Stochastic Structural Dynamics

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Lecture No. # 18

Failure of randomly vibrating systems-2

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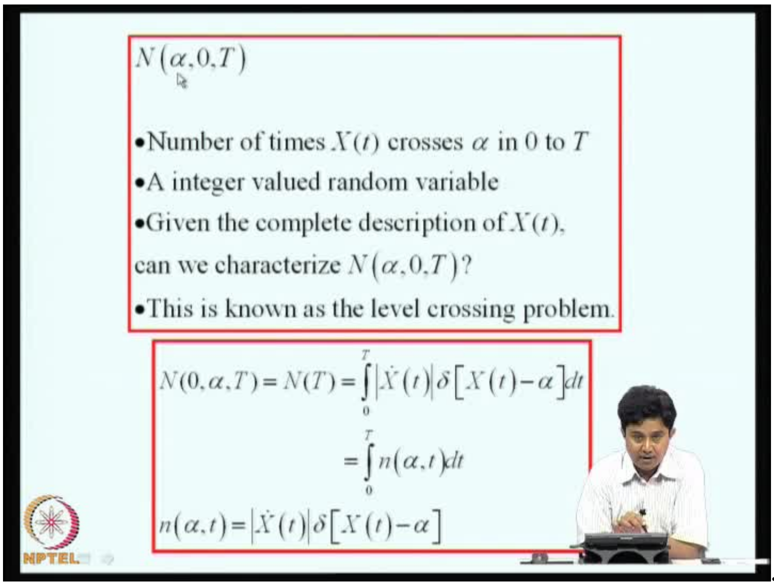

So, we are studying certain properties of random processes, which would help us to characterize failure of randomly vibrating systems. So, in this lecture, we will be studying how to characterize peaks and extremes of random processes. We quickly recall what is, what the things that we did in the last lecture, we characterize the number of times, the level alpha is crossed in the time interval 0 to T.

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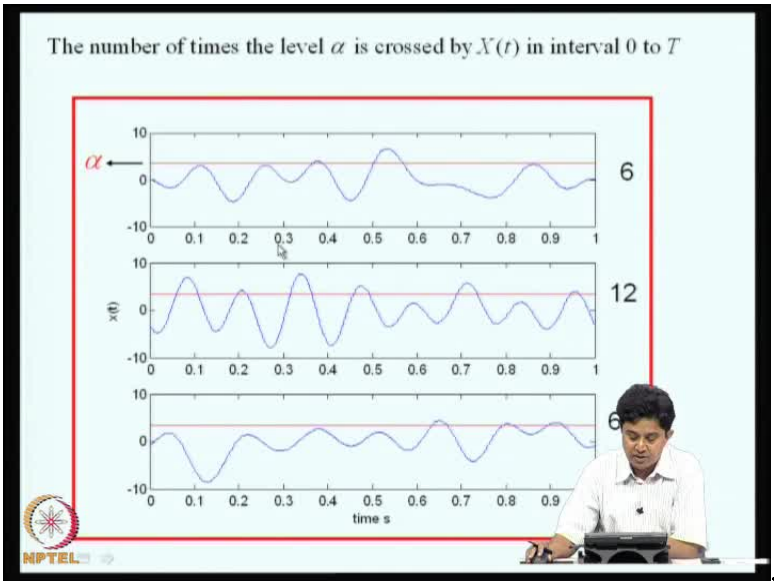

That is, if you consider a few realizations of a random process and if this is the level, then and we focus on an intervals is 0 to 1 second and we ask the question, how many times this level alpha is crossed; you could see that, for this realization the sample crosses level alpha 6 times, whereas here it is 12 times and here it is again 6 times, that would mean, this number is a random variable and where knowing the properties of this parent process X of t, we are interested in characterize in probabilistic obtaining a probabilistic description of these numbers.

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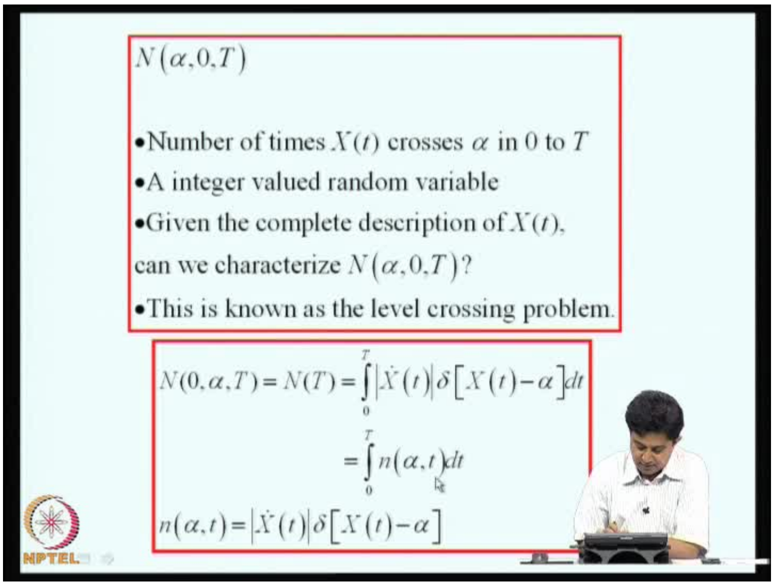

We set up a counter for counting the number of times, the level alpha is crossed and this counter turned out to be a highly non-linear transformation on the parent process and its derivative and the integrant turned out to be the average rate of crossing of level alpha.

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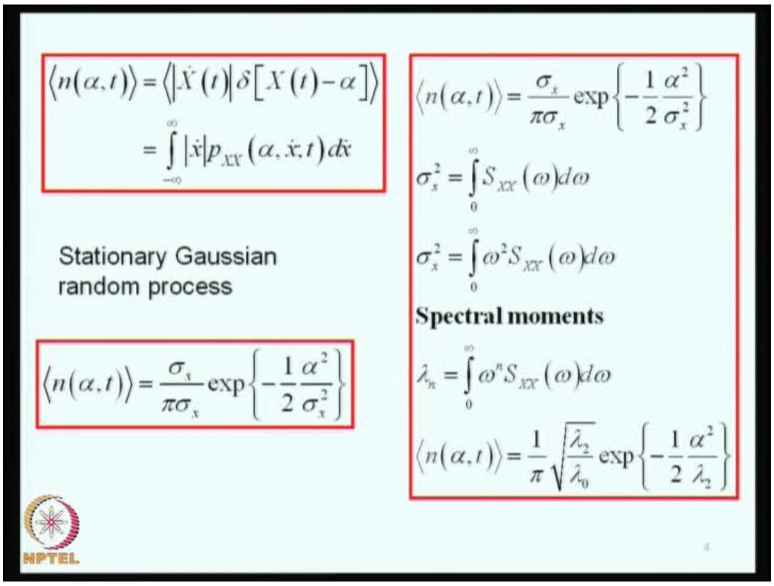

For a Gaussian random process, we showed that the average rate of crossing of level alpha is given in terms of the variance of the parent process and the variance of the derivative process and this was the expression that we got. And the variance of a process and its derivative are related to the moments of the power spectral density function, for example, area under the power spectral density function which is the zero th moment is the variance of the process and omega square into S X X of omega d omega integrated from zero to infinity, which is the seconds spectral moment, is the variance of the derivative process. So, we could express the rate at which the level alpha is crossed, in terms of the power spectral density or more specifically the moments of the power spectral density function.

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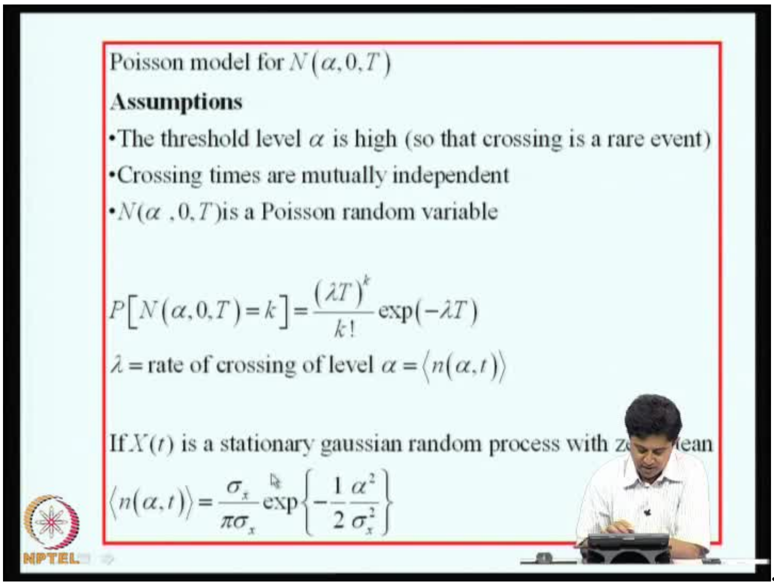

So, we will continue this discussion and we try to, now see if we can make a model for probability distribution function for the number of times, the level alpha is crossed in a given interval. I already pointed out that, we can find the mean of this capital N, with some effort we could find the variance, but finding probability distribution by applying rules of transformation of random variables is not a trivial task, but under sudden heuristic assumptions, for example, if you assume that the threshold level alpha is high, so that, crossing is a rare event and if we also assume that crossing times are mutually independent, then a model for number of times, the level alpha is crossed in intervals can be proposed based on a Poisson model for this random variable. So, Poisson random variable, we know the probability distribution function of the probability mass function is given in terms of a rate; so, this lambda now is actually the rate at which this points arrive and that is nothing but the rate of crossings of level alpha; therefore, we already determined this rate of crossing of level alpha, in hence, we can postulate a Poisson model, the only parameter here is lambda and lambda in terms, in fact would be the average of n (alpha, T), that is for a Gaussian random process stationary Gaussian random process with 0 mean; we have shown that, it is related to variance of the process and variance of the derivative of the process.

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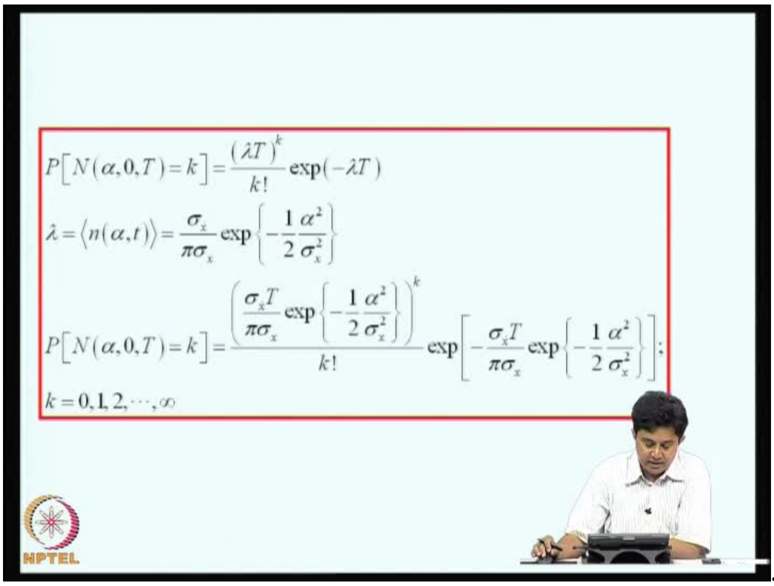

So, consequently we can actually write down the probability distribution function for number of times the level alpha is crossed, that is displayed here. So, this is the Poisson model, so please bear in mind, there are two, there are the assumptions that we are making is the level alpha is sufficiently high, so that crossings are rare events and crossing a different time instance are independent; so, then this model is likely to be acceptable.

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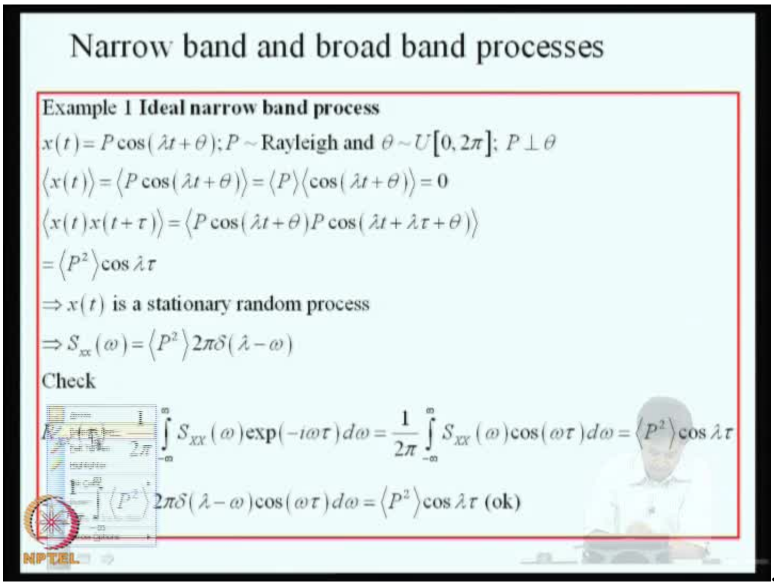

We will continue with our discussion, we will need to now distinguish between what are known as narrow band and broad band processes. So, this band represents a width of a spectrum frequency, so their description here is essentially in frequency domain; so, we are talking essentially about stationary random processes. To clarify what is meant by narrow band process and what is meant by a broad band process, we consider a few examples. Let us begin with a function x of t is P cos lambda t plus theta; let P be a are Rayleigh random variable and theta be uniformly distributed is 0 to 2 pi and P and theta are stochastically independent. Mean of x of t is expected value of P cos lambda t plus theta, since P and theta are independent, I can express them as product of expectation of P and product of cos lambda t plus theta.

The average value of cos lambda t plus theta over a 0 to 2 pi is 0; therefore, expected value of x of t is 0. Now, if you consider the auto covariance function, that is expected value of x of t and x of t plus tau, we can write it as expected value of P cos lambda t plus theta plus into P cos lambda t plus lambda tau plus theta. Now, there are two random variables P and theta are independent; therefore, this can be written as expected value of P square into expected value of cos lambda t plus theta cos lambda t plus lambda tau plus theta and you can show that, that reduces to the term cos lambda tau. So, that would mean, this random process x of t is a stationary random process with 0 mean, because auto covariance of the function of time difference. So, the power spectral density of this is actually the Fourier transform and a power spectral density, for example, if you write it as expected value of P square into 2 pi direct delta of lambda minus omega, you can easily check that the Fourier transform of this is nothing but expected value of P square cos lambda tau, which is the auto covariance that we are looking for, that would mean, this is the relationship between auto covariance and power spectral density; for power spectral density, now I am writing this P square 2 pi delta lambda minus here 1 by 2 pi expected value of P square 2 pi delta lambda minus omega cos omega tau d omega which is this, which is.

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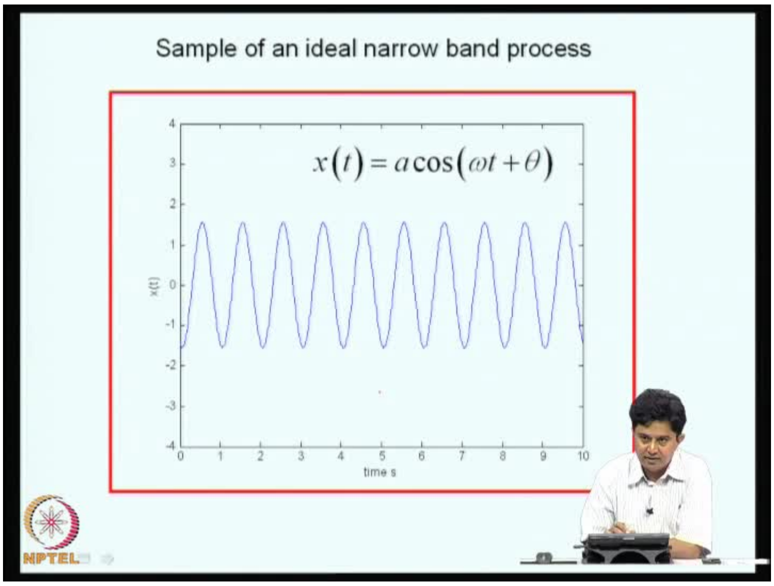

How does the sample of this process looks like; so, I will show a graph here, a sample of x of t equal to a cos omega t plus theta, will be a simply a harmonic function; if a and theta random variables, the amplitude and phase of the next realization would be different; so, distinct realizations will have different amplitudes and phase, but each realization will be a harmonic function.

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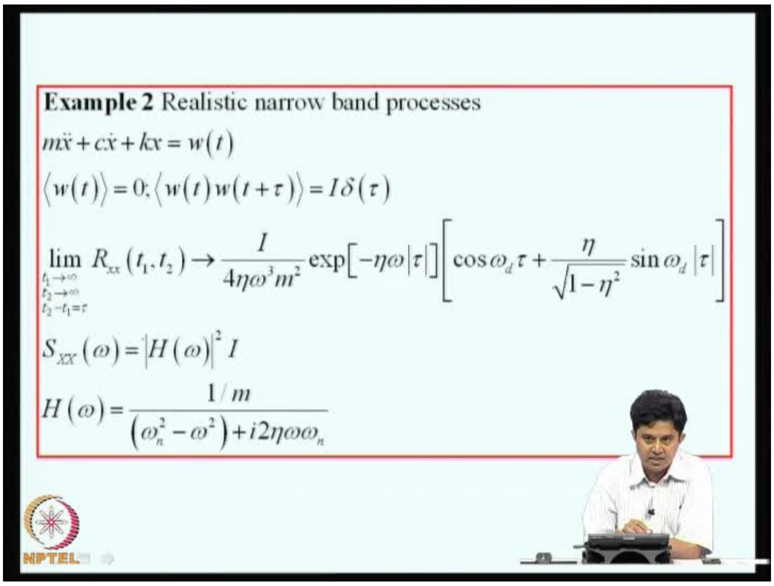

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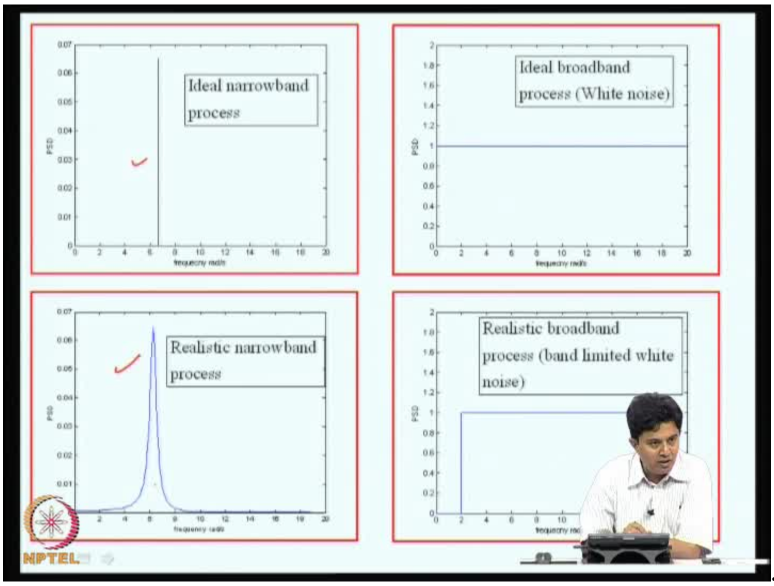

So, if you look at its power spectral density function, it is a direct delta function as shown here. Now, what I will do is, power spectral density is an idealization of what realistically would be a, a slightly different function, so if we now consider a realistic narrow band process, for example, if I pass a white noise through a single degree freedom system, we have already studied this problem and we have shown that, the power spectral density here is H of omega whole square into I. Now, if I plot this power spectral density as the damping parameter eta becomes smaller and smaller; this function tends towards a direct delta function, but for a finite value non-zero value of eta, the power spectral density function would look like this, that means, this is an ideal narrow band process, whereas this is realistic narrow band process.

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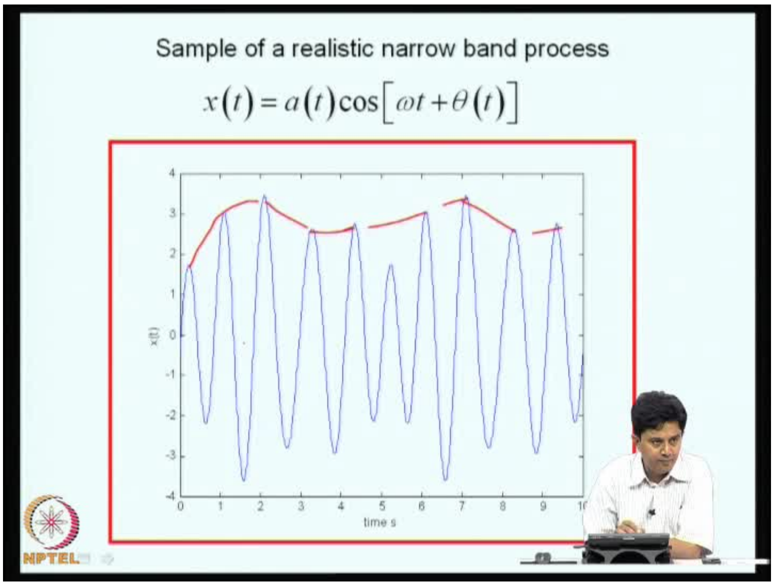

How does sample of a realistic narrow band process looks like? It will look like this; this amplitude will be now a kind of slowly varying function; this, a of t is now a function of time and theta of t also would be slowly varying in time, the sample will be a kind of a modulated harmonic function.

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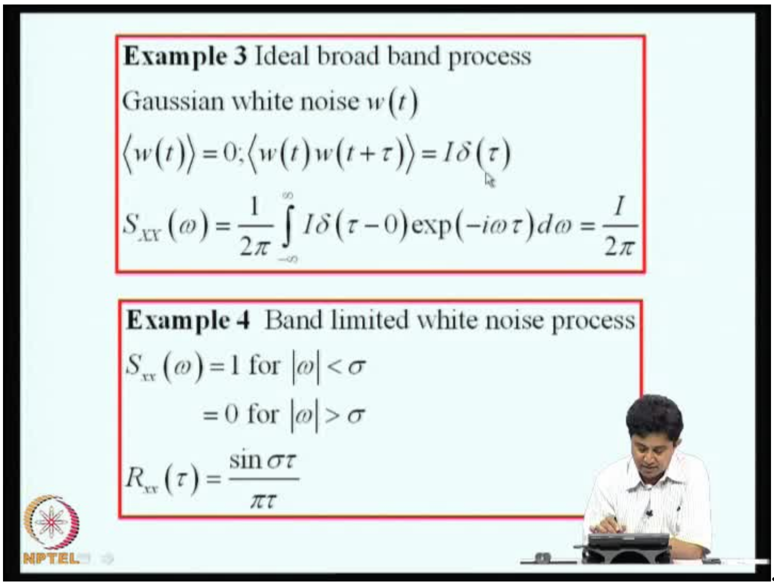

What is an ideal broad band process? An ideal broad band process is a, for example, is a Gaussian white noise, Gaussian white noise can be viewed as a ideal broad band process;

so, the mean of w of t is 0 expected value of w of t into w of t plus tau is a direct delta function; it is an time, it is the direct delta function but in frequency it is a constant.

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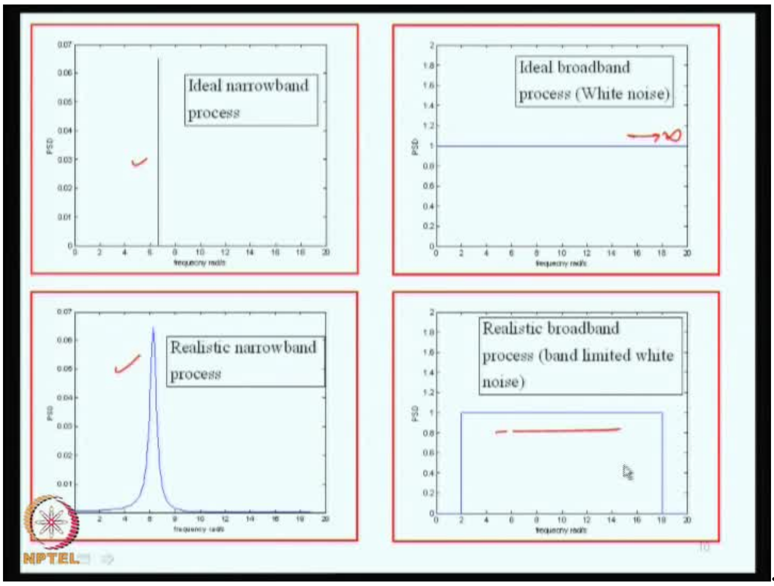

So, if we now look at an ideal broad band process, the power spectral density function is a constant; so, this of course continue, this goes up to infinity, it is constant, for all frequencies that is why it is called white noise, but a realistic broad band process will have a kind of a band limited, it will be a band limited white noise. The variance of this process is unbounded, whereas in reality this kind of power spectral density functions can be idealized as white noise, as far as our interest is focused between, say for example, in this problem between 2 to 18 radiant per second, if our interest is focused only in that frequency band; this power spectral density, this random process can be viewed as a white noise, but in the reality, of course, it is a band limited white noise.

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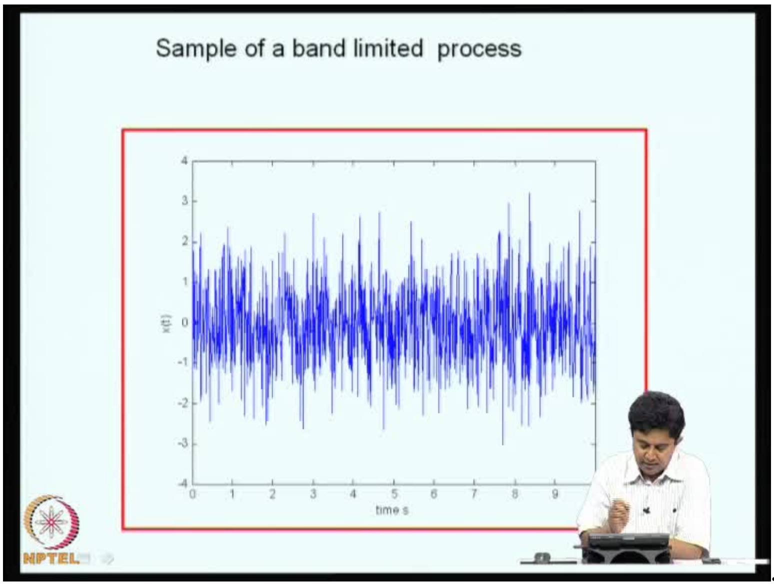

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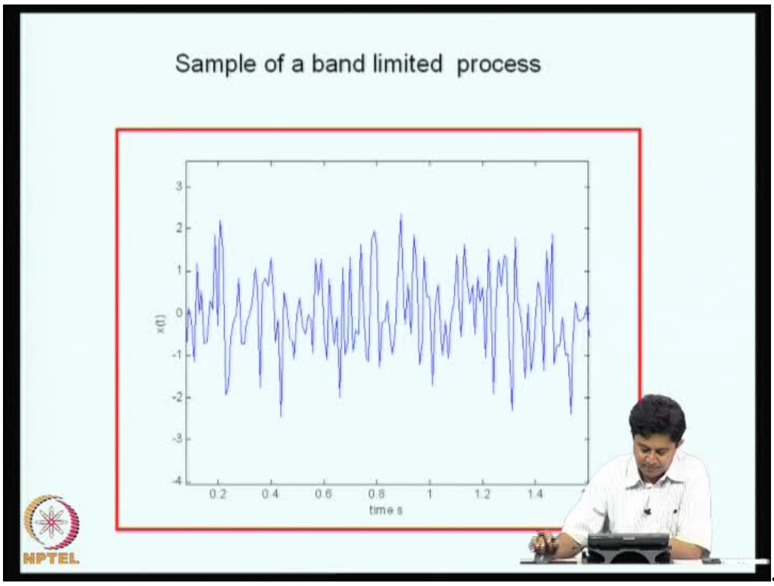

So, how does sample of a broad band process looks like; so, it will be more and more erratic and if I zoom now, say between say 0 to ones 1.6 second, it will have all frequencies are present; therefore, at any resolution, there will be erratic signals, the signal will be erratic.

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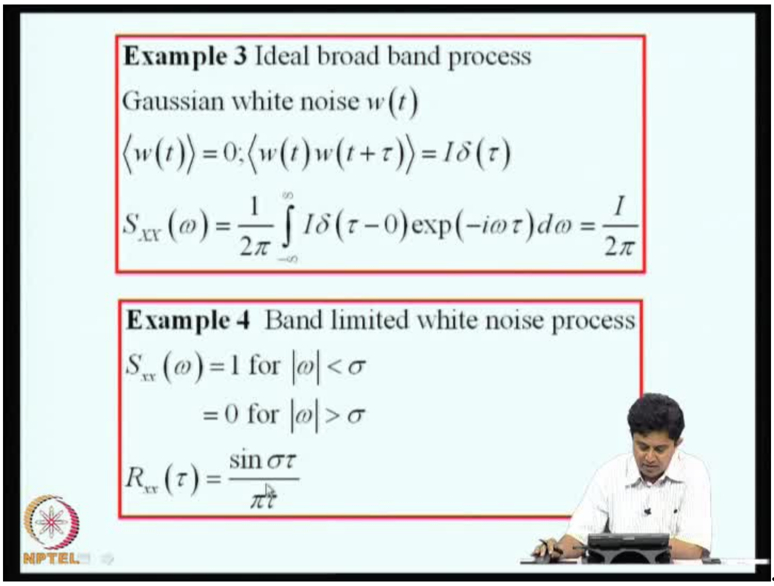

Now, an example, for a band limited white noise would be a power spectral density which is constant for a frequency bandwidth modulus of omega less than sigma and it is 0 outside. If you look at the Fourier transform of this, this is sin sigma tau by pi tau. So, we are looking now at ideal narrow band process realistic, narrow band process and ideal white noise, ideal broad band process and an ideal realistic band limited, you know broad band process, so in reality will be dealing with this and this but for mathematical idealization will be using this and this.

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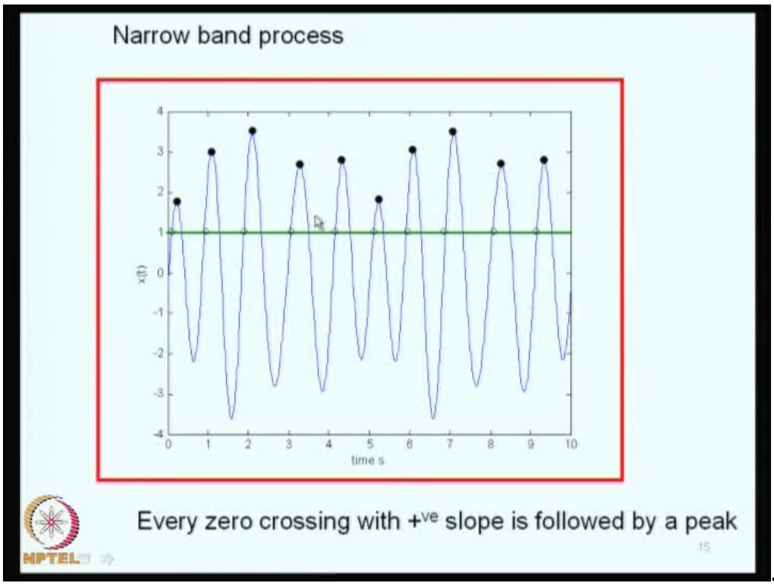

An important feature of a narrow band process is associated with its distribution of peaks and level crossings. If you consider now a realization of a narrow band process and if this is level alpha, the, this green line is alpha; you will see that, every time the level alpha is crossed, there is a peak see. This level is crossed, there is a peak, there is a, this level is crossed, there is a peak. So,, this one of the property of a samples of narrow band random processes, that means, every 0 crossing with positive slope is followed by a peak.

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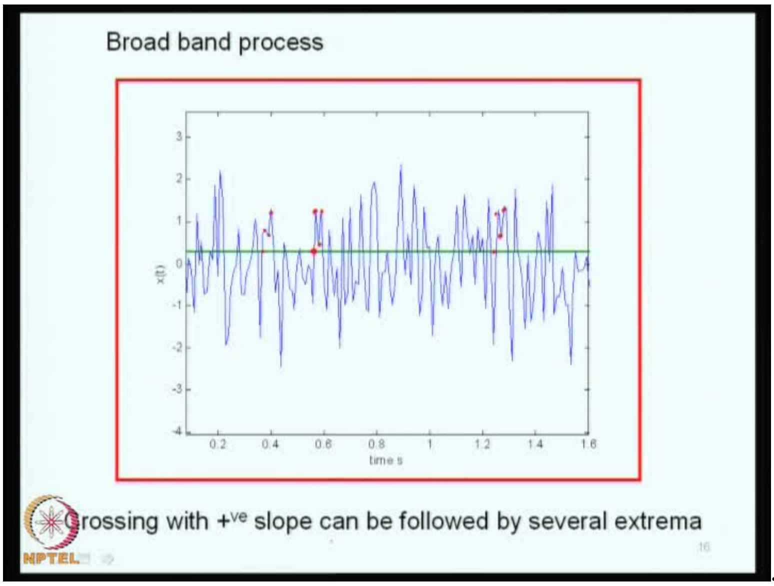

So, if you are interested in studying peaks, you can now imagine that, that is associated with study of level crossings for narrow band processes. This, of course, is not true for broad band processes, for example, this is the sample of a band limited process, you can see here, suppose if you focus on this level crossing, there are three places, where there are extremes, t r maximum, 1 is minimum. So, similarly, here this level crossing is followed by 1, 2, 3, extreme; so, here again you will see 1, 2, 3. So, in ideal broad band process, where the for every crossing of level of alpha, we cannot place a bounded number of peaks, that max exist above the given level; so, this is another property, that should now begin to appreciate, if you are interested in distribution of peaks.

Peaks are associated with highest values of random processes and they are of fundamental interest in engineering. So, for broad band processes, we can make the statement for, as for a sample of a broad band process crossing with positive slope can be followed by several extreme. This is in contrast to sample of a narrow band process, where typically very level crossing alpha is followed by a peak.

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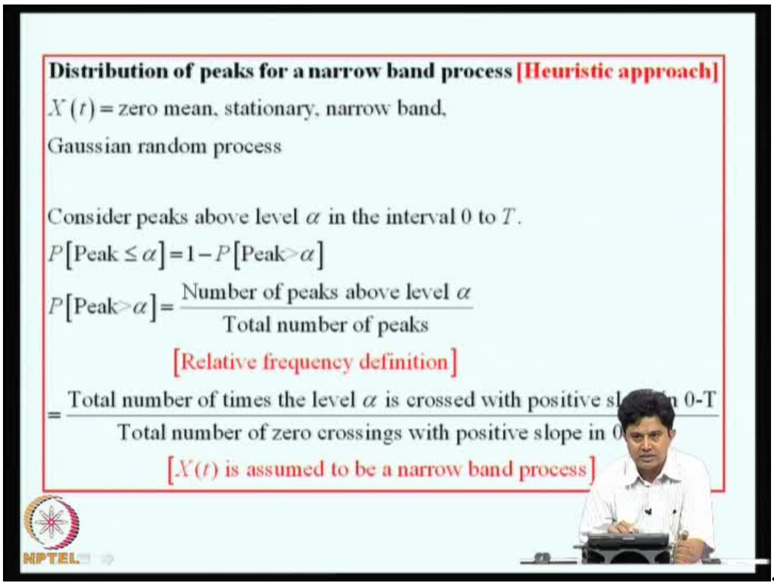

Now, if you focus attention on narrow band process, we can develop a model for distribution of peaks; for a narrow band process, this a heuristic approach, it is not mathematically rigorous, but based on certain heuristic arguments. To explain this, let us consider X of t to be a 0 mean stationary narrow band Gaussian random process; now, let us, consider peaks above level alpha in the interval 0 to T. If I am now interested in the probability, that a peak lies is less than or equal to alpha, this is equal to 1 minus probability of peak greater than alpha. Now, you look at the probability of peak greater than alpha, we can give a relative frequency interpretation for this probability and we will say that, this probability, that peak greater than alpha is given by the ratio of number of peaks above level alpha to the total number of peaks; this is the relative frequency definition. So, this can be elaborated as total number of times, the level alpha is crossed with positive slope in 0 to T, we have shown that, for a narrow band process for every crossing of level alpha, there will be a peak; so, the numerator can be interpreted as total number of times, the level alpha is crossed with a positive slope in 0 to T. What is total number of peaks? The total number of peaks is equal to total number of 0 crossings with positive slope in 0 to T; every time a level 0 is crossed, there will be a peak that gives all the peaks.

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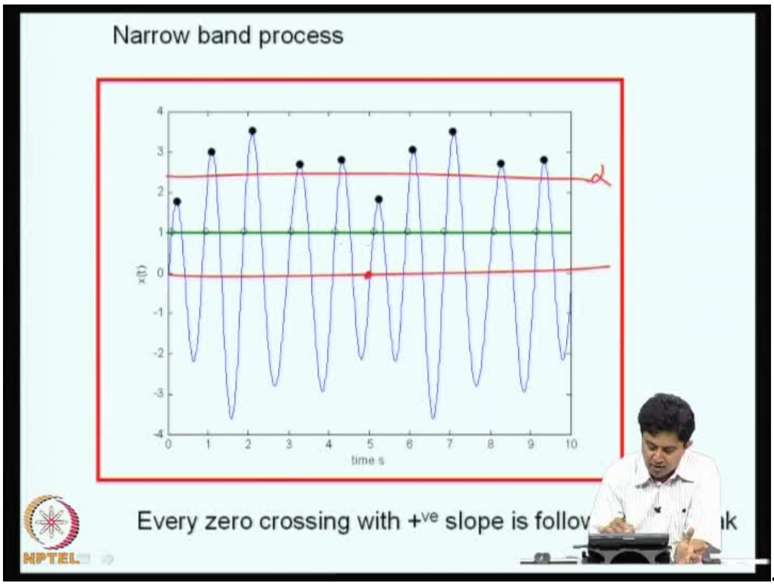

For example, here if I will to set my level alpha here, we see that, this crossing of level alpha does not lead to a peak above level alpha, whereas if you draw this line, the crossing of 0 will lead to this peak. So, when once we are interested in total number of peaks, we have to look at crossings of 0 and you want peaks above alpha you have to look at crossings of level alpha with positive slopes. So, if we do that, we have now the definition for probability of peak greater than alpha, in terms of the number of crossings of level alpha and number of crossings of level 0 and this problem we have tackled; therefore, we should be able to make a model for this probability; so, how do we do that.

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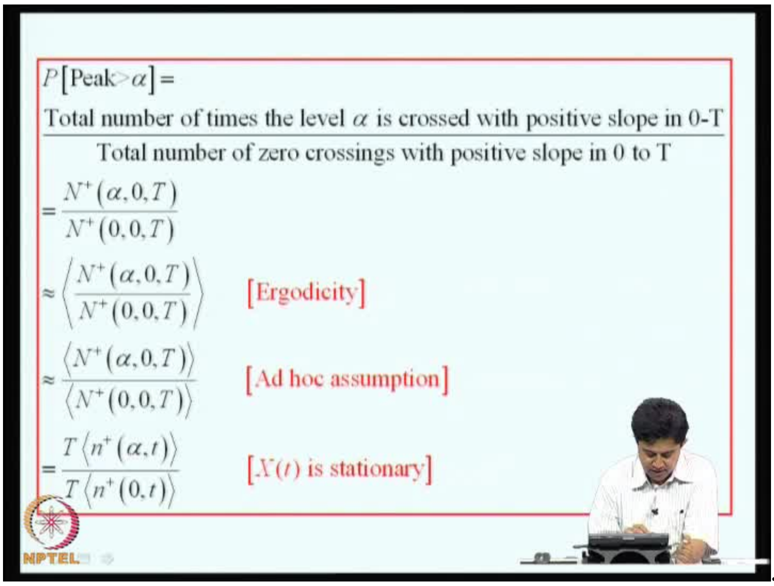

So, probability of peak greater than alpha is ratio of total number of times level alpha is crossed with positive slope and divided by total number of 0 crossing with positive slopes; this is given by N plus alpha of 0 T, that means, number of times level alpha is crossed 0 to T with positive slope divided by number of times, the level 0 is crossed in 0 to T with positive slope. Now, we make an assumption, we replace, this is a random variable, of course, we replace this by its average, so we can say it is some kind of eradicate assumption, where a non-sample, we are now actually replacing in non-sample average, by sample average by non-sample average. Now, if we now make a ad hoc assumption, that the expected value of this ratio is equal to approximately equal to the ratio of expected values, this is not generally true, but if we make this an assumption, we get now numerated as expected value of number of times, the level alpha is crossed in 0 to T, with positive slopes and the denominator, I have number of times, the level 0 is crossed with positive slopes in 0 to T.

Now, since X of t is a stationary random process, this number will be equal to the rate of crossing of level alpha multiplied with a length of the time duration, because this rate is constant, because process is stationary; so, this T in the numerated and denominator gets cancelled and I now have the probability, in terms of ratios of rates of crossing of level alpha and rate of crossing of level 0; that we have already studied.

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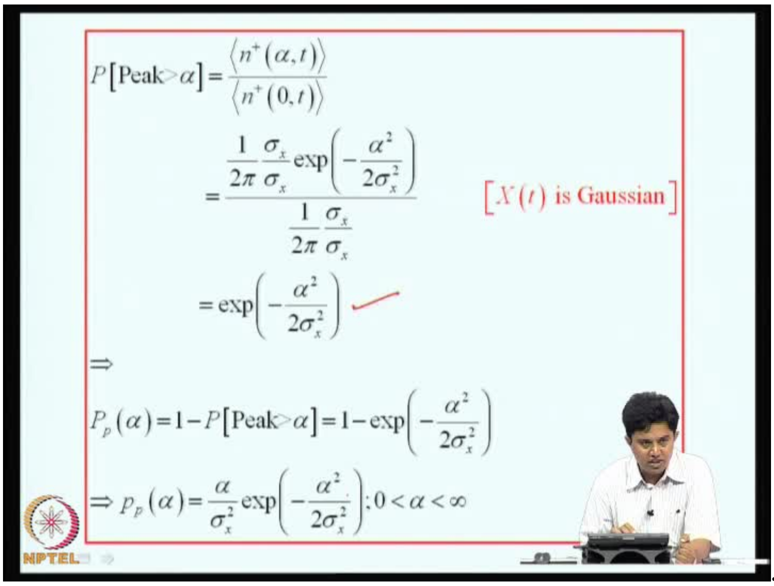

So, we have now for crossing of level alpha, the rate is given in terms of variance of the parent process and its variance of the derivative process, because X of t is Gaussian, I can write this and in this expression, if I put alpha equal to 0, I get the numerator and now some of these terms gets cancel and I am left with for this probability a model, which is exponential of minus alpha square by 2 sigma x square. So, consequently the probability distribution function for the peak can be written as, 1 minus this probability, this is 1 minus exponential minus alpha square divided by 2 sigma x square; you differentiate this with respect to alpha, you get the probability density function and in turns out that, this is a Rayleigh random variable.

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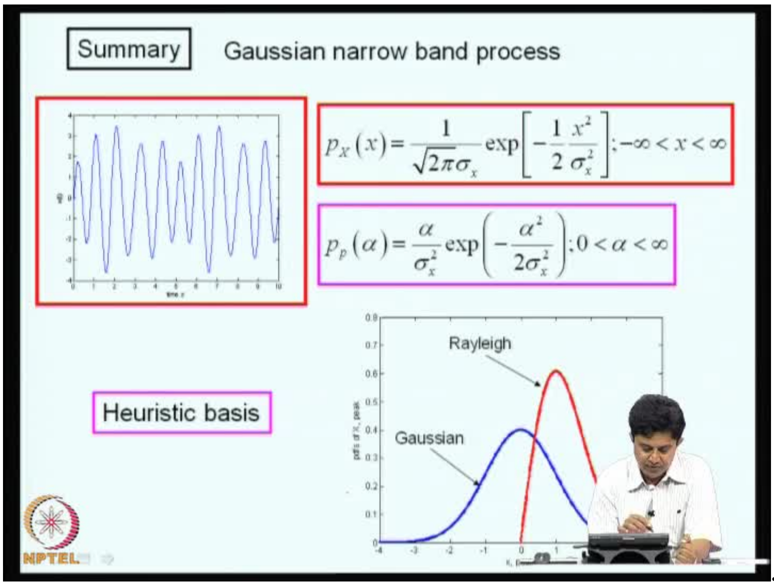

So, we get model for peaks of narrow band Gaussian narrow band processes and according to this model, these peaks are distributed as Rayleigh random variable. So, the summary is, if you consider a narrow random process, this is the distribution of the parent process, this is Gaussian, that is this blue line, this is Gaussian and if you look at the peaks, the peak is Rayleigh, this red line is this peak, this is Rayleigh random variable; this is the preposition based on certain heuristic arguments, this model, of course, is valid only for the situation, where process is narrow band, but how do we characterize the peaks, if process is more general, suppose it is band limited, it is not narrow banded how will you proceed, that is, the next question that will consider now.

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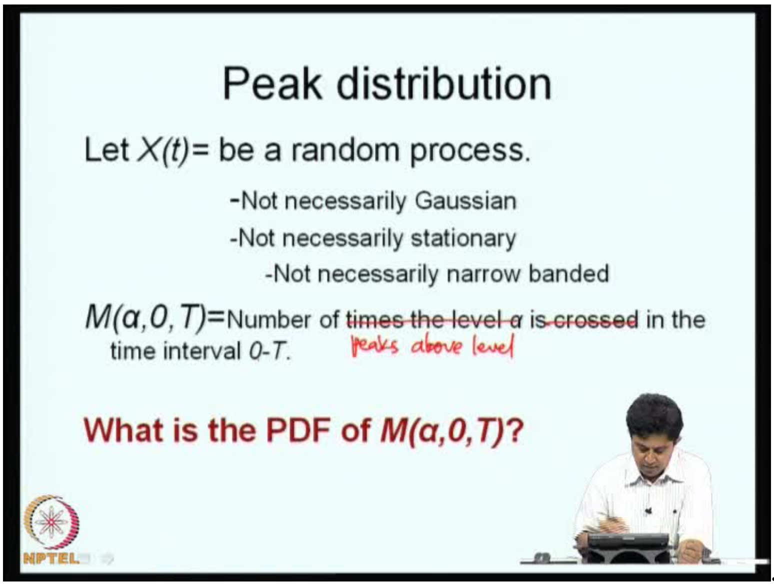

So, that takes us to the problem of establishing distribution of peaks for a general random process; so, we consider a random process X of t, which is not necessarily Gaussian, which is not necessarily stationary and which is not necessarily narrow banded; so, it is a general process. I define this quantity m (alpha, 0, T) as number of times, the level alpha is crossed, no, the number of peaks above level alpha in the interval 0 to T.

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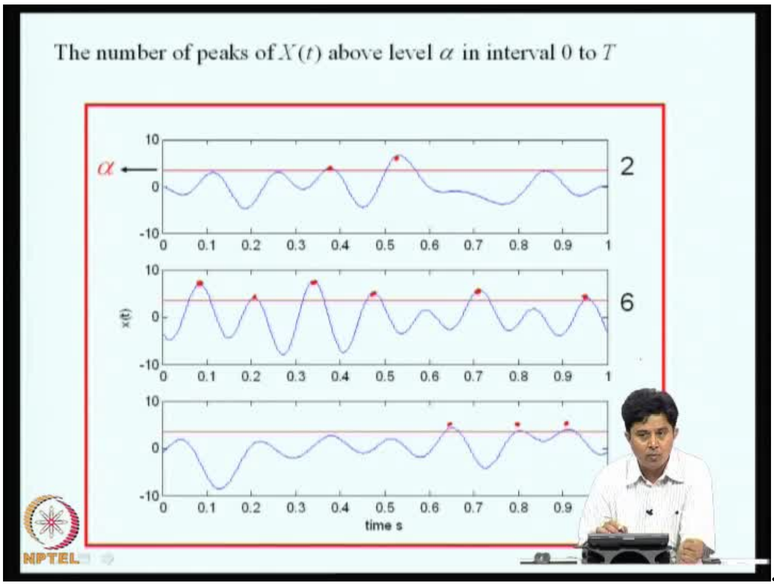

Now, the question is, what is a probability distribution function of M. So, we will again look at a few sample realization of a typical random process; here, how many peaks are there above level alpha, there is one here and there is one here, so for this sample, I get 2, whereas for this sample I get 1, 2, 3, 4, 5 and 6 and for this sample I get 1, 2 and 3; so, clearly, therefore, the number of peaks of X of t above level alpha in a given interval is a integer value random variable. So, if you are given a complete description of X of t, what I can say about this random variable, can I find its probability distribution, can I find it is a moments, mean, variance, what I can do about it.

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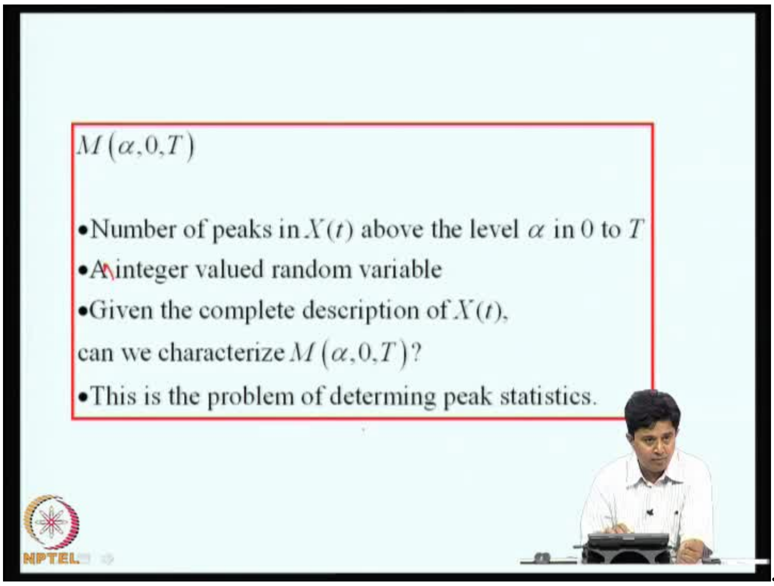

So, M (alpha, 0, T) is number of peaks in X of t, above the level alpha in 0 to T, it is an integer valued random variable. The problem is given the complete description of X of t, can we characterize this random variable, this is the problem of determining peak statistics; mind you I am not talking about a narrow band process, I am not talking about stationary process and I am not talking about a Gaussian random process, it is a general random process.

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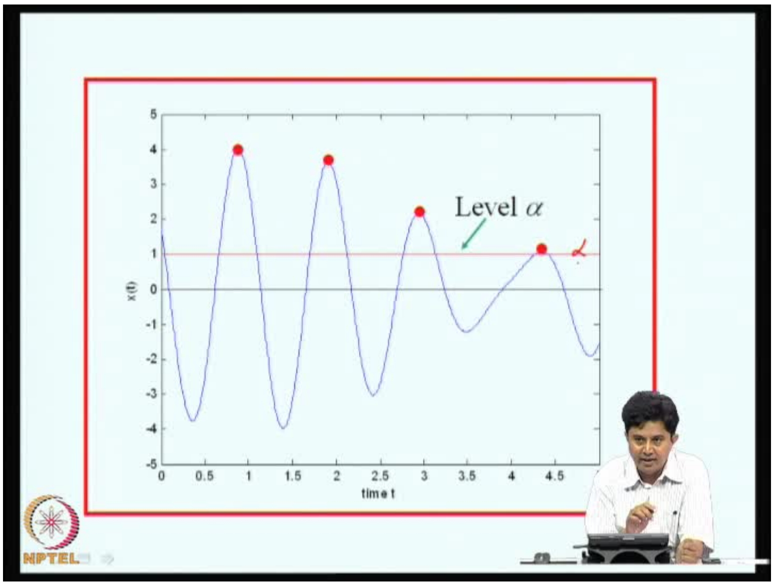

Now, let us look at one realization bit more closely; so, this is the level alpha, now for this realization, in this time interval 0 to 5 seconds or whatever, there are four crossings; now, I need to set up a counter to obtain this four mathematically, what I can count visually, has to be now translated into a mathematical counter; so, to do that, we need to analyze the properties of these four points in some detail.

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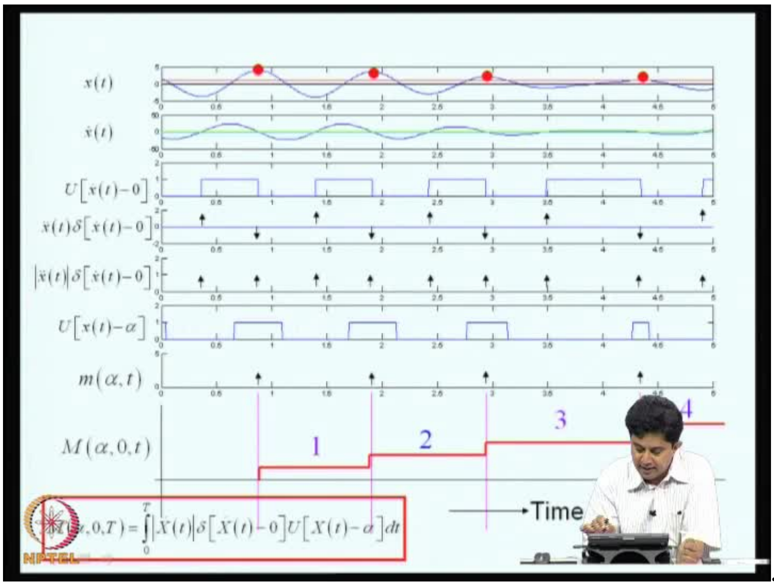

So, how do we do that, now here I have a series of plots; so, let us begin with the top most plot, this is the sample realization of x of t; x axis is time and here it is x of t and the red line here is the level alpha and I am interested in counting the number of times, the level alpha is crossed in the intervals, say 0 to 5 seconds. Actually, this graph is exactly this graph, but shown on a slightly you know different scales.

Now, I define a process which is x dot of t, which is the time derivative of this blue curve; so, whenever there is a peak, there will be a 0 crossing. So, number of peaks in x of t is equal to the number 0 of crossings in the derivative process; so, we will first use that, property to be able to do that, I define the derivative process, so wherever there is a peak, there is now a 0 crossing. Now, I define the process U of X dot of t minus 0, that means, I am now interested in, actually I want to count the total number of peaks about level 0; so, how many times 0 crossings are, 0 crossings occur, therefore, I do this first and then differentiate this, so I get differentiation of step function is the direct delta function; this is direct delta of X dot of t minus 0 and its double derivative; so, I get a spike, wherever there is a crossing, so here 1, 2, 3, 4, 5, 6, 7, 8, 9; 9 crossings are there, 0 crossings are there for X dot of t.

Now, one is positive and another is negative to circumvent, that I take the absolute value; so, I take absolute of X dot of X double dot of t into this; so, I get now 1, 2, 3, 4, 5, 6, 7, 8, 9, these are nothing but the crossings here etcetera. Now, each of these crossing here is associated with an extreme value, here for example, this is a place where X dot of t is 0, so there is a 0 crossing and in this, I am counting that also, but what I need to count is, I do not want to count, the, this particular value, I want to count only these red dots. So, I need to count only these four, so what I do, I define U of x of t minus alpha, that means, I define another process which takes value one, whenever x of t is greater than alpha; so, I get here, if you project, now this point here there is a non-zero function, this is time that processes is spending the above level alpha and it is 1 here; similarly, it is 1 here, 1 here and a small touch 1 here. Now, to get number of peaks during these time intervals, when x of t is staying above alpha, I need to multiply this function with this counter; so, then what happens, I will eliminate this, this I do not want to count, this I do not want to count, this I do not want to count, so I, all, that get eliminated, when I multiply these two, so I get 1, 2, 3, 4.

So, now I got, what I wanted now, I had to simply sum them up, so I integrate this function which is product of this and this, that is modulus of X dot of t into delta of X dot of t minus 0 into U of X of t minus alpha into dt, this integrated from 0 to capital T will give me 1, 2, 3, 4, here is the answer, that am getting this, what will happen here. So, if you carefully see this, the logic will reasonably straight forward, where using direct delta functions and step functions extensively, so that properties of those functions need to be born in mind, while interpreting this.

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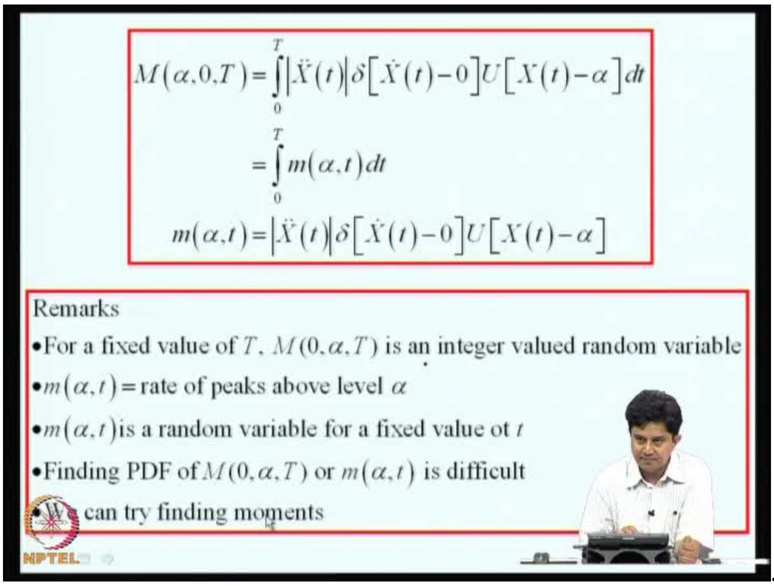

So, I have now the counter ready for number of times level alpha is crossing 0 to T, in terms of the process X of t is time derivative X dot of t and its second derivative X double dot of t; if X of t is displacement, I need properties of displacement, velocity and acceleration. So, the integrant can be interpreted, I denote it as m (alpha, t) this can be interpreted as a rate of peaks above level of alpha in X of t. So, for a fixed value of capital T, m (alpha, 0, t) is an integer valued random variable, because it is a counter 1, 2 3 4 etcetera; this m (alpha, t) is rate of peaks above level alpha, this is also a random variable for a fixed value of this lower case t.

Now, if I am given complete description of X of t, that would mean, I may be able to, I would be able to write the joint probability density function of X of t, X dot of t and X double dot of t given that and using rules of transformation of random variables, it would still be difficult, if not impossible to find the probability distribution of m (alpha, 0, T). So, finding PDF of M or the rate is difficult, if not impossible, so we can try finding a few moments; so, begin by finding the mean.

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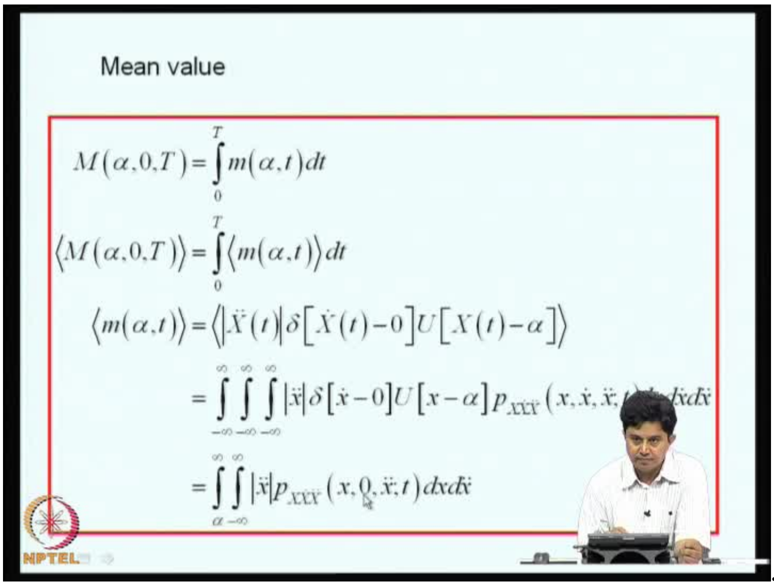

So, what is expected value of number of time p, number of peaks above level alpha n 0 to t is the expectation value, expected value of this and the expected value of integrant is expected value of these terms, involving highly non-linear transformation on the parent process is derivative and is second derivative. This expectation can be determined, if you know the joint density between X of t, X dot of t and X double dot of t, that would mean, here I have to write down a three, four integral with integration on dx, dx dot, dx double dot of P x x dot x double dot and this is the function whose expectations is being taken

Now, since I have a direct delta function integration with respect to x dot will be straight forward and wherever there is x dot, I will write it by replace it by 0, that is, what has to be done here and this, this U of x minus alpha is a step function; so, this can be eliminated from the integrant, if we take care to write the limit of integration for x naught from minus infinity to plus infinity but from alpha to infinity, because anyway it is 0 from minus infinity to alpha, because this is a step functions. So, this step function gets eliminated taken, is taken, into account and this direct delta function is taken into account and this triple integral, now can be recast as a twofold integral or this is nothing but an integrant involving the probability density function of x, x dot, x double dot, but x dot is fixed at value 0 and integration limits for x is alpha to infinity.

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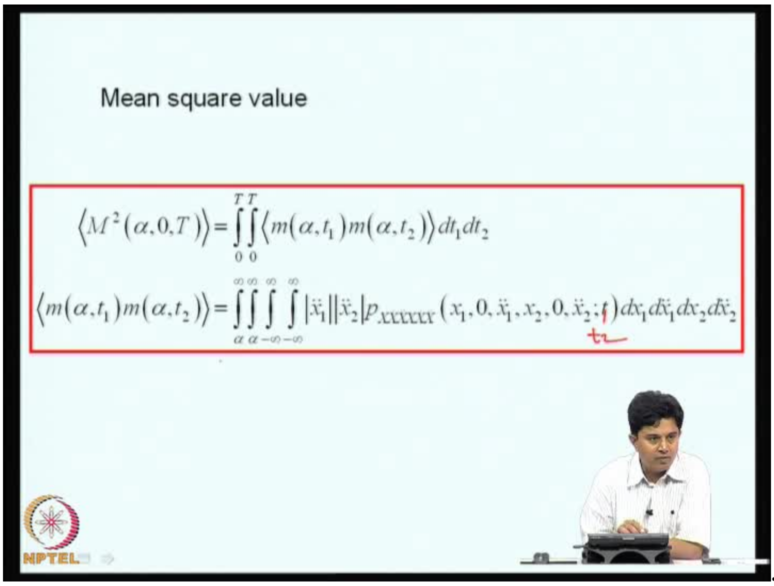

Similarly, we can also find the mean square value, if you are interested in mean square value, you have to find expected value of expected value of M square (alpha, 0, T); so, for one-dimensional integral, I have move to 2-dimensional integrals and this integrant involving m (alpha, t 1) and m (alpha, t 2) can be express in terms of a, actually a six fold integral and two integrals become easy to handle, because there are, there will be two direct delta functions and if you allow for that, I am left with a fourfold integral, involving joint density of x, x dot, x double dot a 2 times instance t 1 and t 2.

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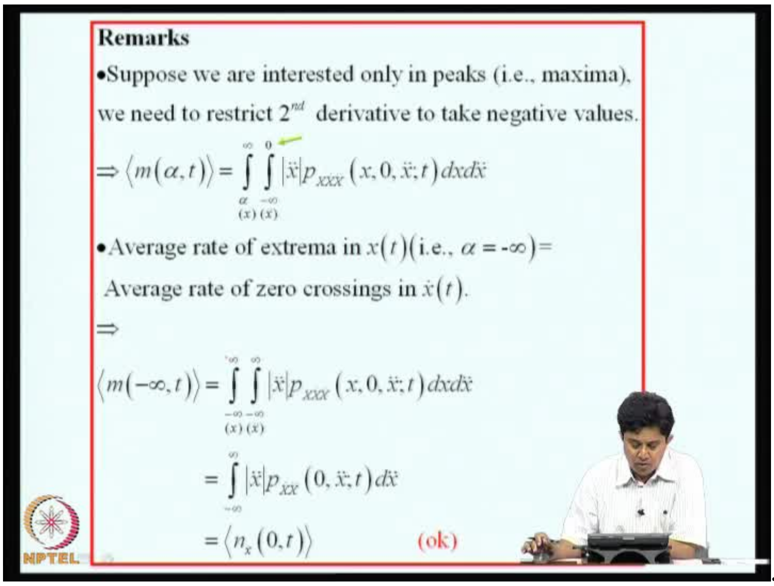

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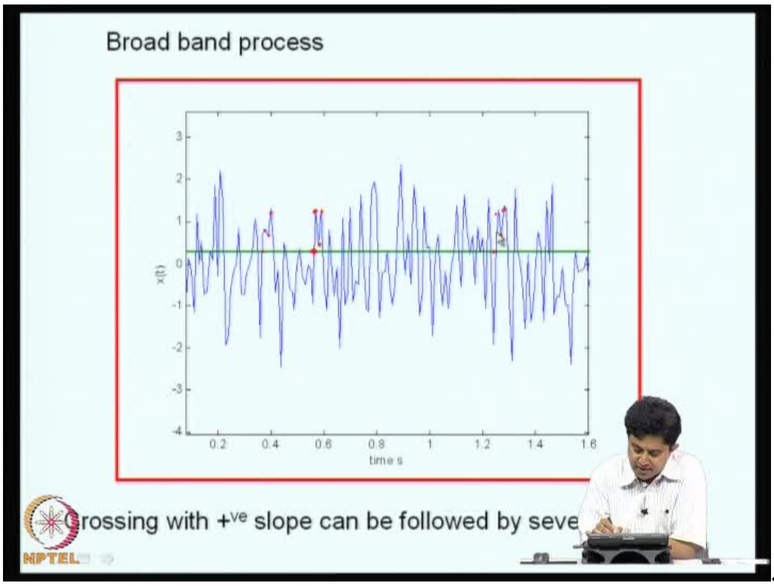

So, this, of course is not a straight forward to integrate, but it is a formal representation. Suppose, we are interested only in peaks, we are not interested in minimum, that means, suppose if you go back to the case of a sample of broad band process, we saw that, here for instance peaks level above level alpha, there are three places, where x dot of t is 0; suppose, I am interested only in this positive peaks, that is maxima and not in these values, then we need to modify the formulas slightly.

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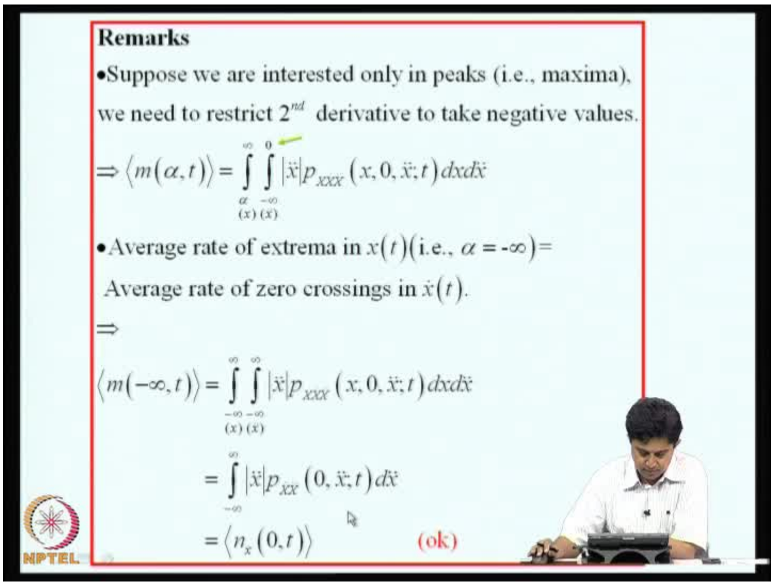

Suppose, we are interested only in peak, that is maxima, we need to restrict second derivative to take only negative values. So, the integration in that case with respect to x double dot will be from minus infinity to 0, this is the only difference; this is the difference that will be there, because of this change in our statement of the problem. Now, average rate of extrema in x of t, that is all, that you know extrema, wherever x dot of t is 0, is average rate of 0 crossings in x dot of t, it peaks, all the peak, the, you know peaks and values show this can be put find found out by its letting alpha to minus infinity, because if level is at alpha equal to minus infinity, it is crossed always; so, you pick up all the places, where the slopes are 0 and we can verify that, if I actually compute this expected value of m (minus infinity, t), if process is Gaussian is not necessary, now to be Gaussian in terms of the joint density of x dot and x double dot will get this and if you now relate this expression to the problem of level crossing, you will see that the average rate of peaks above level minus infinity same as the rate of crossings of level 0 by the derivative process.

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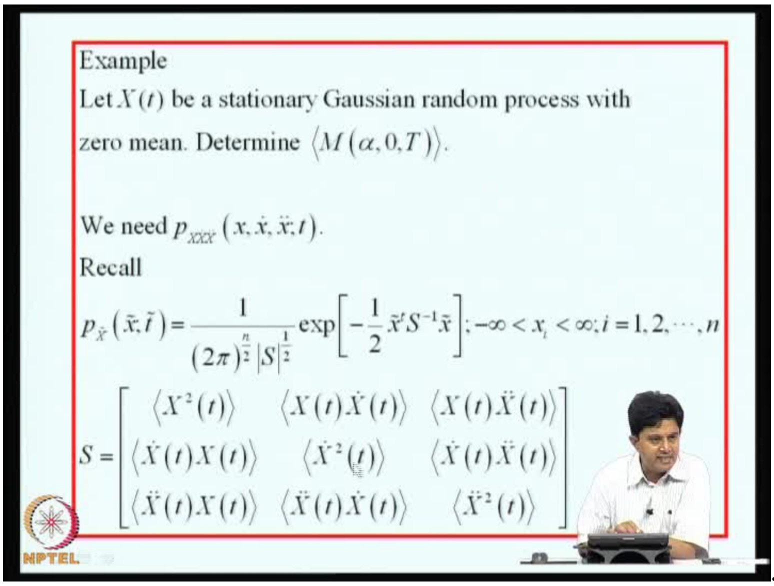

So, this is consistent with what we have to done till. Now, to illustrate the ideas develop let us consider X of t to be a stationary Gaussian random process with 0 mean; the problem on hand consist of determining the expected value of peaks above level alpha in 0 to capital T, for this, we need the joint density of x x dot and x double dot, process is Gaussian; therefore, the probability density function will have this functional form, where this matrix S is a matrix of covariance of the three random variables. The basic three random variable here are X of t, x dot of t and x double dot of t, at the same time t where considering three random processes. So, S will be expected value of X square of t and expected value of X of t into X dot of t X of t into X double dot of t and so on and so forth.

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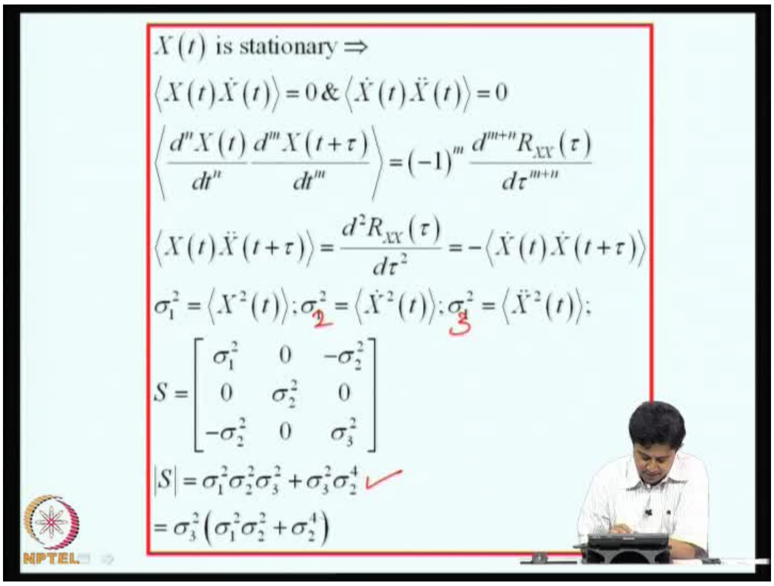

Now, the process is given to be stationary, some of these terms can be shown to be 0. So, if X of t is stationary, you recall X of expected value of X of t and X dot of t is 0 and X dot of t and X double dot of t is 0, the process is a derivative or uncorrelated for a stationary random process, that is, a result that we derived earlier. We have also shown that, expected value of n th derivative of X of t and m th derivative of X of t plus tau is given by this expression formula and that we need to use, now to determine expected value of X of t into X double dot of t plus tau.

So, if we do all these, the s matrix becomes four entries becomes 0 and the other entries can be determined, this is 0, because we are talking about correlation between X and X dot and this is 0, because we are looking at correlation between X dot and X double dot; so, these are zero and this matrix is symmetric. So, it could be this, I am giving a notation, sigma 1 square is expected value of x square of t sigma 2 square is expected value of x dot square of t and similarly sigma 3 square is expected value of X double dot of t whole square.

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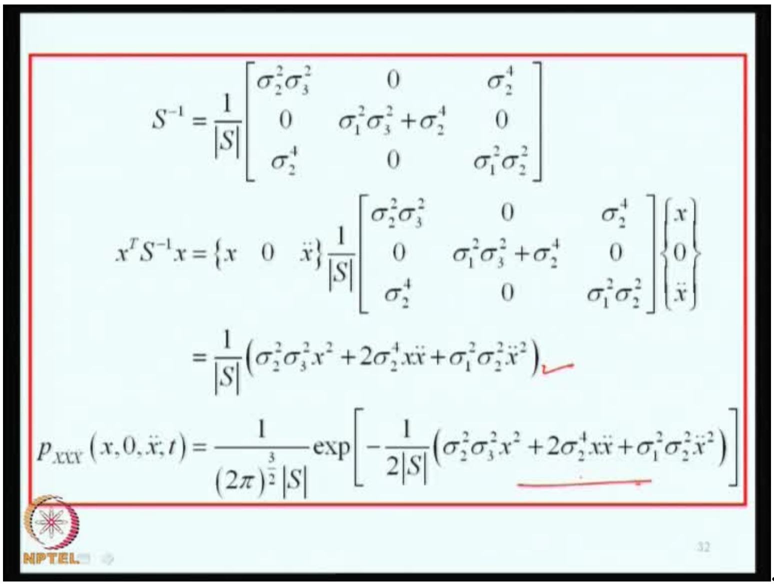

Now, I can expand this, this will be the determinant and this get simplifies to this form and also we need the inverse of this matrix and this quantity x transpose S inverse x, so inverse determinant is found the inverse can be found; following this steps, I get the exponent x transpose S inverse x to be given by this expression. And for x dot equal to 0, the three-dimensional density function between x of t x dot of t and x double dot of t is given by this expression, because x of t is a Gaussian random process; so, I am writing down the three-dimensional joint density function.

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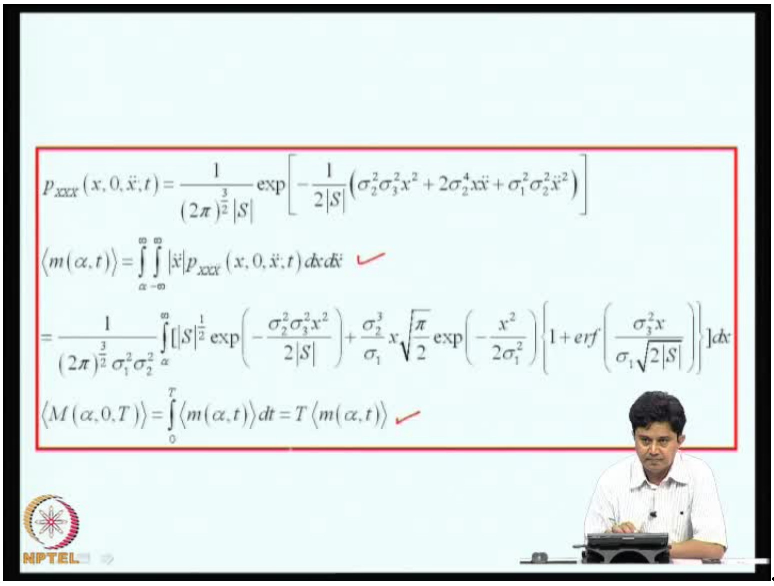

This can now be substituted into the expression for average rate of crossing of level alpha and this is a expression; once you substitute this, one of the integration could be done with some effort, but the integration with respect to alpha perhaps need to be done numerically, anyway with one integration, the expression reduces to this, for the rate of cross peaks above level alpha and therefore, the average number of peaks above level alpha in 0 to t is given by integral of this rate over 0 to capital T. And since process is stationary, none of this quantities sigma 1, sigma 2, sigma 3 are functions of time; therefore, I can multiply the rate by the time duration, over which you are counting the peaks and I get this expression.

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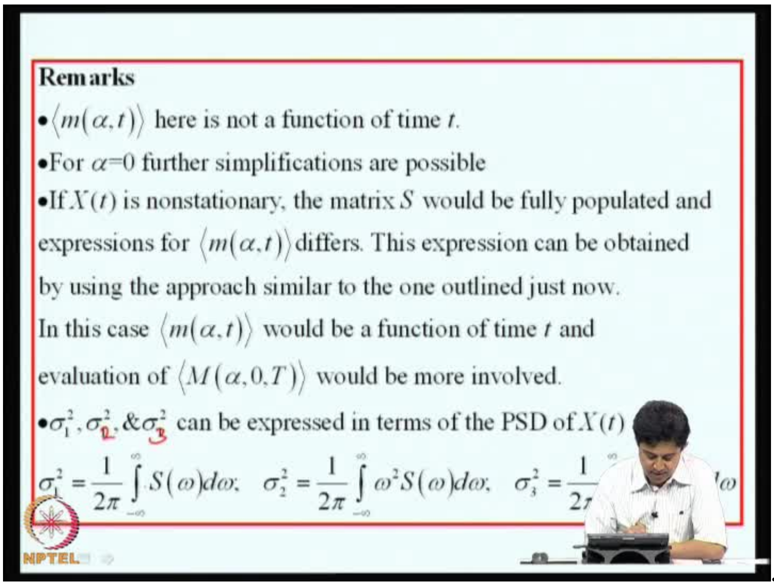

So, the problem is essentially solved; there are few remarks to summarize what we have been saying, this rate of crossing of peaks above level alpha is not a function of time, because X of t is a stationary random process. If you are interested in peaks above level alpha, a few simplifications are possible, but in general that one-dimensional integral that remains when it to be evaluated numerically. If X of t is a non-stationary random process, then what happens is the correlation between X and X dot is not 0 at same time t; so, the s matrix will be fully populated and the expression for this rate of crossing rate of peaks above level alpha differs naturally, but however this expression can be obtained by using the an approach similar to the one, that we outlined just now. In this case, apart from the additional complexity in dealing with fully populated S matrix, the rate of cross peaks above level alpha would be a function of time and consequently, when we find the expected value of total number of peaks above level alpha in o to T, a further integration in time has to be done, that again would require numerical evaluation. Returning the case of stationary random process, we have now used this expression sigma 1 square, sigma 2 square, sigma 3 square to denote variance of X of t variance of X dot of t and variance of X double dot of t; these variances can be expressed in terms of power spectral density of X of t, as variance of the process is density variance of the derivative process is pierced into omega square area under that curve. And variance of the second derivative is omega, area under omega to the power of 4 into the power spectral density; this is the zero th spectral moment, this is second spectral moment, this is the fourth spectral moment.

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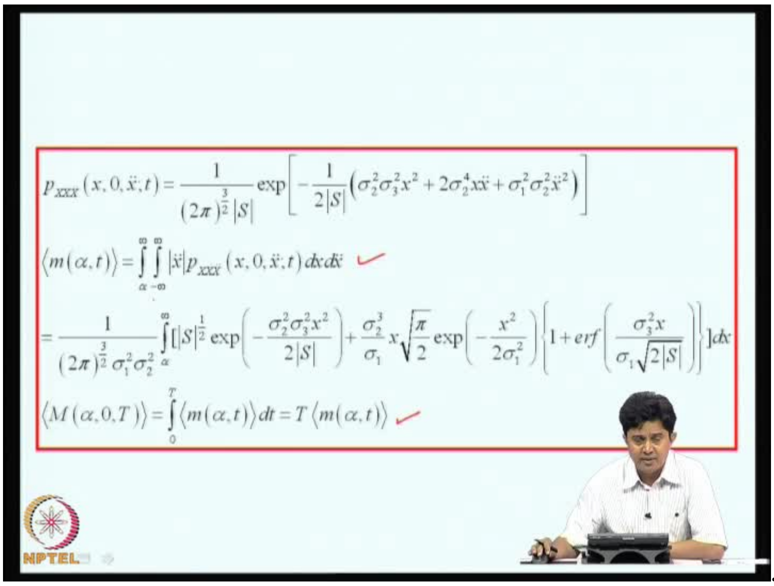

So, that would mean, the rate of peak above level alpha are again getting expressed, so only in terms of spectral moments. You may know the joint density function between x of t, x dot of t and x double dot of t, but if your attention is focused on average rate of peaks above level alpha, only things that matter are the spectral moments zero th second and fourth order spectral moments; so, thus the spectral moments become important descriptors in discussion of peaks.

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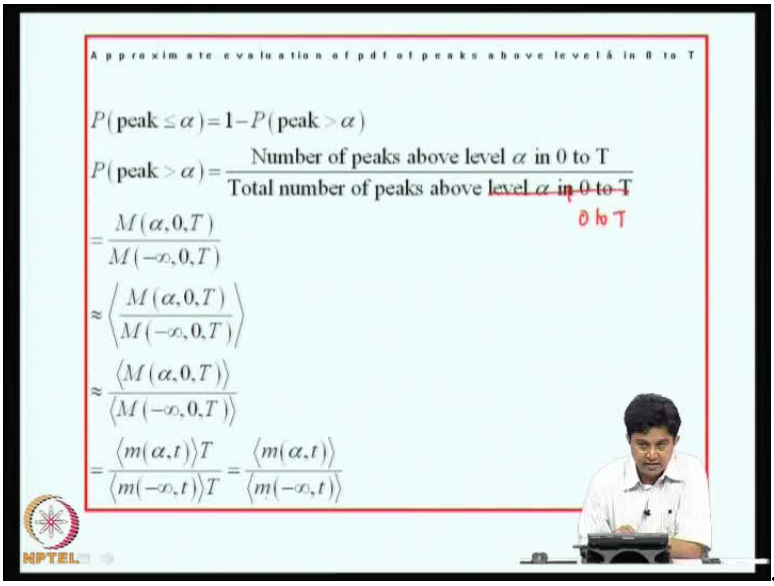

Now, we can develop a strategy for approximate evaluation of probability density function of peaks above a level alpha, in the time duration 0 to T, following the argument, that just now we used for a narrow band process; this is more general, suppose we are interested in probability of peak less than equal to alpha this is 1 minus probability of peak greater than alpha and for the probability peak greater than alpha, if we use a relative frequency approach, we can say that, this is a ratio of number of peaks above level alpha in o to T divided by total number of peaks above level alpha in 0 to T; this is total number of peaks in 0 to T; so, this is given by M (alpha, 0, T) and M (minus infinity, 0, T). Again, we approximate this by an expected value and bring in a heuristic assumption, that expected value of ratio is nothing but ratio of the expected values which is not true, but we invoke this assumption here. I will be able to now write down the expression that we have developed just now, can be now utilized and the requisite probability, now is expressed in terms of average rate of peaks above level alpha and average rate of all the peaks, where alpha is z to minus infinity.

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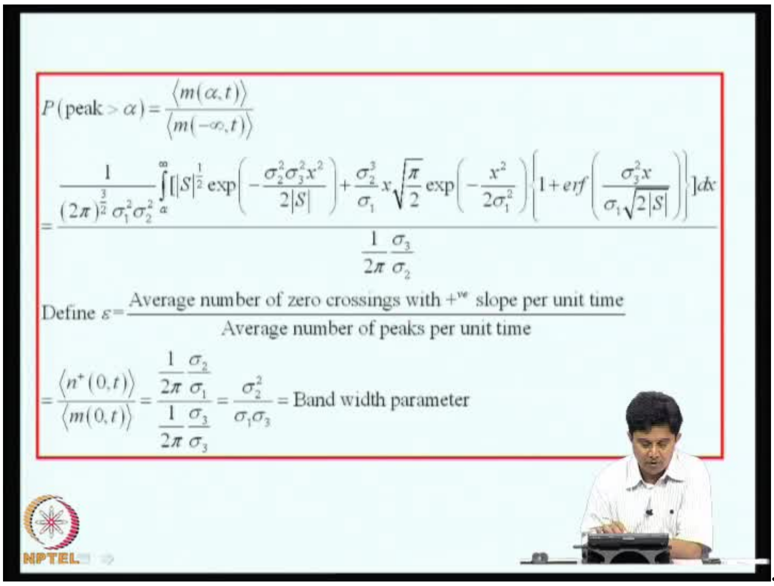

The expressions for this quantity, we have already determined; so, we substitute here and we introduce a parameter denoted by epsilon, we call this parameter average number of 0 crossings with positive slope per unit time divided by average number of peaks per unit. We have now the expressions for these two quantities and in turns out that epsilon using the result, we already have is given by the ratio sigma 2 square by sigma 1 into sigma 3. Now, if process is narrow banded, every 0 crossing will be followed by a peak and what happens to epsilon, epsilon goes to the value of unity. If process is broad banded, in ideal white noise moment is 0 crossing, is a 0 crossing occurs, there can be infinite number of peaks, therefore epsilon goes to 0.

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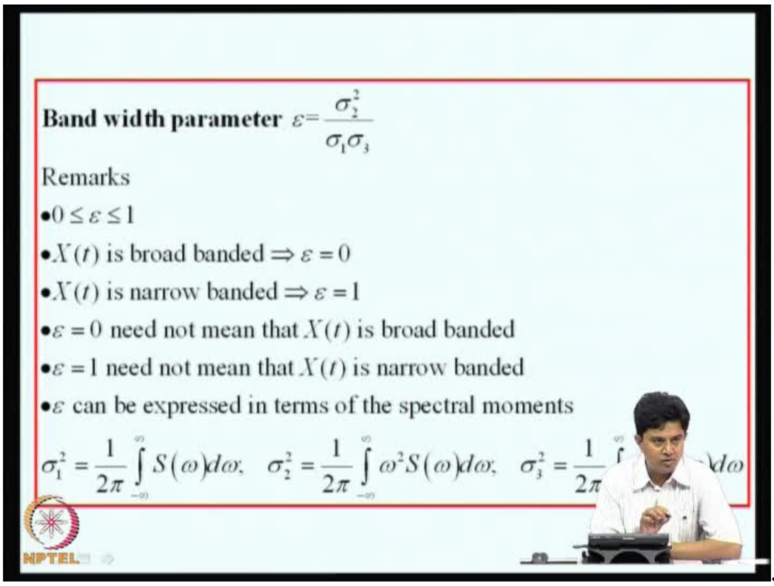

So, this epsilon therefore can be viewed as bandwidth parameter, it takes value between 0 and 1; if X of t is broad banded, it approaches value of 0 and if X of t is a narrow banded process epsilon approaches 1, but we should carefully here to, you know, understand what is meant here, what we are saying is that X of t is broad banded implies that epsilon is 0, but if epsilon turns out to be 0, it does not mean that X of t is broad banded process; similarly, epsilon equal to 1 need not mean that X of t is narrow banded. Again, this parameter epsilon can be expressed in terms of the spectral moments, because sigma 1, sigma 2, sigma 3 are all expressible in terms of moments of the power spectral density.

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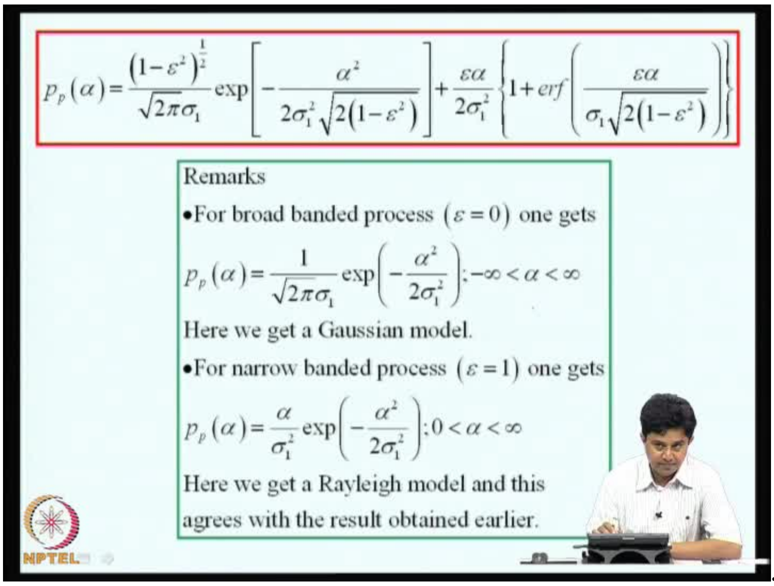

Now, proceeding further and using this notation epsilon and which some after some algebra manipulations, you can show that the probability density function of the peaks is given by this expression, where epsilon is the bandwidth parameter and apart from that, the only other descriptor of the parent process will be in terms of sigma 1, which is the sigma 1square is the variance of the parent process and the properties of the derivative and the next derivative is encapsulated only in the definition of epsilon. Now, in this case, in this expression, if you put epsilon equal to 0, we get the result that the probability density function of peaks indeed turns out to be a Gaussian random process; that means, there is no distinction between the process and it is peaks. For a narrow band process, however as epsilon goes to 1; 1 gets a Rayleigh density function for the peaks and this is quite consistent with what we saw earlier.

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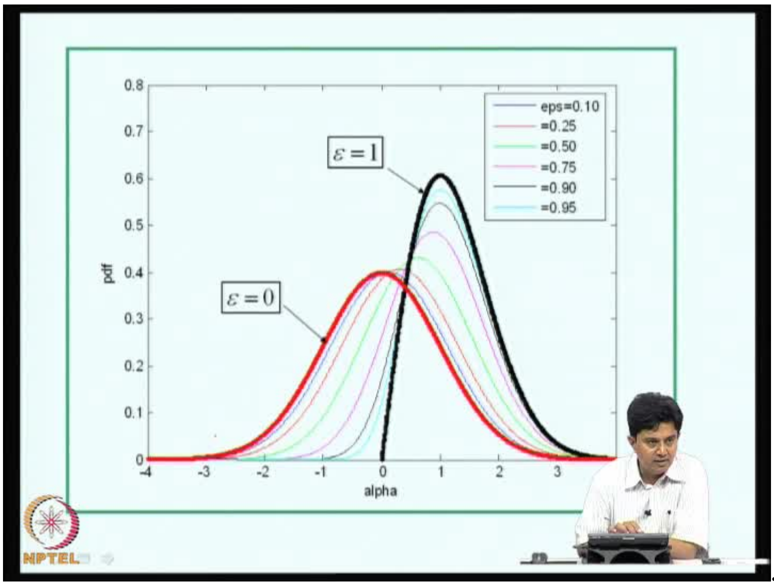

So, how does this graphically look like; so, epsilon equal to 0 is this result, which corresponds to the Gaussian density function and epsilon equal to 1 is the limit of narrow banded process and in between, I have a family of probability density functions, where epsilon takes values from say 0.1, 0.25, etcetera 0.9 and 0.95. So, as we approach one tends, towards become a Rayleigh and as approach 0 that tends to becoming Gaussian. Again, let me emphasize that, this, these results are obtained based on heuristic arguments and one has to be careful, when we use it.

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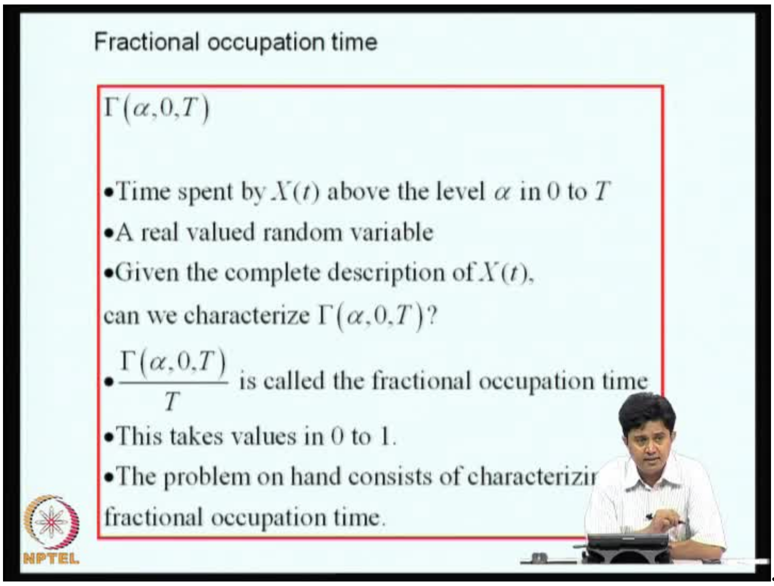

The next description of a random process in which we are interested, the, is related to the time spent by X of t above level alpha in a given duration. We denote this as gamma (alpha, 0, T), this denotes the total time spend above level alpha in duration 0 to T; this is the real valued random variable. Now, again the question is given the complete description of X of t, how can we characterize gamma; if you divide gamma by the total duration, we get a quantity which is known as fractional occupation time, if where entire relation is 0 to t, if level process stage above level alpha, the fractional occupation time is 1. If it never crosses level alpha, the fractional occupation time is 0; therefore, it takes values in 0 to 1; so, we can focus on characterizing this fraction of occupation time and again the question is, if you are given complete description of X of t, can we characterize the probability distribution function of this fractional occupation time; if that is difficult, can we characterize its moments, that is the question.

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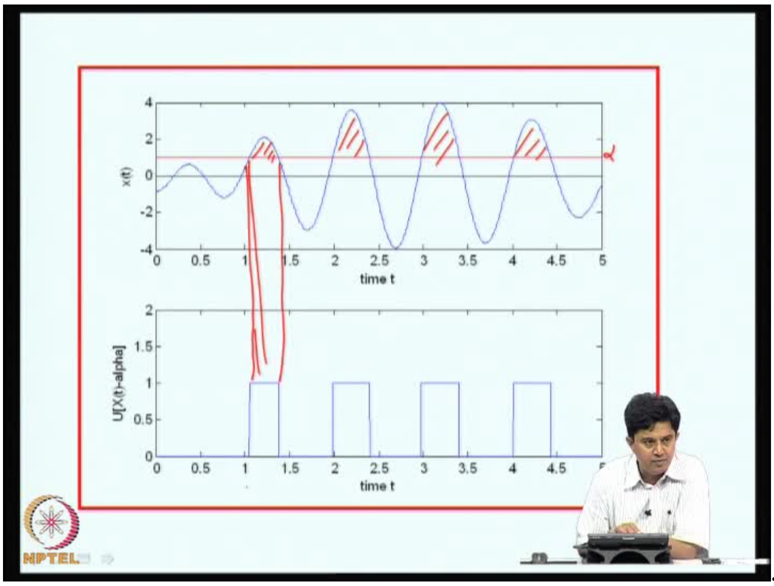

So, what is that, we talking about this blue line, is the sample of random process and this red line is the level alpha; this is level alpha. And we are interested in knowing that, in 0 to 5 second, how much of time is spent above of level alpha. So, one episode of time spent above level alpha occurs here, the next step episode occurs here and the third one occurs here and last one occurs here. So, if you know, project this, these points below, here you see that what I have to defined on the y axis here is a step function u of X of t minus alpha; so, whenever x of t is greater than alpha, a define another random process whose values 1. Now, if I find the area under this function, the height is 1, I will get the time spent by X of t above level alpha; this height is 1, therefore the area of all these pulses will give me the total time spent above level alpha.

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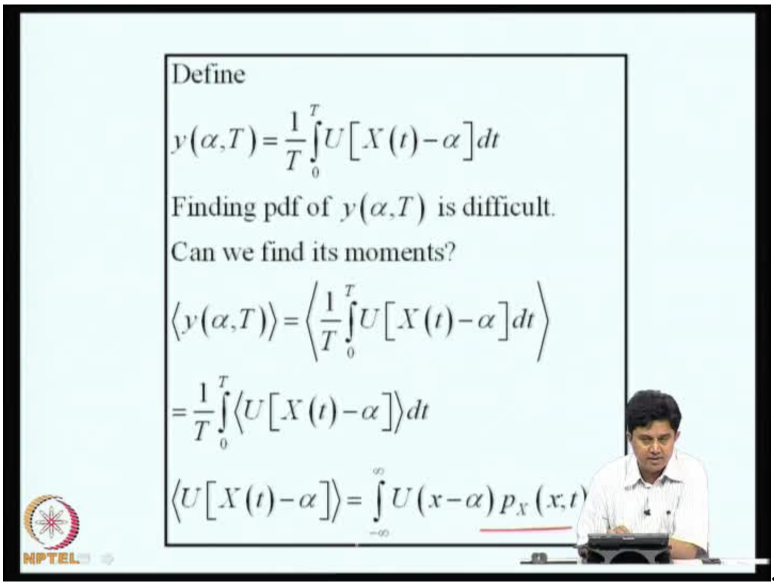

So, y (alpha, T) which is the fractional occupation time is given by 1 of T is 0 to T and step function of X of t minus alpha dt. Given the non-linear transformation implied in the step function, again finding probability distribution function of y is difficult; so, we try to find, say it is expected value or it is a first few moments.

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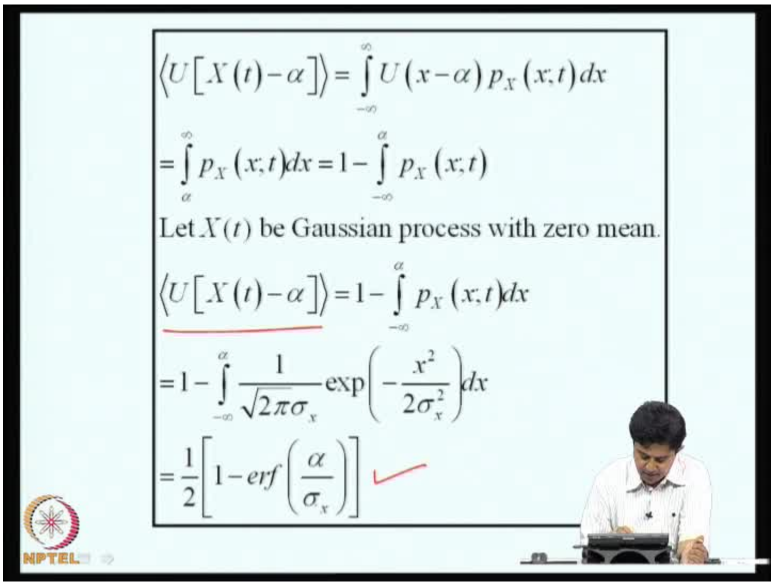

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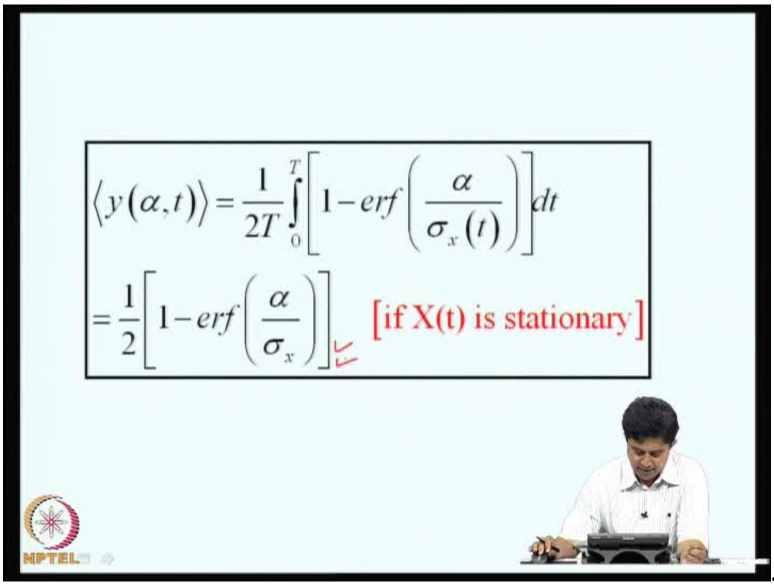

So, if you start with expected value, expected value of y (alpha, T) is expected value of integrant and that takes us into the expected value of U of X of t minus alpha, which is minus infinity to plus infinity U of X minus alpha p x of x comma x colon t d x. So and, this expression of course is valid for X of t, when it not necessarily stationary, not necessarily Gaussian, not necessarily having 0 mean, it is generally valid. Now, if X of t is Gaussian with 0 mean, then expected value of U of X of t minus alpha can be evaluated by evaluating the integral 1 minus minus infinity to alpha p x of x colon t; this is the Gaussian density function, so what is the required is to find the area under Gaussian density function from minus infinity to alpha, that is expressible in terms of error function; therefore, I get the expected value of this function to be given by this. This is in terms of error function and we have this standard division of this parent process appearing in this expression; further simplification, of course, is possible if X of t is stationary, then I can multiply, I mean integrant becomes independent of time and I can pull it out and I will get this expression, the capital T gets canceled and I get the average of the fractional occupation time to be given by this expression.

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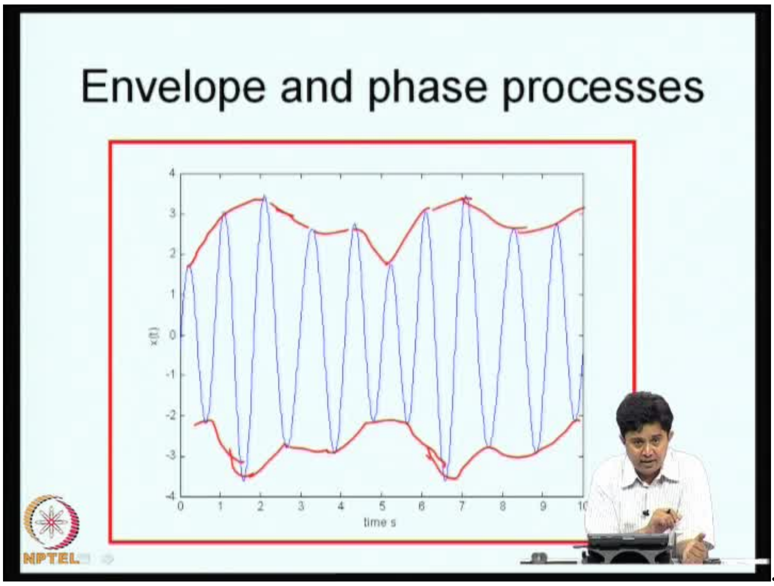

We will see, how it can be used later, when we talk about failures, how this theory would serve charactering failures. The next topic on our list is description of envelope and phase of random processes. To motivate you to the basic problem here, we can consider sample of a narrow brand processes, an envelope would mean that, you have a curve that would pass through the peaks; this is intuitively the notion of an envelope, which is something that can something within which we can encloses, the signal is an envelope something raise. You can see that, the process is oscillating lot more than this envelope; so, envelope is a slowly varying function and we therefore expect that it would be easier to characterize this, then the parent process, that is the expectation.

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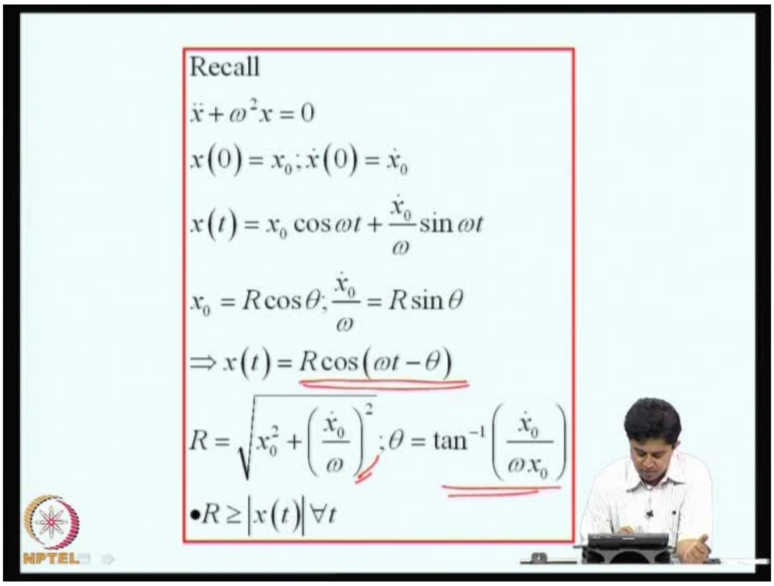

The notion of envelope and phase is not new to us, it is widely used in structural dynamics, for example, if you recall un-damped free vibration of a single degree freedom system, say the equation of motion is x double dot plus omega square x equal to 0, we can construct the solution as x of t equal to x naught cos omega t plus x naught dot by omega sin omega t. Now, if I substitute for x naught x naught is R cos theta and x naught dot by omega is R sin theta, we can express x of t in this form, where R is square root of x naught square plus square of this quantity and theta is tan inverse of x naught dot by omega x naught. This R can be thought of as an envelope of x of t and this theta is the phase; here, R is greater than or equal to the amplitude of the x of t for all t, that would mean, if you plot the sample of x of t, this red line is R of t; actually, this is constant in this case, it is R.

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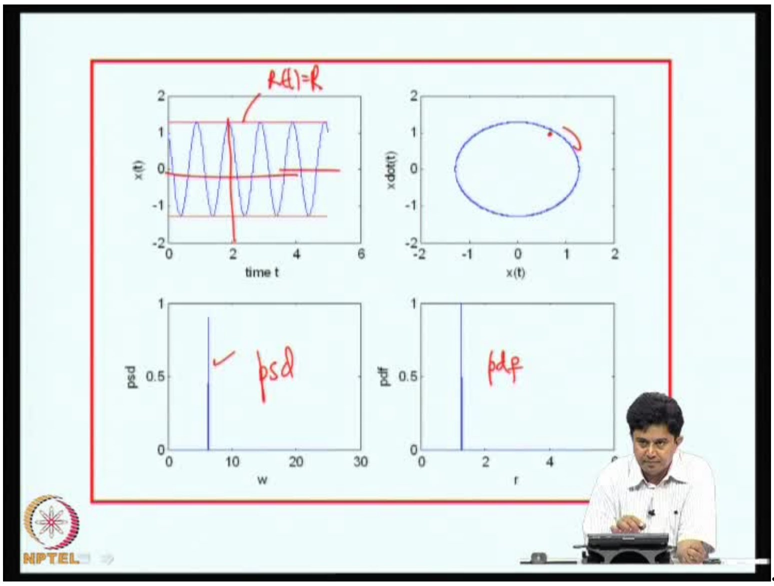

Now, if you plan power spectral density function of this signal, we already shown this is direct delta function. On the other hand, you look at the probability distribution of R, it takes one value and that is again a direct delta function, but this is probability distribution function, this is psd. Whereas, in psd I am looking at frequency distribution, in envelope I am looking at amplitude distribution, that means, in envelope I am looking at value at a given time the highest value; whereas in power spectral density, I am looking across the time, it is a global descriptor. Now, we are moving towards amplitude of the signal, which is encapsulated in a probability distribution function and not in power spectral density function. If we plot now, x dot versus x of t, for this particular sample, we get a close curve, here what you have to do is associated with x of t at x dot x t and for every value of t, I have a pair of numbers x dot of t and x of t and I plot one point here; as a time advance, is this trajectory x of t and x dot of t, we will trace a curve, which will be close curve, because the signal is periodic.

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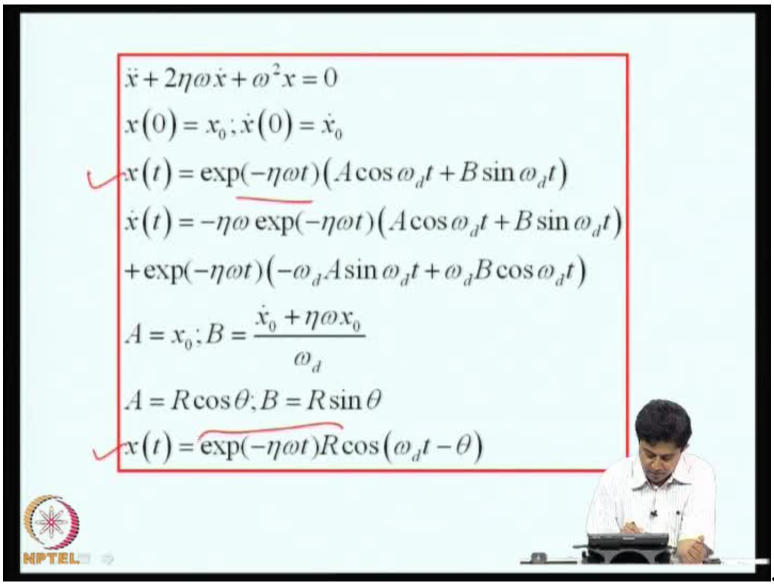

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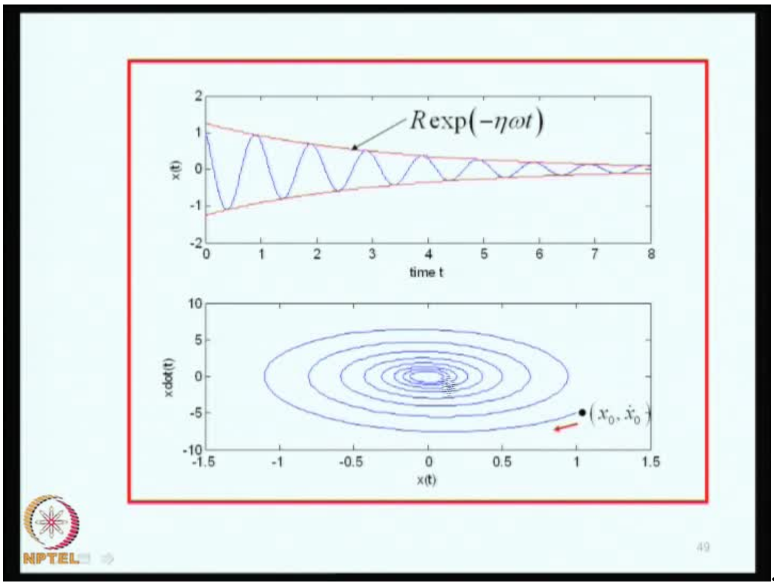

You look at now damped free vibration, we can do a similar analysis, I mean, this is familiar to us; we can express x of t, as you know this is the solution of we get exponential e raise to minus eta omega t a cos omega d t plus B sin omega dt and derivative of this is the velocity and by imposing the given initial condition, we can determine A and B, in terms of initial conditions. Now, if I make a substitution, a as R cos theta and B as R sin theta, x of t can be written as exponential minus eta omega t into R cos omega t minus theta; so, this function can now be interpreted as the envelope, how does it look like, this is R e raise to minus theta omega, which bounds the blue lines bounded by these two pairs of red lines; so, this is the envelope of x of t. In this case, suppose, if you plot a velocity versus displacement curve; this will be a decaying curve, so it spirals down and eventually it comes to origin. This is the initial condition and as time passes, it will propagate along this trajectory and it will finally spiral down to the origin.

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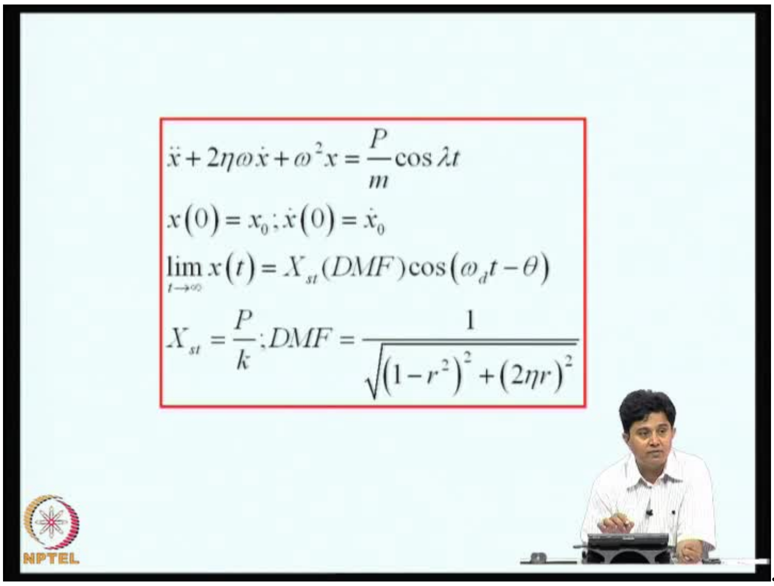

Now, if you are taking a harmonically driven single freedom system deterministic, we know that, the response is characterized in steady state by the static response into dynamic magnification into cos omega dt minus theta and this dynamic magnification is expressed, in terms of the frequency ratio lambda by omega and damping ratio.

So, this quantity x x t into DMF can be viewed as the envelope of response of the system to harmonic excitation and there is an associated phase as well. We will consider in the next class, what happens, if this excitation instead of P cos lambda t, is a general force f of t; how to characterize an envelope and followed by that. We will discuss how to generalize this notion of envelope, if f of t is a random process; that means, x of t is a random process. How do we introduce the notion of envelope and phase, that is the topic for next lecture; so, with this we will conclude this lecture.