Stochastic Structural Dynamics

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Lecture No. # 20

Failure of Randomly Vibrating Systems-4

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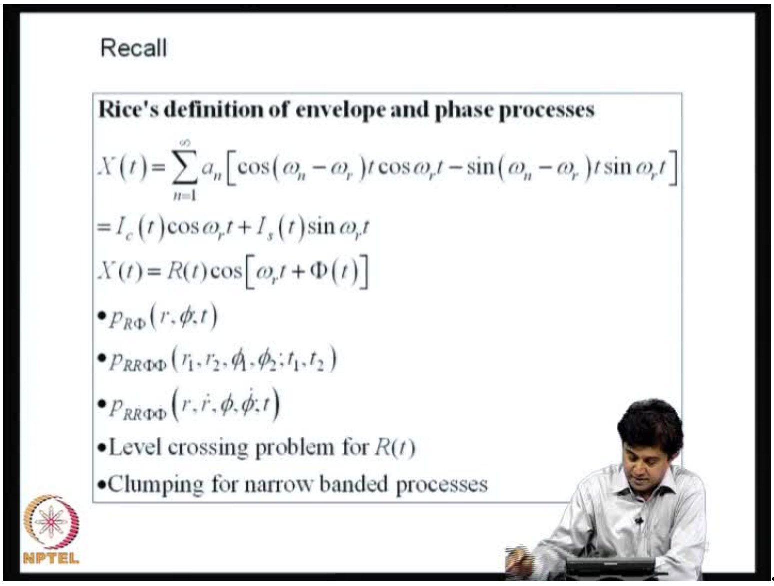

We have been discussing descriptions of random processes, which are helpful in assessing the reliability of dynamical systems, so we will continue with that discussion; so, today we will be talking about extremes of random processes. We will quickly review what we did in the previous lectures; so, in a previous lecture, we considered the definition of envelope and phase processes and according to Rice's definition, envelope is defined in terms of this function R of t. So, we start with the Fourier representation for sample of a stationary random process and we introduce a central frequency omega R, which enables us to write X of t in this form and R of t is square root of I c square plus I s square and this phi of t is tan inverse of I s by I c and we derived in the previous lecture, the joint probability density function between envelope and phase, the joint probability density function between envelope and phase at two times instants and the joint density between envelope and its derivative phase and its derivative. And we also briefly touched upon level crossing problem associated with envelope and issues related to clumping of crossing of levels, in narrow band processes was also discussed.

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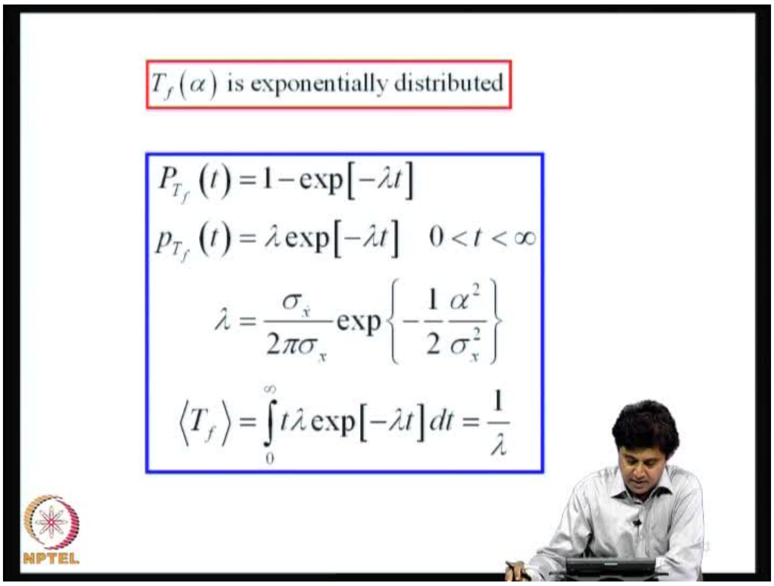

We will use some of these results in due course, but presently, we will have to discuss the problem of maximum value of random processes and this problem is closely associated with the description of first passes time, which again was a topic that we considered in the last class. And we showed that, the time for crossing of level alpha is an exponential random variable and this parameter lambda is related to the level crossing statistics and this is alpha demonstrate, in the previous lecture and we derived the expected value of the first passage time based on this model. So, this essentially uses Poisson model for crossings of high levels and these results are based on that.

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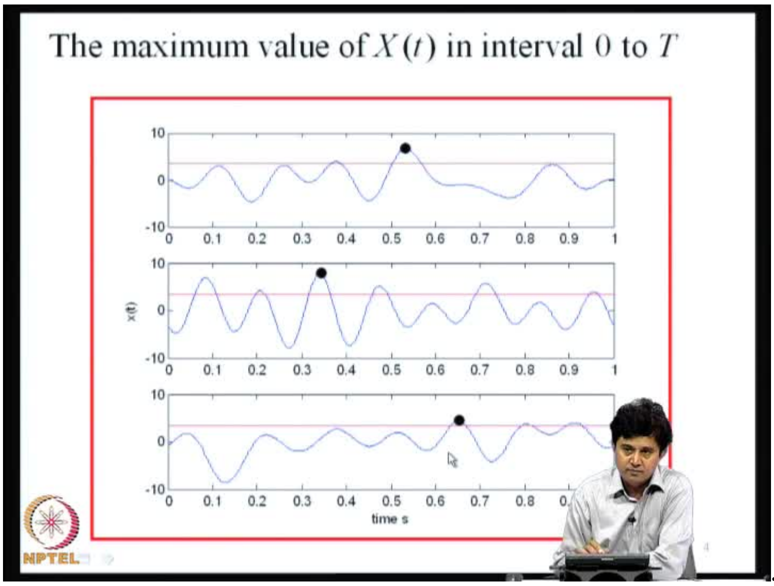

Now, we consider now, the problem of the maximum value of X of t in the interval 0 to T; so, here I have shown three realizations of a random process X of t, for, a, duration 0 to 1 second and I am asking the question what is the highest value of X of t in this duration.

So, for this sample, this is the maximum value; for the next sample, this is the maximum value; for the third sample, this is the maximum value; so, these maximum values therefore can be thought of as outcome of random experiments, as we see different samples and therefore this maximum values themselves are maximum value can be modeled as a random variable.

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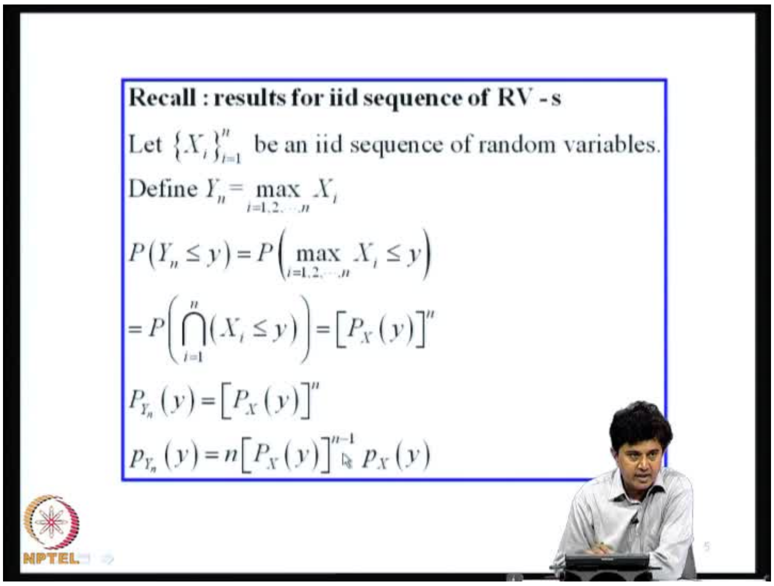

So, the problem on hand is to determine the probability density function of this random variable, in terms of the complete specification of X of t and a property such as level crossing statistics, first passes times, etcetera associated with X of t as a prelude will quickly recall the problem of extremes of sequence of random variables, so if X i, i from 1 to n is an iid sequence of random variables, with a common probability density functions a P X of x. If you define Y n as maximum of X i, where i runs from 1 to n, we have already shown that, this maximum, this random variable the probability distribution of this random variable is given, in terms the parent probability distribution function as shown here, that is the P Y n of y is P X of y to the power of n. Now, if you want the probability density function, we have to differentiate this with respect to y and upon doing so, we get the model for probability density function of y. So, this result is for sequence of n sequence of identical independent random variables.

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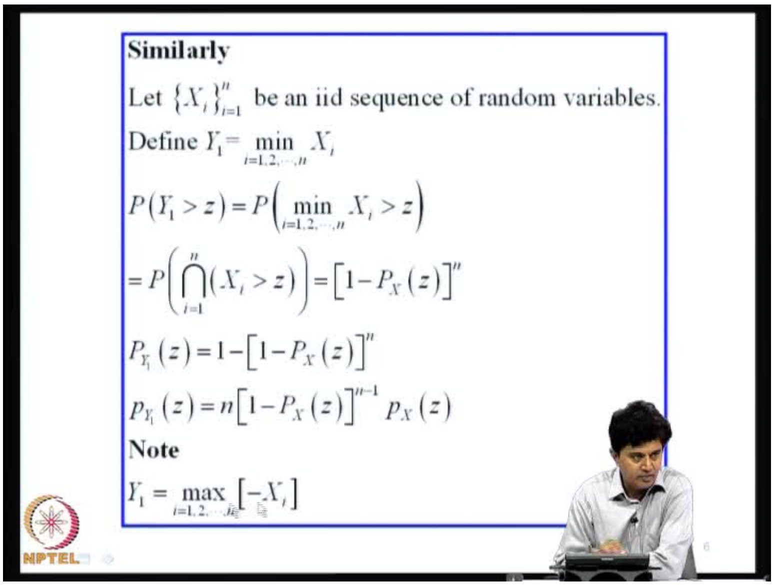

Similarly, we can also consider the problem of finding the minimum, so here we define Y 1 as minimum of X i, where i runs from 1 to n and if you look at probability of Y 1 greater than z is would be probability of minimum of X i is greater than z, this would mean, since X i are independent, etcetera, this probability will be the intersection of the events X i greater than z; assuming independence, this can be shown using the factored exercise are independent, we can show that, this probability is given by 1 minus P X of z to the power n, from which I get the probability distribution function for the minimum as shown here. And again differentiating with respect to z, I get the probability density function and this solves the problem. So, we can just make a note that, Y 1 is minimum of X i, it can also be defined as maximum of minus X i, where i runs from 1 to n; so, once the problem of maximum is solved, we can also get the result based on that.

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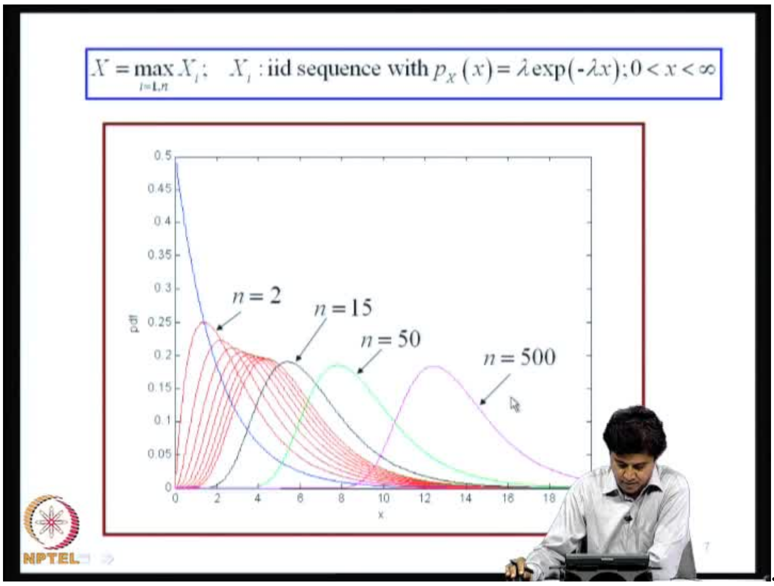

Now, I will show some numerical results to facilitate an understanding of what is the exactly were doing, let us consider X i to be an iid sequence, with the common probability density function which is exponential distribution lambda is a parameter and x takes values from 0 to infinity. So, this probability density function is shown with the blue line in this graph and these initial few red lines, show maximum of, say X 1 X 2 this is a maximum of X 1, X 2, X 15 this is maximum of X i, where i runs from 1 to 50; this is a maximum of X i, where i runs from 1 to 500.

So, you could see that, this is the parent distribution and as n increases its slowly moves towards a right hand tail, which is to be expected; So, the more the number of random variables, you see the higher will be the value, that we will have an opportunity to, you know, see in your realizations.

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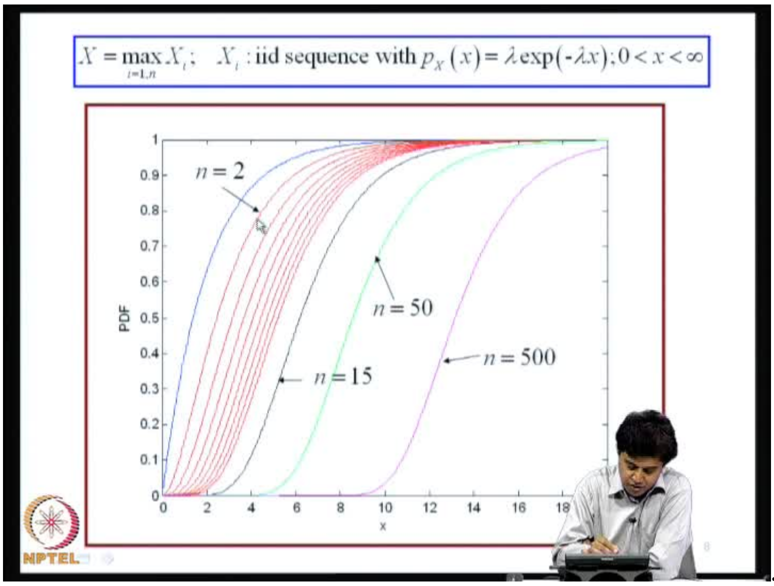

Now, this was probability density function, the same result can be shown for probability distribution function. This is the parent probability distribution function and these are the sequence of probability distribution functions, for the extreme for n equal to 2, 3, 4 etcetera, this is for 15, this is for 50 and this is for 500. These are exact solutions; there is no approximation in this.

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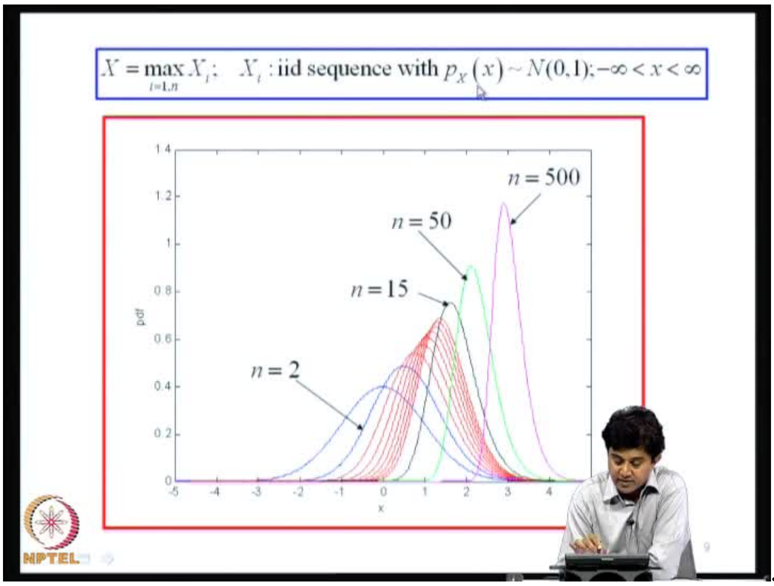

Now, another example, if X i is iid sequence with normal random variables, normal probability density function is the common probability density function is zero mean and unit standard deviation, that is shown here; this is a parent density function and here what is shown, what are shown are the maximum values for, say, n equal to 2, n equal to 15, n equal to 50, n equal to 500; so, since we are looking at maximum values, we are moving on the right hand tail of these density function, I mean, parent density function we are moving on the right hand side.

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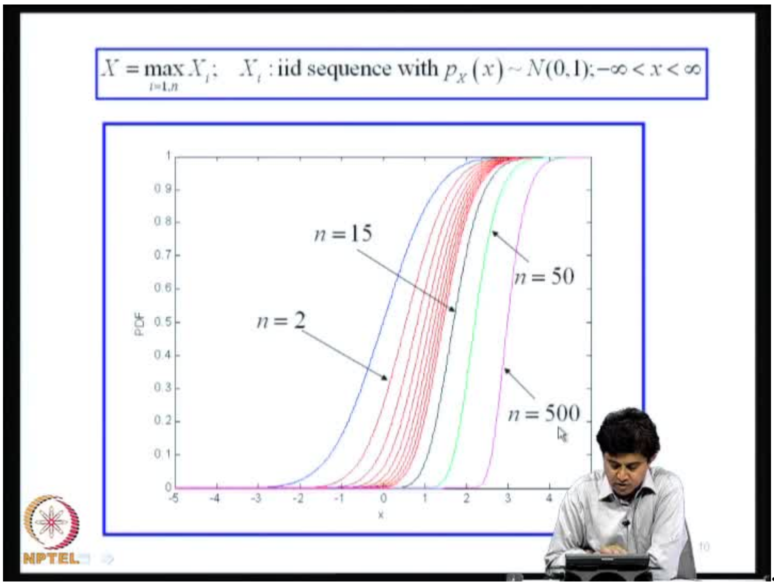

The same results for probability distribution function, this blue line is a parent distribution which is the probability distribution function of a normal random variable with zero mean and unit standard deviation and this is the probability distribution function for maximum of X 1, X 2, this is maximum of X i, i running from 1 to 15; maximum of X i, i running from 1 to 50 and so on and so forth up to 500; so, these are exact solutions for all values of n.

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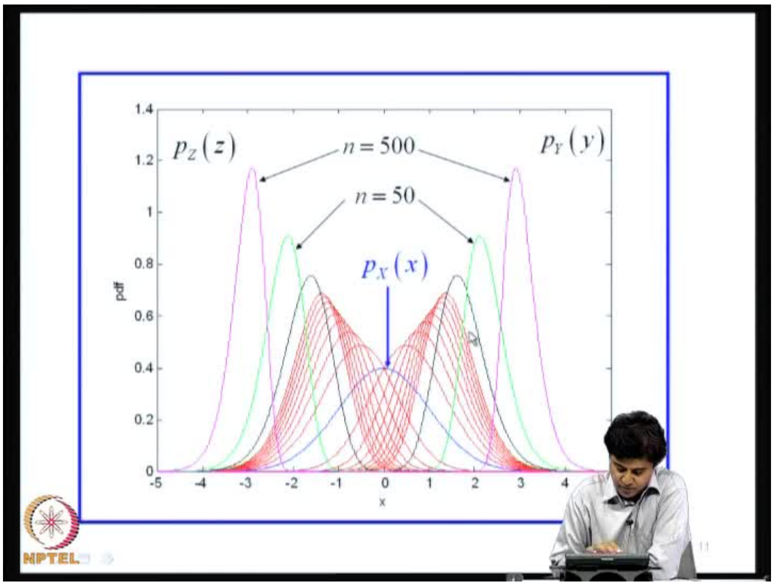

Similarly, we can also get results for minimum, for different values of n; so, I have shown the combined plot here, this is the parent distribution density function P x of x. On the right hand side, we will see the maximums; on the left hand side, we see the minimums; so, this for example, this is pink line here, is the minimum of X i, i running from 1 to 500, where X are iid with a common probability density function which is standard normal. The same result for n equal to 50 and this one is I think n equal to 15 and so on and so forth.

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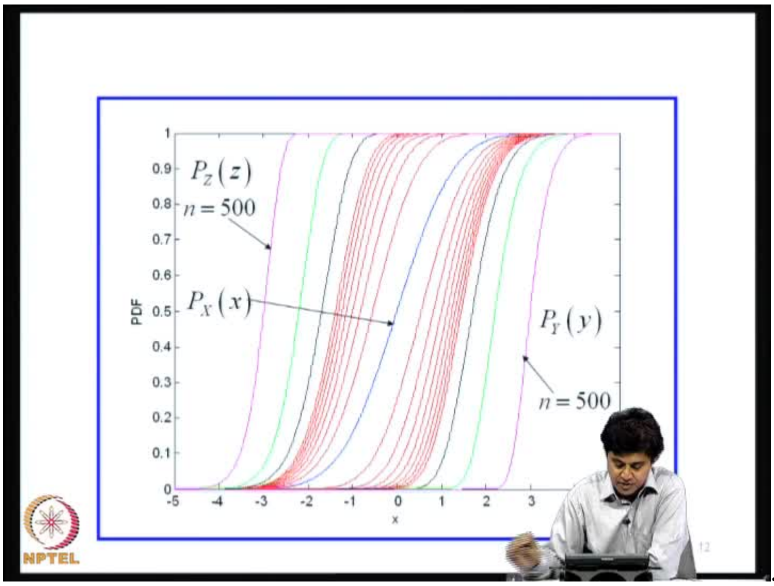

This is again the plot for probability distribution function, this is a parent distribution function, to the left of this, we get the probability distribution functions of minima for different values of n; and to the right, I get probability distribution function of maxima, for different values of n.

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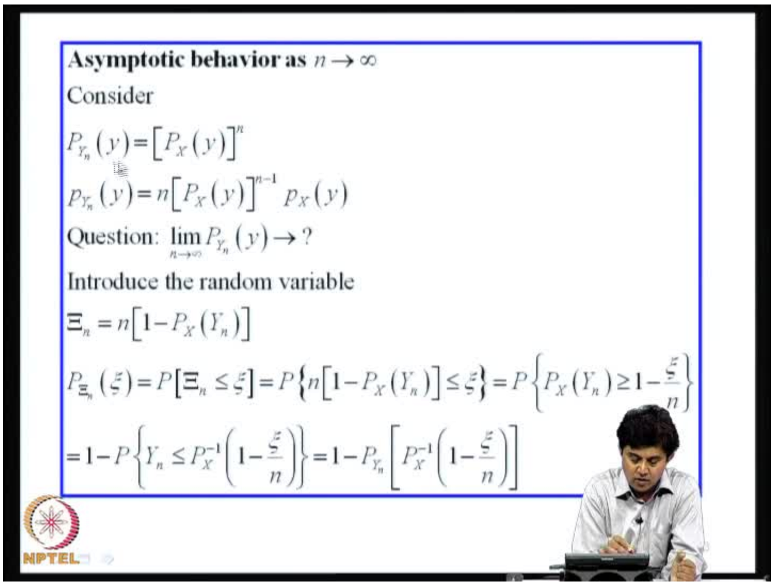

Now, the question that we are interested in is, what happens as n becomes large; now, we consider P Y n of y which is the maximum of probability of X i, i running from 1 to n and this is the probability distribution function and this is the probability density function, so the question we are asking is what happens to P Y n of y as n tends to infinity. Now, to develop and understanding of the issues involved, we introduce a random variable X i n, which is n 1 minus P X of y n; Y n is a random variable here, X i n is a random variable; P X of y n is a transformation on y n, which follows the function P X of x.

Now, let us consider, now the probability distribution function of this X i n; so, this is probability of X i n less than equal to X i is probability of this quantity n into 1 minus P X of y n less than or equal to X I; so, we can invert this relation and we can show that, this requisite probability is nothing but probability of 1 minus probability of Y n less than or equal to P X inverse 1 minus X i by n.

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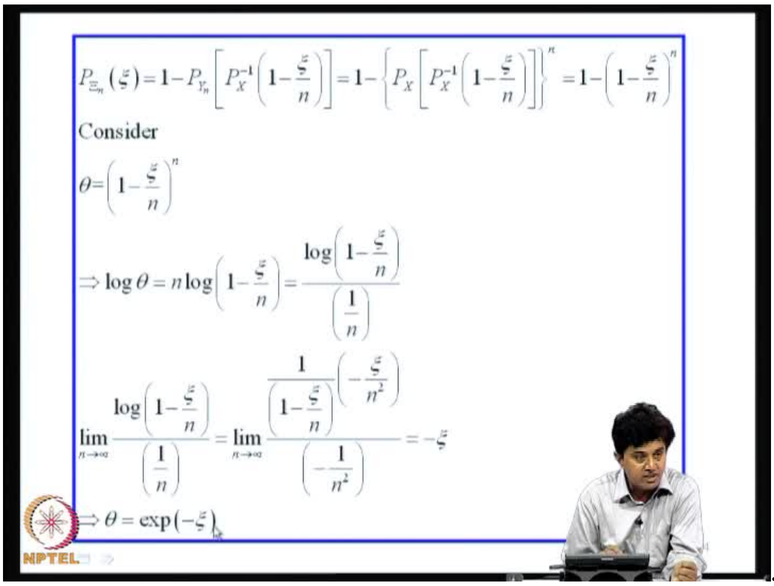

Now, this is nothing but the probability distribution function of P, probability distribution function of Y n evaluated this value of the argument. Now, we know P Y n of y is P X of y to the power of n, so I will use that now; so, I have shown P x i n of x i is this P Y n of x is P X of x to the power of n, so I will substitute that here and the sequence of operation P X into P X inverse will cancel out and I will get 1 minus 1 by X i by n to the power of n.

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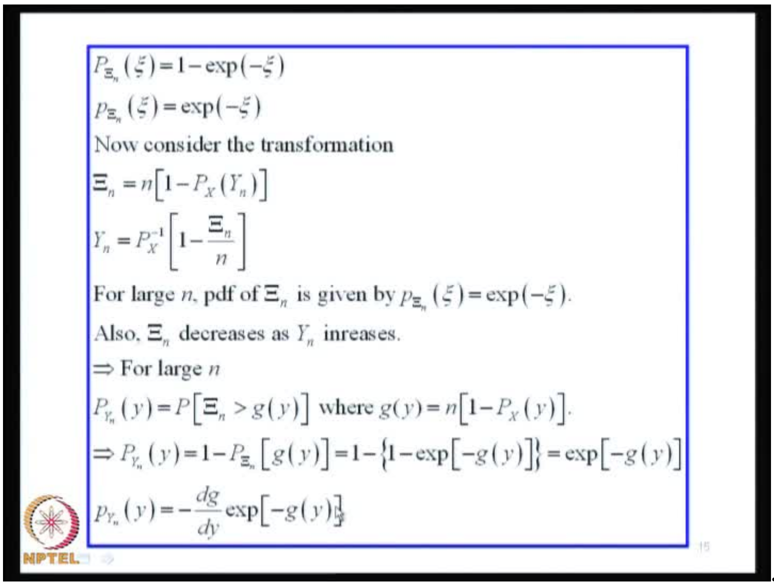

Now, let us consider this quantity, one minus X i by n to the power of n and call it as theta. You take logarithm on both sides, we get log theta is equal to n log 1 minus X i by n, this is nothing but log of 1 minus X i by n divided by 1 by n; now, on this now, I will consider what happens as n tends to infinity. As n tends to infinity, this is we have to apply L'hopitals rule and upon doing that, we can show that this function is actually minus of X i, that would mean, this is nothing but log theta, therefore theta is exponential of minus X i, so this is for larger. Now, I will work backwards and get P x i n of xi is 1 minus exponential of minus X i and the probability density function is exponential of minus X i.

Now, we come back to the transformation, X i n is n 1 minus P X of Y n and from this, I can now get probability density function of Y n, this is known now, X i n is known. So, if I derive this, I will now get the extreme value distribution for large values of n; so, if I do that, I get, already I know for large values of n, X i n is exponentially distributed, so I will exploit that fact and also by examining this, we can see that as X i n decreases as y n increases; therefore, for large n, if I use this facts, we can show that, P Y n of y is nothing but X i n of greater than equal to g of y, where g of y is this function. So, P Y n of y would be 1 minus, 1 minus exponential minus g of y, which is nothing but exponential of minus g y; so this is the extreme value distribution for n turning to infinity. So, P from this, if we can as well get the probability density function, which is the derivative of this, this is minus d g by dy exponential minus g of y.

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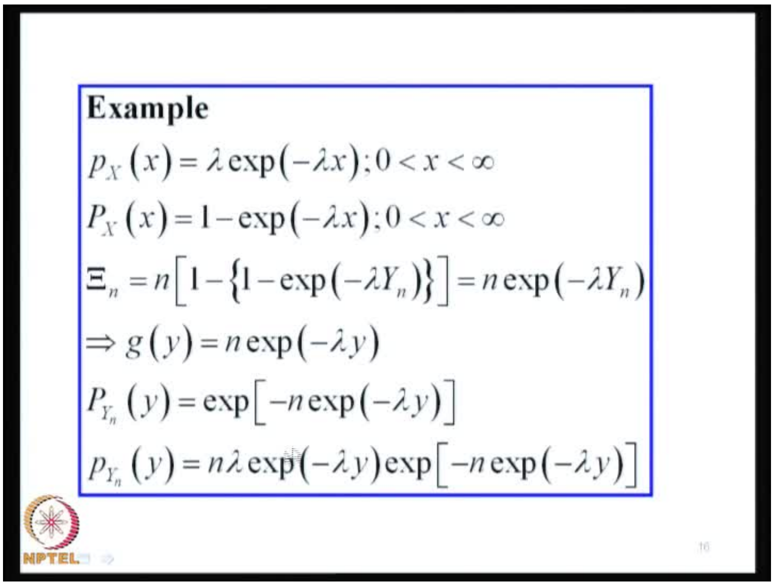

Now, how does it really look like, if we apply to a specific problem; so, let us consider a sequence of random variables, which are exponentially distributed. So, P X of x is density function is lambda exponential minus lambda x, where x runs from 0 to infinity and associated distribution function is 1 minus exponential minus lambda x. So, the transformation for X i n is n into 1 minus 1 minus exponential minus lambda Y n, which is n exponential minus lambda Y n. From this, I get g of y is n exponential minus lambda y. So, P Y n of y is I get exponential of minus n exponential minus lambda y; so, this is a double exponential, by that, I mean, this function. So, from this, I can also get the density function by differentiating this with respect to y and I get this expression.

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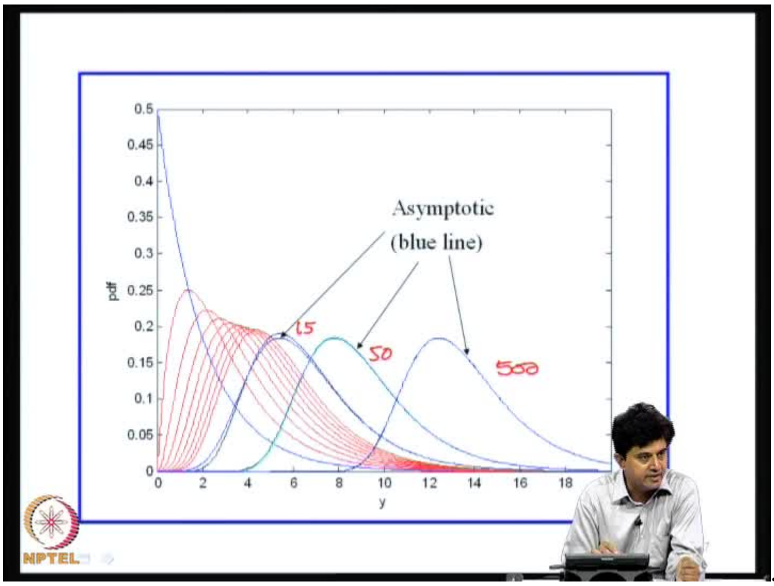

Now, we will redraw the graphs that I showed earlier; now, on this now, I will superpose the result from as asymptotic analysis. So, this is the parent distribution, which is exponential, this is exact extreme value distribution for n equal to 2, 3, 4 etcetera and this is for n equal to this is 500, this is 50, this is 15, the blue line is a asymptotic probability distribution function, for the extreme density function, for the extreme and the black lines are the exact solution. So, you can see that, at 50 you can hardly distinguish between the two different results and 500, of course, there much less distinguish; at 15 they are already good.

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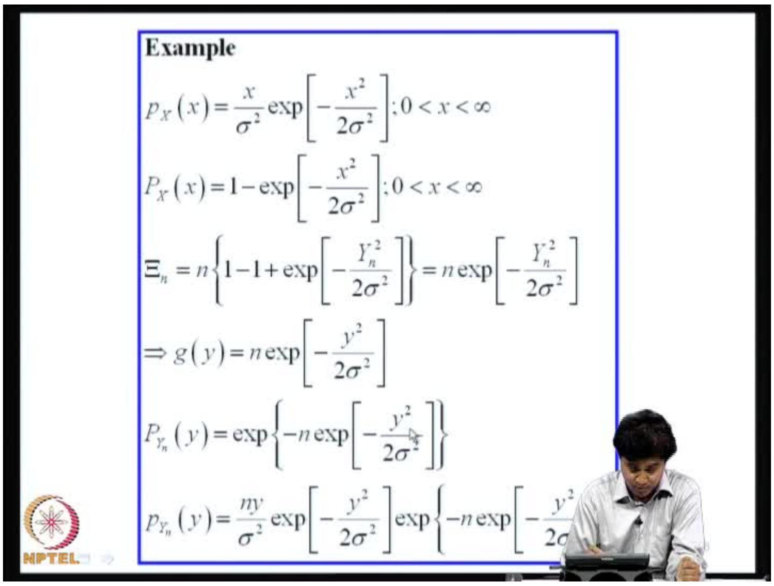

Now, let us remember the form of this probability density function or the distribution function which is exponential of exponential functions; so, this is one of the standard forms of extreme value distributions. Now, let us consider another random variable, say Rayleigh random variable, again I consider a sequence of iid x i, where the common probability density function is Rayleigh and I am interested in maximum values of this x i over i equal to i from 1 to n, where n goes to infinity. Now, I introduce this random variable x i n, that is n minus 1 plus exponential etcetera, this is this; g of y therefore n exponential minus y square by 2 sigma square, from these, the distribution function again is a double exponential, but now the exponent has square of a y, whereas previously we had the exponent which was y to the power of 1.

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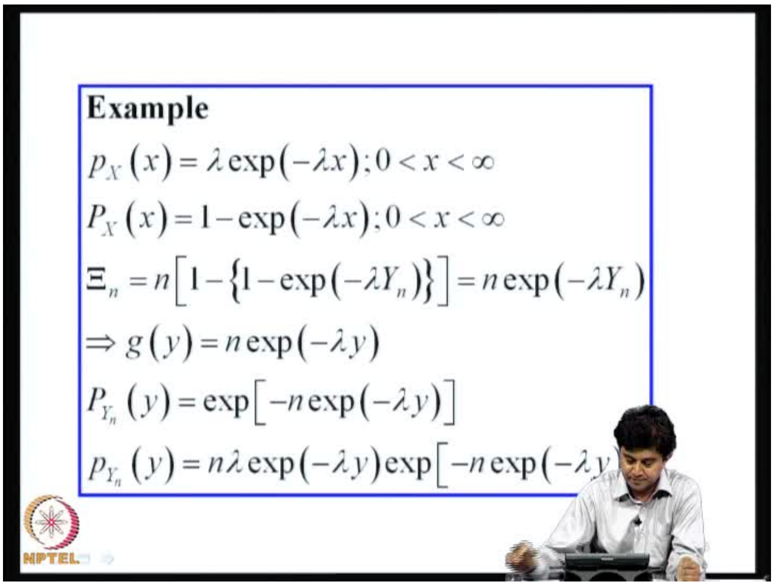

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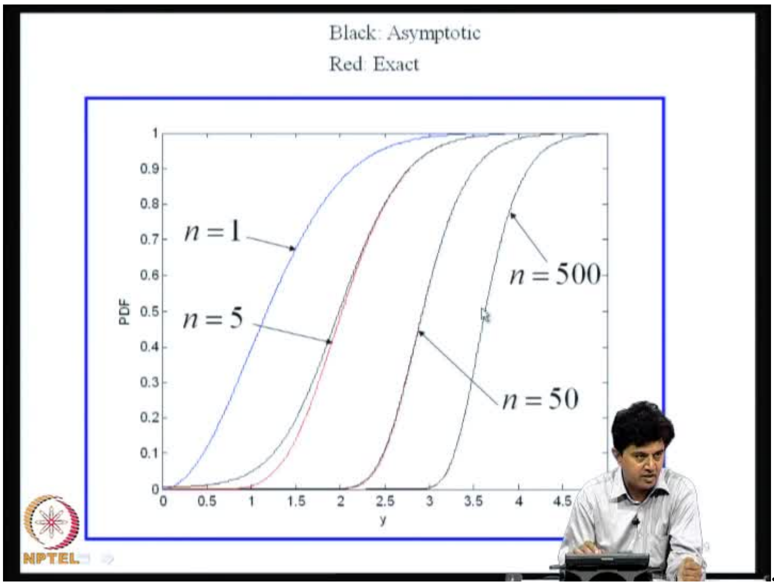

So, here, again we can see the probability distribution function; this is the parent distribution of a Rayleigh random variable and this is for n equal to 5, this 50, 500; so, black line is asymptotic, red is a exact solution. So, for n equal to 5, they tend to agree on the right hand side right tail, but the lower tail there are differences, but as n become 50 and 500, the two result the namely the asymptotic result and the exact result are hardly distinguishable.

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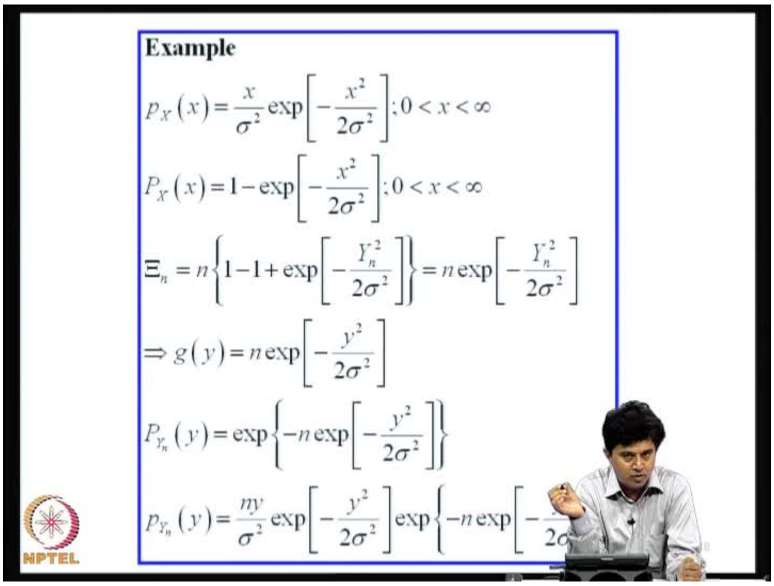

Now, the two examples that I considered, now this is Rayleigh, the tail goes all the way up to the infinity; similarly, exponential the tail goes all the way up to the infinity. So, we are interested then the tail region of interest is on the right side, because we are looking at maximum values.

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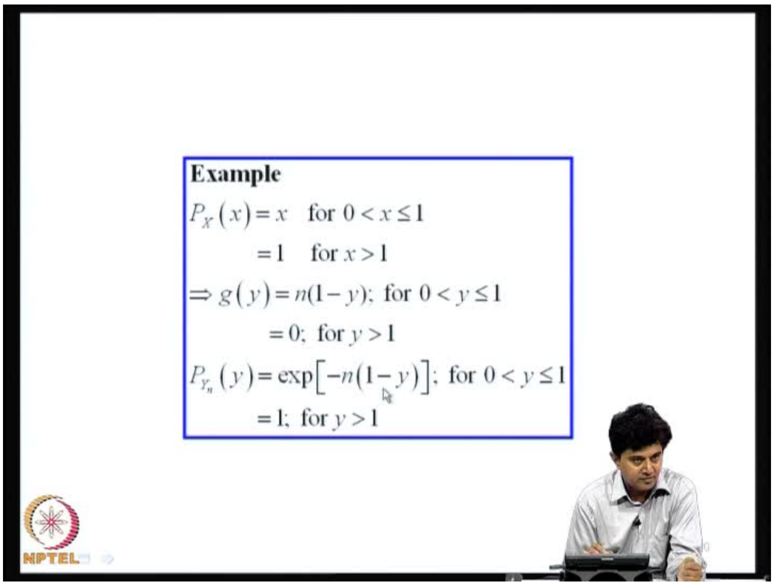

Now, let us consider one random variable, where the tail stops at on the right hand side, so a good example for that is a uniform distribution between 0 and 1; so, the probability distribution function is a linear function between 0 and 1 and it is 1 for x greater than 1 and based on this, I get g of y, as this for y between 0 to 1 and later on, it is 0. So, here I get the extreme value distribution is an exponent of y, this is on double exponential, any more now is exponential of minus y.

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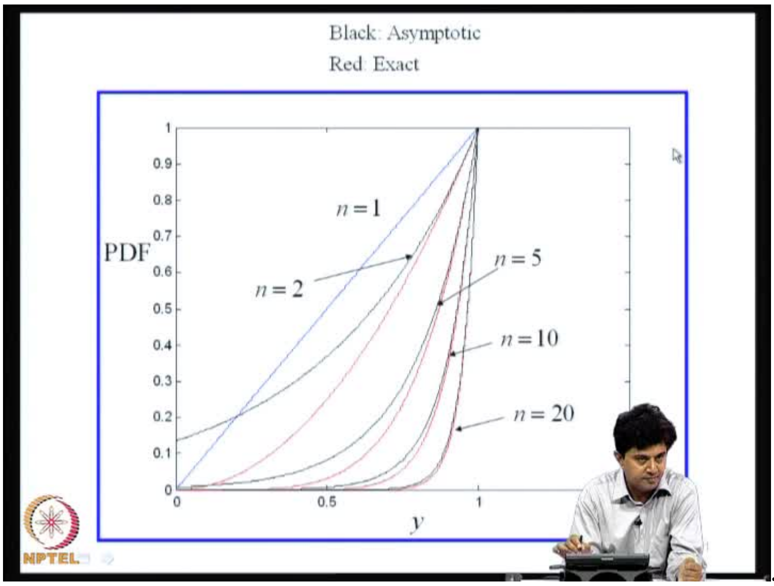

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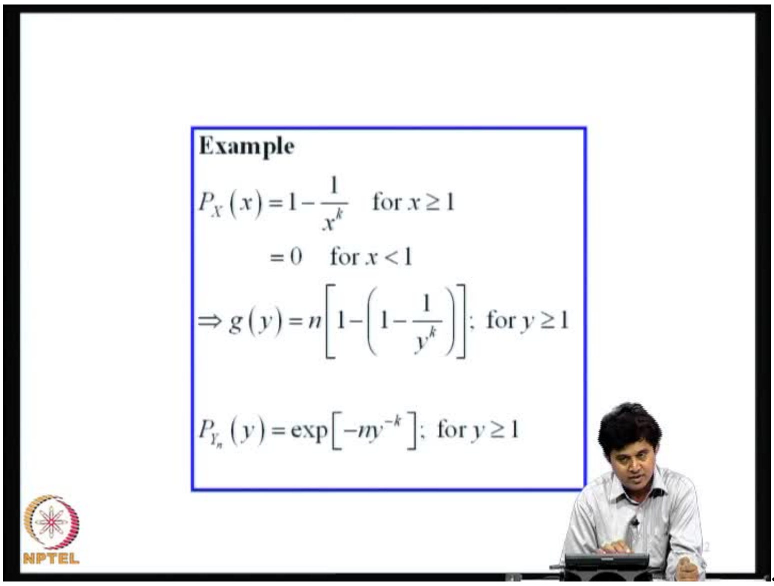

So, if you look at that, again this blue line is the parent distribution, red is the exact solution and black is the asymptotic solution; you can see that, for n equal to 2, the differences are quite pronounced; n equal to 5, the differences start reducing, but for by time, we reach n equal to 20, the differences are almost negligible. This distribution is again bounded on the right hand side. One more example, P X of x is 1 minus x to the power of k, whenever x is greater than or equal to 1, it is 0 for x less than 1. So, here I get g of y as this and from which I get the asymptotic distribution for the highest value and here I am getting exponential of y to the power of minus k; this is an asymptotic form for this type of random variables. So, we are getting double exponentials exponential of y exponential of y to the power of minus k and so on and so forth.

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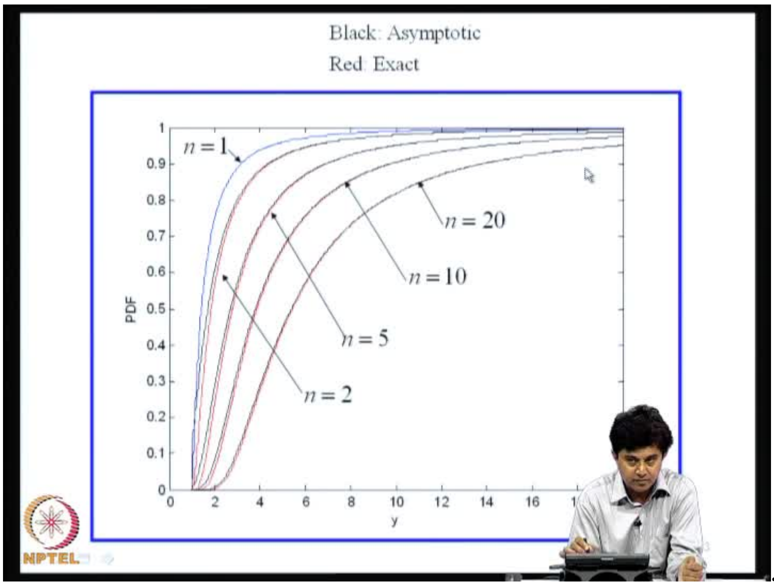

So, before we discuss that, we will again see some numerical results; this n equal to 1 is the parent distribution and this is maximum for n equal to 2, n equal to 5, n equal to 10 and n equal to 20, black is asymptotic red is exact and we see here that for even for n equal to 2, the difference between the exact solution and asymptotic solution is, you know, much smaller and for n equal to 5 and n equal to 20, they are almost in distinguish; here, the tail goes all the way up to infinity again.

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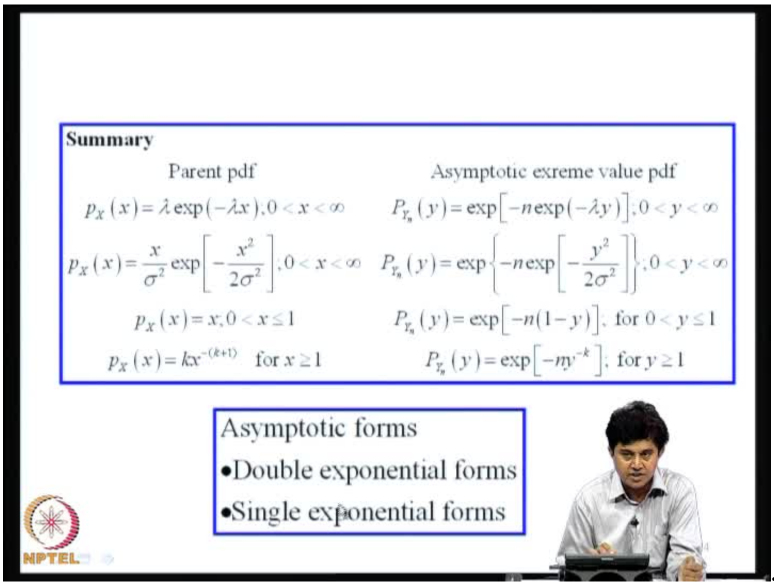

So, in summary of the four examples that we considered, first we started with an exponential, the tail goes all the way up to the infinity and here, I get a double exponential and the exponent is y Rayleigh tail goes all the way up to the infinity. And here, again we get double exponential but the exponent is y square. Uniform distribution, we get exponential of minus y or plus y and this particular distribution, we had exponential of y to the power of minus k. So, we get basically double exponential forms and single exponential forms for the extremes.

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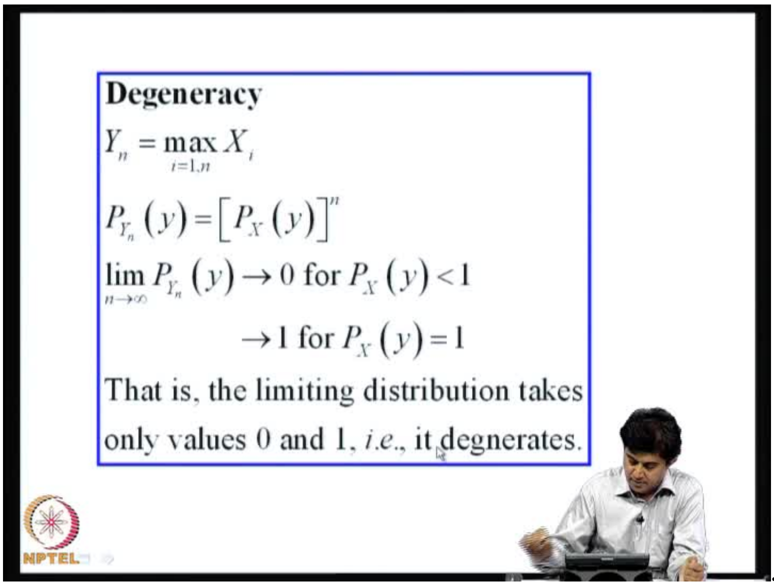

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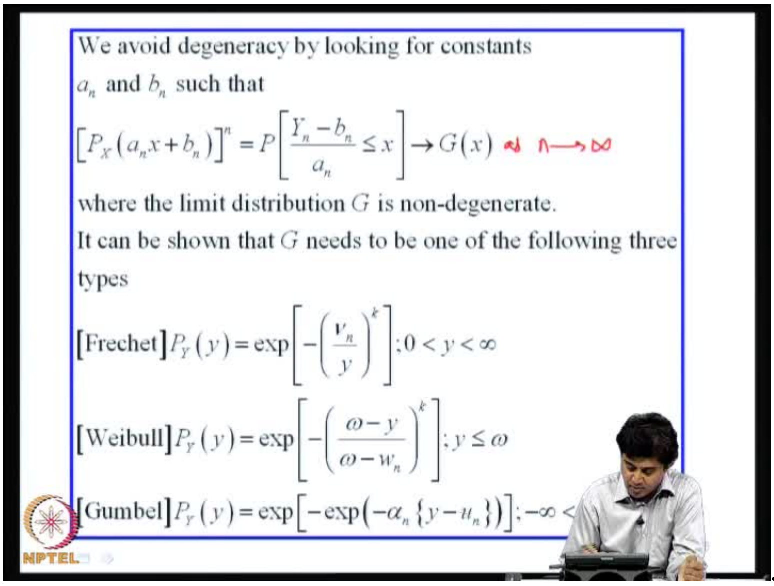

Now, if we look at, again let us look at Y n is maximum of X i, i running from 1 to n and P Y n of y is P X of y to the power of n, now if we look point wise, if P X of y is less than 1 as you raise its power to n as n tends to infinity that number should go to 0, say for example, if it is point 0.5, 0.5 square, 0.5 cube, 0.5 to the power of 10, 0.5 to the power of 100, so the value goes to 0. If it is 1, then P X of y to the power of n stays at 1; so, we see, to get that the limiting distribution takes values only 0 and 1 or in other words, degenerates. Degeneracy does not interest us, we want to know is, there any way of is, it the only way that extremes behave or are they any other possibilities. To explore that, what we do is, we consider a transformation on Y n and we look at Y n minus b n by a n, so we shift it and scale it and look at the behavior of this random variable, as n tends to infinity, if that we are looking at we are looking at P X of a n x plus b n to the power of n.

So, as n tends to infinity, G of x in this distribution function converges to another distribution convergence in distribution, this can be non-degenerate; if such non-degenerate G of x are possible, it has to belong to only three classes, that is one of the results in extreme value analysis; that means, it can be shown that, g needs to be one of the following three types; these are Freshet, Weibull and Gumbel distribution functions.

So, as n tends to infinity, this function will become this form, we say then it is Frechet. Here, you are saying y to the power of minus k that we were showing, I was illustrating with specific example, here it is y to the power of k single exponential here; these two are single exponentials and we are getting double exponentials, this is none as Gumbel.

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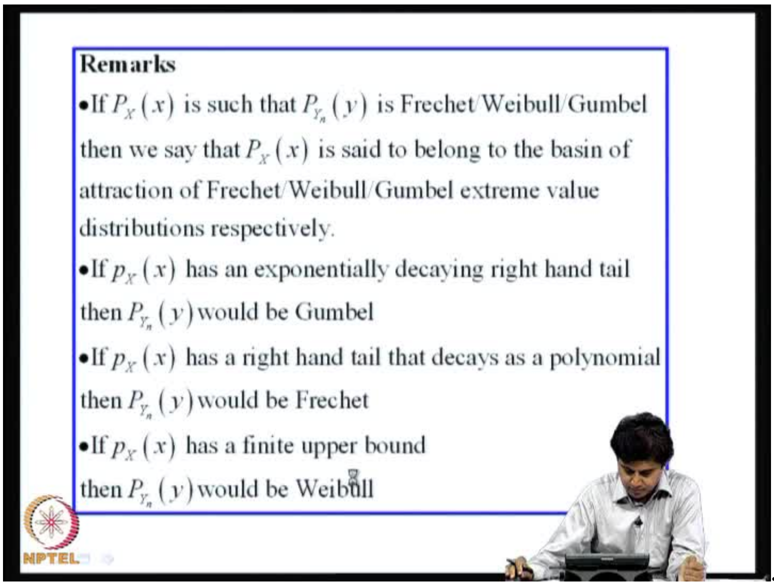

Now, we will make some remarks; if P X of x which is the parent distribution is such that P Y n of y is Frechet or Weibull or Gumbel, then we say that P X of x belongs to the basin of attraction of Frechet, Weibull or Gumbel, that is, if P X of x is such that the asymptotic P Y n of y is Frechet, then we say that P X of x belongs to the basin of attraction of Frechet extreme value distributions. Similarly, if P X of x is such that P Y n of y is say Weibull, then we say that P X of x belongs to the basin of attraction of Weibull right. So, there are mathematical theorems, which help us to deduce few facts, which I am stating here without proves; if P X of x that is the probability density function of x has an exponentially decaying right hand tail, then the asymptotic extreme value distribution would be Gumbel. If P X of x has a right hand tail that decays as a polynomial, then P Y n of y would be Frechet; that means, an asymptotic extreme value distribution would be Frechet.

If P X of x has a finite upper bound, as in case of uniform distribution, as we saw just now, then P Y n of y would be Weibull. So, these three random variables Frechet, Gumbel and Weibull are widely studied and considerable information is available about use of these models, use of these probability distributions in modeling extremes.

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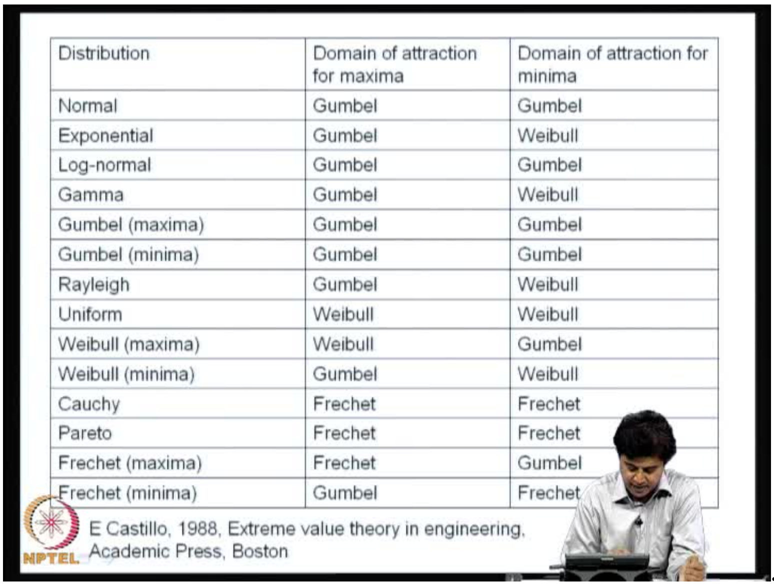

Distribution	Domain of attraction for maxima	Domain of attraction for minima
Normal	Gumbel	Gumbel
Exponential	Gumbel	Weibull
Log-normal	Gumbel	Gumbel
Gamma	Gumbel	Weibull
Gumbel (maxima)	Gumbel	Gumbel
Gumbel (minima)	Gumbel	Gumbel
Rayleigh	Gumbel	Weibull
Uniform	Weibull	Weibull
Weibull (maxima)	Weibull	Gumbel
Weibull (minima)	Gumbel	Weibull
Cauchy	Frechet	Frechet
Pareto	Frechet	Frechet
Frechet (maxima)	Frechet	Gumbel
Frechet (minima)	Gumbel	Frechet

Now, here I have tabulated the basin of attraction for different distributions, for example, distribution is normal, the domain of attraction for maxima is Gumbel. So, the arguments that we use still now was on the highest value, we can construct similar arguments for the lowest value also and that leads us to models for minima and in which case, we have to look at tails on the left hand side instead of right hand side right. So, for a normal distribution which has tails, which goes to minus infinity as well as plus infinity as exponentials, the models for extrema or for maxima it is Gumbel, for minima it is Gumbel. For exponential, the, maximize is Gumbel, but minima when we look at left hand side, it is bounded, it is Weibull.

So, for similarly for Rayleigh, a maximum is Gumbel minimum is Weibull; for cauchy the maxima is Frechet and minima is also Frechet. If the random variable and their question is Gumbel maximum, its own maximum will be Gumbel, minimum will also be Gumbel. So, some of this results can be proven, so that is a good book which would help you to understand this in a deeper way, that is book by Castillo extreme value theory in engineering.

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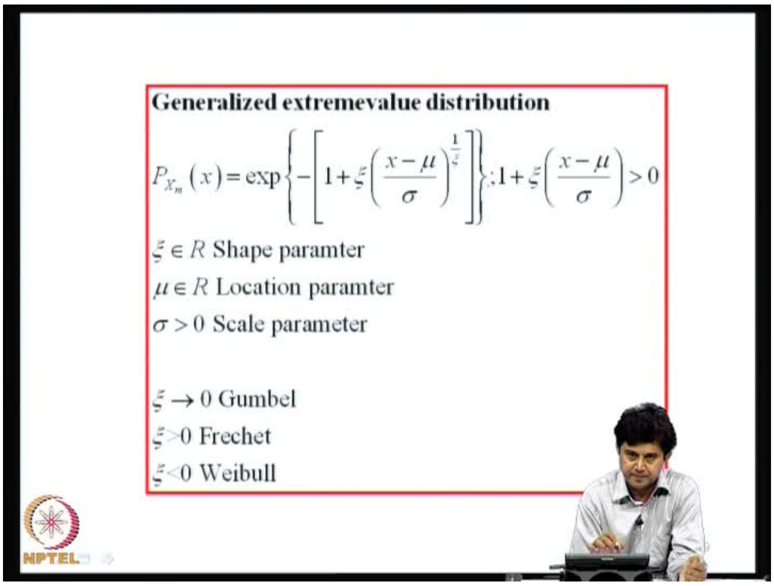

Now, all these three asymptotic extreme value distributions can be put into a single model and this is known as generalized extreme value distribution, which is mathematically shown here. The distribution function P X m of x exponential of this 1 plus X i x minus mu by sigma to the power of 1 by X i etcetera, where this x takes value, such that 1 plus X i into x minus mu by sigma is greater than 0. So, there are three parameters here epsilon, mu and sigma epsilon is a real number; this is known as a shape parameter; mu is a real number known as location parameter; sigma is a scale parameter which has to be positive real number and in this model, as X i tends to 0, we get Gumbel model; as X i if X i is greater than 0, we get Frechet model and X i less than 0, we get Weibull model. So, these three models are embedded into a single mathematical expression and in estimating system parameters, this could prove useful.

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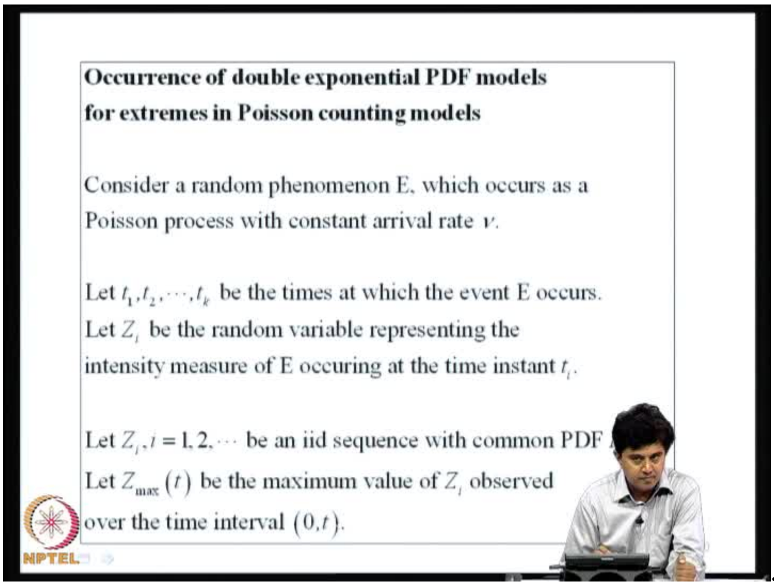

Now, we saw that this double exponential PDF models occur in finding maxima of random variables, whose tails decay exponentially on the right hand side. We can recall something that we discuss in one of the lectures, when we are discussing Poisson models, we again encountered double exponential models; so, I am quickly recalling the context here. So, let us consider a random phenomenon E which occurs as a Poisson process with constant arrival rate nu; this could be occurrence of earth quake in a given location whose magnitude is, say greater than 5 or wind with velocity greater than as specified velocity etcetera.

Now, let t 1, t 2, t k be the times at which the event e occurs and let Z i be the random variable representing the intensity measure of E occurring at the time instant t i; it could be, as I said Z i could be the magnitude of an the earth quake, one earth quake occurs at t 1, another at t 2, the magnitude could be 4, 4.5 and so on and so forth; so, Z i is a random variable. Now, let us assume that Z i is an iid sequence with the common probability distribution function P Z of Z; we are interested in Z max of t, which is the maximum value of Z i observed over the time interval 0 to t.

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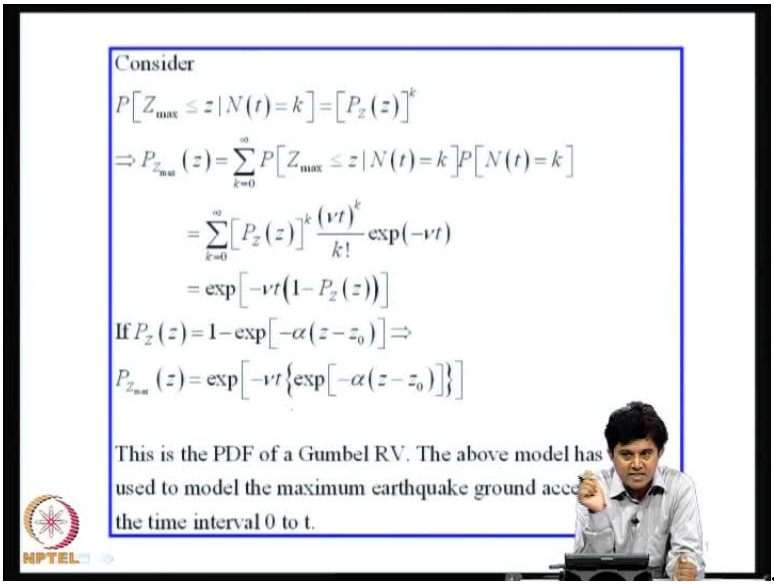

So, how do we get properties of this Z max; so, what is probability of Z max less than or equal to Z, condition on the fact that N of t is k N of t is a Poisson process, if it is condition N of t equal to k, then it is P Z of z to the power of k. So, the unconditional probability distribution function, we have to some over the probability density distribution of Poisson model, Poisson PDF, that would lead to this expression, which is k from 0 to infinity, this conditional probability into probability of N of t equal to k; this is known to be, you know, nu t to the power of k by k factorial exponential minus nu t this is P Z of z to the power of k.

So, I get now here exponential of minus nu t 1 minus P Z of Z; now, if P Z of Z is exponentially distributed, I get the double exponential model which is the Gumbel model in this case. So, this is the PDF of a Gumbel random variable, the above model has been use to model the maximum earth quake ground acceleration as a time interval 0 to t. So, the issue that I am trying to point out through this example is that, this kind of double exponential for extremes can also be derived through a Poisson model for an associated counting process; so, we will see how we can use this shortly.

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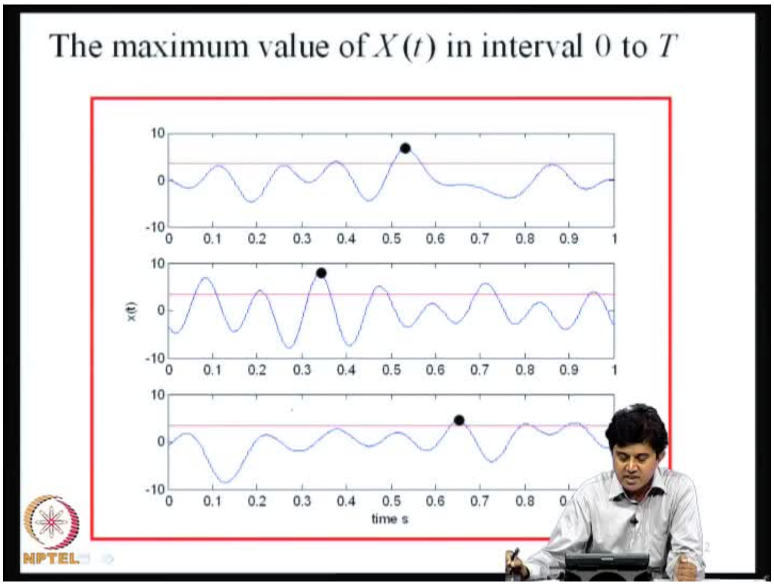

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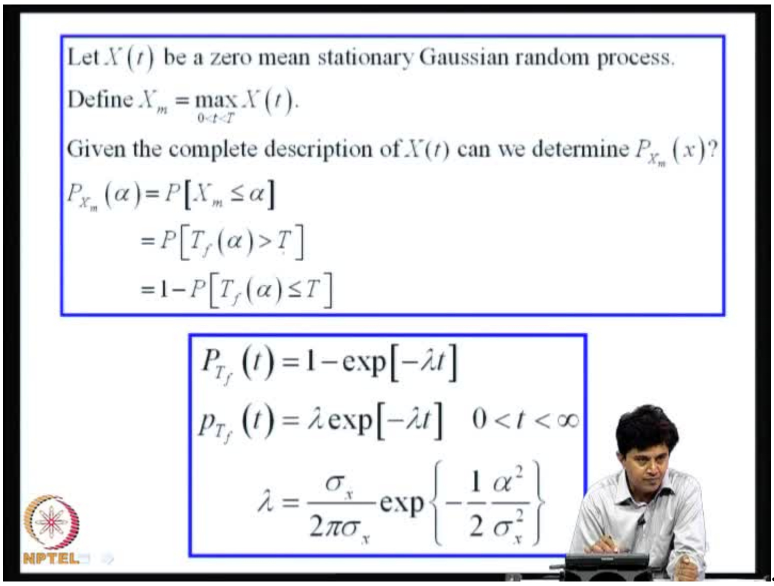

Now, we will now return to the problem that we were stating at the outset; what is a maximum value of X of t in the interval 0 to capital T. So, again I showed three realizations and these three are realization of the extremes; so, extreme is the random variable. So, we define X m as maximum of 0 maximum or the interval 0 to t of X of t; problem is given the complete description of X of t can we determine P X m of x. What is probability distribution of P X m of alpha, which what is probability distribution function of X m it is probability of X m less than or equal to alpha. Now, this probability is intimately related with first passes time that is the time taken by X of t to cross level for the first time. The fact that probability of X m less than or equal to alpha is equal to probability of T f of alpha greater than T is something that we have to carefully consider and I can help you with some illustration.

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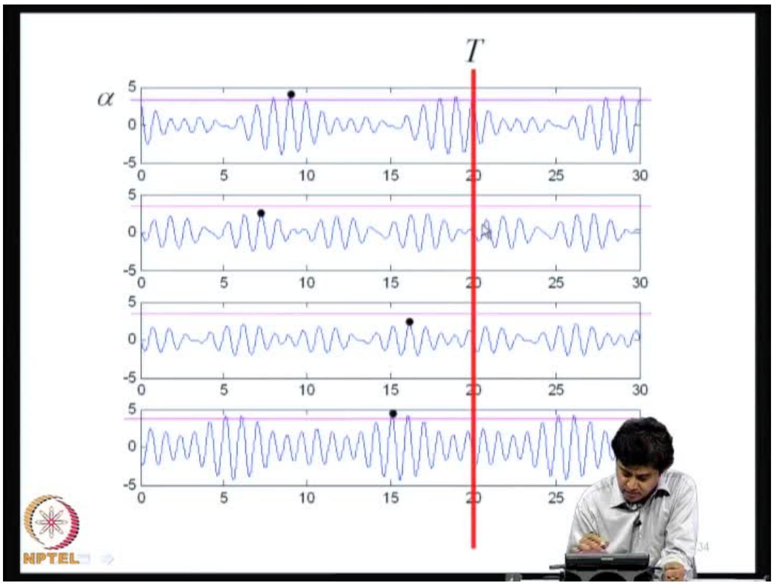

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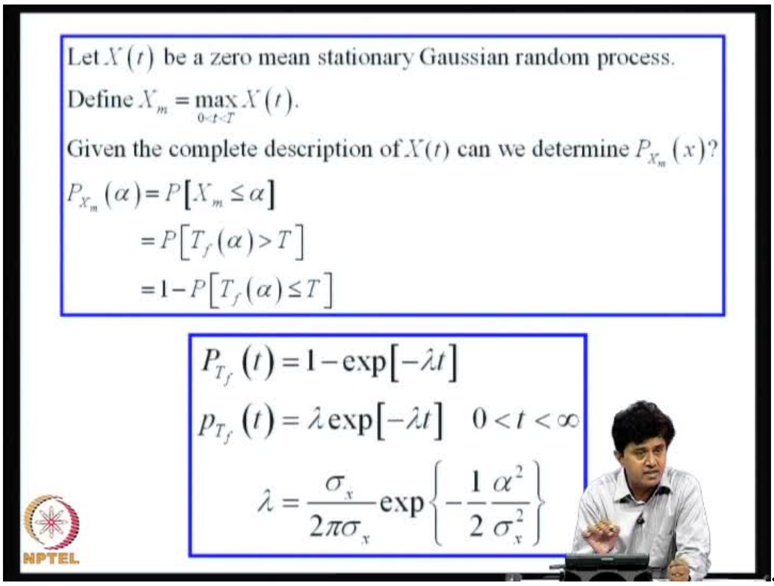

Suppose, we consider X of t to be a random process and I have shown here four realizations of X of t and this is 0 to capital T, which is the time duration of interest. So, what I am interested in, is the highest value of X of t, in the time interval 0 to 20 seconds; so, these are these black dots, this is for the first realization, this is for the second realization, this is for the third realization, this is for the fourth realization.

Now, what is the question we are asking, this is level alpha shown in the pink line here, here and here, the question we are asking is, what is the probability that X m is less than or equal to alpha; so, for this realization is X m less than or equal to alpha, no, for this realization, yes; for this realization, yes; for this realization, no; now, when we got a no, we can also see that what is the first passes time is the first passes time less than capital T yes, no, the first passes has not occurred between 0 and t; therefore, it is bound to a happen later, here again answer is no, here answer is yes. So, we go back to the statement that the event X m less than or equal to alpha is same as the event T f of alpha greater than or equal to T.

Now, T f of alpha we have already characterized. So, I have now model for T f of alpha which I have shown that, it is an exponential random varies for a gaussian stationary random process we have shown that, this lambda is given by sigma x dot divided by 2 pi sigma x exponential minus half alpha square by sigma x square, that would mean, the moment I have solved the first passes time problem, I also have an handle on the problem of extremes. How, did we solve this first passes problem; we first solve the level crossing problem, we computed the average number of times the level alpha is cross in 0 to t and then assuming that level the crossing levels that we are interested are high, so that crossings are rare, we postulated a Poisson model; from the Poisson model, I, we derived this exponential model for the first passes time.

Now, that now further helps us to arrive at the model for extremes. As engineers, we are interested the most fundamental question, that we are interested in answering is this, what is the probability distribution of X m, which is maximum value of a random process over a given duration, that is the fundamental quantity of interest. If X of t is response, obviously I would be interested in knowing the highest response if a random process is evolving in space, like strength of a beam or etcetera, I am interesting in the least value of strength, highest value of the load effect; so, we are always interested in extremes.

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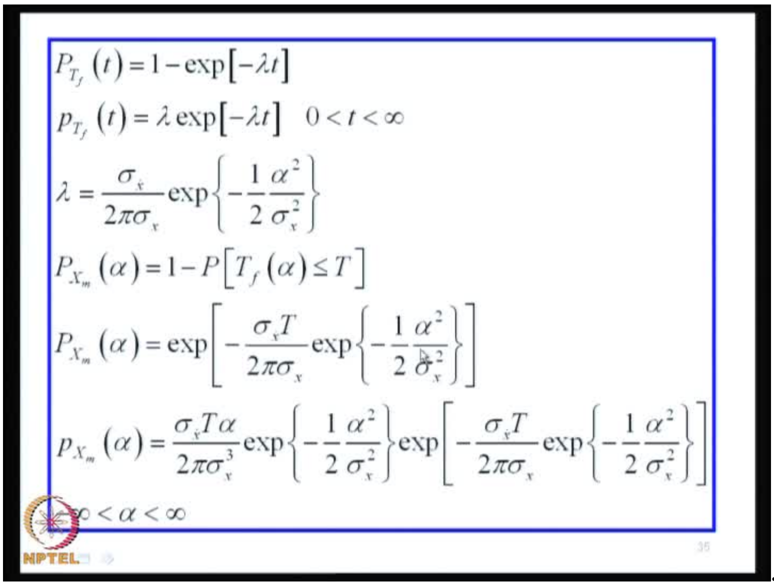

So, we will run through this calculation; now, we already have a model for the first passes time, now let us put that into our model for extremes and P X m of alpha is 1 minus T f of alpha less than or equal to T and based on that, I get the model for probability distribution of the extreme, which is a double exponential function in this problem; for a Gaussian random process, I am getting a double exponential. So, you differentiate this with respect to alpha, I get the probability density function and here alpha takes values from minus infinity to plus infinity.

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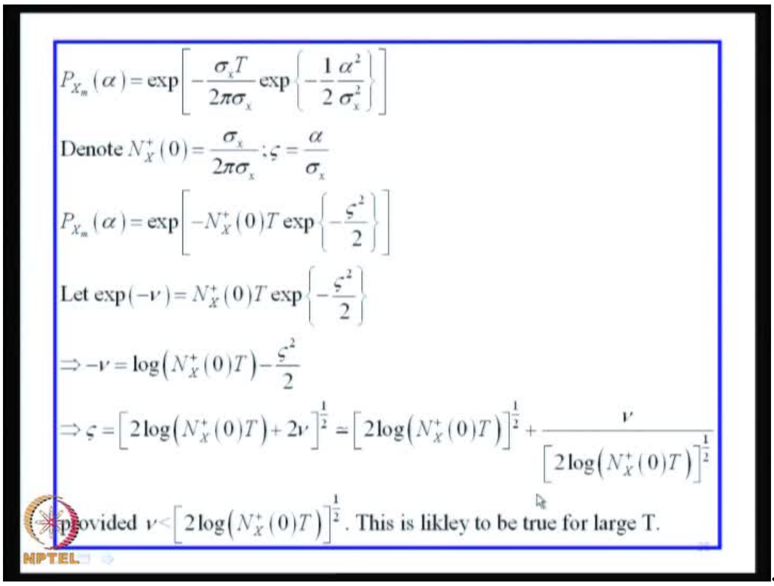

Now, this is not a Gumbel model, [per se] Gumbel model has alpha is a double exponential model alright, but in the inner exponential, we do not have non-linear function of alpha but a linear function of alpha. Now, we will simplify this expression a bit and see if we can get the Gumbel model under what conditions it is possible; so, this is the model for the probability distribution function of the extremes and in this, I denote sigma x dot by 2 pi sigma x which is this is nothing but the average rate of 0 crossing with positive slope; so, I give this notation N X plus 0 for this quantity and this quantity alpha by sigma x which is non-dimensional, I give a notation zeta.

So, in this notation, I can write P X m of alpha as exponential of minus N X plus 0 into capital T exponential zeta square by 2. Now, I will introduce a new variable nu and defined as exponential of minus nu is this exponent; so, you take logarithms and rearrange the terms and get zeta in terms of nu and we get this expression.

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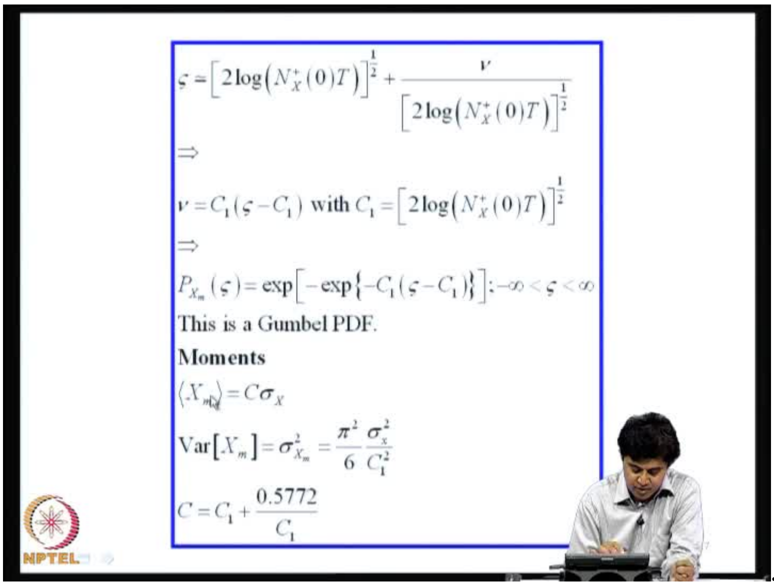

Now, here if capital T is assumed to be large, we can do a Taylors expansion and retain only first two terms and we can show that, this is approximately equal to this; I do not want non-linear terms in nu, so I retain this and this is possible, if nu is less than equal to this, which is likely to be true for large values of time, that means, we are interested in maximum over a reasonably long duration of response. So, if these assumptions are acceptable, then I get zeta in terms of nu and I can introduce some notations to simplify the representations, I introduce a quantity which is C 1, which is square root of 2 log whatever is inside this parenthesis and with that, I get now the model for extremes which is the double exponential form exponential of minus exp minus C 1 zeta minus C 1, where zeta runs from minus infinity to plus infinity; this is the form of the Gumbel probability distribution function; we can derive its mean, we can derive its variance. If this assumption is not acceptable, that means, you do not want to make this assumption that nu is less than or equal to this. We already have the exact solution, which does not make any of those assumptions, but they are, this is not in the standard Gumbel form.

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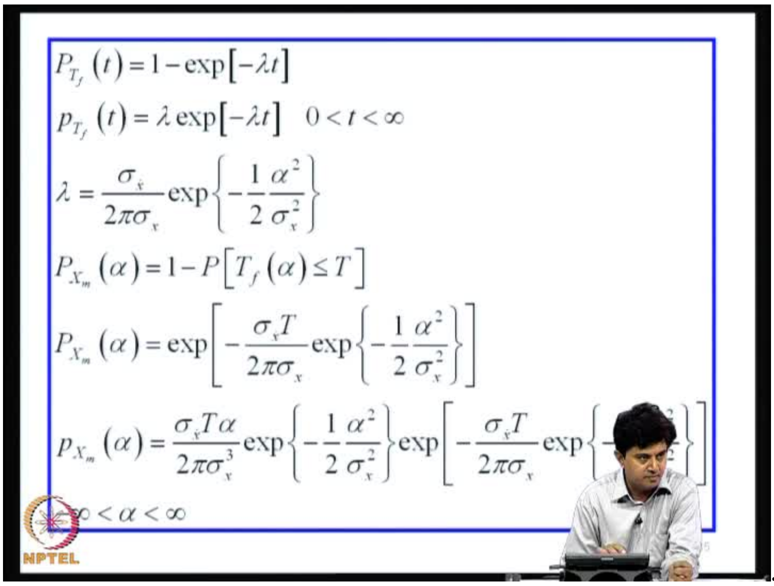

So, that would mean based on the level crossing statistics, the model for extremes under sudden simplifying assumptions become Gumbel. The result from level crossing result itself does not directly lead to the Gumbel model, we need to invoke as an additional assumption to get the Gumbel model.

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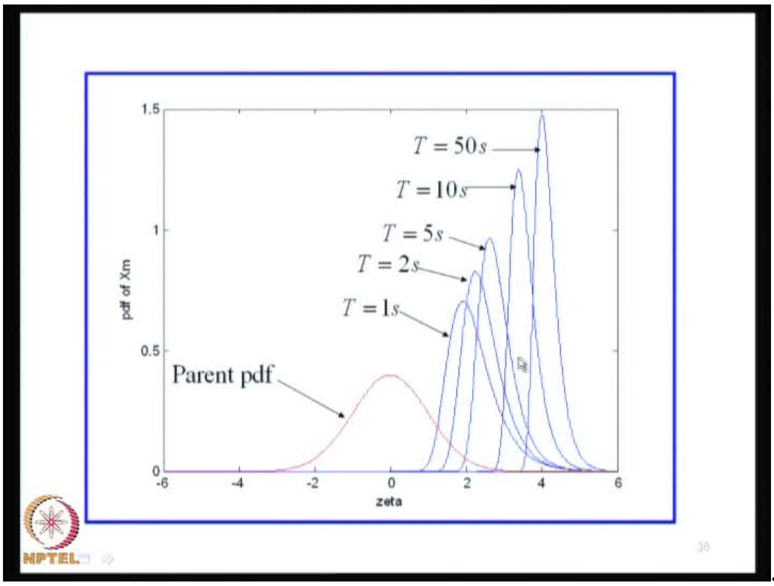

Again I have illustrated this basic result, in terms of a graph here, where the red line is a parent probability distribution function, which is the normal probability distribution function and this is the Gumbel model for capital T equal to 1 second, 2 second, 5 second, 50 second, so the you can see that maximum value is shifting towards right and zeta is already normalize with respect to standard deviation; so, if you are looking at maximum or 50 seconds, you are looking in regions, where we are looking at states which are 4 to 5 times the standard deviations. It is something like three point, may be 3.5 sigma to nearly 6; sigma in that region how they extremes are distributed.

So, these are large effects that we are interested in; then this is the density function. So, one could make a similar analysis for minima which is not displayed here and the probability distribution function for minima would appear to the left, on the left tail and it will be quite similar to this, for this example.

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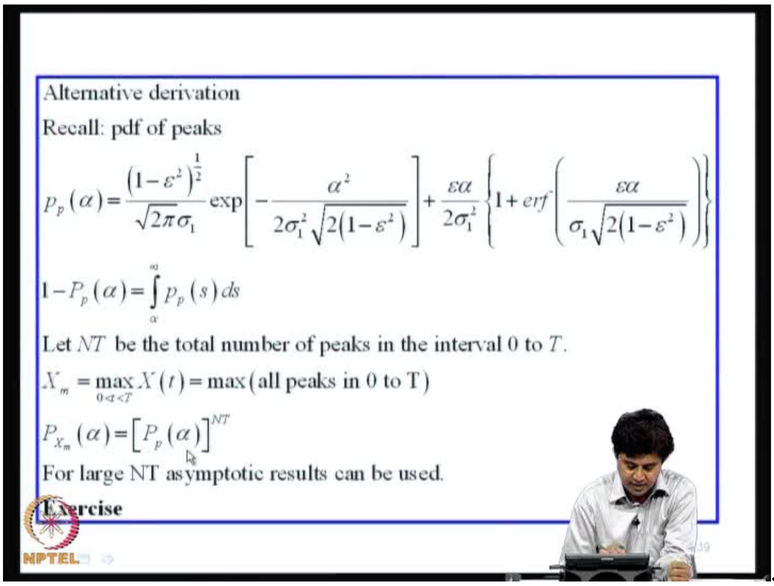

Now, there is an alternate derivation possible here, we have derived the probability density function of peaks in this form, where epsilon was a band width parameter and from this, I can also derive the probability distribution function and this is the probability distribution of peaks. So, if I assume that, there are NT number of peaks in the interval 0 to capital T, then X m which is maximum of X of t, where 0 to T itself can be viewed as maximum of all peaks in 0 to T; so, if peaks are taken to be iid sequence of random variables, then I can use results from maximum of iid sequences and get the model for X m; so, X m would be X m of alpha which is P P of alpha to the power of N T.

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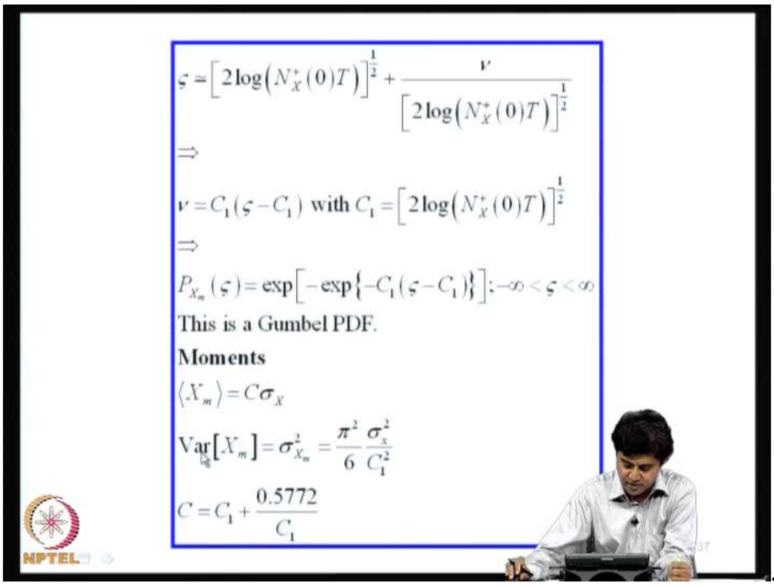

So, this is the another route to derive the extreme value distribution and for large N T, we can use asymptotic results which I derived, a introducing the capital X I N make the transformation and you will get that and define g of y etcetera, you will get that. And on that asymptotic result, if you linearise the exponent term, you can show that the result from this analysis will exactly match with the Gumbel model that we have obtained here. And that I will leave it as an exercise and we will see, how, you know, later in a problem solving session, we can see, if we can get into some of those details.

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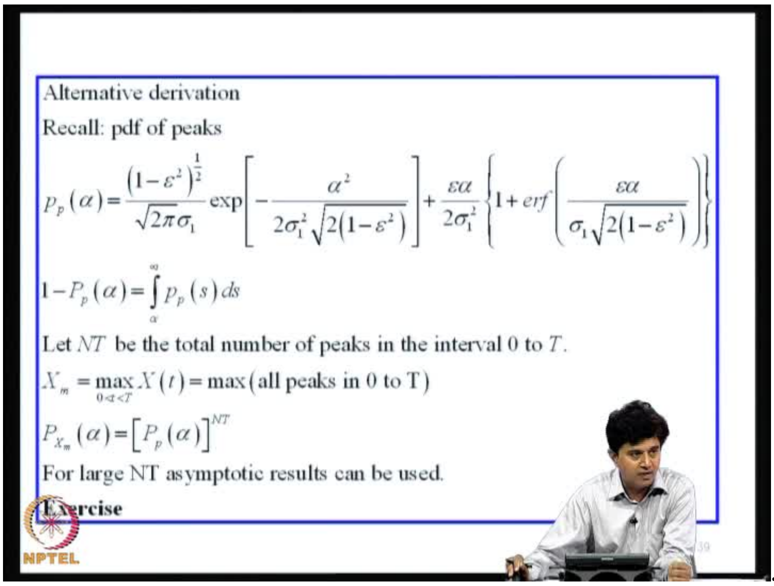

So, this is an alternate route for deriving extremes by solving problem of peaks, but solution of probability distribution of peaks itself is again obtained based on level crossing problem; so, the level crossing problem seems to be the fundamental problem which leads to all these important solutions.

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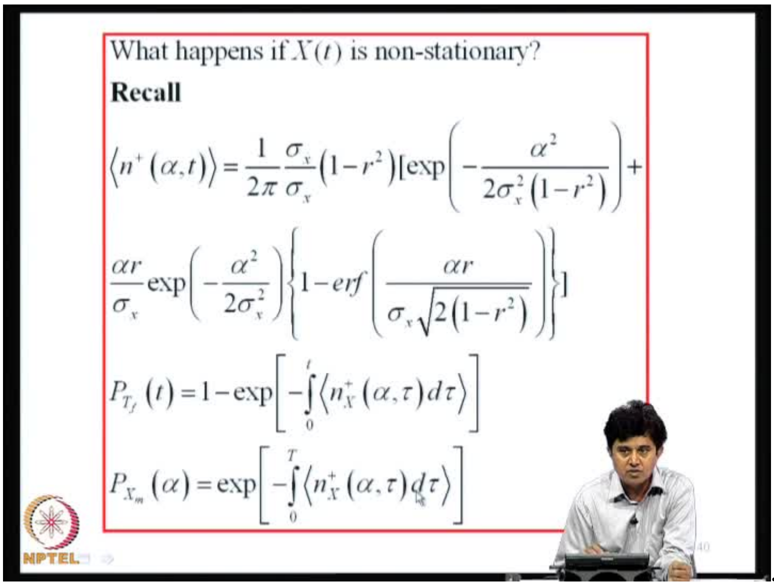

So, what happens if X of t is non-stationary, no serious conceptual difficulty here, if we recall that, the average rate of crossing of level alpha for a non-stationary Gaussian random process with zero mean, we have already derived this. Here the quantity sigma x, sigma x dot and r are all functions of time. So, the problem of first passes time has already been solved and we have shown that, this is 1 minus exponential minus 0 to t, this function of time instead of writing lambda t, we have to write now integral 0 to t lambda of tau d tau, where lambda of tau is a rate of crossings which is given here, average rate of crossings

Now, once a first passes time, a problem of first passes time is solved the problem of extreme is also automatically solve and we have at least notionally, the solution as displayed here. So, if you want to really derive this details of the solution, you have to carry out these integration in time and for that, you have to have the time histories of sigma x, sigma x dot and r which is the correlation coefficient between process and its derivative. For a stationary random process, the process and its derivative are uncorrelated at the same time instant; so, r would be 0, in that case and sigma x and sigma x dot could be independent of time, but if process is non-stationary not only sigma x and sigma x dot would be functions of time, but the correlation between process and its derivative at the same time instant could not be 0 and that you have to model.

So, this for example, if X of t is response of a say single degree freedom system to a random excitation through a standard vibration random vibration analysis, we can get the time history of sigma x, sigma x dot and r and we can substitute into this and from based on that, we can get the results on extreme value distributions also.

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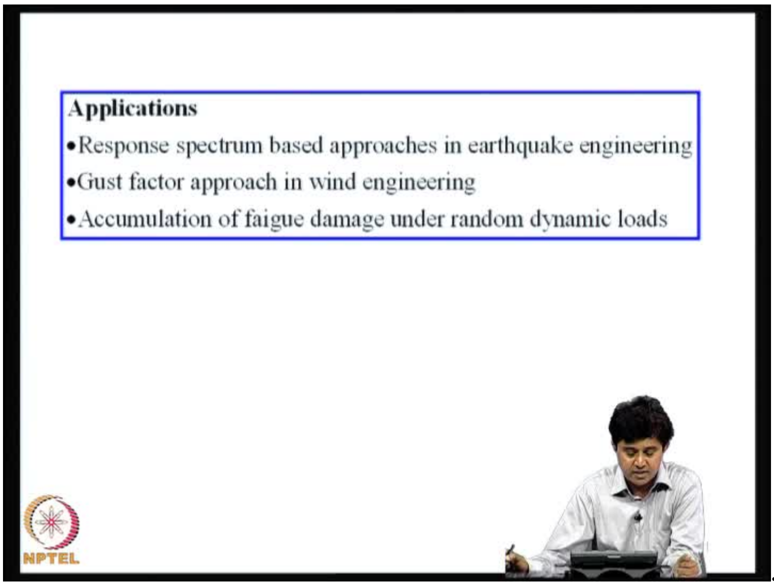

Now, where do we apply this theory of extreme value analysis? I will be discussing three major applications, first would be in the area of earthquake engineering, where we will see what is the relationship between random vibration analysis and analysis based on response spectrum; response spectrum based methods are widely used in earthquake engineering, there essentially deterministic methods, but if you model the earth quake ground acceleration as random processes, we need to carry out a random vibration analysis and the question would naturally arise, what is the relationship between such a response spectrum based analysis and random vibration based analysis. There will be two questions that I will be interested in; one is what is the relationship between response spectrum of ground acceleration and its power spectral density function that will be the one of the question that will answer; the second question is the response spectrum base method, when applied to multi degree freedom systems, if you recall is based on model combinations, the concept of response spectrum is essentially with respect to single degree freedom systems; so, when we applied to multi degree freedom systems, the normal, we have to formulate the problem in the natural coordinates and we get a set of uncoupled generalized coordinates and they need to interpret as a single degree freedom systems, so that, on each of those modes, we can use response spectrum base methods, but to get the response in the physical coordinate, we need to sum it up in some manner and there I will show that rules of model combination like s r s as c q c etcetera, we will come to that are essentially based on random vibration principles.

The next application I would be discussing would be in the area of wind engineering as what is known as gust factor approach; so, we will model wind as a Gaussian random process and use extreme valued theory of random processes and see how we can develop methods for analyzing response of structures subjected to wind load.

The last application would be in the area of modeling accumulation of fatigue damage in randomly vibrating systems. So, due to continuous reversal of stresses, the strength of the structure depurates and that is known as fatigue and if the vibration that is crossing the cyclic reversal of stresses, is the essentially a random in nature, then we need to develop stochastic models for rate at which fatigue damage accumulates and even in that, we will see that the theory based on extreme value analysis and level crossing and peaks etcetera would be of significant use.

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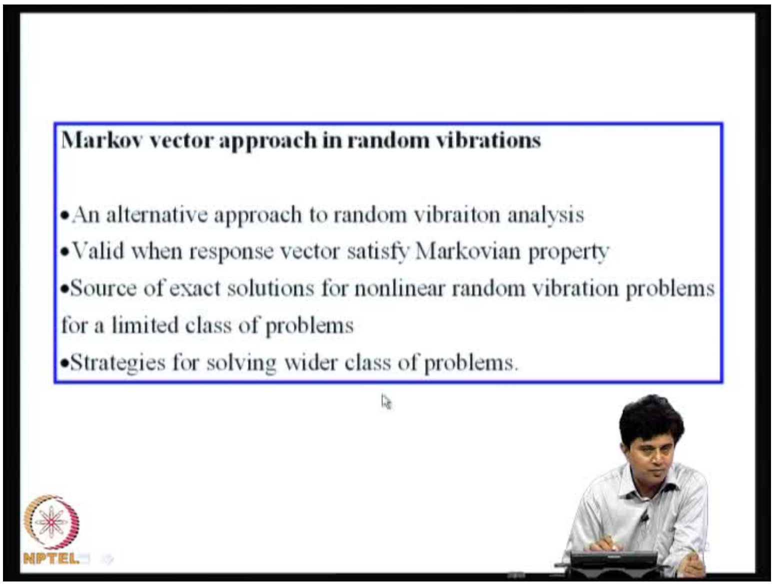

So, these applications I will take up in due course, but in the next part of our lectures, I will be considering a new topic what is known as Markov vector approach in random vibrations; this is an alternative approach to analyze random vibrations, it does not depend on Duhamel's integral and calculation of moment etcetera. The essence of this method is that, just by as by applying Newton's laws and Duhamel's principles, we get equations of motion for evaluation of displacement, velocity etcetera. We can also derive an equation of motion like equation, for the evaluation of the probability density function X of t is the response process, say of a single degree freedom system, then P X of t uses first order probability density function; so, can we derive the equation of motion which governs P X of t.

So, these provide a newer perspective and the basis for, that is, allies in what are known as what is known as Markov property of the response process. So, here, this method is applicable, one response vector satisfy Markovian property, I have do, I have briefly touched upon what is Markovian property when we discussed a random processes; we will return to that in greater detail now. The merit of this approach is that the Markov vector approach is a source of exact solutions for a class of non-linear random vibration; so, the structure being linear or non-linear is not so crucial in this approach, but certain class of non-linear problems, you can, exactly you can solve the governing equations exactly; so, this is a very valuable information that we get through this approach.

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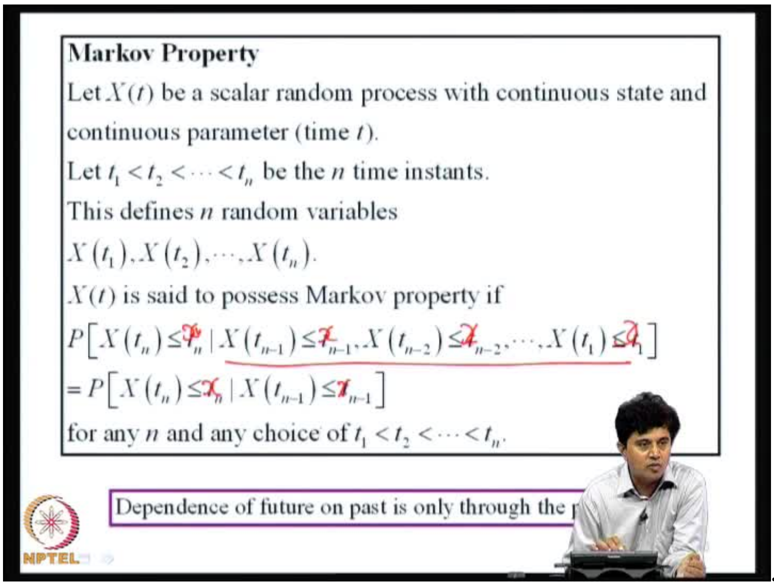
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Even for problems, where exact solutions are not possible, within the framework of this Markov vector approach, we can develop newer strategies to develop approximate, solutions. So, this is broadly the scope of Markov vector approach in a random vibration; so, during next couple of lectures, we will develop the basins of this approach; to do that we should start with defining what exactly is a Markov property as I mentioned already, we have briefly touched upon this issue earlier; so, I will quickly recall now. So, let X of t be a scalar random process with continuous state and continuous parameter and that parameter let it be time t.

If I now select n time instants t 1, t 2, t n this choice would result in definition of n random variables namely X of t 1, X of t 2 and X of t n; these n random variables are completely specified in terms of nth order joint probability density function. Now, if you now consider the probability of X of t n less than or equal to x n x n minus 1 x n minus 2 x of t n less than equal to x n conditioned on x of t n minus 1 being less than x n minus 1 x of t n minus 2 less than x n minus 2 etcetera, if this conditional probability density function is this, that means, you may have information on the state of the system at t n minus 1, t n minus 2 and up to t 1, but the condition this conditional probability density function depends only on X of t n minus 1 less than are equal to X n minus 1, that means, dependence of if you assume that t n minus one is the present t n is the future and t n minus 2 t n minus 3 up to t 1 are the past, then dependence of future on past is only through the present.

What happens tomorrow depends on what is happening now, today and not what happened how today was reached, that means, there is a one-step memory; such processes are said to have, when they are known as Markov process; it is a Markov property, it is a property of conditional distribution function, when I say a process is a normal random process, I referred to its probability distribution function, but when I say a process is a Markov I am talking about its memory, it has one step memory. So, there is nothing like a Markov probability distribution function that I am referring to here.

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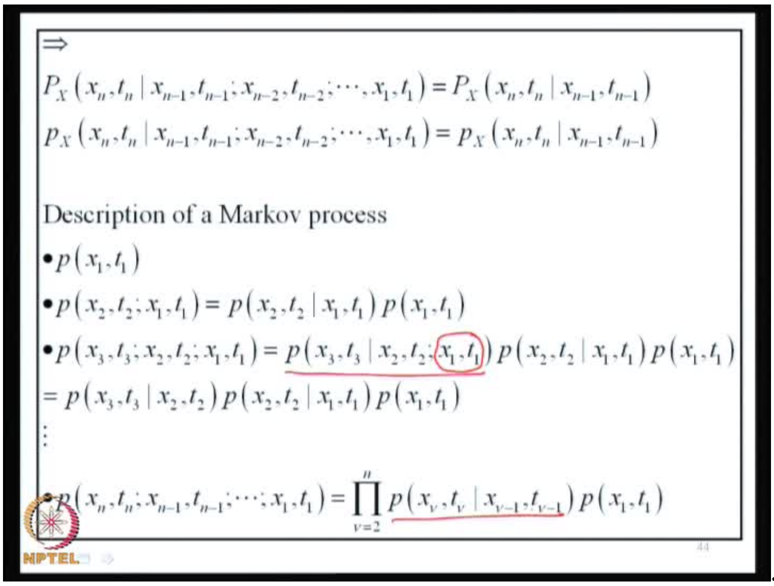

That would mean, if you consider the joint the conditional probability distribution function of x n at t n condition on x n minus 1, t n minus 1, x n minus 2, t n minus 2 etcetera x 1, t 1, this is equal to p x of x n, t n conditioned on x n minus 1, t n minus 1. Similarly, associated probability density function also has this problem, then we say that x of t has Markov property.

So, how do we describe a Markov process, is there any simplification possible or are we still stuck with n th order probability density function. The first order property distribution function is density function with p (x 1, t 1). The second order probability density function can be written as p of x 2, t 2 conditioned on x 1, t 1 into probability of x 1 probability density of x 1, t 1. The third order density function can be written as p of x 3, t 3 conditioned on x 3, t 3, x 1, t 1, into PDF of x 2, t 2 conditioned x 1, t 1 into p x 1 t 1. Now, you look at the first conditional probability density function, if x of t is Markov, this dependence would go away and I will write this as p of x 3, t 3 conditioned x 3, t 3; since, x of t is Markov, this x 1 t in the knowledge of x of t at t 1 is not helpful in writing this density function.

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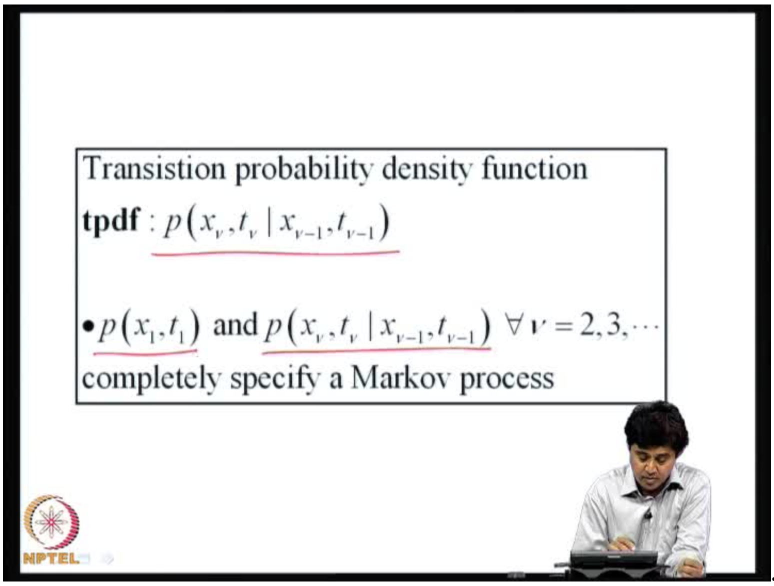

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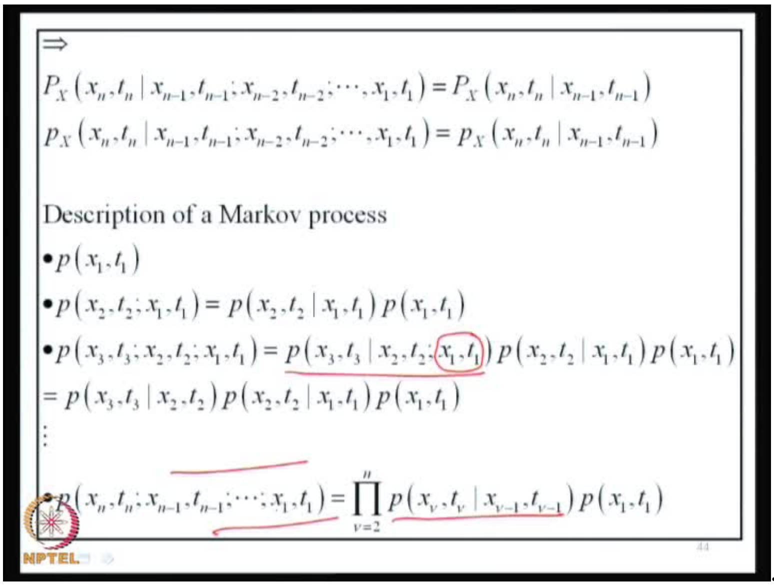

So, I get p (x 2, t 2) x 1 t 1 p (x 1, t 1). Now, if you similarly I extend this argument and consider n th order joint probability density function, we will see that what we need is this condition probability density function x nu, t nu conditioned on x nu minus 1 t nu minus 1 into p of x 1 t 1. This function is known as transitional probability density function. So, for we give a special name to this, it is called tpdf transitional probability density function; so, for x of t, if you know p (x 1, t 1) and this transitional probability density function for nu 2, 3 etcetera, then x of t is completely specified, because based on the knowledge of these two quantities, for all these nu, I can go back and use this expression this expression and get the required n th order probability density function; so, X of t gets completely specified.

Now, in the next class, we will see what other properties this Markov processes have and how to use that in analyzing dynamical systems; so, that is what we will come up in the next lecture and but we will close this lecture at this point.