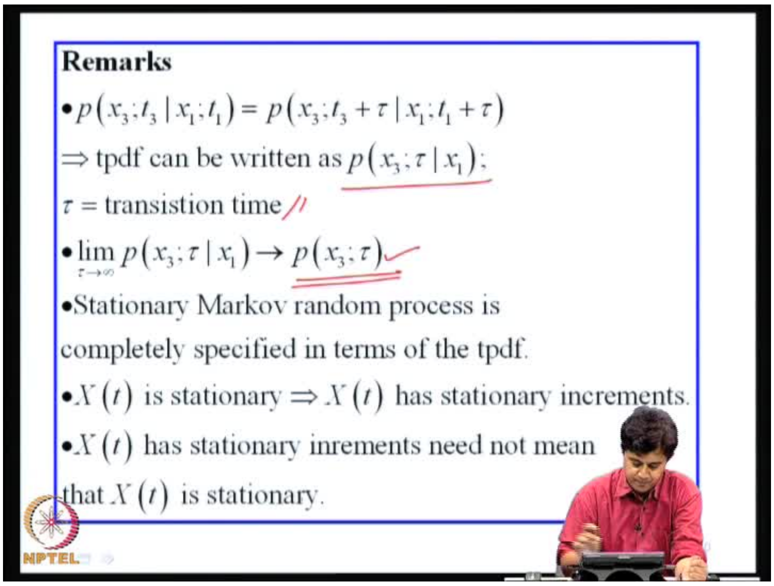

So, in such a case situation, the TPDF can be simply be written as p of x 3; tau conditioned on x 1, where tau is a transition time and as tau goes to infinity, the transition PDF now depends only on tau x 3; tau and the memory of x 1 is lost and this is one of the properties that we need to use.

So, Stationary Markov random process is completely specified in terms of the TPDF in this case, because at a tau tends to infinity, it is defined only in terms of the transition probability density function. You should notice that X of t, if X of t stationary, X of t has stationary increments, but it is does not mean that, if X of t is stationary increments, this statement need not mean that X of t is stationary. So, you can again verify, these are small points that would help you understand the notion of stationary, stationary increments.

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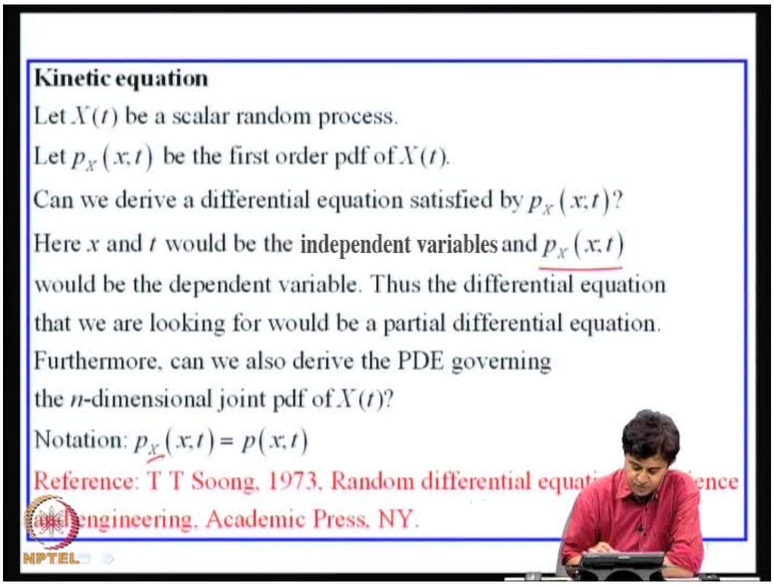

Now, our basic interest in using Markov property would be to derive an equation for evolution of the probability density function, so how do we go about doing that. This equation is known as kinetic equation; so, let X of t be a scalar random process and p x (x; t) be the first order probability density function of X of t; so, p x this PDF depends on two variables x and t. Can we derive a partial differential equation which describe this evolution in x and t that is a question, how can we do that. If we can do that, x and t would be the independent variables and the dependent variable will be the PDF.

Thus the differential equation that we are looking for would be a partial differential equation with p x dependent variable and x and t as independent variables. Furthermore, can we also derive the partial differential equation governing the n-dimensional joint PDF of X of t, if we can do it for one-dimensional PDF, you may think of doing it for n-dimensional. And so, from now onwards, what I will do is, this p x (x, t), I will simply write it as p (x, t), this subscript X of t I will omit, for sake of simplicity; this is a reference that you may find useful for the topic, that I am going to cover in this part of the lecture; how do we proceed.

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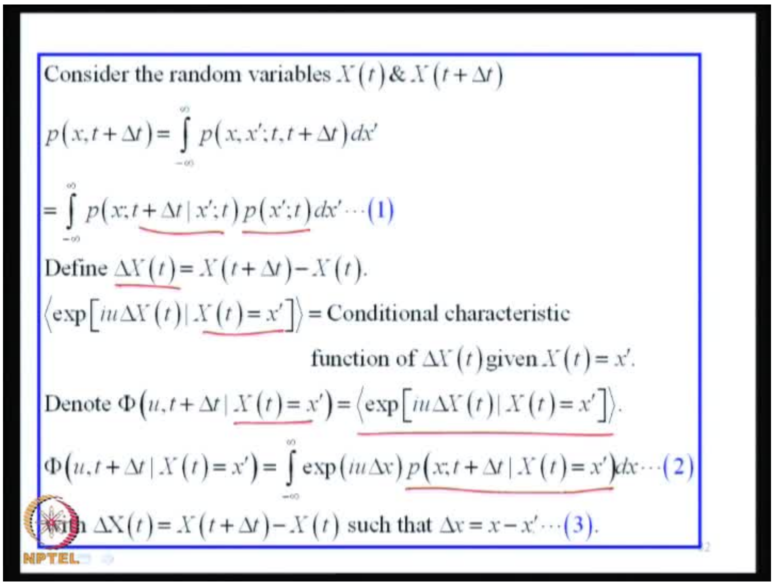

Now, let us consider two random variables X of t and X of t plus delta t; so, the probability density function at t plus delta t can be obtained as the marginal density of the joint PDF at t and t plus delta t this is a simple statement. This itself can be express in terms of product of a conditional probability density function and a first order probability density function.

Now, let us define the increment delta X of t as X of t plus delta t minus X of t; let us consider the conditional characteristic function of delta of X of t conditioned on X of t being equal to x point, so that would be expectation of i u delta X of t conditioned on X of t equal to x phi.

Now, let us denote this as phi u, t plus delta t conditioned on X of t equal to x phi; so, this is the actually expectation and in terms of the density function it is given, this is the definition of the characteristic function, where this delta X of t is a X of t plus delta t minus X of t such that delta x is x minus x phi.

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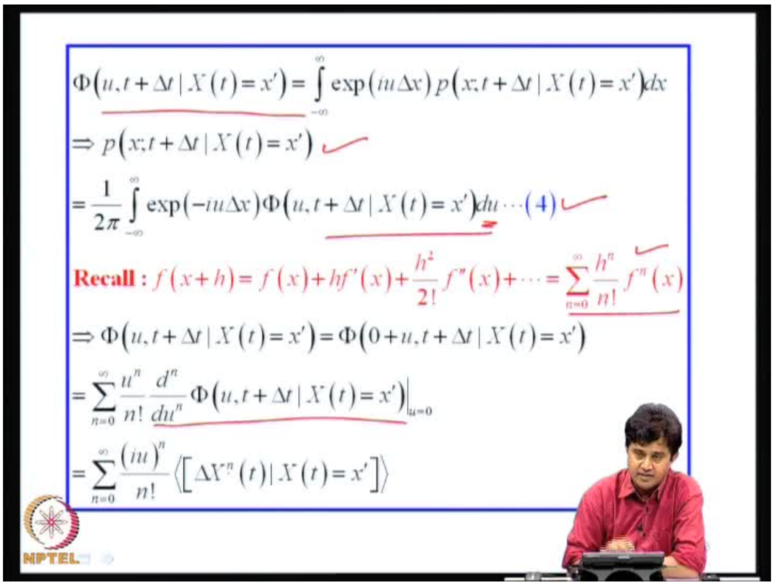

Now, we will do some manipulations on this; if this is a definition of characteristic function, the conditional PDF can be obtained by inverting this through using inverse Fourier transform, so I get this equation, now the integration is with respect to u; now, you recall f of x plus h can be express in terms of tailor's expansion in this form, that we will be using. Now, let us expand the characteristic function around origin u equal to 0, so I will write phi u, t plus delta t as 0 plus u t plus delta t etcetera and then expand this around the region u equal to 0 and if we do that, we get this expansion and we know that through the property of characteristic functions, this is nothing but the moments of the random variable. So, in this case, we are talking about delta x conditioned on X of t equal to x prime and I get this as delta of X n expectation of that.

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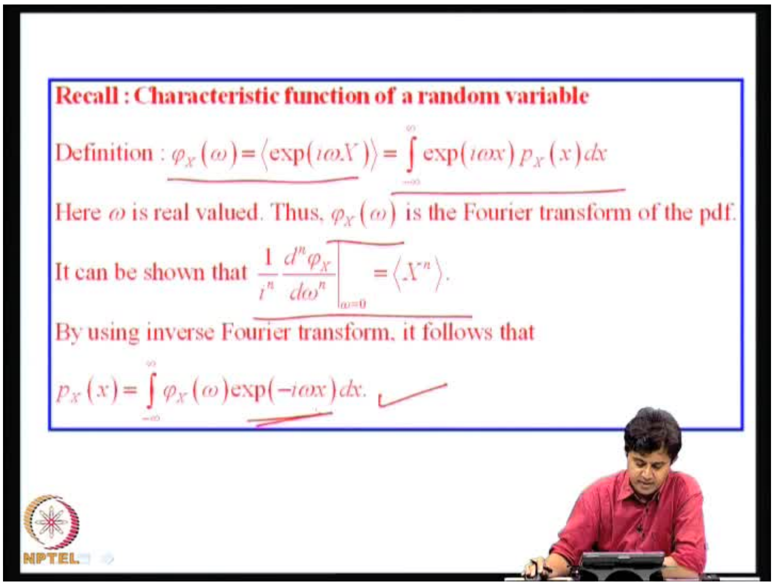

Now, quick recall, we know the definition of a characteristic function of a random variable is the expectation of e rise to I omega x and phi x of omega is actually the Fourier transform of the probability density function and we have shown that the characteristic function has this property, which I am using in fact. And the inverse Fourier transform is given, in terms of this and that leads to definition of probability density function.

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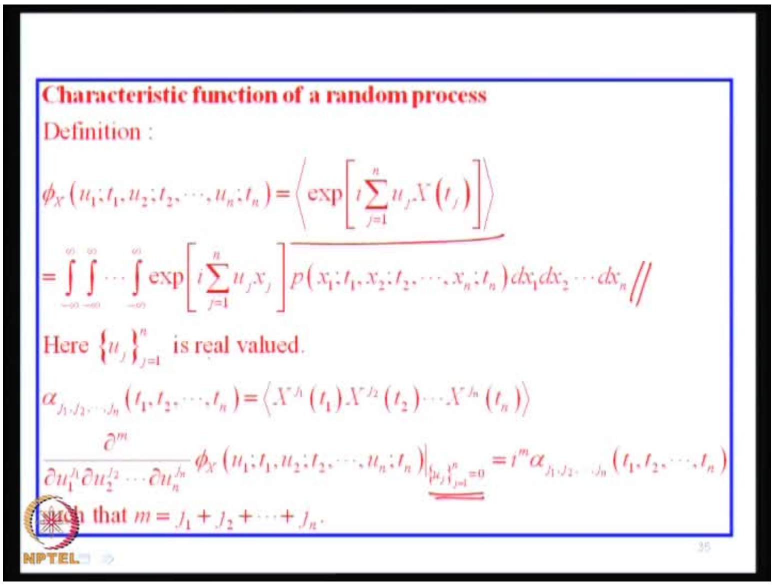

Similarly, we can define characteristic function for a random process, suppose if we consider n random variables X of t 1, X of t 2, X of t n, I can write the expression for the joint characteristic function of these random variables through this expectation and that is nothing but this integral. Here, this u 1, u 2, u 3, u n are all real valued and the moments X of t 1 to the power of j 1, X of t 2 to the power of j 2 etcetera, can be obtain in terms of the derivatives of this characteristic function evaluated at the origin, following this rule; this is the extension of the definition characteristic function for a random process. So, the red font, I am using to recall what we have already done; so, these are thing we are recalling.

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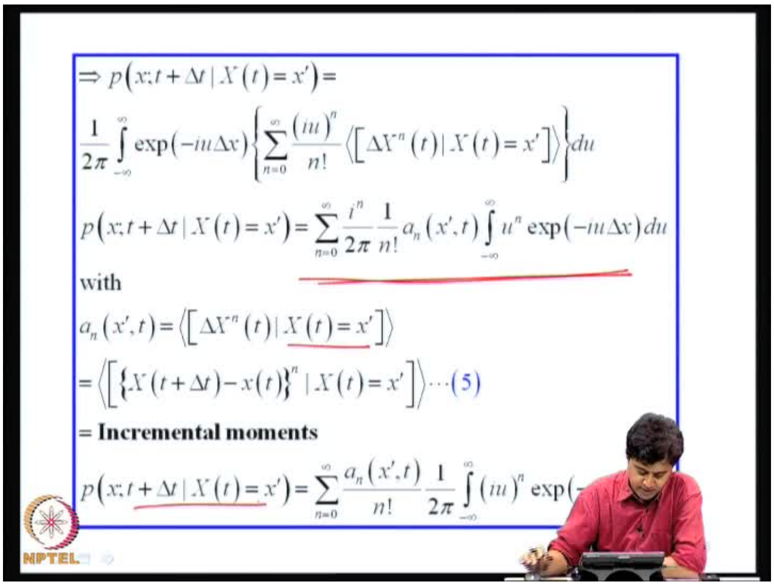

Now, let us go back and write in the expression for the inverse Fourier transform of the density function, the expansion that we have used and we get this expression, where this quantity a n x prime, t is this moment delta X n of t conditioned on X of t equal to x prime and these are known as incremental moments. So, in terms of incremental moments, I have got, now, the expression for this probability density function.

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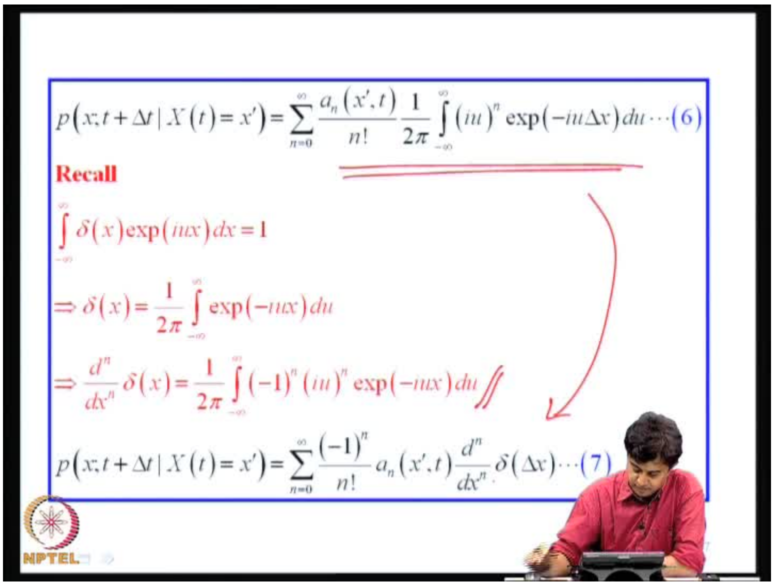

Now, I need to redesign this expression, recalls this expression slightly in a different way; so for that we recall if we take the Fourier transform of a direct delta function we get a constant. So, a direct delta function, therefore can be expressed as inverse transform of e raise to minus i u x, I have shown here; now, if you differentiate this, it can be express in terms of differential of the on the right hand side; so, we get this expression. So, what we are getting here turns out to be related to the derivatives of direct delta function and we can use that result here and we get this expression.

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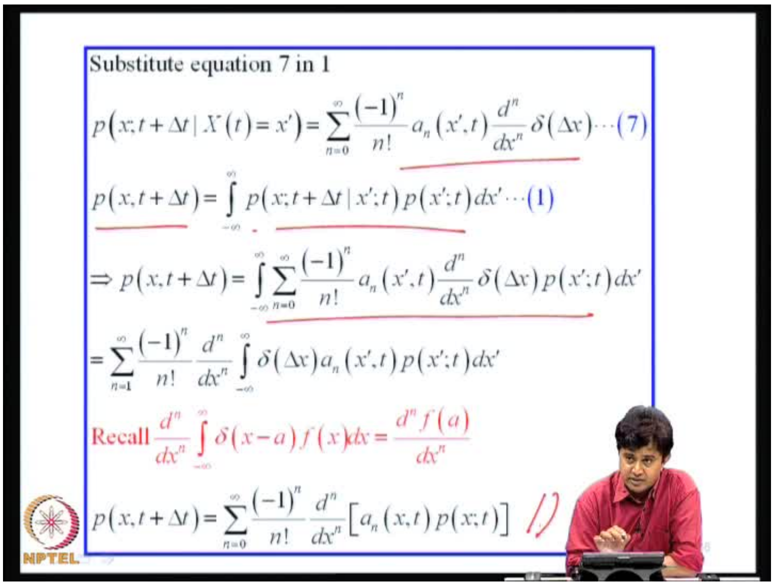

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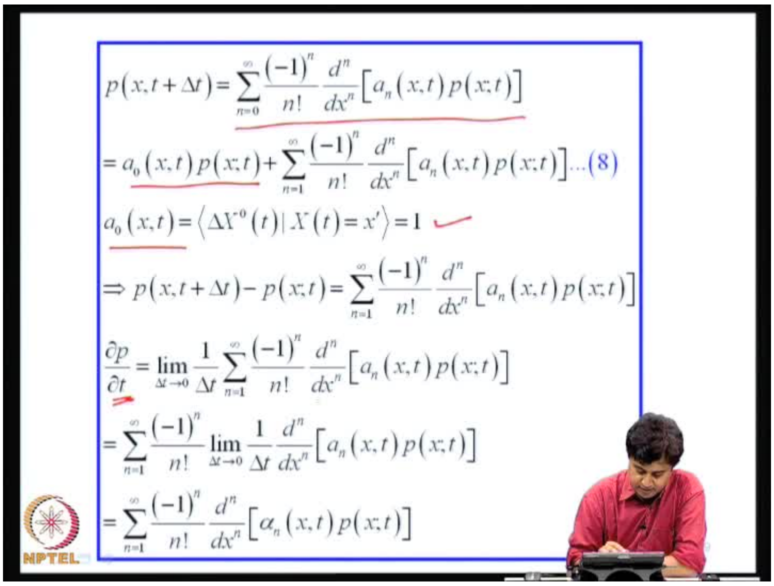

Now, a slight amount of rearrangement of these terms and splitting, so we have substituted in terms of direct delta function and we return now to the first order density function this is the integrand, so this expression will come in the integrand and I get this expression and again rearranging these terms and using this fact that n th order derivative of a integral like this is nothing but d n d x to the power of n f of a and if we do that, I get this expression; this derivation is bit t d s, but I am trying to highlight some important steps, so you need to work through this bit patiently, then only we follow this. So, what we do now is a summation on the right hand side, I will right the first term that is n equal to 0 separately and then take the summation from one to infinity.

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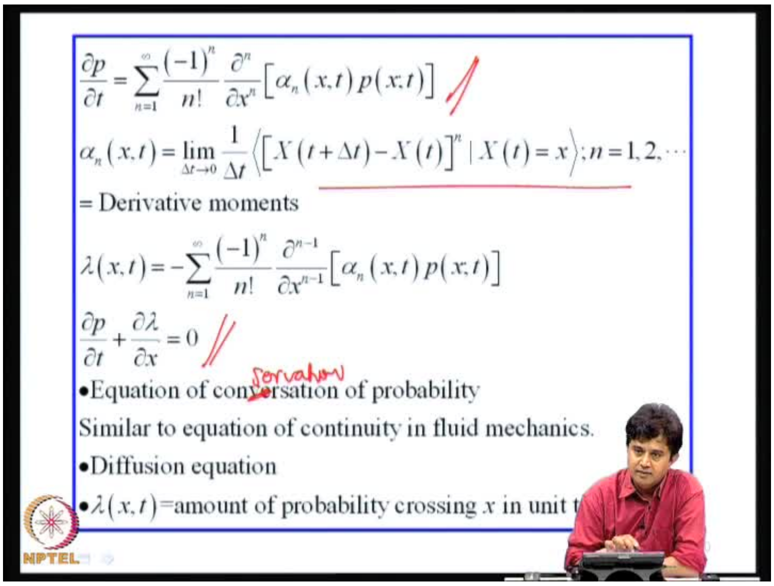

So, in this I get this expression a naught x, t and this is nothing but delta x to the power of 0 of t conditioned on X of t equal to x prime and this is one. Now, that would mean, I can take this to the left hand side and now I am in a position to define the derivatives, I will divide this by delta t and take limit delta t to 0 and on the left hand side, I get dou p by dou t and on the right hand side, I get these expressions, which can be simplified to get a partial differential equation as displayed here.

So, this is what I am looking for, this is actually the evolution of probability density function of X of t, in terms of what are known as incremental moments or derivative moments as shown here. We can call this right hand side, that is, the we take dou by dou x outside and call the what remains, that is n equal to 1 to infinity minus 1 to the power of 1 divided by n factorial and n minus 1 derivative of this, as lambda (x comma t), then this equation can be written as dou p by dou t plus dou lambda by dou x equal to 0. So, this takes a quite elegant form now, so this can be this can be viewed as equation of conservation of probability; so, this similar to the equation of continuity in fluid mechanics. So, this is a diffusion equation and this quantity lambda (x, t) can be interpreted as amount of probability crossing x in unit time in the positive direction; so, this is analogy with fluid mechanics that would be alpha.

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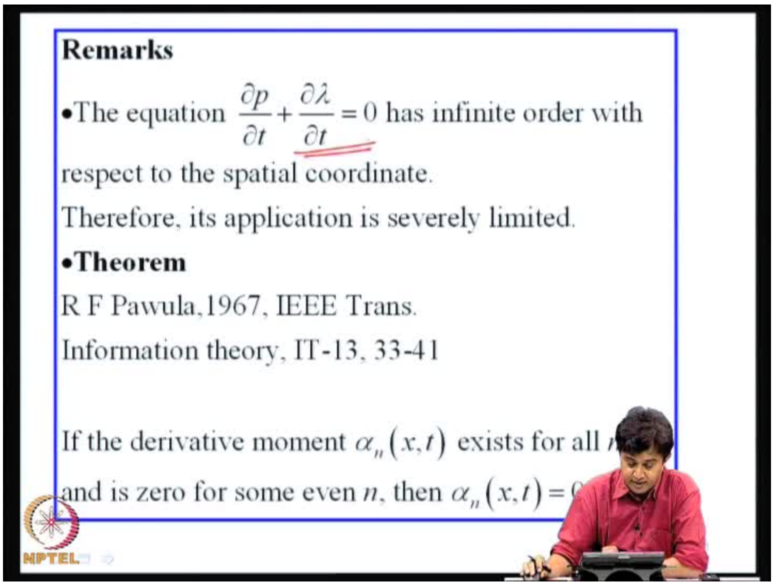

But this equation, although, it has nice form if you look back, this lambda itself has terms running from n equal to 1 to infinity, which has derivatives to the power of n, where n is running from 1 to infinity; so, that would mean, that this equation has infinite order respect to the spatial coordinate; therefore, it is not an partial differential equation which can be solved this only a formal representation.

But there is one nice result, which would help us to simplify this and that is what I will briefly mention, the what this results say is, if the derivative moment alpha n exists for all n and is 0 for some even n, then it automatically means, alpha n is 0 for all n greater than or equal to 3; what that means is, this differential equation has infinite number of terms, if it is going to have finite number of terms, it has to stop at n equal to 2, there cannot be an equation which is valid for n equal to 5,7 etcetera, if it is not 2, it has to be infinity, there is nothing in between.

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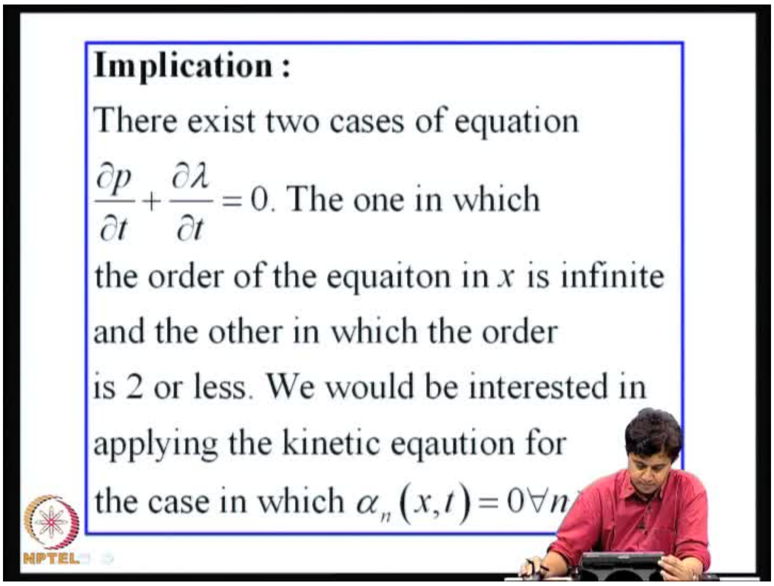

So, there exist two cases of this continuity equation, the one in which the order of the equation in x is infinite and other in which the order is 2 or less. We would be interested in applying the kinetic equation for the case, where this alpha n equal to 0, for n greater than or equal to 3 and we will show that for a Markov process that is indeed true.

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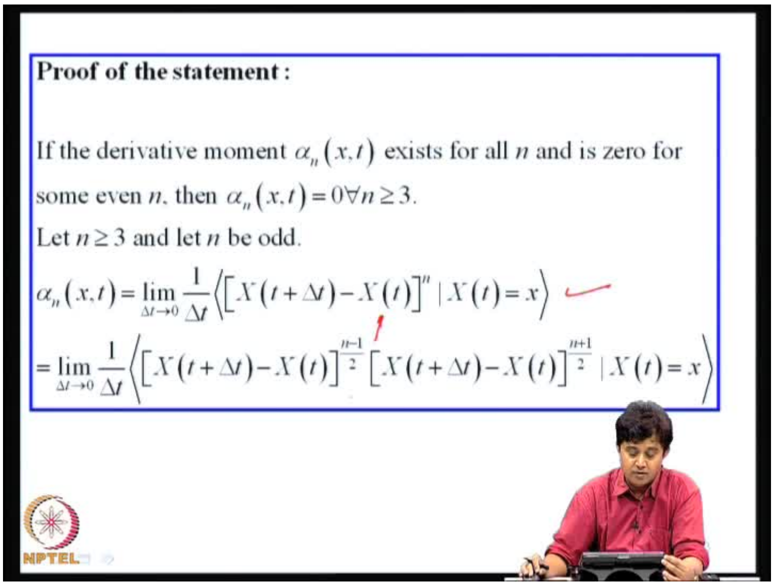

Now, there is the brief proof for this, I will just highlight the steps, so that you could go through bit more carefully. So, if the derivative moment alpha n exist for all n and is 0 for some even n then alpha n x, t equal to 0 for n greater than or equal to 3.

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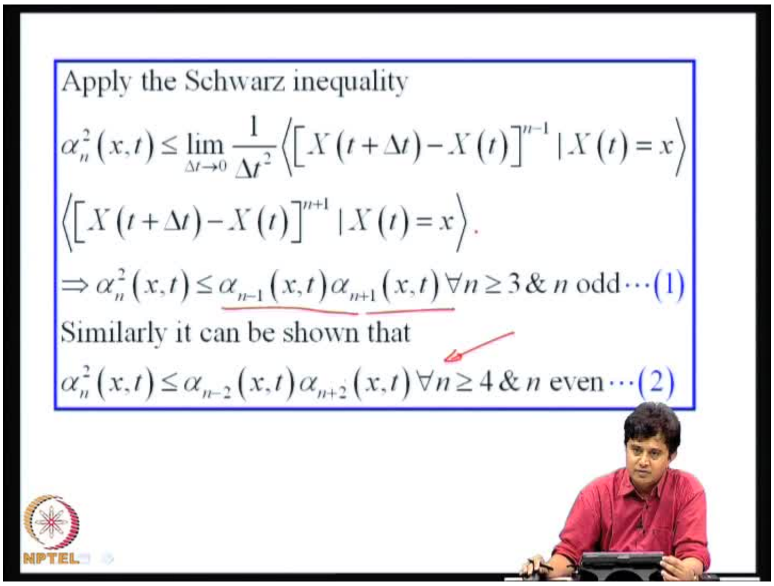

So, let us consider the case where n is greater than or equal to 3 and let n b odd, so alpha n x, t is actually this expectation and I can write this as product of this quantity inside the bracket to the power to the power n minus 1 by 2 and n plus 1 by 2, if you multiply this, we will get this term. Now, this is a permissible splitting, on this now I will apply Schwarz inequality, you recall Schwarz inequality, we use to show that, I have correlation coefficient is between plus 1 and minus 1, earlier the same inequality I am using, I will get alpha n square x, t must be less than or equal to product of 2 expectations, that is alpha n square x, t must be less than or equal to product of alpha n minus 1 x, t alpha n plus 1 x, t for n greater than or equal to 3 and n odd; for n even, we can show that, we get another similar inequality.

Now, if you carefully consider these two inequalities, you can show that, if alpha n is 0 for some n greater than or equal to 3, then all alpha should be equal to 0.

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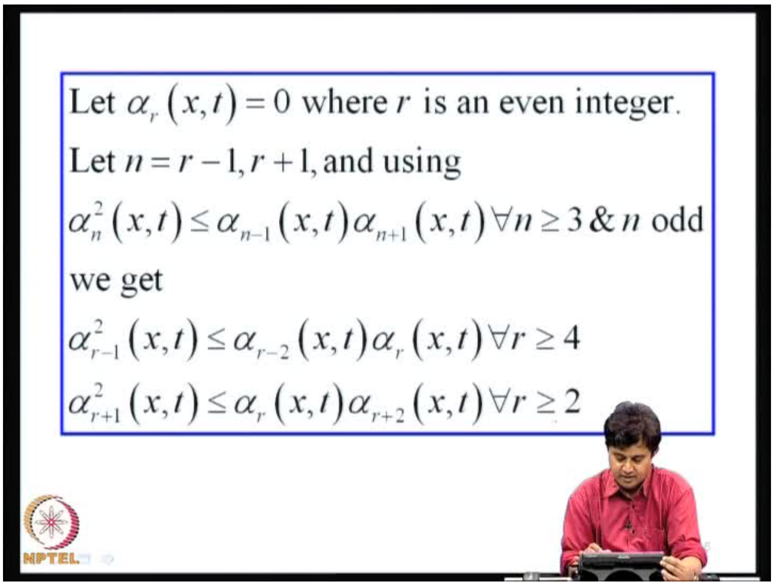

So, let alpha r (x, t) equal to 0, where r is an even integer; so, let n equal to r minus 1 r plus 1 and using this we get these two inequalities, for r greater than or equal to 4 and r greater than or equal to 2.

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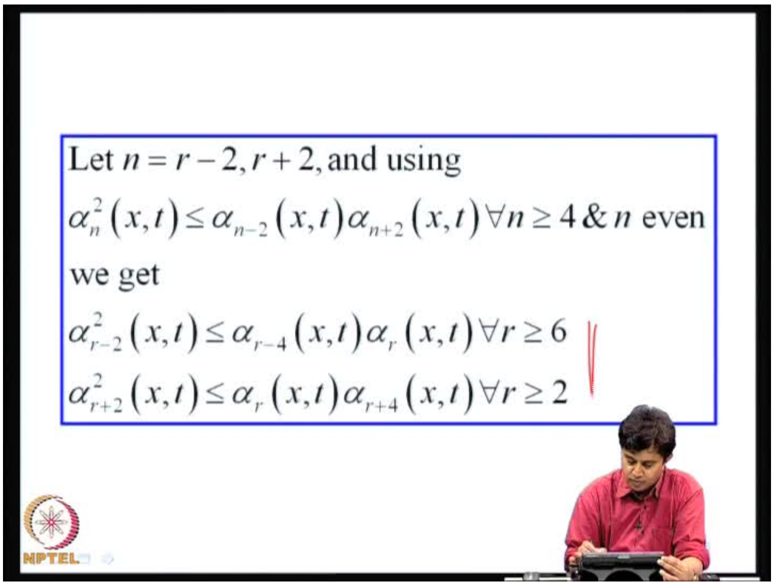

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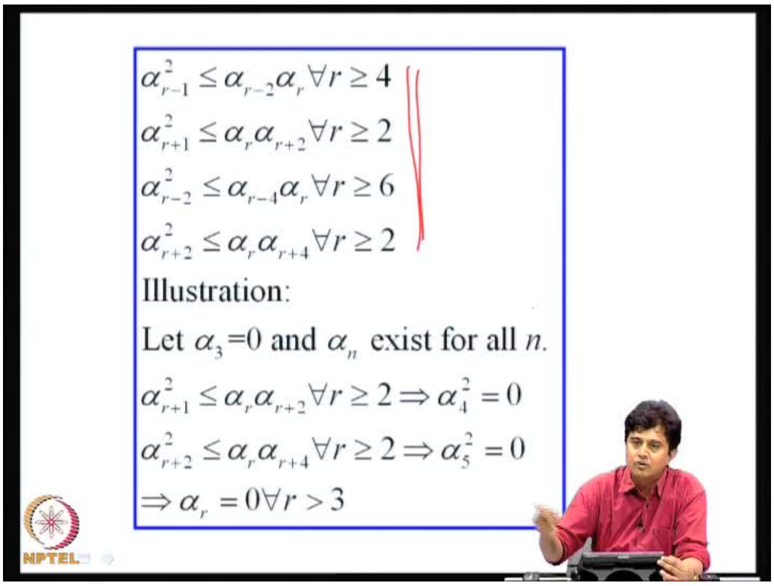

Similarly, we get for n r minus 2 and r plus 2, using for n even, we get these two inequalities and if I put all of them together, I have these four inequalities. Now, an illustration, let alpha 3 be 0 and alpha n exist for all n; now, you can use these inequalities in sequence and show that, if this is true alpha 4 must be equal to 0 and alpha 5 must be equal to 0, if r 4 and 5 are 0, 6 and 7 must be 0, if 6 and 7 is 0, 8 and 9 must be 0; so, you can show that, all n greater than or equal to three would be 0.

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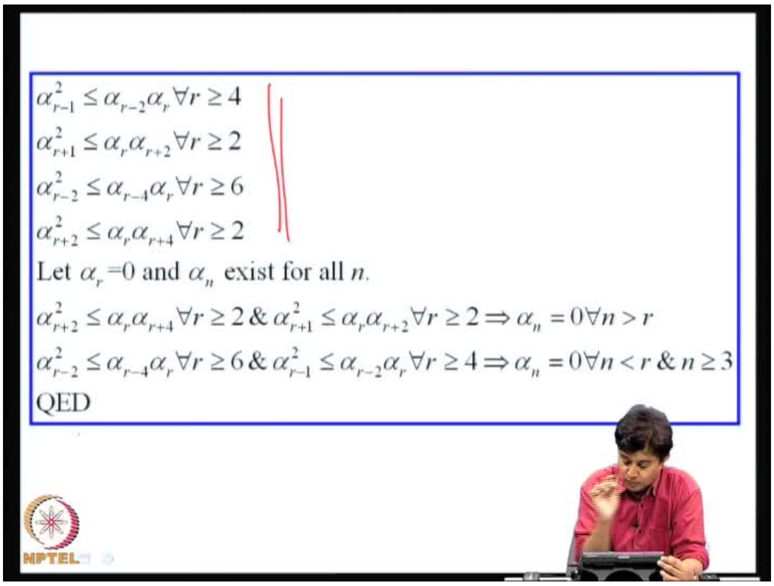

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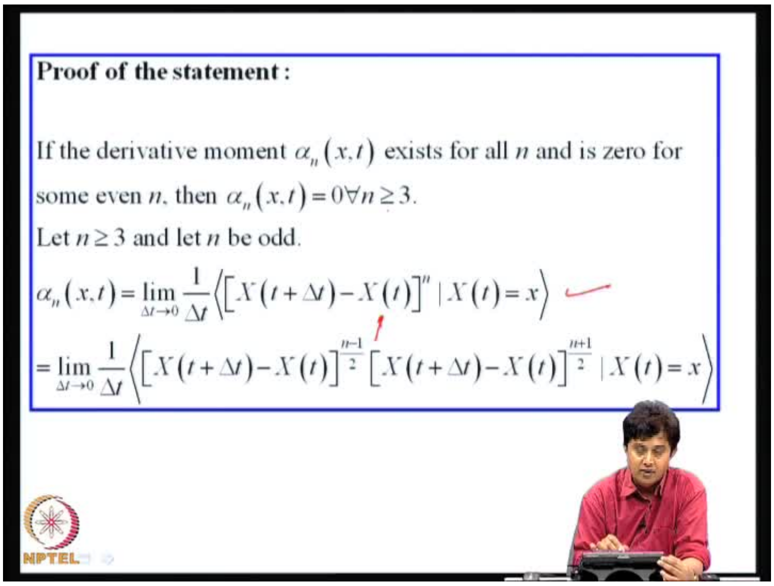

So, that is constructed here and this require some moment of reflection, there is no complex issue here, you have to simply understand, what is being said through these inequalities and if you appreciate that, we would have proved the statement that we made that if alpha n greater than or equal to alpha n for n greater than equal to 3 is 0, no, what is the statement, the statement is if the derivative moment alpha n exist for all n and its 0 for some even n, then alpha n is 0 for all n greater than or equal to 0, for all n greater than or equal to 3.

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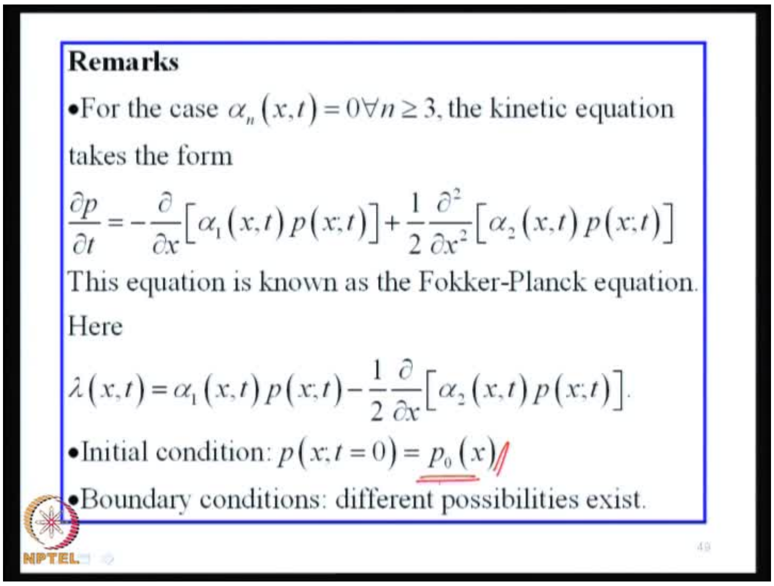

If that is the case, we are considering the kinetic equation, now becomes a simpler equation differential equation dou p by dou t is minus dou by dou x alpha 1 into p plus half dou square by x square alpha 2 into 2 p, this equation is known as the Fokker Planck equation. Here, this lambda is express in terms of alpha 1 and alpha 2 and the initial condition for this equation would be that t equal to 0 x of 0 can be a random variable and its probability density function is specified.

Now, boundary condition different possibilities exist, I mentioned about absorbing barriers and reflecting barriers and so on and so forth; so, there is a body of a literature on, what are the admissible boundary conditions and when do solutions exist etcetera, etcetera, so I will not be getting into although details.

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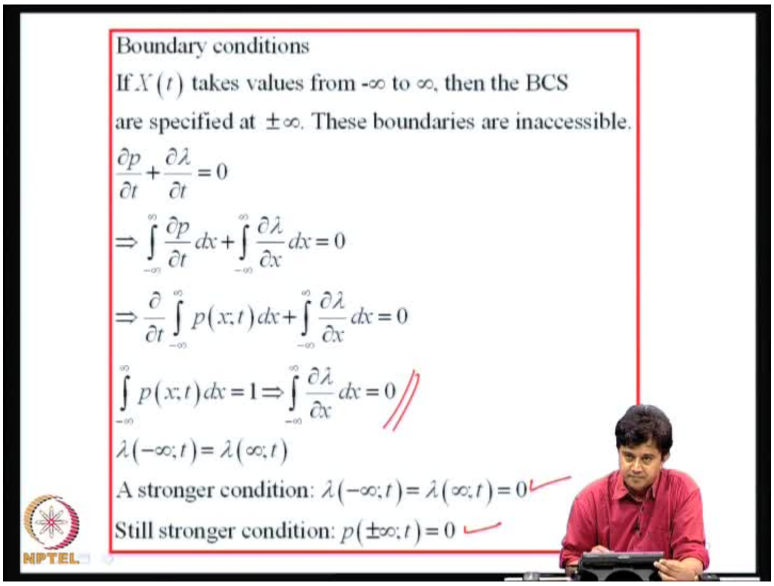

Now, if X of t takes values from minus infinity to plus infinity, as in the case of say a Gaussian random process, then boundary condition is to be specified at plus minus infinity; these boundaries are inaccessible. So, if we now consider the kinetic equation and integrate from minus infinity to plus infinity, we can manipulate this expression a bit and we know that t of area under the probability density function is 1, therefore the first term would be 0 and that leaves us with the condition that dou lambda by dou x into dx must be equal to 0; this requires lambda (minus infinity; t) must be equal to lambda (infinity; t); a stronger condition would be that these two are independently equal to 0 by a much stronger condition would be that probability density function itself is 0.

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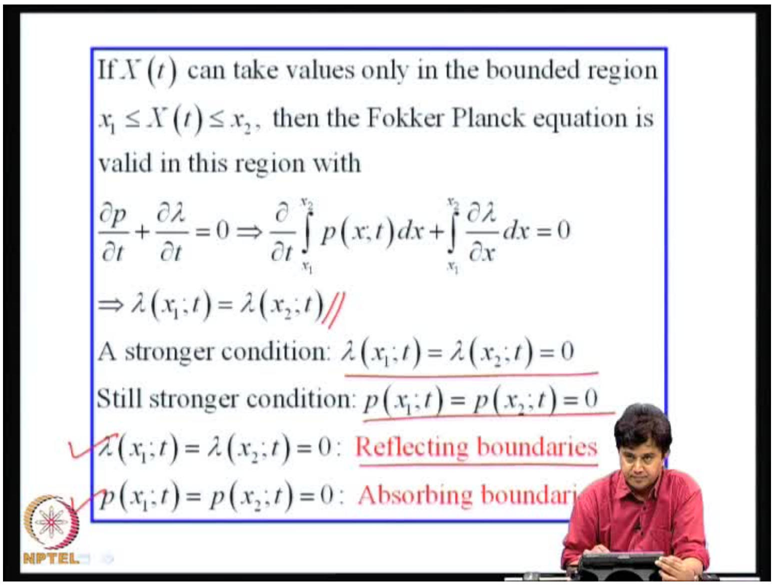

So, this we can show that corresponds to an absorbing boundary and this corresponds to a reflecting boundary. If X of t can take values only in the bounded region x 1 to x 2, then the Fokker Planck equation is valid in this region, with, you know, you have to manipulate this equation again form if you integrate from x 1 to x 2, we get this condition and a stronger condition would be this and much a stronger condition is this and as I was mentioning, the this boundary condition corresponds to a reflecting boundaries and this stronger condition corresponds to absorbing boundaries.

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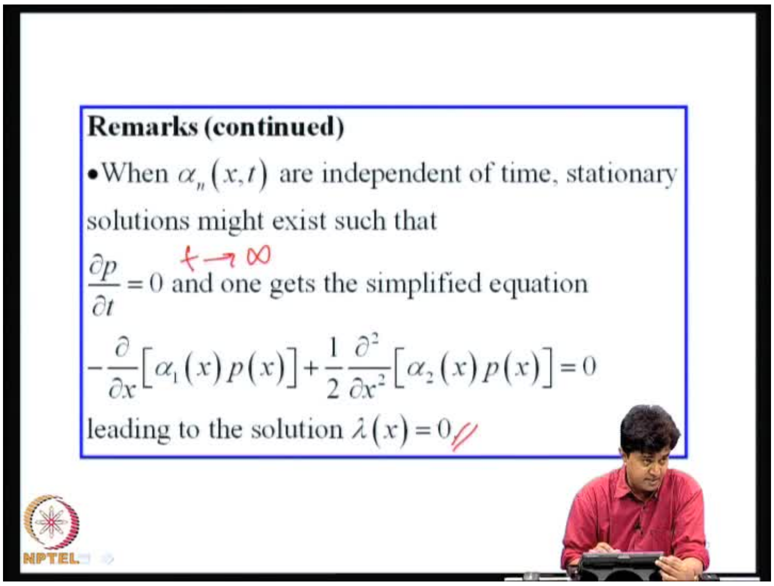

Now, if alpha n are independent of time, stationary solutions might exist, such that dou p by dou t become 0 as 1 as t goes to infinity and one gets the simplified equation, which is, there is no time variable; this is, in fact, an ordinary differential equation which leads to the solution lambda of x is equal to 0; so, this is one way to find out steady state solution.

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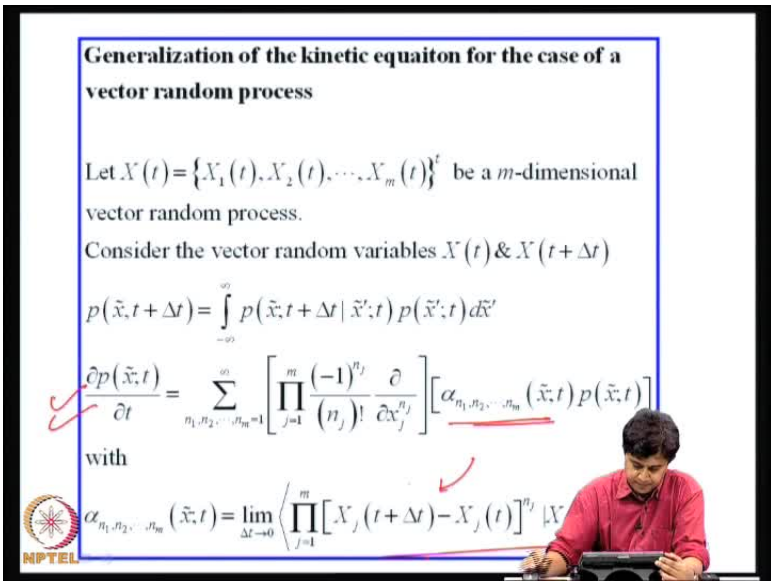

Now, the generalization of a kinetic equation for a case of vector random process is quite straight forward conceptually, but it is bit tedious to express, but I leave that as an exercise, this will be the form of the kinetic equation; here, now I get the derivative moments in slightly more involved manner and this is reflected here.

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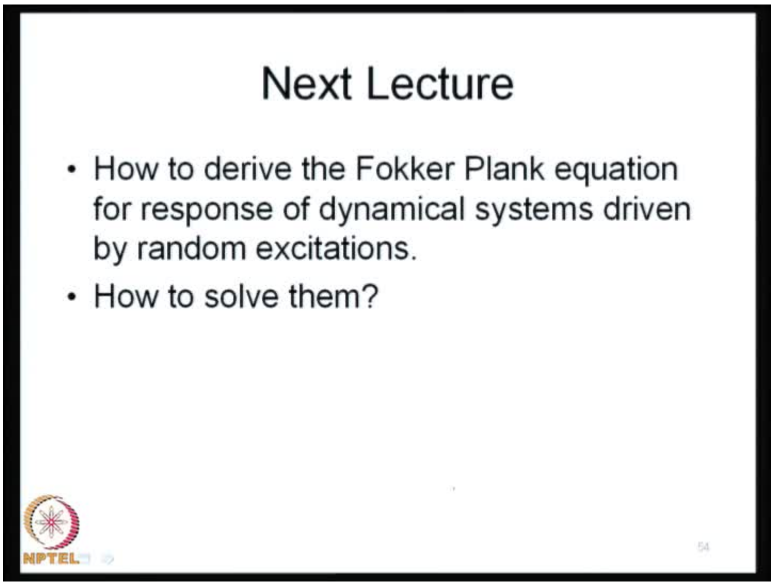

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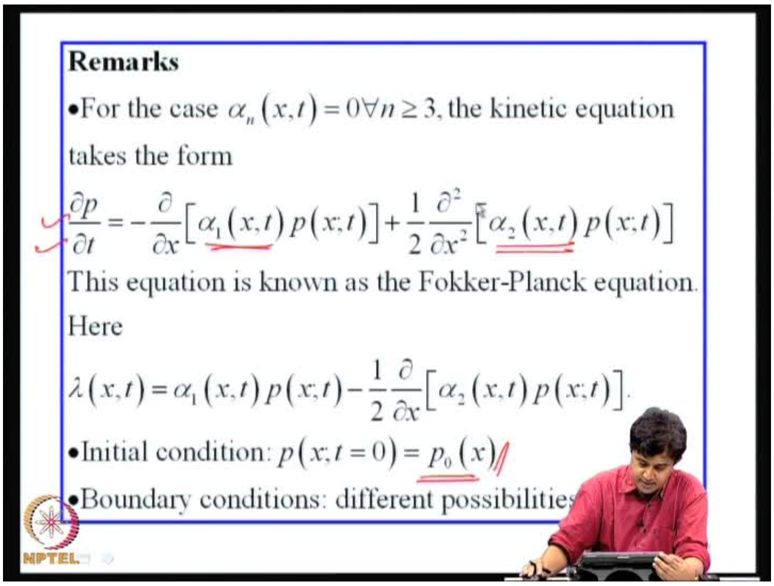

So, till now what we have done is, we have derived the governing partial differential equation, for the evolution of probability density function. Now, the question that we should be answering is to set up this kinetic equation, for example, if you want to setup this equation, I need to determine this parameters alpha 1 and alpha 2.

Suppose, I look at a dynamical system and I formulated its equation of motion starting from that, can I derive alpha 1 and alpha 2, in terms of, for example, mass stiffness and damping property of the system, can I derive this derivative moments, I will explain how that can be done in the next lecture, but if that can be done, then this equation that we are seen here can be thought of as the equation of motion, for the dynamical system with depended variable being the probability density function.

We have derived till now the equation motion for the samples of a random process, from that we could derive the mean, the covariance etcetera. Now, can we derive the equation of motion directly for the probability density function itself, if that is so, how do we relate this equation to the governing deferential equation of motion for a dynamical system?

If we derive this equation, the next question would be on the associated initial conditions, the associated boundary conditions, when what are admissible and when do we hope to get a solution. The next issue would be how we can solve this equation, so it turns out, that for linear dynamical systems with additive Gaussian white noises; this differential equation can be solved exactly, not just that but that is not our major use of Markov process theory. The Markov process theory becomes quite useful, because this equation of motion can be derived even for non-linear systems for cases, where there are non Gaussian excitations, for cases when the excitations are parametric; there can be parametric excitations and external excitations simultaneously acting on the system, this forms an avenue for formulating the governing equations of motion for the probability density function and under certain situations, the governing equations even for non-linear systems and even for parametrically excited systems can be solved exactly.

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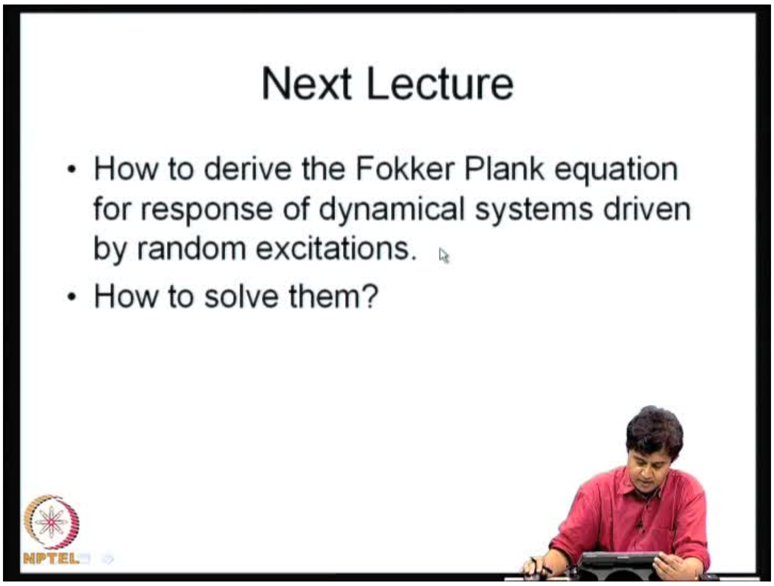

So, the Markov process approach provides a means for obtaining exact solutions for certain class of non-linear random vibration problems. It also provides a strategy to formulate approximate methods, when such exact solutions are not possible. So, this is what, is essence of Markov vector approach and in the next lecture, we will consider some of these questions in greater detail and will conclude this lecture at his juncture.