Stochastic Structural Dynamics

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Lecture No. # 19

Failure of randomly vibrating systems-3

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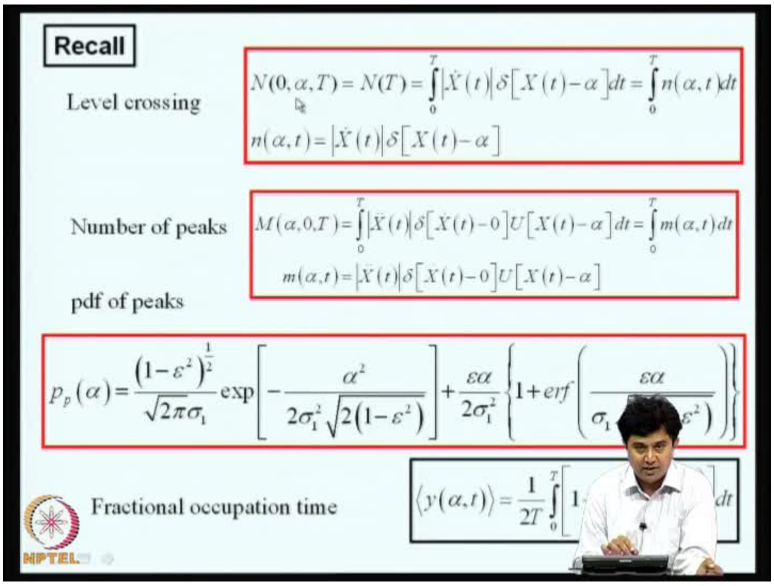

We have been discussing the development of certain descriptors of random processes, which help us to model failures of randomly vibrating systems. So, in this lecture, we will be discussing more on envelope and phase processes associated with a given random process. Before that, we quickly recall what we have been doing; we have solved this problem of characterizing, the number of times a level alpha is crossed in 0 to T, by a random process X of t and this is a counter that we setup and this lower case n (alpha, t) gives a rate or crossing of level alpha. And when X of t is a Gaussian random process, we have been able to characterize, the some of the lower order moments of these rates.

We also ask the question on number of peaks above a given level alpha and again we setup a counter and were able to characterize its properties for Gaussian random processes. Based on certain heuristics assumptions, we also derived the probability density functions of peaks for both narrow banded and broad band processes and this was the expression that we obtained. Here, epsilon is a bandwidth parameter that helps us to characterize with other processes, narrow banded, broad banded or somewhere in between, we also characterize the so-called fractional occupation time, that is the fraction of time, that a random process spends above a level alpha in a given duration 0 to T and we were able to derive its expected value, for a Gaussian random process.

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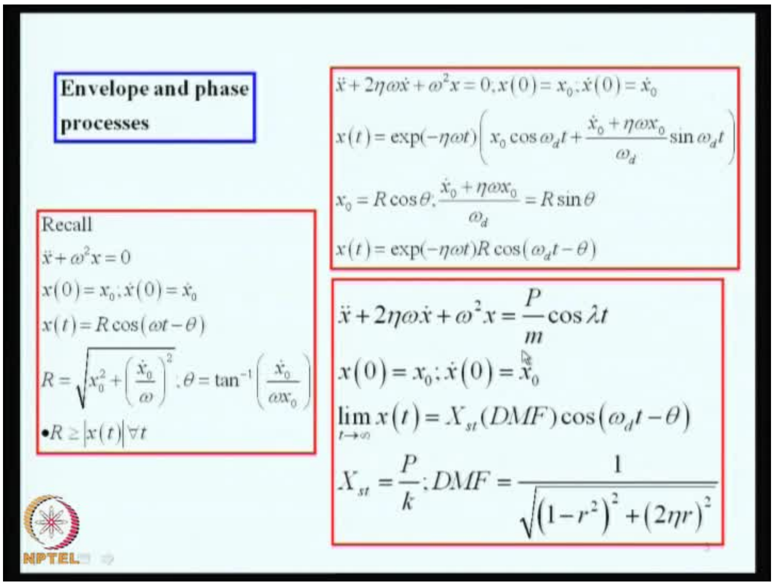

I also talk briefly about the notion of envelope and phase processes, during the last lecture. So, we consider, for example, an un-damped free vibration of a single degree freedom system and the equation of motion is x double dot plus omega square x equal to 0 and if system start from initial conditions x naught and x naught dot, we can write this solution as x of t is R cos omega t minus theta, where R is the amplitude of x of t and which is the function of the initial conditions and the natural frequency of the system. Similarly, the phase angel theta is the function of initial conditions and the natural frequency and this R has a property, that it is greater than or equal to modulus of x of t for all t and this is called the envelop of x of t, in this case.

Similarly, for a damped free vibration problem, we could show, that the response can be written as e raise to minus eta omega t R cos omega d t minus theta, where R is again described in terms of system natural frequency damping and initial conditions and the quantity R into e raise to minus eta omega t, can be thought of as the envelop for this response. Now, if the same system is driven harmonically, again we can show, that the response in steady state can be written as X s t, which is a static response into a dynamic magnification factor into cos omega d t minus theta and this quantity X of s t into DMF can be thought of as the envelop of response for this system. And similarly, theta is the phase angle for this system and it is dependent on system natural frequency damping and the driving frequency.

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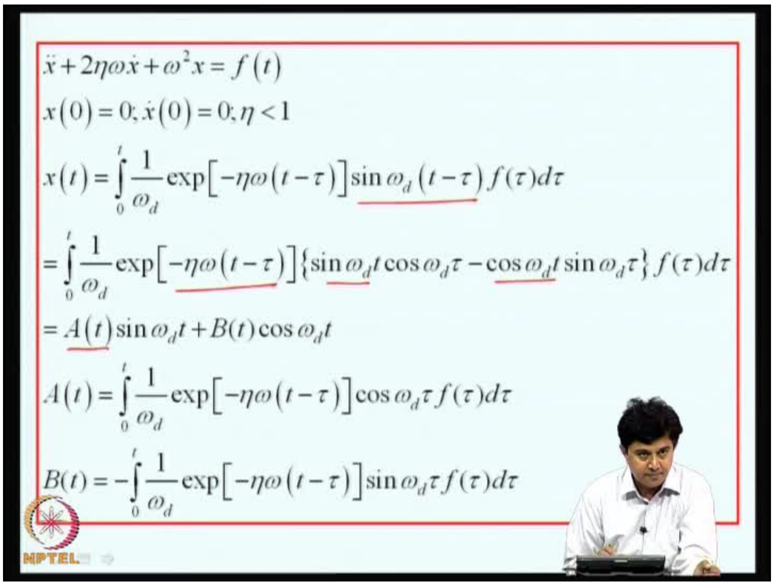

Now, what happens is the system is now driven by an arbitrary force f of t, can we get an envelope representation for the response, in this case. So, we start with the case, where the system starts from rest, that initial displacement is 0, initial velocity is 0 and we assume that system is under damped. So, the complete solution of this equation is given by the Duhamel integral 0 to t h of t minus tau into f of tau d tau and the h of t minus tau is given by the first two terms here and f of tau is an excitation. Now, what we could do is, we can expand this sin omega d t, this term and write it as sin omega d t cos omega d tau minus cos omega d t sin omega d tau f of tau d tau, now the integration with respect to tau; therefore, terms involve in time can be pulled out of this and I can write this integral as A of t into sin omega d t plus B of t into cos omega d t, where A of t and B of t are these integrals, A of t is the 0 to t 1 by omega d e raise to minus eta omega t into cos omega d tau d tau f of tau d tau and similarly B of t is given by this.

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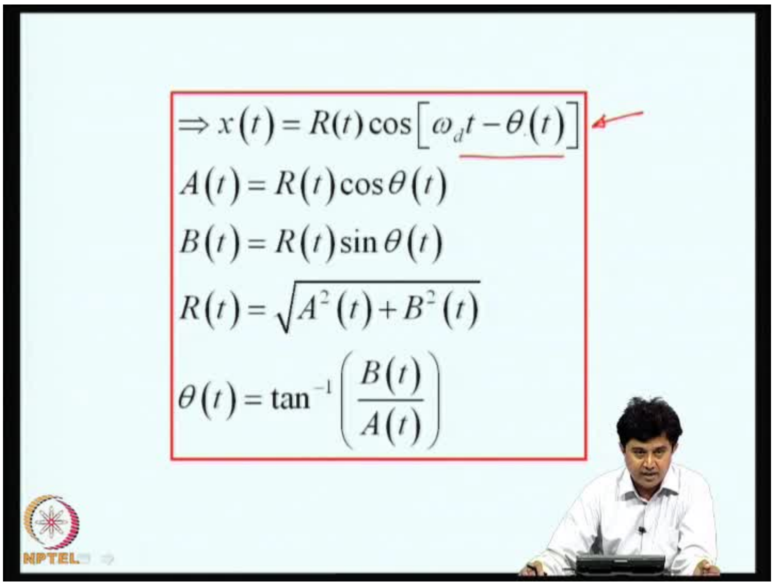

So, from this expression, we can proceed further and write x of t as R of t cos omega d t minus theta of t. Here, A of t, that is, that integral just now I showed, is written as R of t into cos theta of t and B of t is written as R of t sin theta of t. So, R is square route of A square plus B square and theta is tan inverse B by A; so, that would mean, even in this case, we can write the response in terms of an envelope R of t and a phase theta of t. So, this kind of representation is quite useful in characterizing the dynamic response and question would naturally arise, how to use such descriptions in charactering random processes. In alternative interpretation for the envelope can be obtain by considering R square of t as x square of t plus x dot square of t divided by omega d square.

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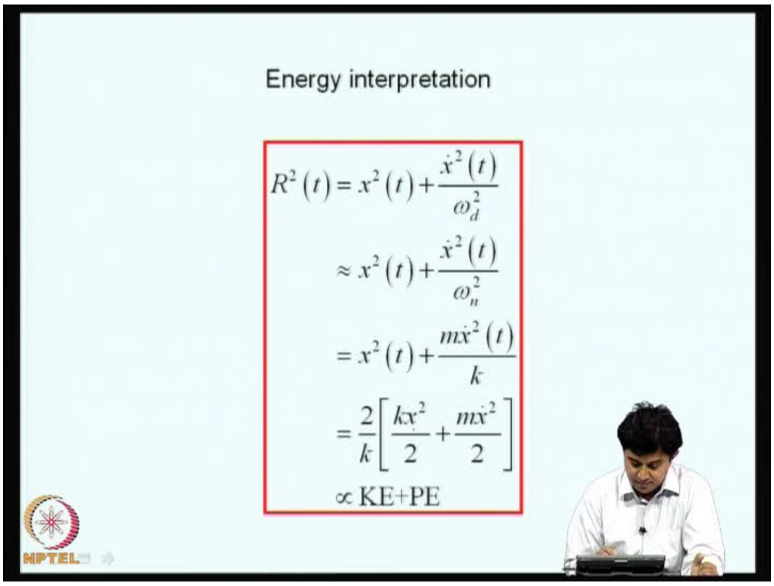

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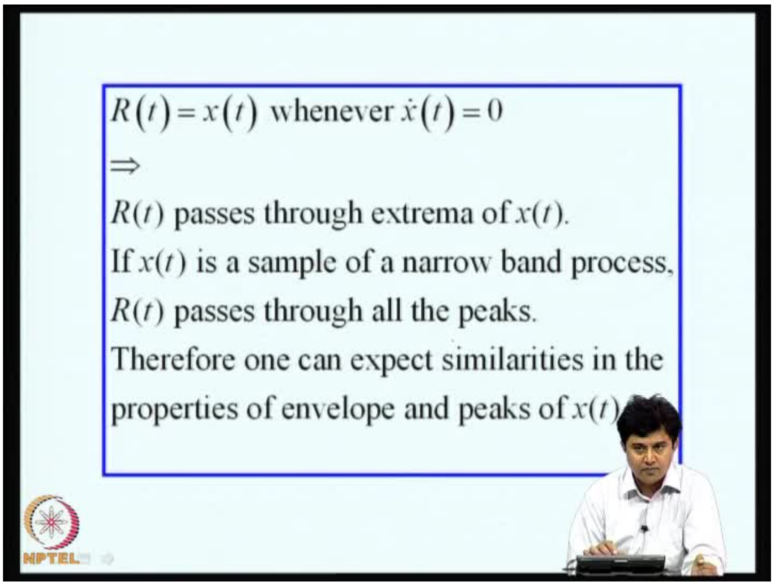

So, this damped naturally frequency can be approximated by the un-damped natural frequency and we can write for omega n square k by m and you can rewrite this as 2 by k into k x square by 2 plus m x dot square by 2. Therefore, R square of t can be taken to be proportional to the total energy, which is sum of kinetic energy plus potential energy. Now, whenever x dot is 0 or whenever x is maximum, R of t passes through the maximum values of x of t, that is, R of t is x of t, whenever x dot of t is 0; the condition x dot of t equal to 0 is the condition for x of t to reach its extreme values, so that would mean, R of t passes through extreme of x of t.

If x of t is a sample of a narrow band process, R of t passes through all the peaks; so, we can expect that since we have already studied peaks, you could expect that properties of an envelope and properties of peaks, in some sense would be similar, but that has to be actually verified; in fact, when we characterize the probability density function of peaks, we had used a heuristics argument which was not mathematically rigorous, but we could obtain an expression for probability density function of peaks which could prove useful, if it is acceptable. But doing the course of the following discussion, we will show that by following a more rigorous approach, we can show that R of t indeed shares some of the properties of the PDF of peaks that we obtained heuristically.

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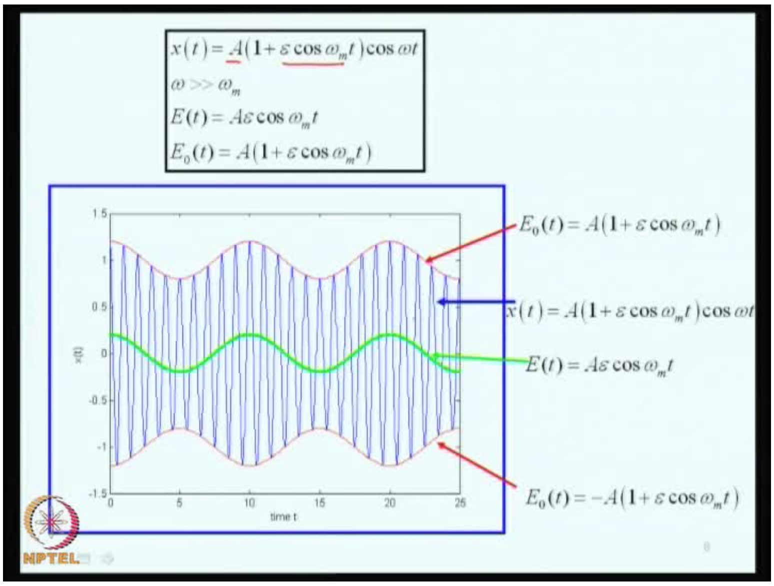

To clarify, the notion of an envelope further, we can consider a signal x of t is A into 1 plus epsilon cos omega m of t into cos omega t; the blue line that you see here is actually this function x of t. Now, if you look at the multiplier A into 1 plus epsilon cos omega m t, that is shown in the red line here. This is the actually the envelope, this is e naught of t is A into 1 plus epsilon cos omega m t which multiplies cos omega t and this line is minus of that. So, there are pairs a pair of curves, which actually bound the function x of t; this green line shows only this component E of t which is A epsilon cos omega m t, this part, a into epsilon that is a green line. So, you can see that, this is much slowly varying than the x of t itself and it bounds the x of t; therefore, if you are interested in highest values of x of t and so on and so forth, it may be much easier to study an envelope than the blue line.

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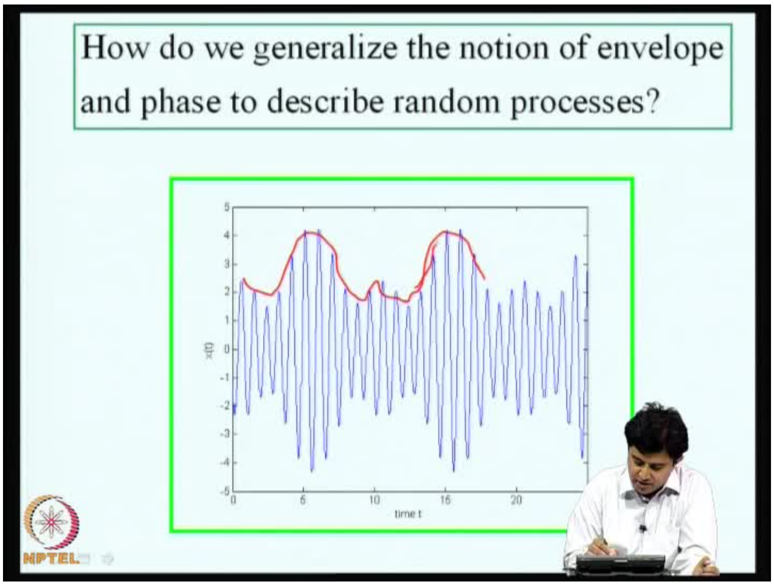

So, now, we will pose this question, how do we generalize the notion of the envelope and phase to describe random processes. So, again you see, this is the sample of a narrow band process, so the envelope should pass through, you know something call, it will release to should pass through all this peaks, that is what intuitively we expect, but now we need to formalize this notion.

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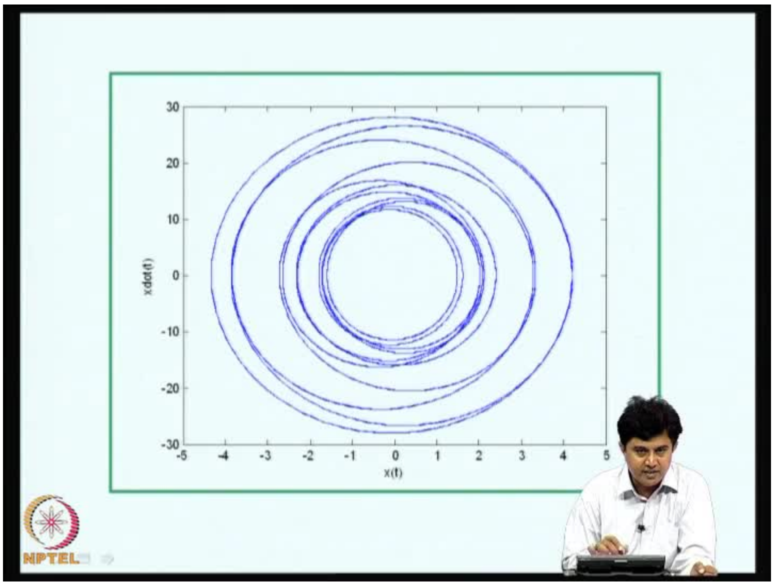

If you look at the plot of x dot of t verses x of t, a narrow band process has this type of character, it does not fill up the entire space, it occupies, you know, certain space which is not, if x of t is a broad band process, it will simply fill up this space; so, this another feature that we need to bear in mind.

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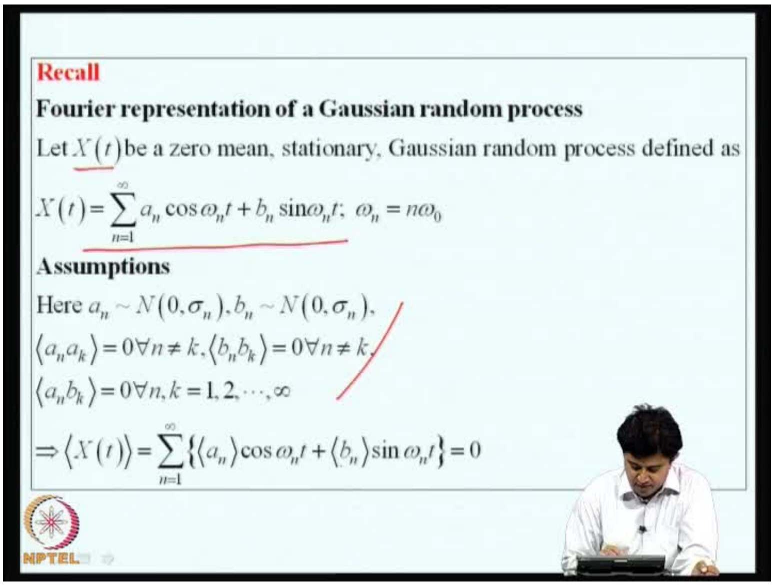

Now, to obtain an envelope representation for a random process, we begin with X of t, let it be a zero mean stationary Gaussian random process and we will represent this random process, in terms of a Fourier series as shown here. This we have discuss in one of the earlier lecture, I am recalling what we discussed; here a n and b n are random variables and we can assume them to be normal distributed zero mean and say standard duration sigma n; a n and b n are mutually independent and identically distributed; so, that properties clarified here and using these properties, if you want, say, mean of X of t, you take expected value of this; we know expected value of a n is 0, expected value of b n is 0; therefore, expected value of X of t is 0.

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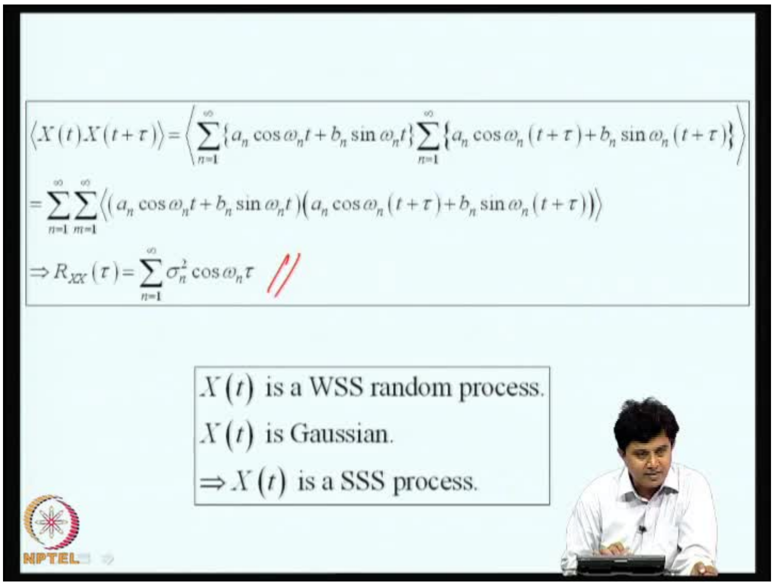

Similarly, we can find the auto covariance expected value of X of t into X of t plus tau and we can show that, that auto covariance is given by a function, which is function of only tau, that would mean, the process is a wide sense stationary random process, but since X of t is Gaussian, because a n and b n are Gaussian and we are adding Gaussian random variables X of t also would be Gaussian and therefore, X of t is the strong sense stationary process.

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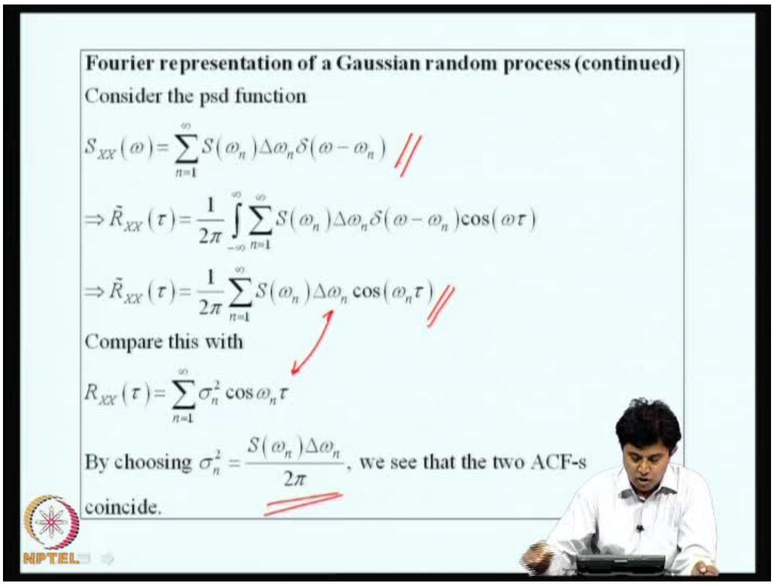

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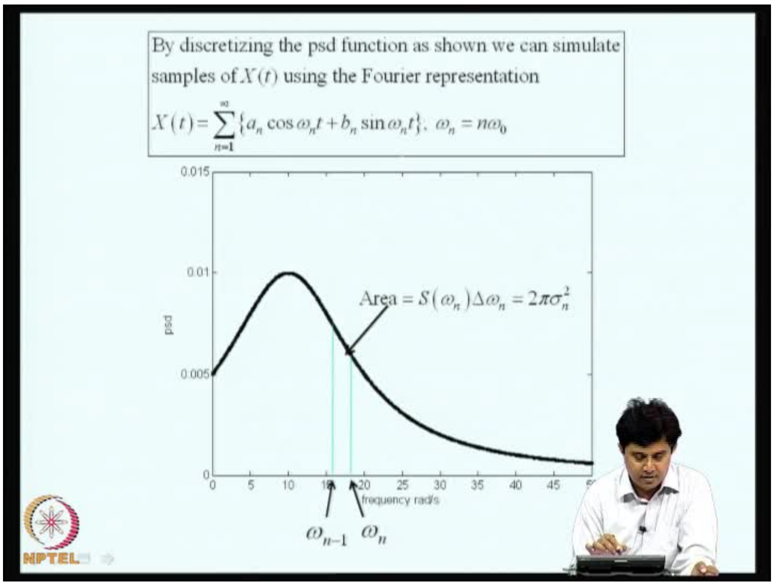

Now, I also shown in the previous lecture, that if we now start with a power spectral density function made up of a set of Dirac delta functions and if we compute the auto covariance of this, it has this form by using the Fourier transforms and if we compare this form with the auto covariance of the signal, that we just now described. We can see that, these two definitions will agree, if sigma m square is chosen to be this; that means, for the process that we described here, this process the power spectral density function will be of this form. So, if we are given a continuous power spectral density function like this and if we discretize this into… If you discrete frequencies, we can represent a Gaussian random process, in terms of a Fourier series with random amplitudes; so, that is, the, you know result that I will be using shortly.

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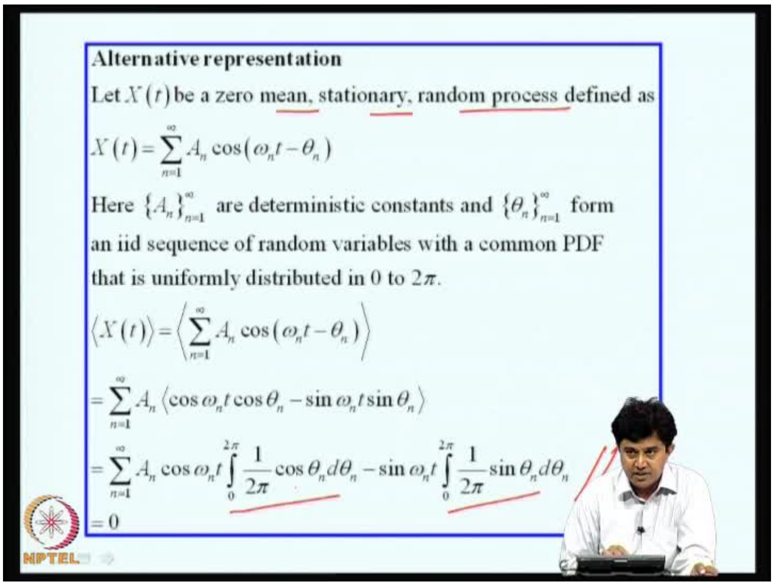

Now, before I proceed, we can also notice that there is an alternative representation slightly different from, the one that I described just now. So, to clarify that, let us consider X of t to be a zero mean stationary random process, defined as X of t n equal to 1 to infinite A n cos omega n t minus theta n; here, this A n are deterministic constants, they are not random variables, the only quantity, that is random on the right hand side are this theta 1, theta 2, theta 3, etcetera we assume that these theta n are form an iid sequence of random variables with a common probability distribution function, which is uniformly distributed in 0 to 2 pi.

Now, let us study the property of this random process; suppose, you are interested in mean of X of t, you have to take expectation of X of t and to do that, if you expand this cos omega n t minus theta n using this identity, we can show that, this expected value is given by this expression, where the expectations of cos theta n and sin theta n need to be evaluated and since theta n are uniformly distributed in 0 to 2 pi, these two integrals are 0, that would mean, mean of X of t is 0.

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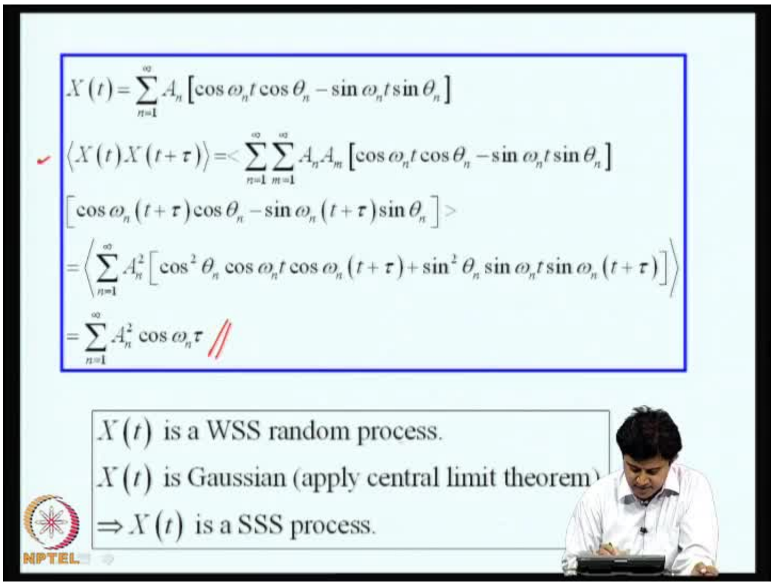

Now, following the definition of auto covariance of X of t, we find now the expected value of X of t into X of t plus tau. So, since A n are deterministic and theta n are iid sequence, we can manipulate this expression and show that, the auto covariance is indeed given by A n square cos omega n tau summed from n equal to 1 to infinite; so, this process also has a similar structure of auto covariance as we studied just, where there was summation of A n cos omega n t plus b n sin omega n t, where A n and B n were random. Similarly, the Fourier transform of this, which will give the power spectral density, will also be a sequence of Dirac delta functions centered at omega n; so, the two representations at the level of auto covariance and psd yield the same results.

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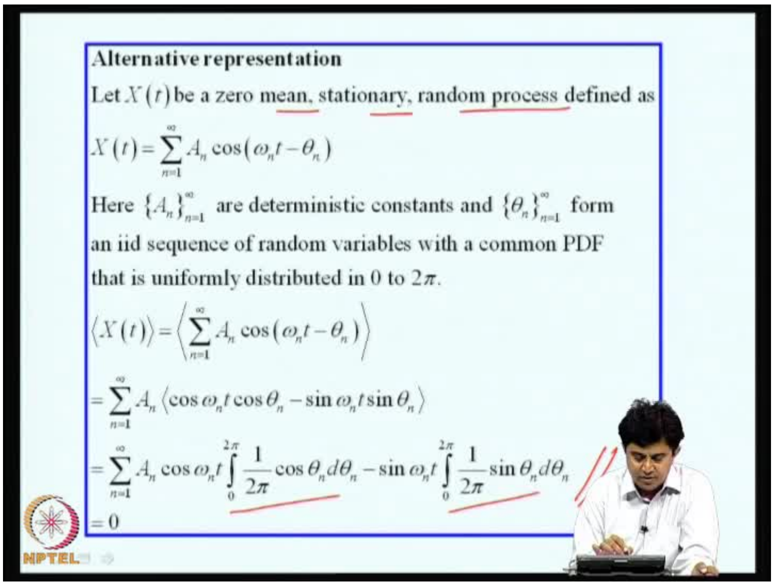

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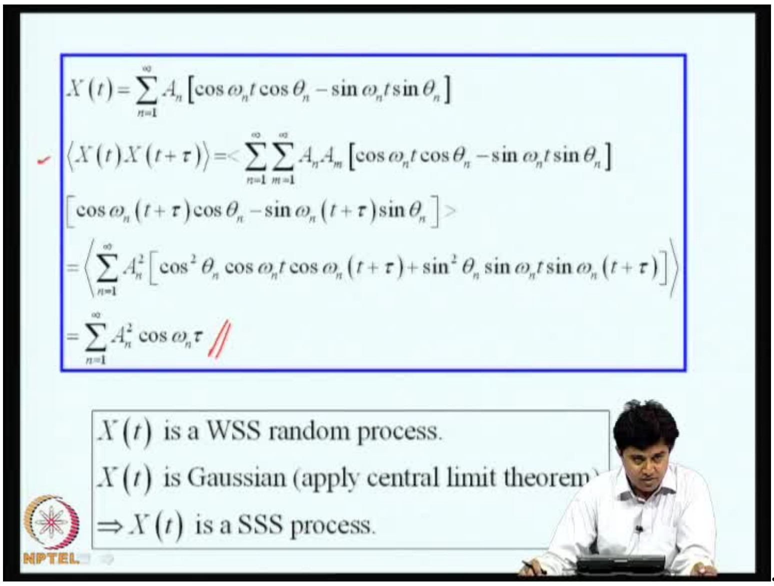

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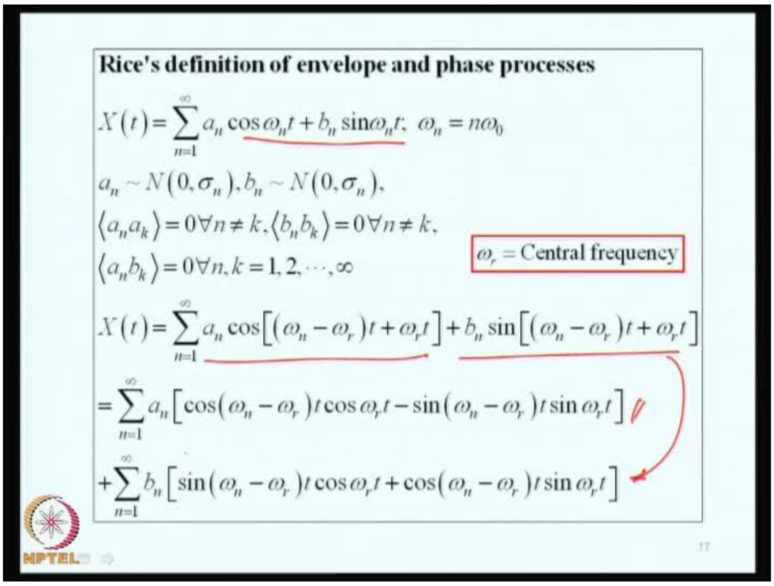

So, X of t in this case, again is a wide sense stationary random process and we can show that X of t is Gaussian, because we are adding random variables which are identically distributed and which are independent and we can expect that X of t would be Gaussian indeed, that would be the case and therefore, even this process, would be a strong sense stationary process. Now, based on these definitions of X of t, we can now introduce the notion of envelope and phase process for a random process; this definition follows the one that is, proposed by Rice's in nineteen forties. So, we begin by using the Fourier representation X of t is n equal to 1 to infinite a n cos omega n t plus b n sin omega n t and A n and B n are Gaussian random variables mutually independent and omega naught is one of the basic frequencies, that is the parameter in this model.

We rewrite this terms cos omega n t sin omega n t as cos of omega n minus omega r t plus omega r t; similarly, b n sin omega n minus omega r t plus omega r t, where omega r t, omega r is a central frequency, we will clarify the meaning of this in due course. Now, I can now manipulate this expression, I can expand this cos of omega n minus omega r t plus omega r t using cos of a plus b identity; so, I rewrite this expression in this form. So, the first term correspond to the first terms correspond to this and the second term corresponds to this; now, the summation is on n, therefore terms involving omega r can be pulled outside.

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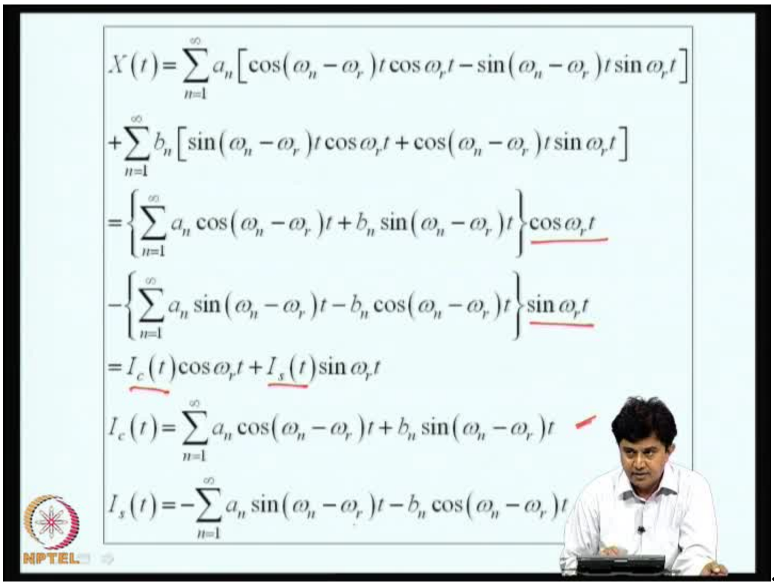

So, I can rewrite this as coefficient of cos omega r t is collected in one place, that is what is contended in this braces and coefficient of the sin omega r t is collected in one place. so the first term inside the brace, I call it as I c of t and the second term, I call it as I s of t right, where I c of t and I s of t are indeed this summations as depicted here. I c of t is again a Gaussian random process, because a n and b n are Gaussian and I s of t by the same argument is also a Gaussian random process, having zero mean and you can show that stationary also.

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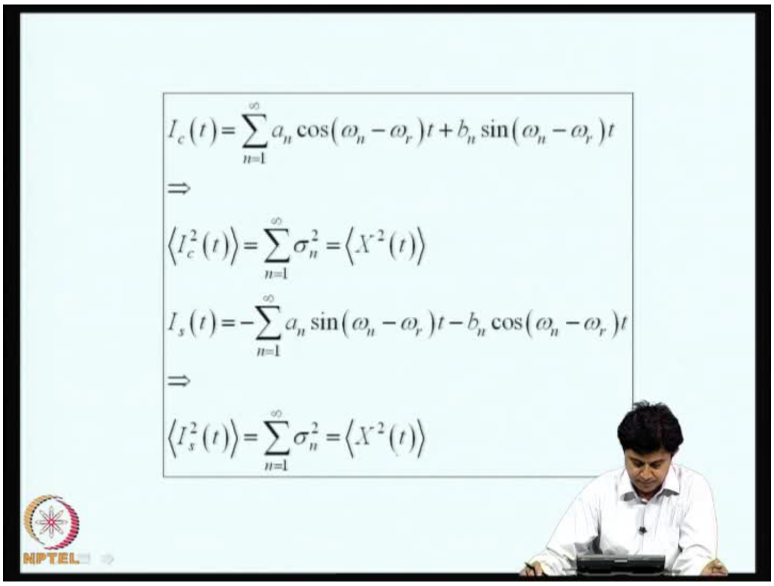

Now, I c square of t, if you mean is 0 so for variance if you want to find it is expected value of I c square of t, you can show that this is same as the variance of x square of t, similarly I s of t is given by this and based on this we can show that variance of I s of t is again equal to x square of t.

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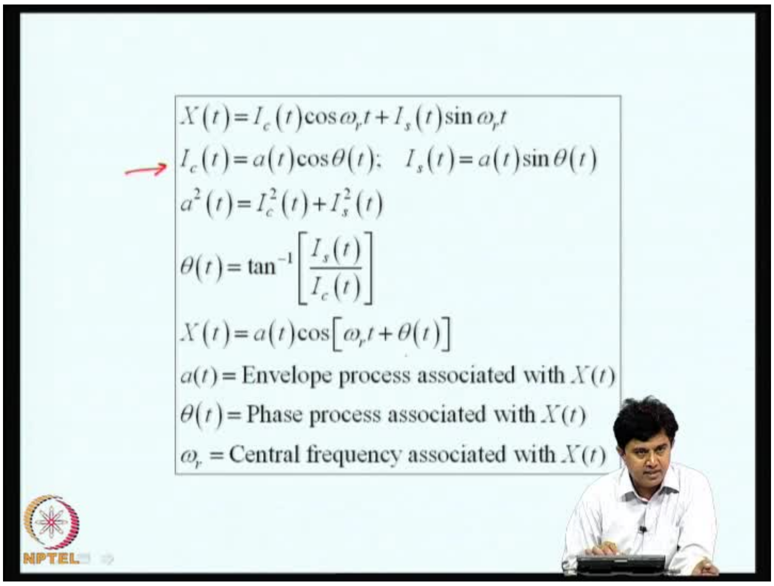

Now, I now introduce this substitution I c of t is a of t into cos theta of t; I s of t a of t into sin theta of t, where a square is I c square plus I s square and theta of t is tan inverse I s by I c; using these notation, now I am able to write as X of t as a of t cos omega r t plus theta of t; a of t is the envelope process associated with X of t theta of t is the phase process associated with X of t; a of t is square route of plus I c square plus I s square, that would mean, it is a non-linear transformation on to Gaussian random processes, so a of t would be non-Gaussian. Similarly, theta of is a non-linear transformation on ratio of two Gaussian random processes; therefore, theta of t also would be non-Gaussian. Omega r is a central frequency associated with X of t; so, this is the envelope representation for a random process.

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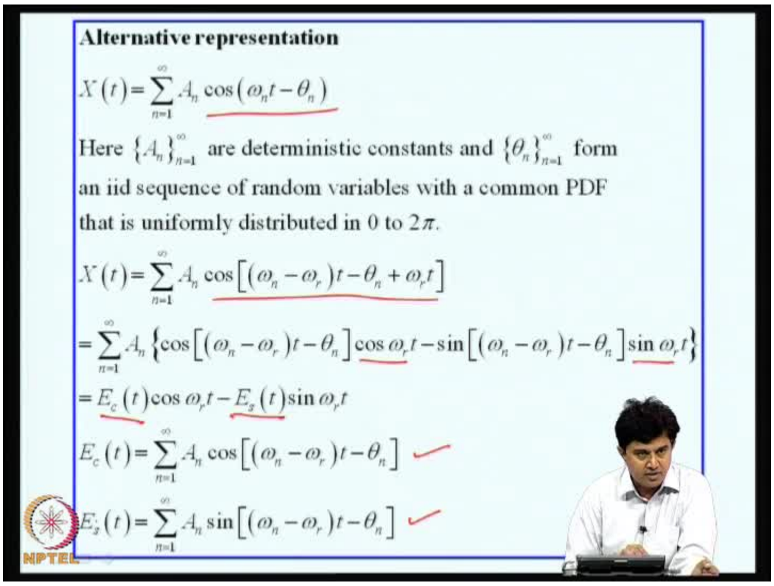

We will quickly consider the alternative representation that we used, where X of t a was written as n equal to 1 to infinite A n cos omega n t minus theta n, where A n were deterministic. Here again what I will do is, I will rewrite this as cos of omega n minus omega r t minus theta n plus omega r t; again expand, collect terms which multiply cos omega r t and sin omega r t and I will be able to write this as E c of t into cos omega r t plus minus E s of t into sin omega r t; these are again two summations E c of t is this summation, first term and E s of t is a second summation, these are again stationary random processes, having properties quite similar to that of X of t.

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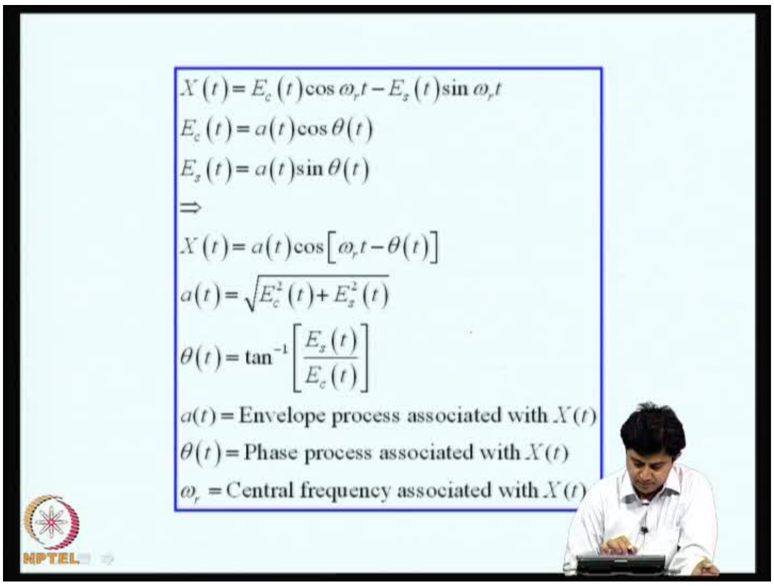

Now, if I now introduce the notation E c of t is a of t cos theta of t and E s of t is a of t sin theta of t, I can write X of t as a of t into cos omega r t minus theta of t, where a of t is square root of E c square plus E s square, which is the envelope process; theta of t is tan inverse E s of t divided by E c of t, this is the phase process. So, using the two alternative representations, we get similar representation for the envelope; they may differ in some details, but in essential in essential they are quite similar.

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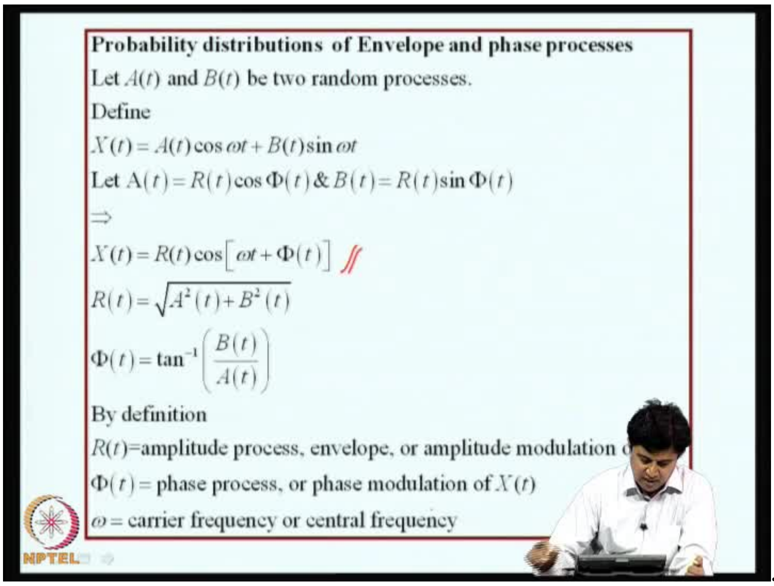

The question now is we have defined envelope and phase processes, they are non-Gaussian, even when X of t is Gaussian. So, we are we seem to be making the problem complicated, the notion of envelope and phase processes would be useful, if we can determine their probability distributions. The basic idea is that, envelope varies lot more slowly than the parent process, therefore, describing a slowly varying function is easier than describing a rapidly varying function; so, that is a basic expectation, but that expectation would be met, only if we are able to determine the requisite probability distribution functions of these two random processes.

A random processes is completely described in terms of its joint probability distribution and density functions and unless, we are able to say something useful about damp. This notion of envelope and phase which essentially introduces non-linear transformation on the parent process, likely to be not helpful, but fortunately the problem of finding probability distribution of envelope and phase processes is solvable, especially when X of t is a Gaussian random process and we will see later, that this could be done even for a few non-Gaussian random processes. So, to see that, we will start with the following definition; we introduce two quantities A of t and B of t, which are random processes and we define X of t as A of t cos omega t plus B of t sin omega t and further more we put A of t is R of t cos phi of t and B of t is R of t sine phi of t, so X of t itself now can be written in the form of R of t cos omega t plus phi of t, where R of t is square root of A square plus B square and phi is tan inverse B by A.

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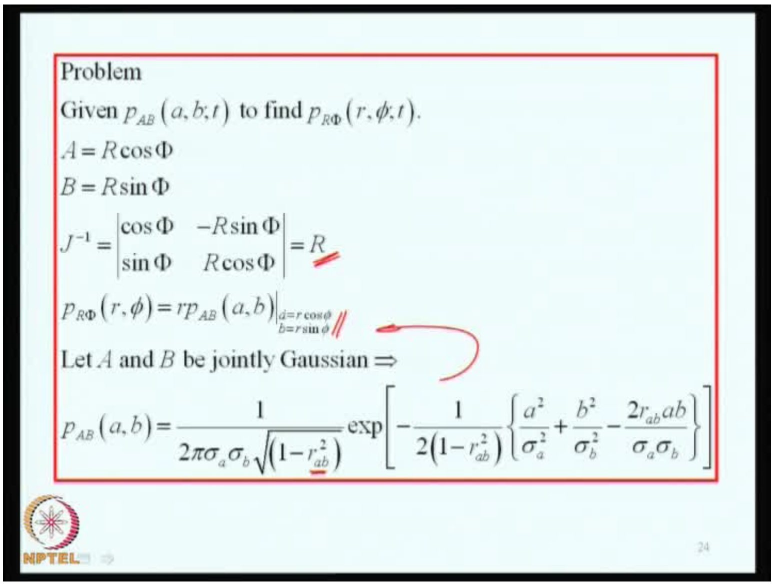

By definition, we say that R of t is amplitude process envelop or amplitude modulation of X of t, they are all synonyms; phi of t is also is called a phase process or phase modulation of X of t; omega is known as a carrier frequency or carrier frequency or central frequency. Now, what is the problem, now the problem is we started with definition of A and B, suppose, we are given the joint probability distribution function of the process A of t and B of t, can we find the joint probability distribution function of R and phi, which is envelope and phase.

So, we know A is R cos phi and B is R sin phi; this transformation of random variables can be handle using the rules of transformation of random variables, this is not very complicated, so we find the Jacobean or its inverse and we show that this is J 2 power of minus 1 is 1 by J is R and consequently, we get P of R phi as r P A B (a, b) with a and b evaluated r cos phi n r sin phi n. If A and B are jointly Gaussian, I can right the two-dimensional probability density function, in terms of standard deviation of A standard duration of B and correlation coefficient between A and B and in that, is of this form ,we are taking that A and B have zero mean.

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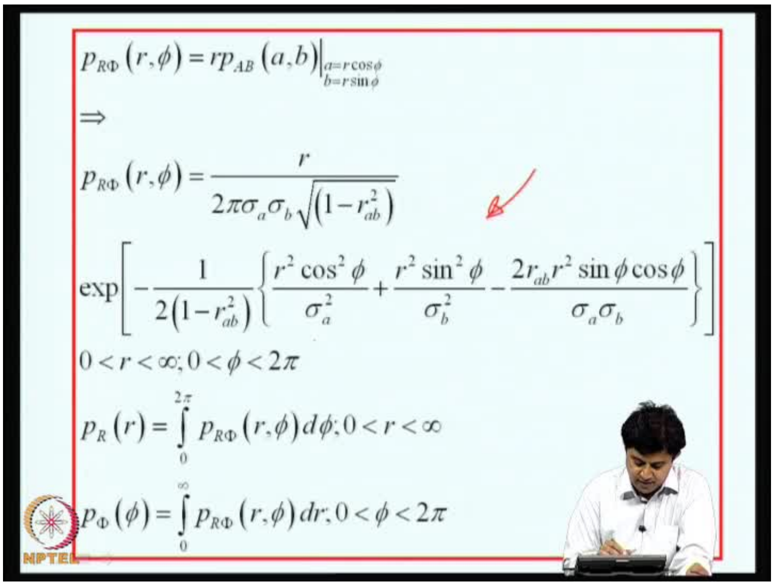

So, now we can substitute this expression into this identity and try to get the probability density function between joint probability density function of r and phi and if we do that, we get this expression, which is a joint probability density function between r and phi. If we want the marginal probability density function of r, you have to integrate the joint density function between r and phi with respect to phi; r takes values from 0 to infinity, phi takes values from 0 to 2 pi. Similarly, you want marginal density function of phase, this is 0 to infinity P R phi r , phi dr and phi varies from 0 to 2 pi; so, the problem is in, in some sense, the, at this level is solved.

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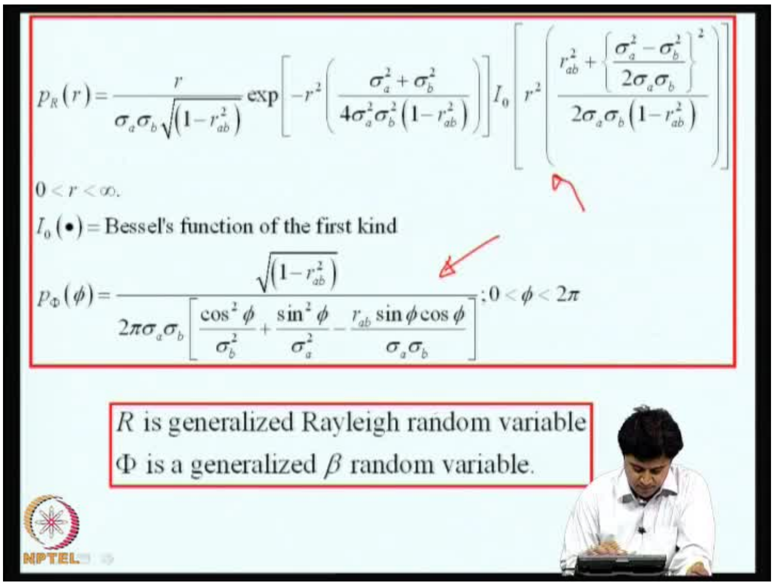

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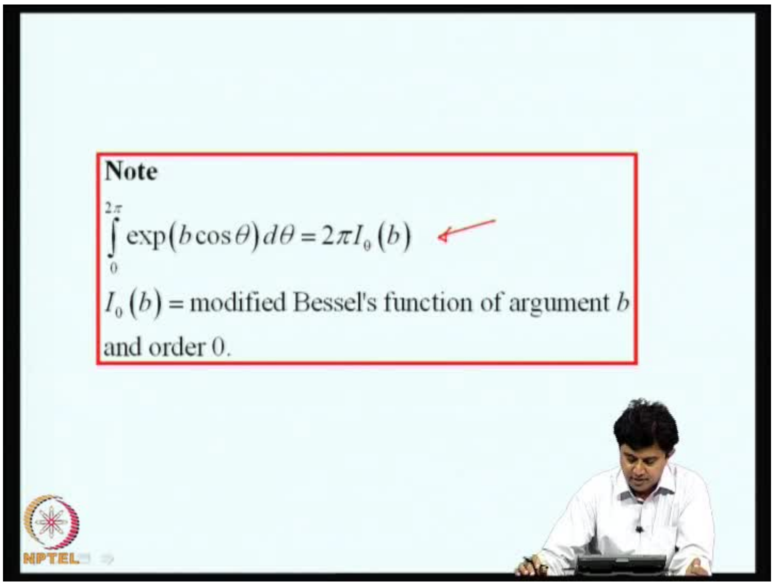

Indeed for the this particular joint probability density function, we can evaluate these integrals and we can show that, the envelope process, the first order probability density function has this form and the phase process has this form; it is not uniformly distributed between 0 and to 2 pi, in that sense, is not there it is always characterless, it has some properties. Now, this I naught is a Bessel's function of the first kind and this distribution we called it as a generalize Rayleigh distribution; so, r is a generalize Rayleigh random variable and phi is a generalize beta random variable; this is a beta distribution. This I naught incidentally is the definition of I naught is displayed here, it is integral 0 to 2 pi exponential of b cos theta d theta is 2 pi I naught of b, where I naught of b is a modified Bessel's function of argument b and order 0. This is a tabulated function, so you can obtain the value of I naught of b with reasonable effort.

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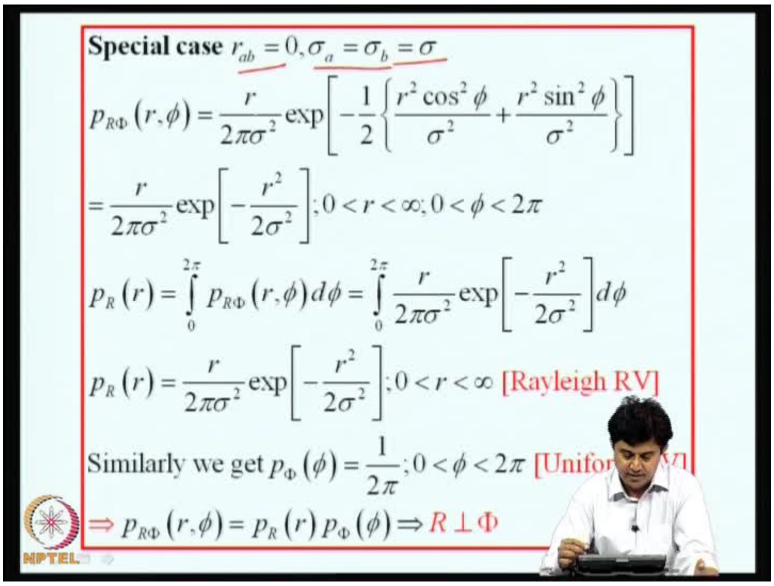

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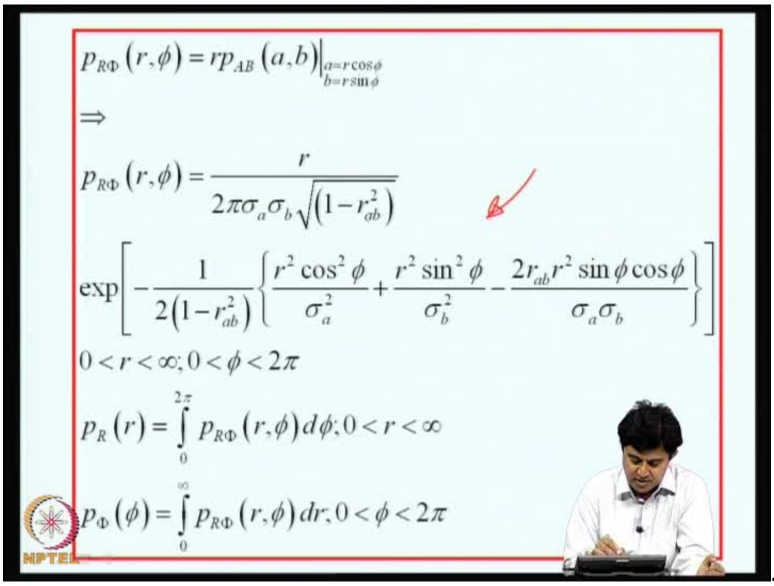
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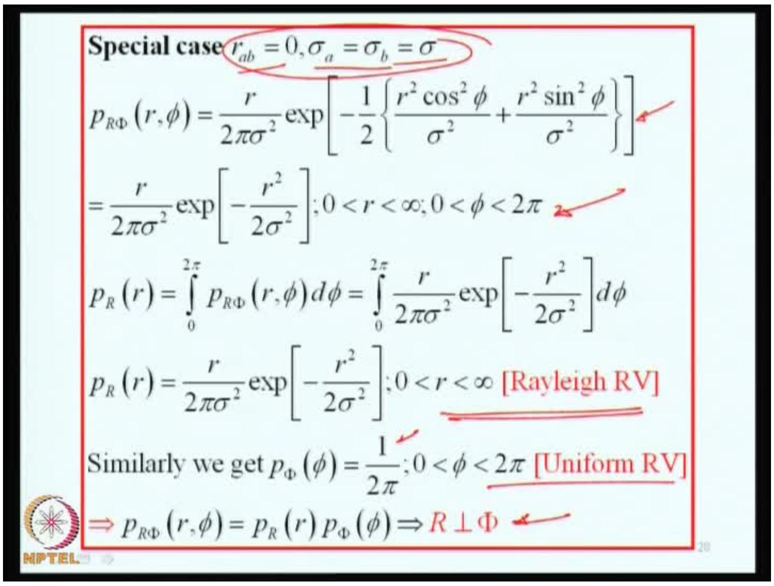

We can now consider a special case, where we can assume that a and b are uncorrelated and they are identically distributed with same stand deviation. In this case, the joint density function is given by this, because here moment r a b becomes 0, some of these terms drop of and it is possible to simplify that and we get this expression. Now, cos square plus cos square phi plus sine square phi is 1, therefore we really get an expression which is lot simpler than the case when r a b is not 0. And if we now find the marginally probability distribution function of r, we get Rayleigh random variable and if we find the marginal distribution of the phase angel, we find that the phase angle is uniformly distributed and also, we can show that, the envelope and phase are statistically independent.

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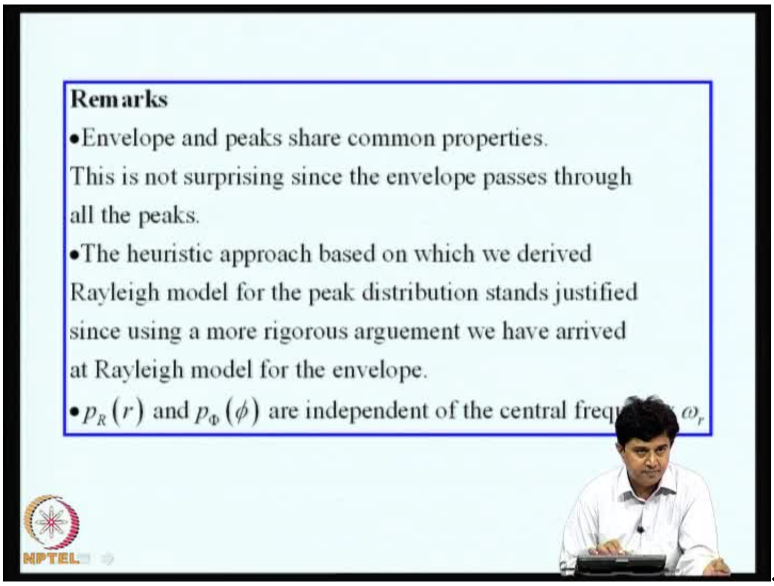

So, mind you this is for this special case, where a and b are uncorrelated and sigma a is equal to sigma; the more general result has been obtained previously. As I said at the beginning of the lecture, envelope and peaks share common properties, because envelope passes through all the maximum values of X of t and therefore, it is not surprising, that for the envelope, we obtained a Rayleigh probability distribution function, because the same result was obtained earlier by studying peaks and by using heuristic argument, a valid for narrow band random processes. The heuristic approach based on which we derived the Rayleigh model for the peak distribution, thus in a way stand justified since using a more rigorous argument; we have arrived at Rayleigh model for the envelope, in a way, that the ad hoc assumption that we made seen to be justified.

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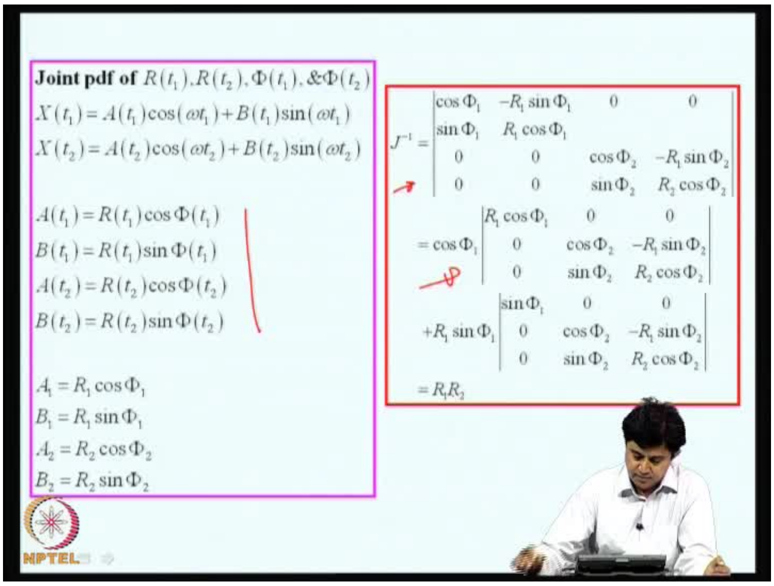

Another important thing that we should notice is the first order probability distribution of properties of amplitude and phases are independent of the choice of central frequency. We can ask slightly more involved questions, for example, can we find the joint probability distribution function of R of t 1, R of t 2, phi of t 1 and phi of t 2, that means, R of t is a random process, can be determined the second order probability, characteristic second order moments or second order probability density function; so, to do that, we consider X of t, at t 1 and t 2, so I get A of t 1 cos omega t 1 plus B of t 1 sin omega t 1 and so on and so forth. So, the quantity A of t 1, A of t 2, B of t 1 and B of t 2 are related to R of t 1, R of t 2, phi of t 1 and phi of t 2, through these four equations. So, we will consider this as A 1, B 1, A 2, B 2 and R 1 phi 1, R 2 phi 2 and we rewrite this, in this from and the here, again this is the problem of transformation of random variables, we are considering this transformations at two time instance, there are four random variables transform to produce four more random variables; so, we can apply the rules of transformation, we need to evaluate the Jacobean, which is determinant of a 4 by 4 matrix and in this case, it terms out, that the some of the intermediate steps are displayed here, 1 by j turns out to be R 1 into R 2.

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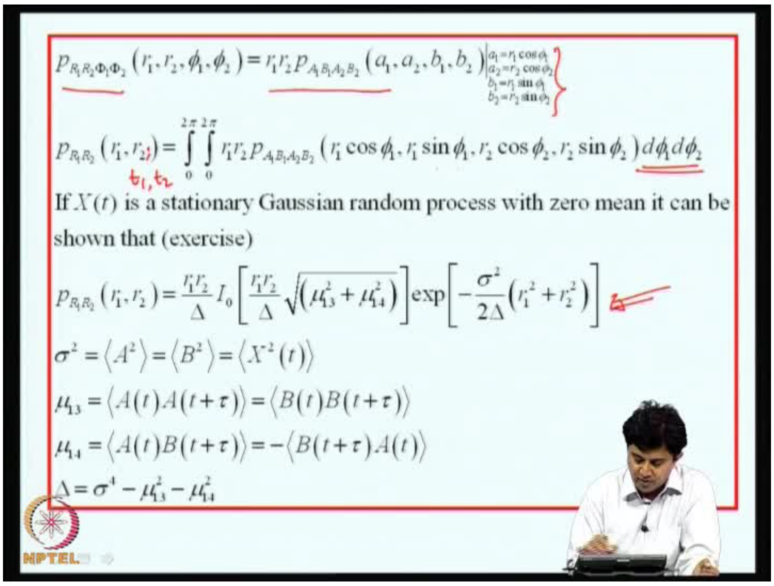

Therefore, now, the joint density function between r 1, r 2, phi 1 phi 2 is obtained in terms of joint density of a 1, b 1, a 2, b 2 using this identity r 1 r 2 is 1 by j and we for a 1, a 2, b 1, b 2 we have to use these transformations; so, from this, if I want now the joint density second order probability density function of the envelope process. This can be obtained by finding the marginal density of this joint density, we need to integrate with respect to phi 1 and phi 2 over 2 to 2 pi; actually, this strictly speaking, this has to be written as t 1, t 2, because r of t is a random process and we are considering two time instants. Now, if x of t is a Gaussian random process with zero mean and a and b become Gaussian; in fact, this integration can be done and one can show, that the second order probability density function is indeed given by this joint density function. These details can be worked out, that I leave it as a matter of an exercise for you to verify. So, any case, the basic result, is that, we are able to find out first order and second order probability density functions of envelope and phase process, although I have displayed here the result for the amplitude process, you can also get the second order probability density function of phase process also by a similar exercise, where I integrate from 0 to infinity, this quantity with respect to r 1 and r 2.

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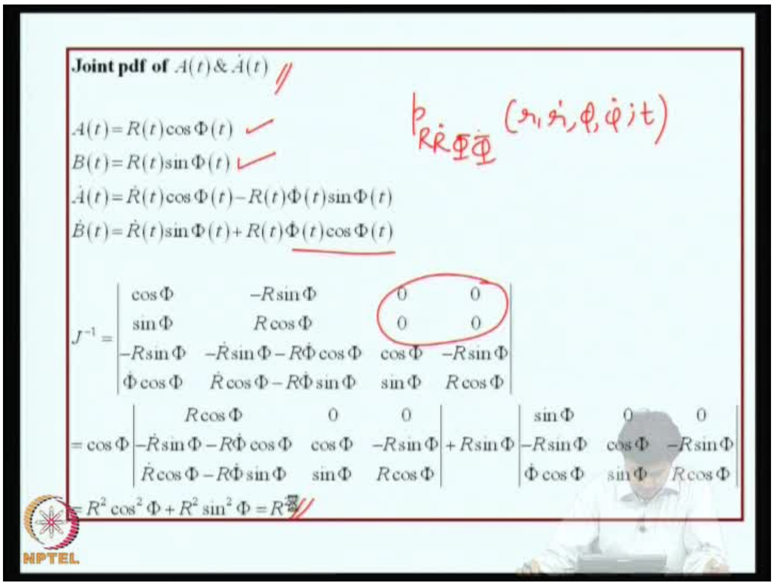

Now, in our studies on level crossing and peak etcetera, we found that, if X of t is a random process, the number of times the level alpha is crossed depends, if you want characterize, that we need to get the joint probability density function between the process and its derivative; so, that leads us to the question, if I am now interested in finding, for example, the number of times the envelope process crosses a level alpha, then I will need the joint probability density function between the envelope and its derivative at the same time instant; so, it is a fairly complicated question, mind you A of t is a non-Gaussian random process. Now, however the nature of transformations involved are not very complicated, therefore a solution to this could be obtained and that is what I will briefly out line. So, I have A of t is R cos phi of t, B of t is R of t sin phi of t, from this, if I now evaluate A dot of t it will be R dot cos phi minus R phi dot sine phi of t. And similarly, B dot will be R dot sin phi of t plus R of t phi dot cos phi of t, that mean, I am differentiating sin phi of t and I get these terms.

So, there are now four random variables which are transformed through a set four non-linear equations leading to four new random variables and we can now do the address, the problem of finding the joint probability density function of (Refer Slide Time: 35:00) p R R dot phi phi dot, all of that evaluated at the same time instant t, this is doable, you have to now evaluate the Jacobean, which is 1 by j, in this case, which is determinant of a 4 by 4 matrix and we can go through this calculation; initially, it may look quit complicated, but you should notice that, there are several zeros in this and the expansion of determinant is lot more simple, then what it appears at the first side and you can show that, 1 by j is indeed r square. So, I have shown some intermediate steps to assist you in verifying this.

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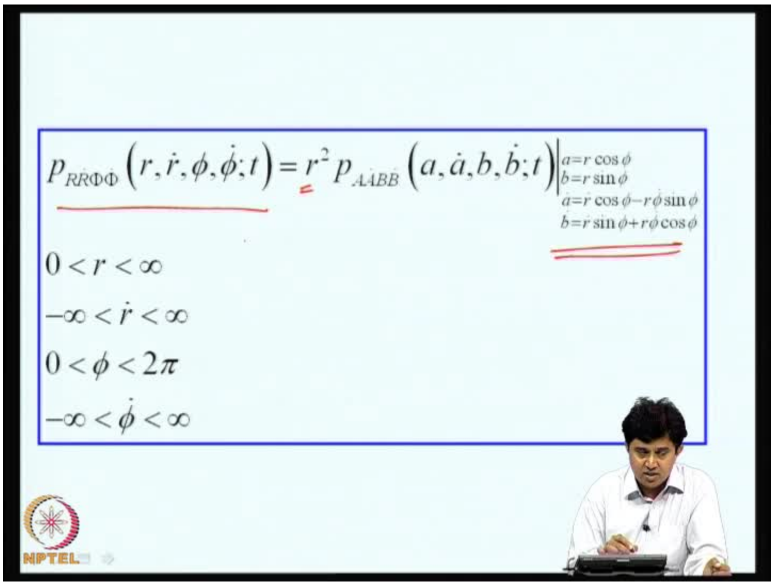

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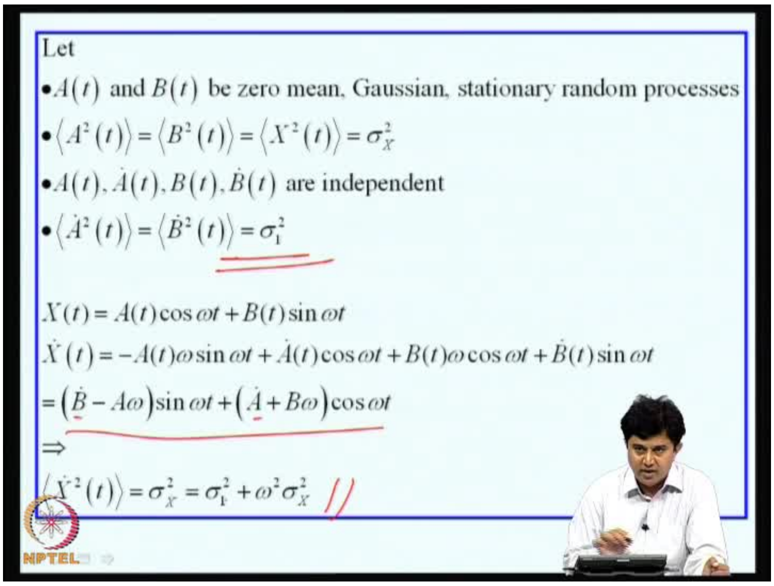

So, we have this now the formal solution, joint density of r r dot phi phi dot evaluated at t is given, in terms of r square which is 1 by j p of A A dot B B dot, where a b a dot b dot are related through these relations. So, in principle I have obtained the four-dimensional joint probability density function between r r dot phi phi dot; now, if I want only r joint density function of r r dot, I have to carry out a twofold integration with respect phi and phi dot these are tedious, but do especially if certain simplifications are made on properties of A of t and B of t. Now, if A of t and B of t are zero mean Gaussian stationary random processes, such that A square expected value of A square expected value of B square are equal and that we have seen a while before, that there indeed equal for A X random process X of t and if we further assume that, A of t, A dot of t, B of t and B dot, B dot of t are independent and we impose a condition expected value of A dot square of t and B dot square of t is sigma 1 square, which is not the variance of the velocity derivative process, it is something different.

We can now consider, for example, if we take now X of t A of t cos omega t plus B of t sin omega t, X dot of t, I can write in this form and we can actually evaluate X dot square of t, which I need here as sigma X square is equal to sigma m square plus omega square sigma X square, where sigma 1 square is related to, you can show this, this is sigma 1 square, where in evaluating this, you will see that A dot and B dot are sitting here and that is why we get sigma 1 square here.

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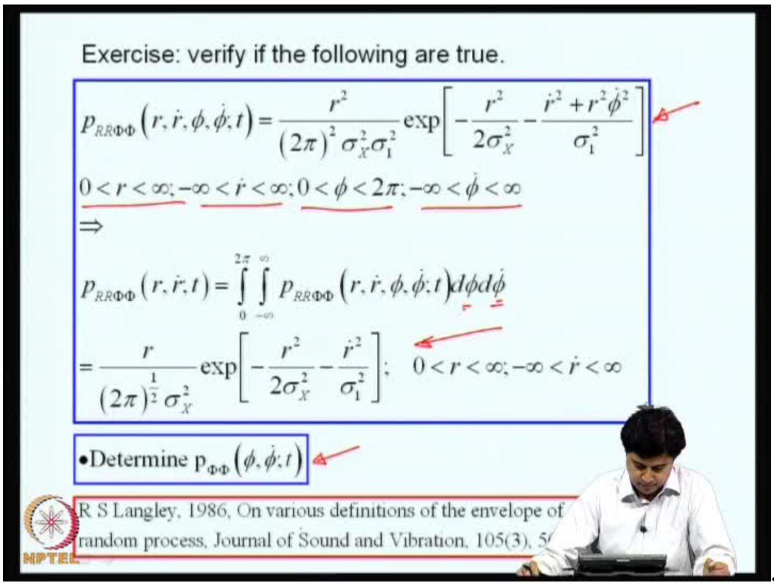

Now, I leave it as an exercise fairly, Langley exercise, for you to verify that this fourth order join density function is given by this; you have to verify whether these statements are true. The range of r is 0 to infinity r dot is minus infinity to plus infinity phi is 0 to 2 pi phi dot is minus infinity to plus infinity. You can, after determining this, you could find the marginal density of r r dot by integrating from 0 to 2 pi for phi, minus infinity to plus infinity for phi dot and if you do this, you get this non-Gaussian two-dimensional probability density function; you could also determine the second order probability density function of phi phi dot evaluated at same time t.

There is one research paper by Langley in 1996 which appeared in journal of sound and vibration, where some of these issues are discussed in greater detail; so, if you would like to solve this address, this exercise, attempt this exercise, I would encourage you to go through this paper.

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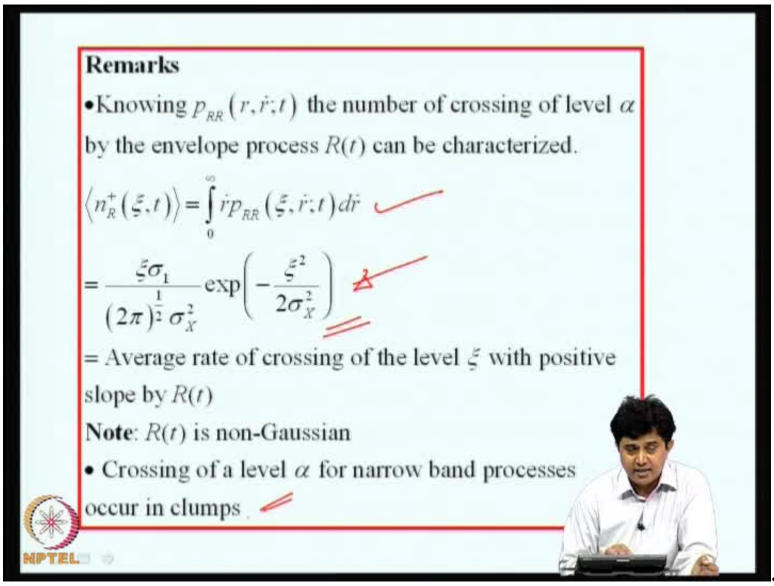

Now, as I said at the beginning, if we know the joint density function between process and its time derivative and the same time instant, we can characterize the number of crossing of level alpha by the envelope process R of t, I mean, we can characterize, in fact, the average rate of crossing of level x i by the random process R of t, where the crossings are taken to be with positive slopes is given by this expression and we could use the result, that we derived just now and show that, the this rate is indeed given by this expression; please notice that, R of t is non-Gaussian, so this expression is not similar to what we got for a Gaussian random process slightly different.

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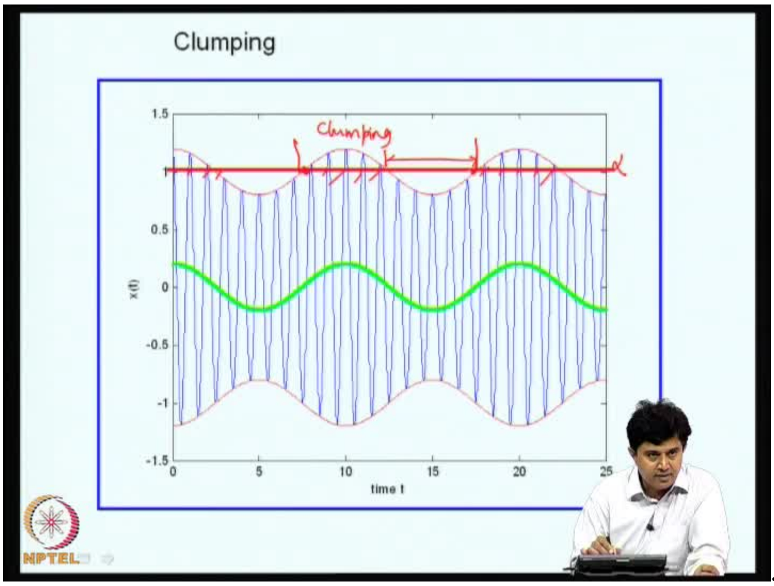

Now, one more thing that we should notice is when we are talking about crossing of level alpha by a random process X of t, if a process is narrow banded, the crossings tend to occurring clumps, what does it mean. Now, you consider this blue line, which is you can say that is a sample of narrow band process and suppose, you have interested in crossing of this level alpha, now you follow the crossing of this red line by the blue curve, here there is one crossing and you see that moment one crossing occurs, there are four crossings, after that, there is a line, you weight here for this time and again there are crossings; similarly, here crossing is in a clump, so this is known as clumping. Whereas, you look at the envelope here, this crossing, the next crossing with positive slope occurs here, that means, the time between two crossings is well separated for the envelope process than for a narrow band process.

Now, what is the signification of this result; when we were modeling the number of times the level alpha is crossed by the propose X of t, we proposed the use of a Poisson random variable, that counting process be modeled as a Poisson random process was our preposition. In Poisson model, we, the events are taken to be independent. Now, the assumption of independents is more likely to be valid, for an envelope than for the parent process, because for a parent process, moment this is cross, that there, it going to be several crossings, that would mean, this crossing and this crossing are unlikely to be stochastically independent, there is a element of dependence, because they occur in clumps; whereas, here for the envelope, there is no such restriction, because a time spent between two successive crossings is longer and therefore, the assumption of independence is likely to be more acceptable here.

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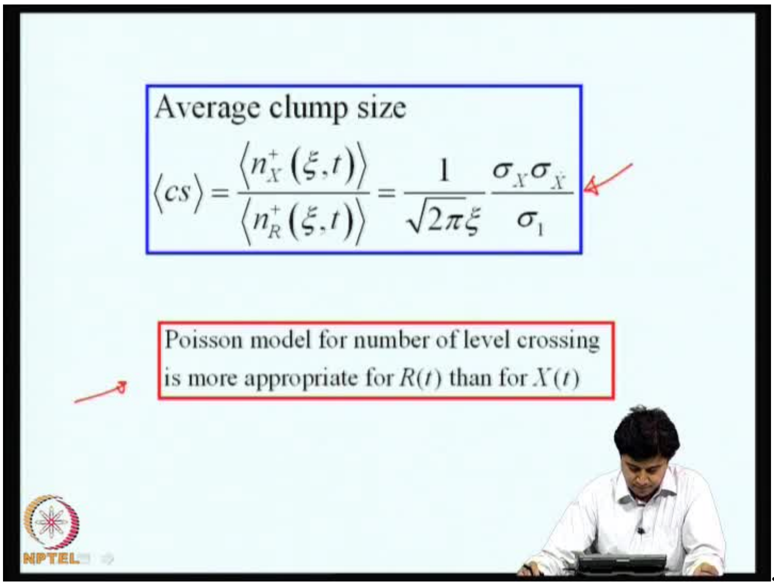

Later on we will see that their implications of these features; to characterize is clumping effect, we define what is known as clump size and that is the average clump size is defined as the ratio of rate of crossing of level x i with positive slopes by the parent process to the rate of crossing of the same level, by the analog process with positive slopes and for the process, that we have been studying Gaussian random process, the expression for this is obtained as shown here. So, as I said, the Poisson model for number of level crossings is more appropriate for R of t than for X of t.

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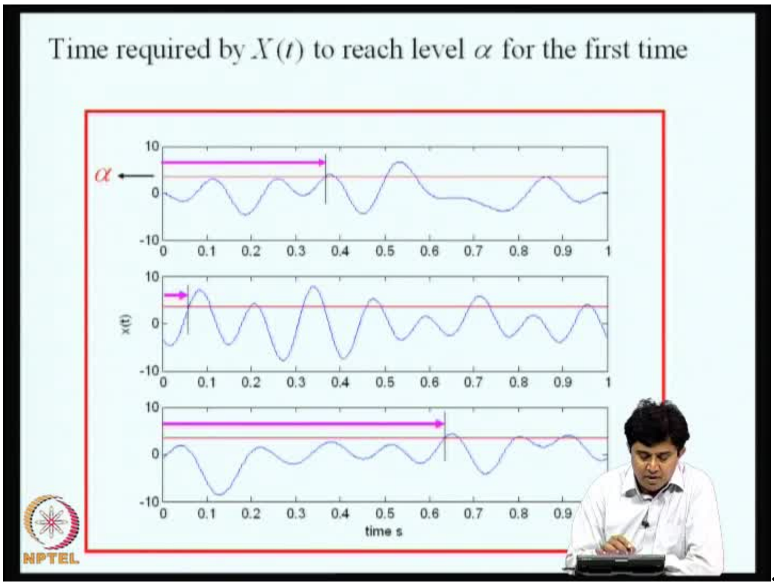

Now, we now move to the description of random process by at another criteria, we consider now the time required by X of t to reach a level alpha for the first time. So, quickly let us recall, suppose this is X of t, the blue line is X of t, this is X of t and the red line is the level alpha. For this trajectory, the time required for X of t to cross alpha for the first time, we shown by this pink line; for the next realizations, this crossing occurs fairly early and the time required for first occurrence of crossing is much less, whereas here it takes quite a long time or in another words, for every sample realizations, if you observe the time required for first crossing of level alpha, you will see that those observations can be interpreted as outcome of a random experiment and therefore, that itself is a random variable.

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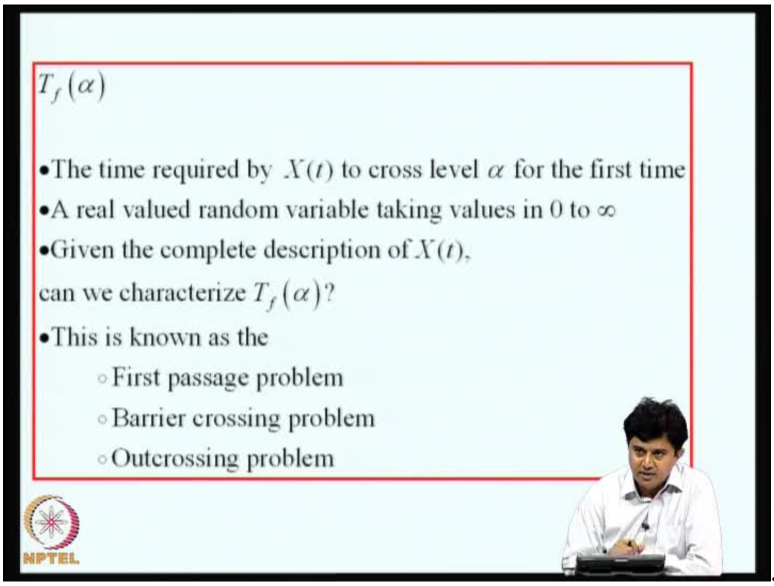

So, we probably is, therefore, we introduce that as T f of alpha, that is time required for crossing of level alpha for the first time, is a real valued random variable taking values in 0 to infinity. Now, the question is, given the complete description of X of t can we characterize this random variable, can we obtain its probability distribution function or its movement or what we can do about it; this problem is known as the problem of first passage problem, barrier crossing problem or out crossing problem, they are all synonyms and it is a very important problem, in the study of reliability of dynamical systems, because this time for first passage can be interpreted as a life time of the system. So, the level alpha could be the crossing of some prescribed stress metric at a given point and if that stress test is level crossed, we define that as failure, so how much time the structure takes to cross that level for the first time; so, that tells us what the life time of the structure is.

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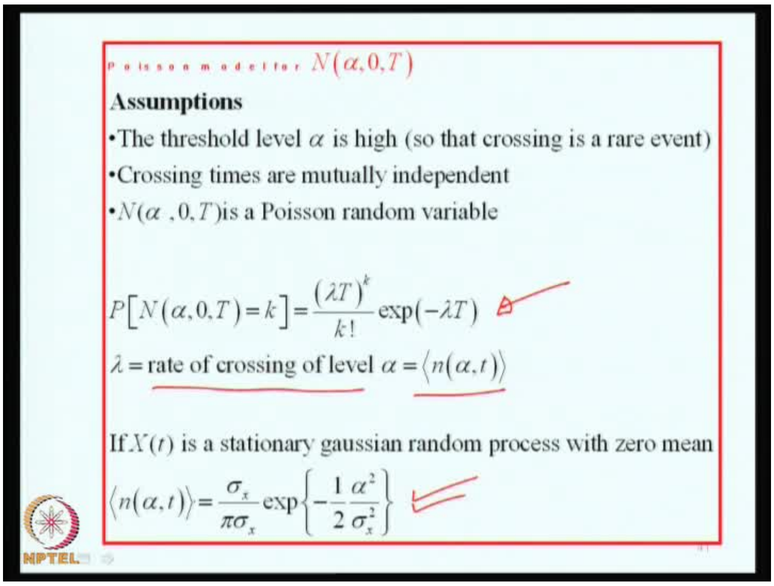

Now, we use now Poisson model for this number of crossing of level alpha by 0 to T, this we have seen earlier. So, X of t is a random process and alpha is a level and we assume that, the threshold level alpha is high, so that the crossing is a rare event and crossing times are mutually independent. And under these assumptions, we show that, we can use the model that N (alpha, 0, T) is a Poisson random variable with this probability distribution function. The parameter lambda here is a rate of crossing of level alpha and for a random process, this we have already determined to be N of expected value of N (alpha, T) and we have derived this for Gaussian random process and also for the envelop process. If X of t is a stationary Gaussian random processes with zero mean, we have shown that, this is the expression for this rate parameter lambda.

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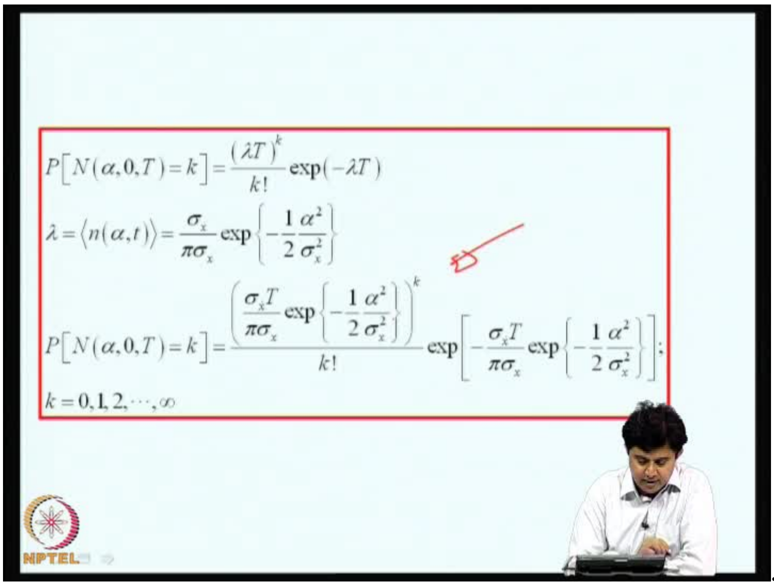

So, the probability distribution function for the number of times the level alpha is crossed, that is probability that N equal to k, is given by this expression essentially the Poisson module with alpha lambda the parameter lambda given in terms of expected value of N (alpha, T).

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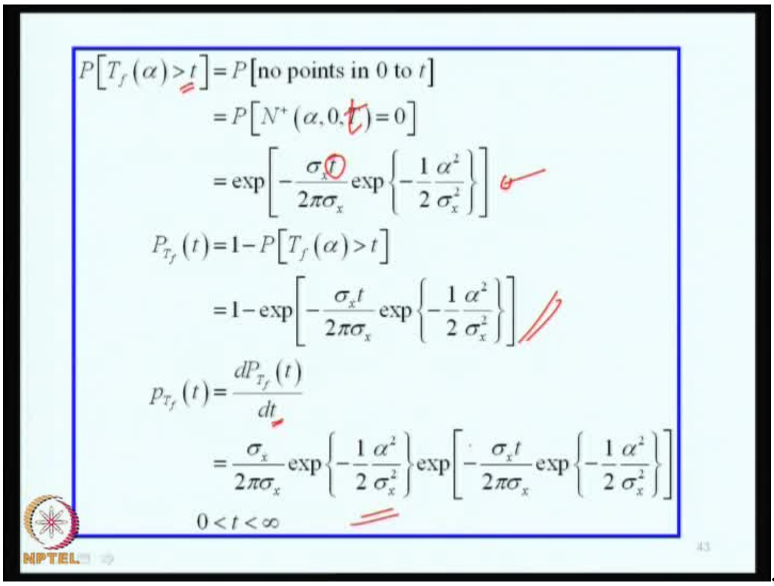

Now, what is its relation to the problem of first passage times; now, if you look at probability of first passage time being greater than or equal to t, this probability is same as the probability, that there are no crossings of level alpha in 0 to t or in other words, probability that crossings with positive slopes of level alpha is actually equal to 0; if the first passage time is greater than t, this should be equal to 0. Now, we have no postulated a model for it and this is what we get, in terms of rate of crossing of level average rate of crossing of level alpha; from this, now we can get the probability distribution function which is 1 minus P of t of f alpha greater than or equal to t; mind you, that this lower case t, which appears here is actually the state variable now; t f is a random variable, t is a state variable and this is a distribution function which is given here. If you want the probability density function you have to differentiate that with respect to t, this lower case t and we get this as the model for the probability density function for the first passage time. So, what are the parameters involved here, the level alpha the variance of the parent process, the variance of the derivative process and of course, the time t with the state.

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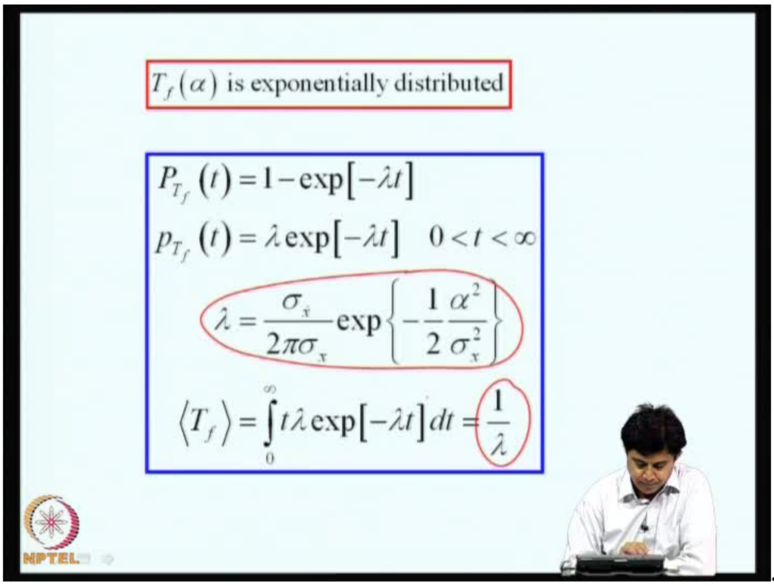

This is actually nothing; this density function actually corresponds to the probability density function of an exponential random variable. So, probability distribution function is 1 minus e exponential minus lambda t and probability density function is lambda into exponential minus lambda t, where lambda is this rate. So, the expression just now I showed, this expression is essentially this written with lambda in these places. So, T f under these hypothesis, that is a using Poisson model for level crossings, we get exponential model for the first passage time; we could, of course evaluate moments of this first passage time its variance and so on and so forth, for example, the expected value of first passage time can be shown to be given by 1 by lambda.

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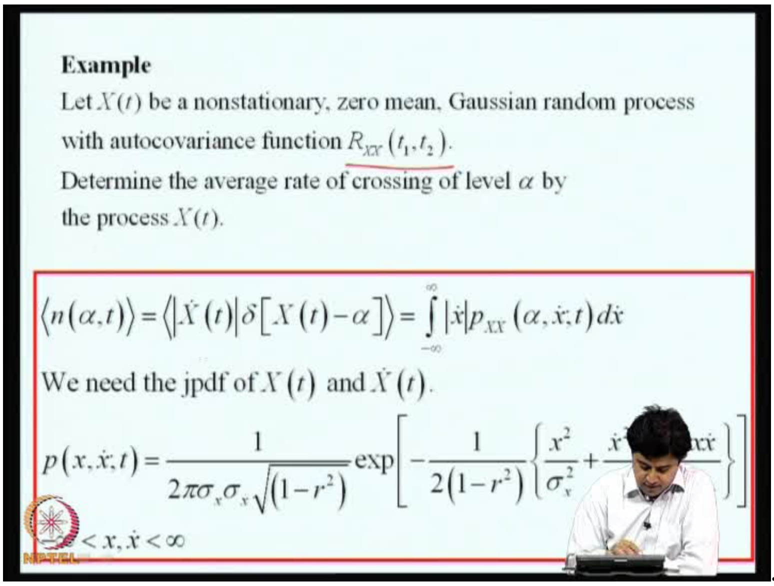

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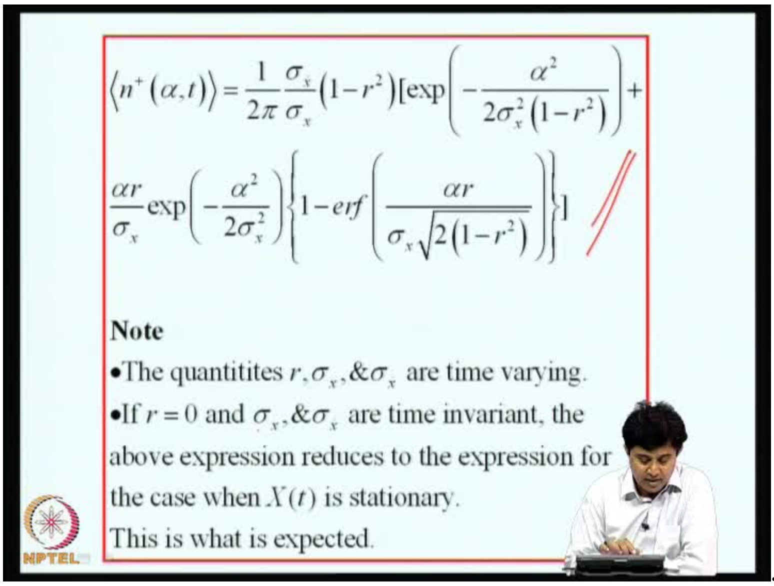

Now, what happens if X of t is a non-stationary zero mean Gaussian random process; so, let us consider that, let X of t be a non-stationary zero mean Gaussian random process with auto covariance R x x (t 1,t 2). Now, we have solved this problem, how to find the average rate of crossing of level alpha by the process X of t, this is the problem that we have considered before and we got this is a expression, that we need to solve and we have shown that, this rate is given by this fairly complicated formal. The parameter sigma x, sigma x dot and r which is the correlation coefficient are all now time varying; so, this rate itself is time varying because process is non-stationary.

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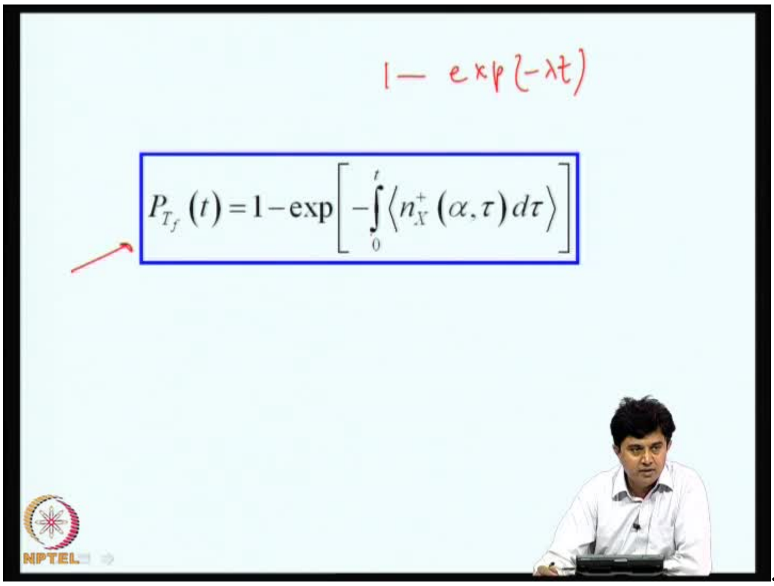

Now, how do you, I, find the first probability distribution function of the first passage time here; we had one minus exponential of minus lambda t, where lambda was a constant, now, we have to replace it by an integral it is one minus exponential 0 to t, this rate into d tau. So, this is fairly involved, because parameters inside, that are, that are present in the expression for a n X plus are function of time which characterize the non-stationary trend of the random process X of t and that need to be evaluated, I mean, that integration has to be performed over time, to evaluate this probability distribution function. So, obviously this is doable, but it is more complicated than the case of a stationary random process.

So, the problem of first passage time therefore can be tackled, if you can correct as level crossing problem. So, when we started talking about level crossing problem, this connection to first passage time was not very obvious, but if you now trace back the argument, that we have used we started with level crossing problem and for high levels of crossings, we use Poisson model and based on Poisson model, now we are able to solve the problem of first passage times.

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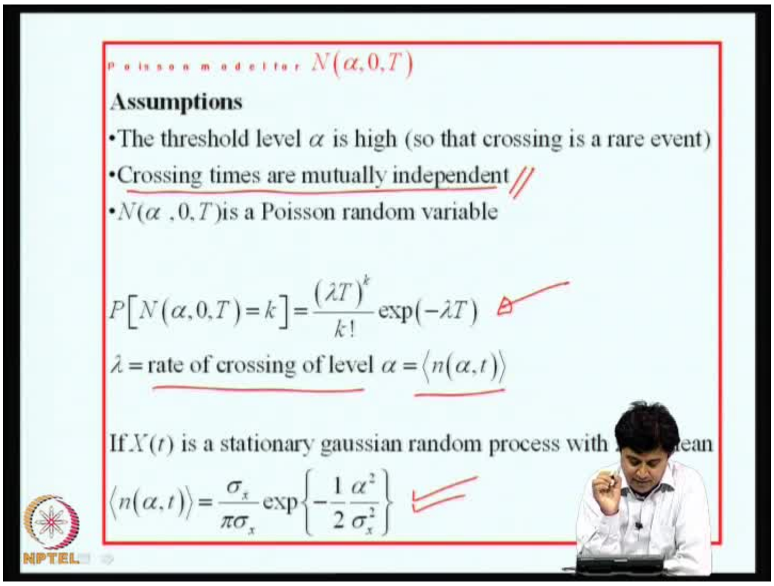

Now, one of the assumption that we made as I mention, this is the important assumptions that is crossing times are mutually independent; now, we noticed, when I discussed about clumping facts in narrow band random processes, the assumptions that crossing times are mutually independent is unlikely to be valid for a narrow band process, because of this occurrence of clumps, but on the other hand, for the same process, the envelope process can be thought of as, I mean, this assumption of crossing times being mutually independent is likely to be more valid for envelope, because the time difference between crossings are longer and a more random for envelope process.

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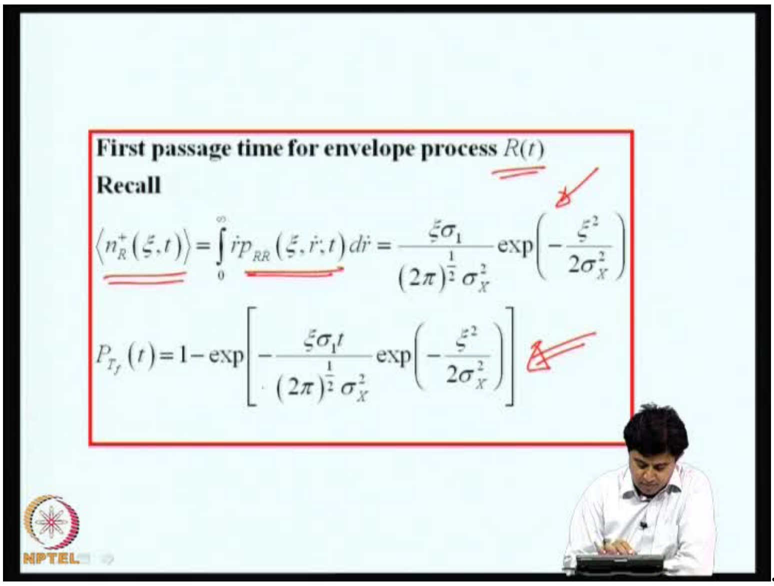

So, based on that, we could look at first passage time for envelope process R of t; so, we have actually derived this rate for a actually for a specific case; in general, since we know the expression for the join density between the envelope and its time derivative, in principle, this rate can be evaluated, but for under certain simplified assumption, we have shown that, this rate is given by this; therefore, the first passage time of this non-Gaussian random process right, this a fairly complicated question, has been tackled and we get this model for first passage times. This is much likely to be, much more realistic than the previous model that we obtained here.

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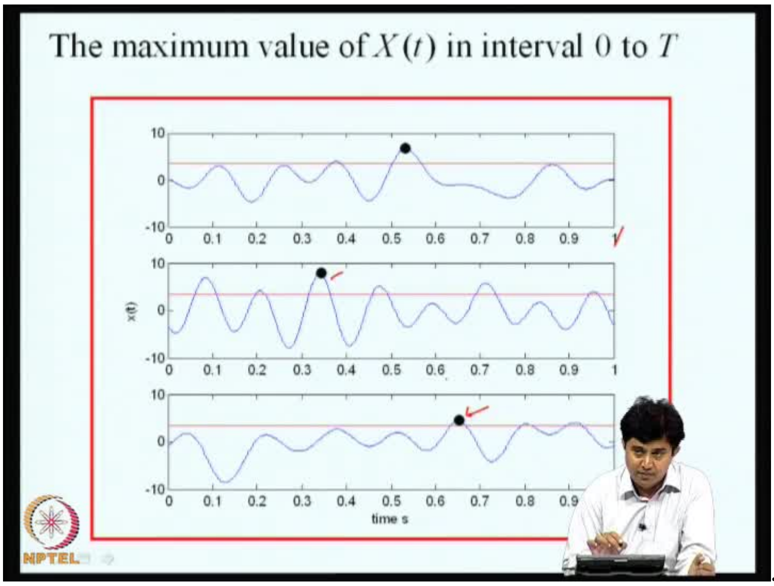

Now, in the next lecture, I will be considering the most important descriptor, namely the maximum value of a random process in a given time interval 0 to T, this is what primarily we are interested in engineers; suppose, we are considering time duration of say 0 to 1 second and we are interested in the highest value of X of t, this black dot shows, that number for this realization; this is for the second realization; this is for a third realization. Clearly, for different realization of X of t, this highest value can be thought of as an outcome of a random experiment; therefore, it is a random variable itself and it is a continuous random variable. So, the problem is, given the complete description of X of t, what is the probability distribution function of this extreme value.

I will show in the next lecture, that the solution to this problem is again intimately connected to the problem of level crossings Poisson model for level crossings for high levels and the solution of first passage problem; all these are inter linked and this is what we will consider in the next lecture and we will conclude this lecture at this stage.