Stochastic Structural Dynamics

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Lecture No. # 17

Failure of Randomly Vibrating Systems-1

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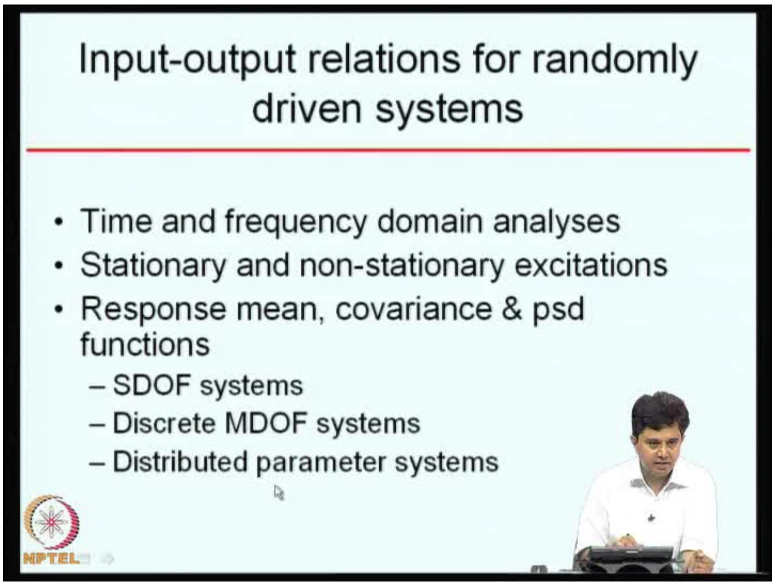

Today, we will start new topic, we will discuss how to describe failures in randomly vibrating systems; a quick recall of whatever, what we have been doing till now, is that, we have established now input-output relations for randomly driven systems, our attention is a limited to linear systems, linear time invariance systems.

So, we have established input-output relation in time and frequency domain; in time, we have impulse response functions, and in frequency domain, we have the frequency response function. We have considered stationary and non-stationary excitations; for stationary excitations, both time and frequency domain descriptions are possible, for non-stationary excitations, generally, time domain description is possible.

We have derived expressions for mean of the response, covariance of the response and prospect density system of the response. We have considered single degree freedom systems, discrete multi-degree freedom systems and distributed parameter systems. For single degree freedom system, the impulse response of scalar function; similarly, the frequency response function is also a scalar function. For discrete multi degree freedom systems, the impulse response function becomes a matrix, we get matrix of impulse response functions; similarly, we have a matrix of frequency response functions, these are square matrices

For distributed parameter systems, the impulse response function and frequency response function become function of space variables; so, it is a scalar for single degree, a matrix for discrete multi-degree freedom system, whereas for distributed parameters systems, typically it is functions of special variables, the drive point as well as measurement point.

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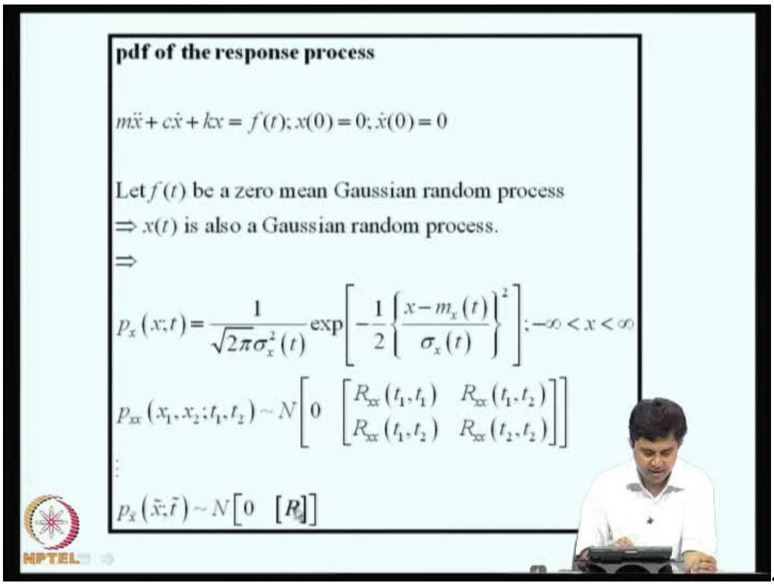

We have restricted our attention to characterizing the mean and covariance and power spectral density function of the response, and if we now ask questions on probability distribution function of the response, what is the nature of probability density function of the response; then we need to specify for the input, the probability density function characteristics.

Suppose, if you consider a single degree freedom system excited by a random excitation f of t and if you assume f of t be a Zero mean Gaussian random process, we can show that x of t is also a Gaussian random process; we have discuss this previously, because this transformation is linear, this statement is true.

Now, if x of t is a Gaussian random process, then the first order probability density function of x of t is shown here; this mean, parameter m x is a function of time, the standard deviation is also a function of time, that would mean, I am allowing for the fact that x of t could be non-stationary.

Similarly, if you consider two time instants t 1, t 2, we will get two random variables; the joint density function of these two variables is again normal 0 mean and covariance matrix given by this matrix as shown here.

Now, if you consider n time instants t 1, t 2, t 3 so on and so for t n, then the resulting vector of random variables x of t 1, x of t 2 and up to x of t n will have a multidimensional Gaussian density function with 0 mean and a covariance matrix which we can denote as capital R.

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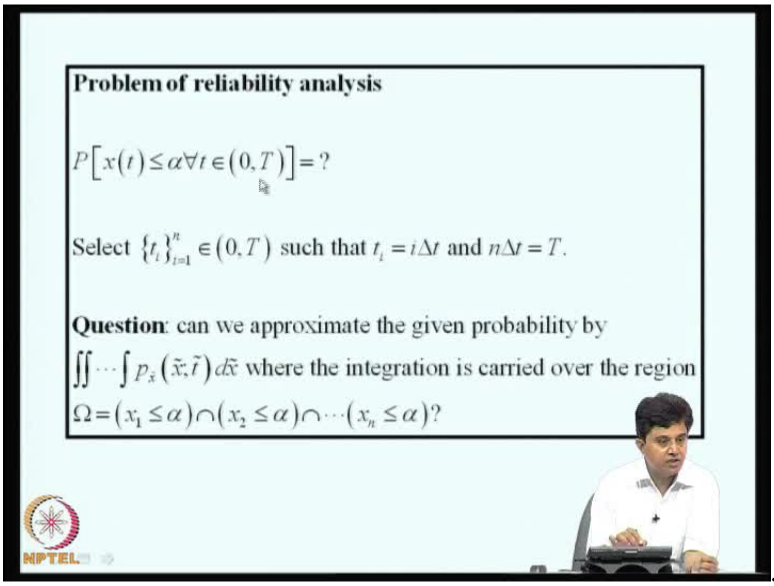

Now, typically we are interested in asking the question, what is the probability that the response case bellows safe limit alpha over a time duration 0 to t? Now, the question is based on what we have learned so far, can we answer this question, suppose we begin by considering n time instants t 1, t 2, t 3, up to t n belonging to 0 to t, such that t i is i delta t and n delta t is T. We get associated with these time instant n random variables x of t 1, x of t 2 etcetera and we have already derived the joint density function of these n random variables; now, this probability our objective is to evaluate this probability. Now, suppose if I consider can the evaluation of this probability over the region omega x 1 less than or equal to alpha intersection x 2 less than or equal to alpha and so on and so forth, the evaluation of this integral over this region, would this, serve as an answer to this question, that is the question.

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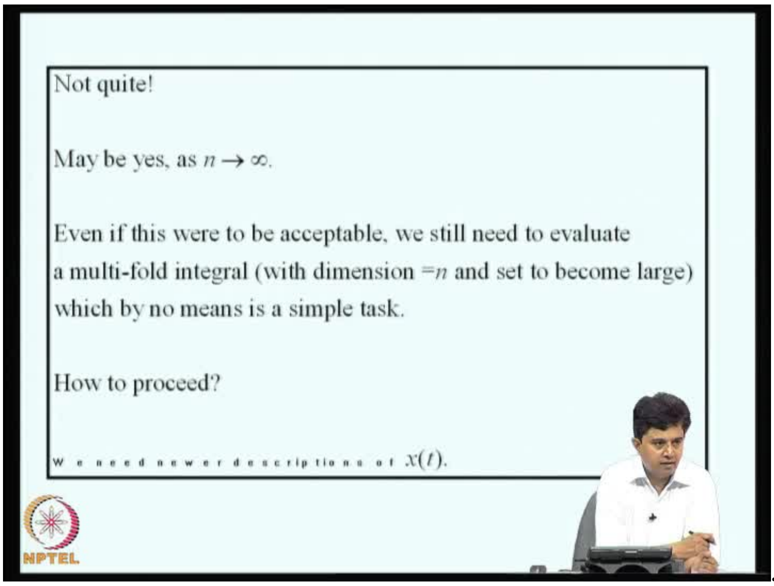

Not quite, may be as n tends to infinity, but even if we succeed in evaluating the n-dimensional integral as n becomes large, it would not be a satisfactory answer, firstly because as n becomes large, we will have the impossible task of evaluating a multi fold integral with dimension n, which is set to become large and this is not by no means a simple task.

So, this seems to be an impassive, here the question is now how we should proceed, the suggestion is that, we need to develop newer description for x of t, what these descriptions are is the topic for today's lecture.

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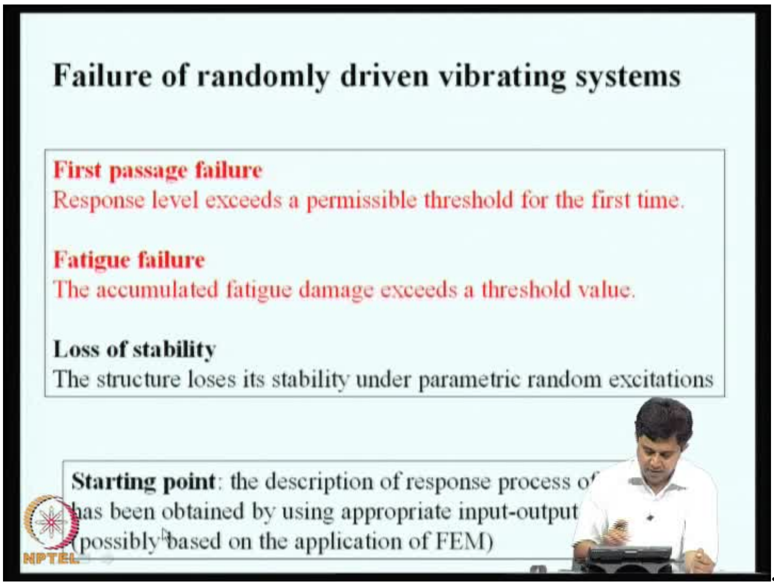

So, we are taking about failure of randomly driven vibrating systems, one can look at this problem from three perspectives, the three types of failures that we could expect, one is vertices known as first passage failure, here the response level exceeds a permissible threshold for the first time, for example, an electric bulb burns for such so many n number of powers and it burns out.

So, once it burns out, its life has ended and there is no further purpose of analysis; this is known as first passage failure. We can also have a failure known as fatigue failure, I will elaborate slightly on this shortly, here the accumulated fatigue damage exceeds a threshold value and the structure fails, there could be other modes of failure where the structure loses stability, for example, the structure could lose its stability under parametric random excitations.

We are not going to discuss this loss of stability due to random excitations, but we will restrict our attention to description of first passage failures and fatigue failures. Now, the starting point for this would be the description of response process of interest based on the analysis methods of analysis that we have learned so far.

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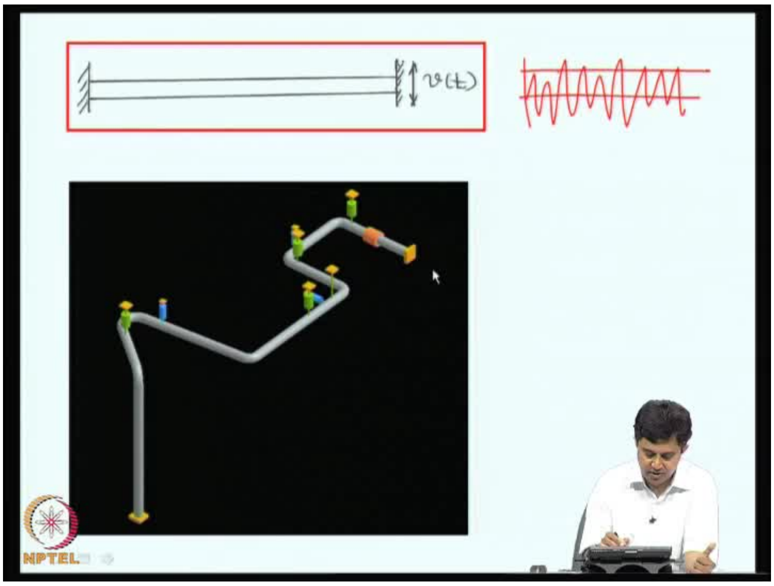

Just to describe what could be the modes of failure, we will consider, for example, a structure which is subjected to a random support displacement; here we may be interested in, for example, reaction transferred to the support or principle stresses crossing certain thresholds or a one metric stress on a metric, one metric, one mises stress metric crossing a threshold value so on and so forth.

Suppose, x of t is some characteristic response, this could be displacement reaction, one mises stress etcetera, so there will be a threshold value which we are willing to tolerate; so, this may exceed. Similarly, in a multi supported piping system, the stress in the bend could exceed permissible values or reaction transferred to the supports, could exceed certain values and so on and so forth.

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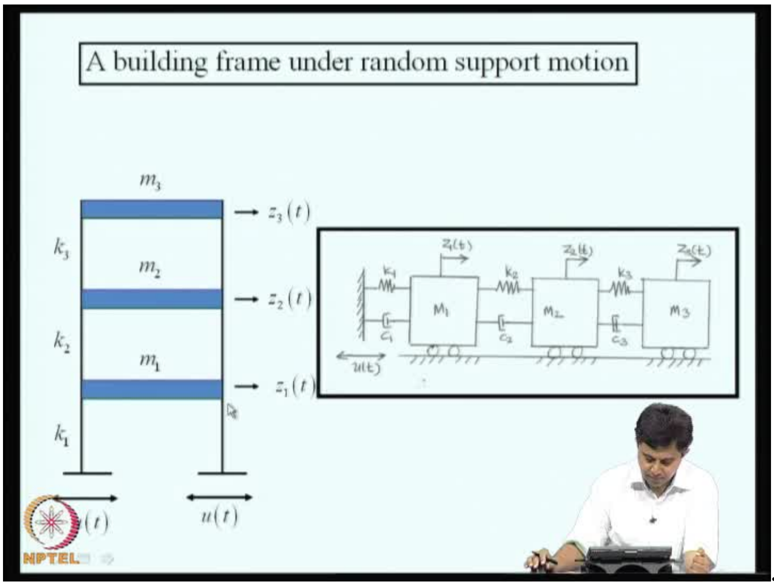

In a building frame under earth quake like support motion, there could be several failure criteria, for example, the basher transferred may exceed the certain acceptable values or inter story drifts could exceed certain values or the tip displacement could exceeds certain values, which are not acceptable; so, we define a random process that would characterize this failure and look at the questions on crossing of acceptable limits.

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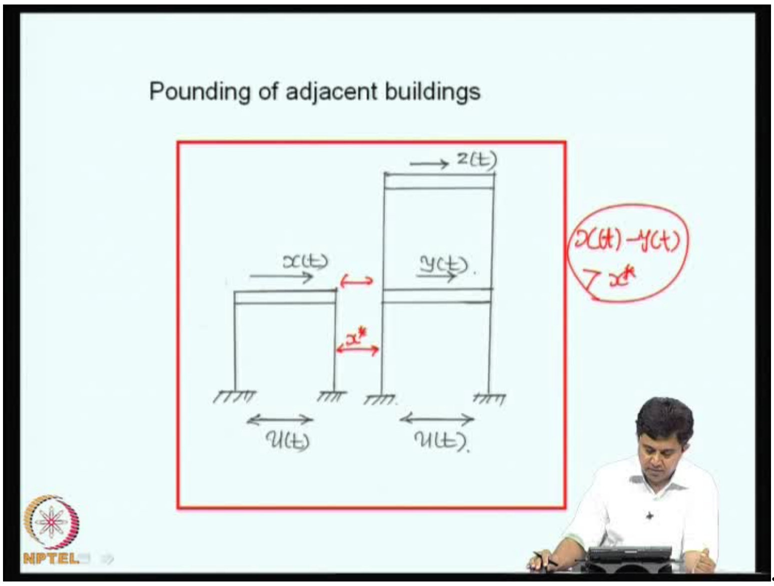

There could be other types of failure, for example, if you have two adjacent building frames subjected to say earth quake ground motion, each of this structure in isolation could be well designed, in the sense, they can with stand this load without crossing any permissive any permissible limits, but on the other hand, if these two structures are constructed close to each other, then there is a risk of pounding, that means, these two slabs may hit each other and that impacting due to pounding may cause destruction.

So, we would be interested in knowing, for example, x of t minus y of t crossing, this say the distance x star; x of t and y of t are produce basically due to the same excitation, but their characteristic depends on the elastic and mass properties of this structure, damping property of this structure as well as this structure which will be invariably be different. So, x of t and y of t will have different characteristic; so, evaluation of this event is of interest. So, here the failure occurs because the process x of t minus y of t crosses the threshold value x star.

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We talked about fatigue failure; any structural material on some scale will have certain imperfections. Due to application of cyclic stresses, there will be a reversal of stresses and because of that the amount of defects that are present invariably present grow.

So, in a cubic volume of the material, if the amount of defects grows, the strength of that material comes down and this phenomenon that is loss of structural integrity, due to reversal of stresses is known as fatigue. So, typically this kind of problems arise, for example, in gusset place at the junctions in railway bridges; so, as these type of loads are repeatedly applied, there will be a stress fields, cyclic stress fields in the body of the bridge and typically at the junctions and certain other critical points, there will be a accumulation of fatigue damage, and in the example like this, one could easily imagine that the payload of the train could be random, the speed at which the train moves could be random and there could be guide way unevenness, due to, for example, the fluctuations in a gage or fluctuation in vertical profile, so on and so forth. So, the accumulation of fatigue damage here will be occurring, under a random environment and the question is how do we characterize this?

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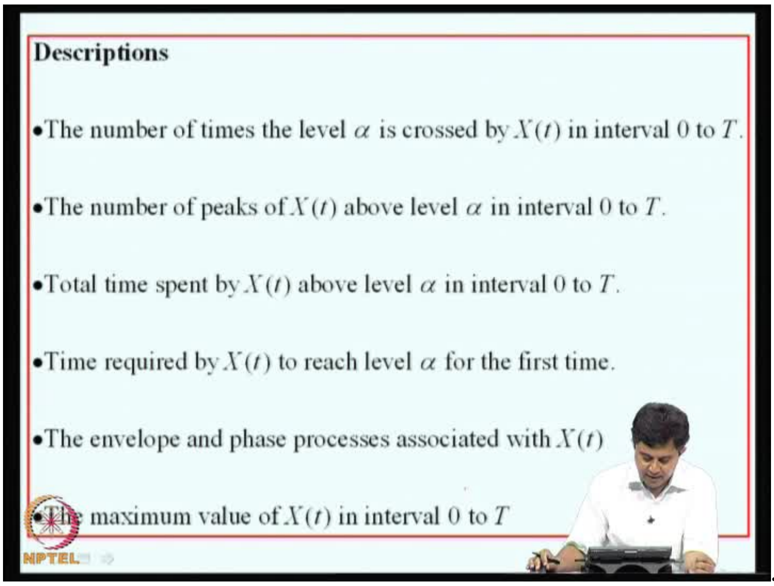

Now, we hope to be able to find answers to the questions that we pose still now, but we should return to certain theoretical formulations, before we could develop the necessary tools to answer these questions.

So, let X of t be a random process and let us assume that we have a complete description of X of t in terms of a n th order joint density function, probability density function. Now, we will ask the few questions about X of t; the first question we ask is what is the number of times the level alpha is crossed by X of t in a interval 0 to T; then, how many peaks are there in X of t above a level alpha in a time duration 0 of t 0 to t; how much time does X of t spend above level alpha in a time interval 0 to T.

Similarly, what is the time required by X of t to reach a level alpha for the first time; then we talk about what are known as envelope and phase processes associated with X of t, I will describe this shortly and finally we are interested in knowing what is the maximum value of X of t in a interval 0 to T. So, these are some of the questions that will be asking, I will briefly take up each one of this short and explain what exactly the problem is.

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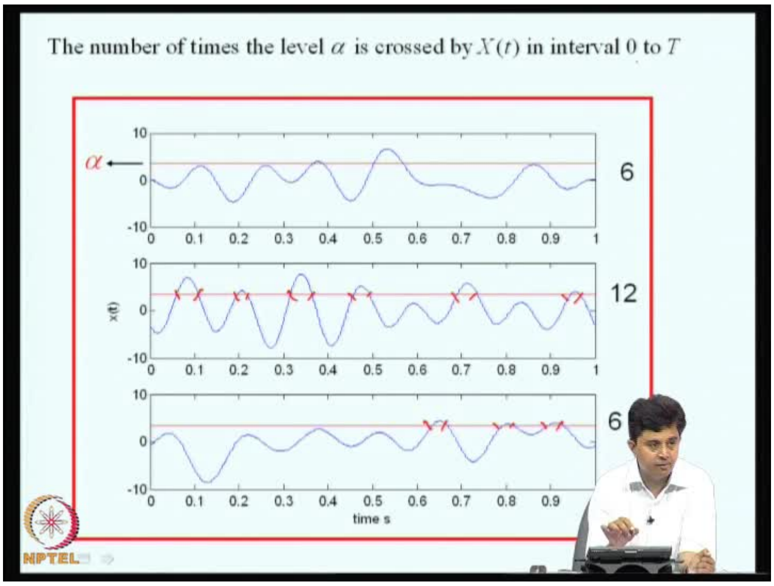

In this figure, you see three samples of a random process X of t, this is a critical level alpha shown here; in the first question, that we are asking, we are trying to find out the number of times the level alpha is crossed by X of t in the interval 0 to T, suppose we consider the interval 0 to 1 second, that is 0 to T.

So, in this sample, if you see how many times level alpha is crossed, we have a crossing here, we have a crossing here, here, here and probably twice here; so, there are six crossings, here in this realization of X of t. Now, another realization of the same random process as shown here, we have one crossing here, here, here, here, and here; so, there are 12 crossings, yet another realization there are 1, 2, 3, 4, 5, 6.

So, if you consider one realization of X of t and count the number of times level alpha is cross in 0 to 1 second, you get a number which can be viewed as outcome of a random experiment or in other words, this number itself is a random variable, that means, the number of times, the level alpha is cross by X of t in interval 0 to T, is the random variable.

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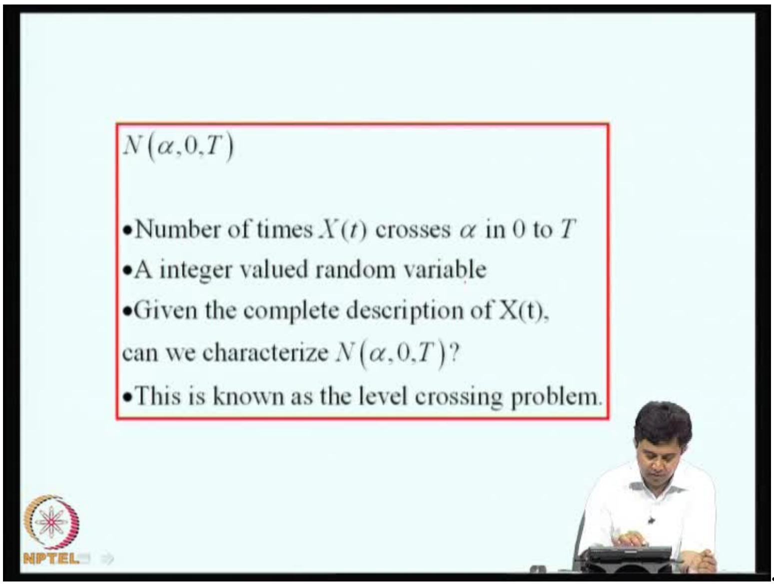

We denote this by N (alpha, 0, T) that is number of times level alpha is crossed in the interval 0 to T; it is an integer valued random variable. Now, the question is, if we are given the complete description of this random process X of t, what is the probability distribution of this random variable N, can we derive it or can we estimate at least its mean and variance, what we can do about it; so, this problem is known as level crossing problem. This is the first problem that we will be considering, before we get in to the details of the level crossing problem, we will run through all the other descriptors that I just mentioned.

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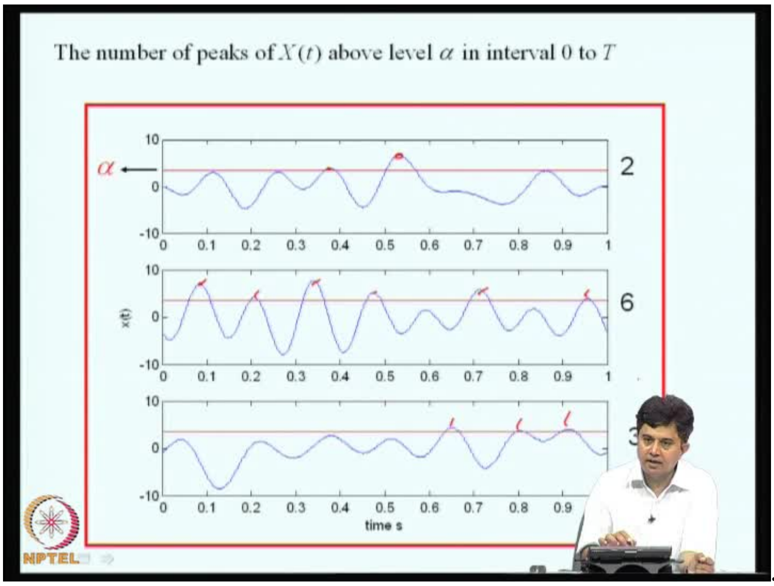

So, the next in our list is the number of peaks of X of t above level alpha in interval 0 to T, again I consider the same three realizations of random process X of t, how many peaks are there, now above level alpha here, I see 1 here, I see 1 here, now it is 2 for this sample, for this sample there is 1, 2, 3, 4, 5, 6 whereas for this sample it is 1, 2 and 3; of course, X of t will have many more realizations and associated with each realization, I will have different numbers. Again counting this peaks above level alpha for each realization, you know what the count that we get can be viewed as outcome of a random experiment; therefore, this number itself is another random variable.

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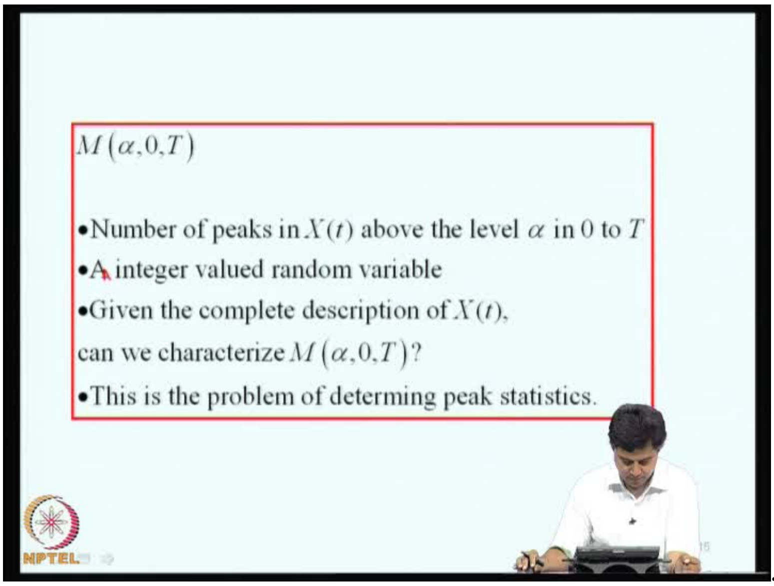

We denote this random variable by M of alpha of 0, T; so, this is the number of peaks in X of t above the level alpha in 0 to T; it is an integer valued random variable. Again the question is given the complete description of X of t, can we characterize this quantity m? Can we find out its probability distribution function or its mean or variance what we can do about it? This is the problem of determining the peak statistics of X of t.

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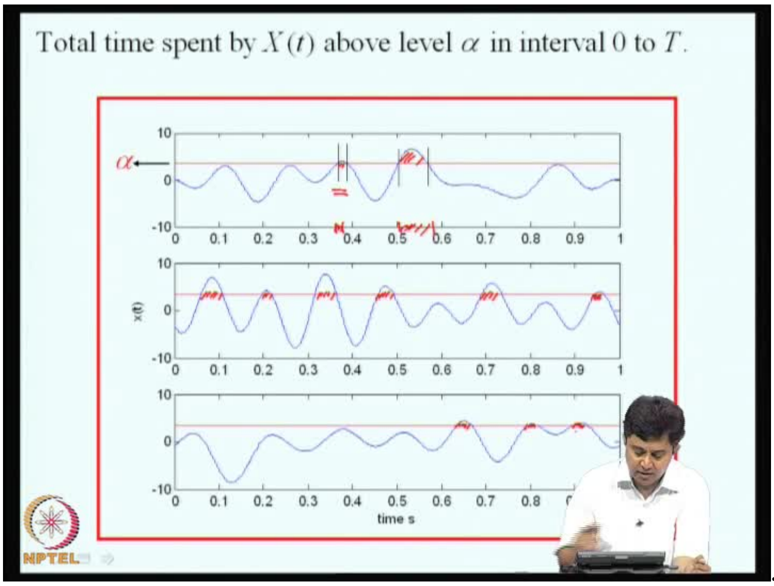

The next problem that would be of interest, is the total time spend by X of t above level alpha in the interval 0 to T. How much time X of t spends above level alpha, for example, here in this sample I have drawn two vertical lines here, you can see that in this time interval X of t is above level alpha; similarly, here in this time interval X of t is above alpha, so this region, this region, if you project it down, this is the time, that we are, if you add these length of these two intervals, we get the total time that it as spend level above alpha in this realization.

Here, of course, there is this, this, these are episodes where X of t is excursing about alpha and short stay here; so, if you take union of this, I get again the time that X of t has spent above level alpha; so, here I am measuring time it is a real valued quantity that I am measuring, but for each realization I get a different number, different real number; so, here, again here, for example, this is this, this is this, this is this.

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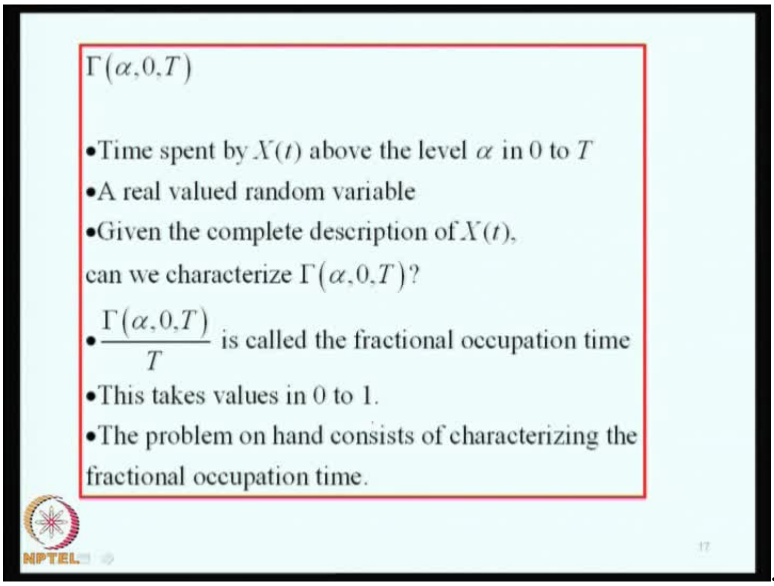

This we denote by gamma (alpha, 0, t), this is time spend by X of t above the level alpha in 0 to t; it is a real valued random variable. Now, the question is given the description of X of t can we characterize gamma? We can simplify this problems slightly by defining the ratio gamma by capital T, this is non-dimensional; this quantity is called fractional occupation time. This fractional occupation time takes values in 0 to 1, it is a random variable taking values between 0 and 1; it is a real valued random variable. The problem on hand consist of characterizing the fractional occupation time, in terms of its say probability distribution function or moments or whatever we can do about it, what we know is properties of X o t.

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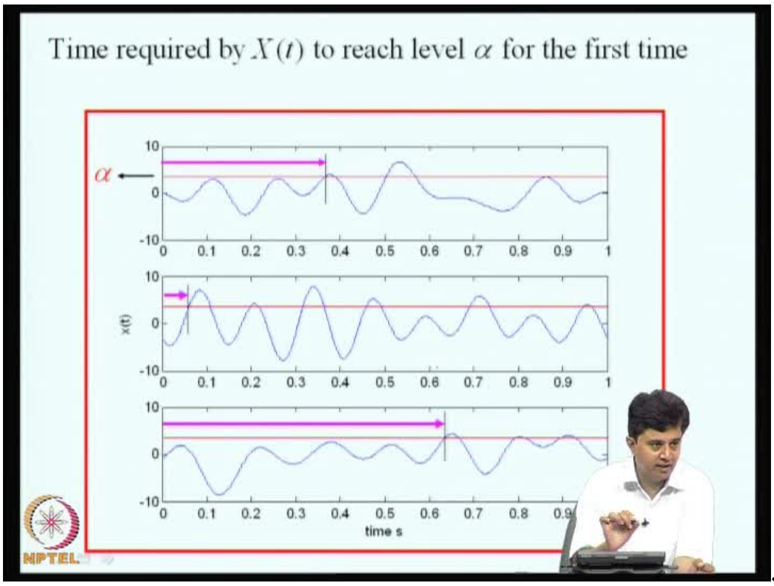

Now, what is the time required by X of t to reach a level alpha for the first time? Imagine, this alpha is some critical threshold and X of t is the response, so within the time frame, that we are looking at in this sample X of t, takes this much time shown in pink arrow; this is the time needed for X of t to cross level alpha for the first time, whereas for this, it is this whereas for the next sample, it is this. So, for each sample, we get different timed lines; so, these numbers that we observe are real numbers, they take values in 0 to infinity, the level alpha may never be crossed or it may be cross immediately after you start recording the time.

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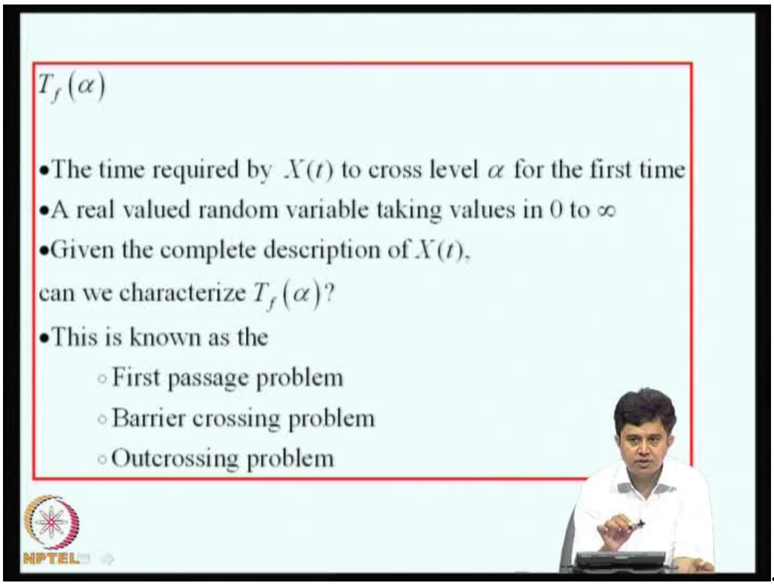

So, this is the time required by we denote this by T f of alpha f is the first passage, is the word that we would use; so, the time required by X of t to cross level alpha for the first time. It is a real valued random variable taking values in 0 to infinity.

Again the question is given the complete description of X of t, can we characterize the first passage time? This T f of alpha, there are different names determining probability distribution T f of alpha is known as first passage problem or barrier crossing problem or out crossing problem, all these terms are used in the existing literature. So, in terms of characterizing the usual life of a structure, the first passage time is a very crucial descriptor.

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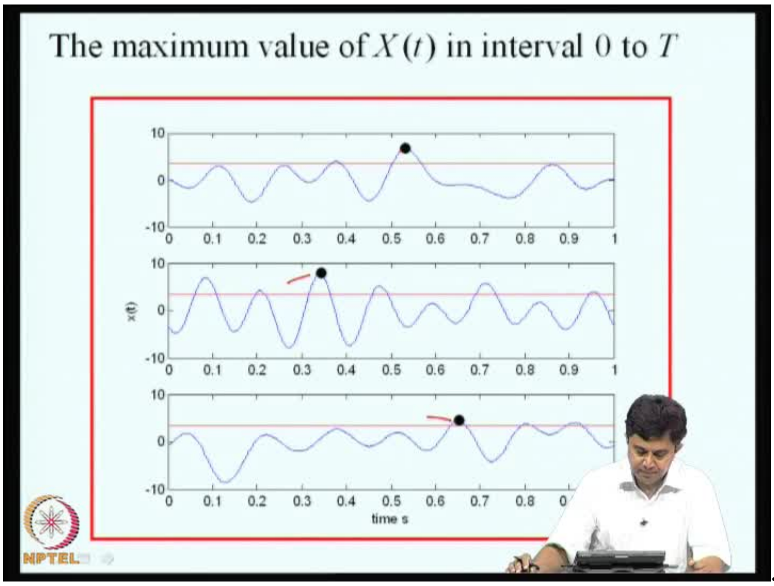

Next, in a given time interval of interest what is the highest value of X of t; this is, of course, the most important descriptor of a random variable from the random process, from the point of view of reliability analysis. If the maximum values takes below alpha, the entire response would have state below alpha; so, this as we will discuss shortly, if you find answer to this question, the problem of determining the probability that X of t stays below alpha over a time duration, where we, when we started discussing we ended up with a multi fold integral, because there are large number of random variables that we need to handled, but if you can get the probability distribution of this maximum value, then that multi fold integral collapses to a single integral; so, its simplifies the problem enormously. So, again, here for this sample this is the maximum value; for this sample, this is the maximum value; and for this sample, this is the maximum value; so, this is again a random variable, real valued random variable.

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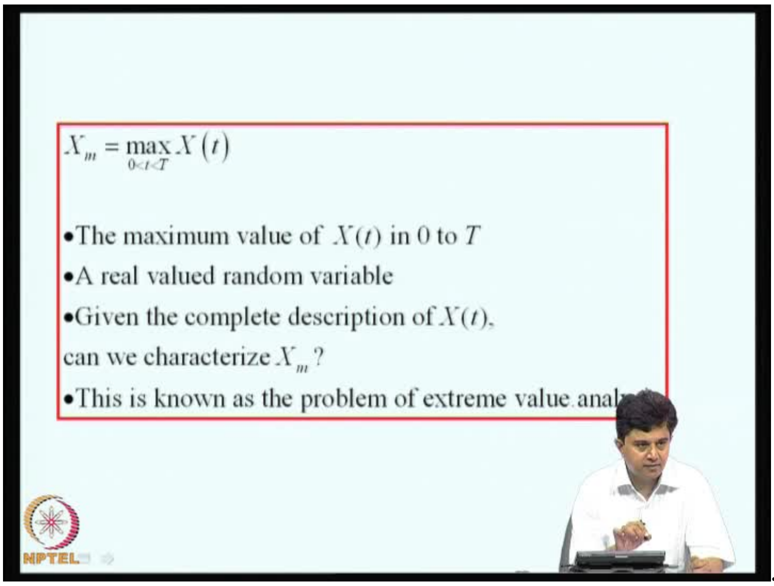

And we denote this as X m which is defined as maximum of X of T over the interval 0 to capital T; it is a real valued random variable. Now, the again the question is given the complete description of X of t can be characterize X m, can be find out its probability density function or its mean or standard deviation, what all we can do about it; this problem is known as the problem of extreme value analysis; this lies at the heart of reliability analysis. So, we need to now develop theories to answer these questions.

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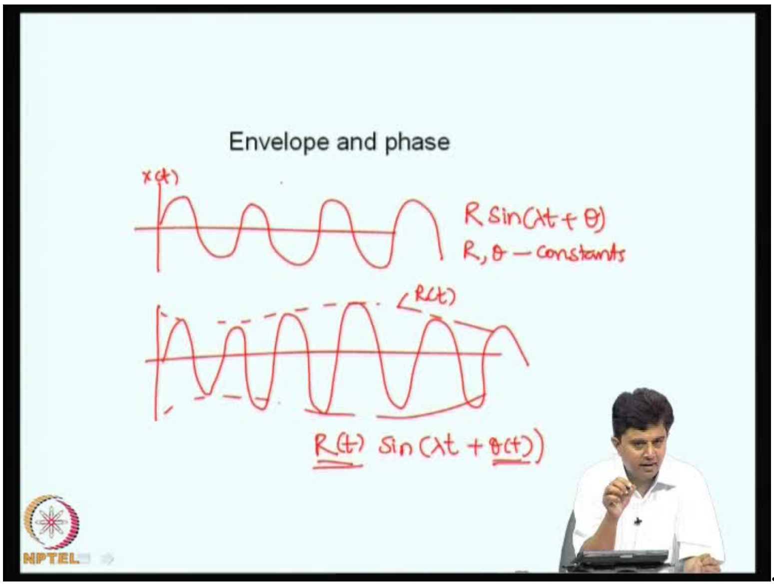

Now, I talked about quantities known as envelope and phase, I will just briefly explain what it is; suppose, if you have a sinusoidal function, this function can be represented as R sin lambda t plus theta, where r is, R and theta are constants, but on the other hand, if X, this is X of t, if this is deterministic, if X of t is a sample of a random process, it could vary in this manner, where we get, we follow the same representation and write this as sign lambda t plus theta of t, that means, this quantity R of t and this quantity theta of t are now functions of time presumably, slowly varying and we call this quantity R of t as an envelope and theta of t as a phase.

Since, R of t varies slowly, it is evident that description of X of t can be simplified, if we describe R of t; it is much, it should be much easier to describe R of t than X of t, because R of t varies much slowly than X of t. Similarly, here phase is constant, the phase will be slowly varying, it should be easier to describe a slowly varying function than a process X of t, which is, which could be rapidly varying.

So, the envelope and phase descriptions simplify the description of parent process X of t and they could serve as useful tools for elaborative modeling; again, we will see some applications of this idea as we go along.

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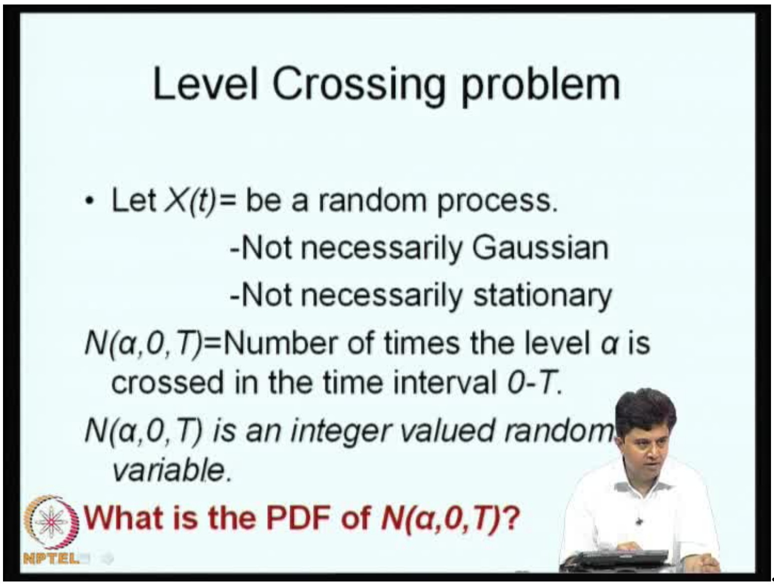

Now, I have stated these problems 6 or 7 problems, now we have to develop mathematical solutions to answer the question, that we have post just now. So, I will begin with discussion of what is known as level crossing problem. So, we will consider a random process X of t which need not be Gaussian which need not be stationary; I define N (alpha, 0, T) as number of times the level alpha is crossed in the time interval 0 to t, as we have seen N (alpha, 0, T) is an integer valued random variable and we are interested at the outside, then trying to find out what is the probability distribution function of this N.

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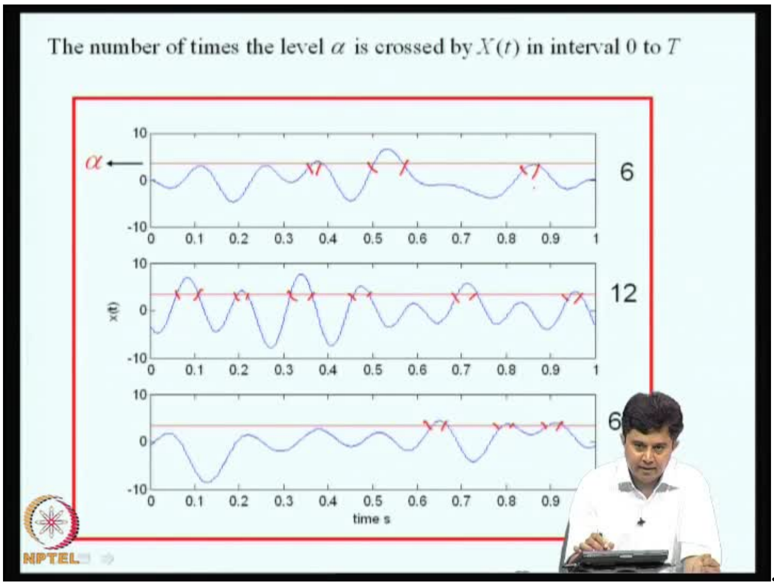

Now, when we looked at this pictures and when we counted how many times level alpha is crossed, I said this for example, here the number of crossing is 1, 2, 3, 4, 5, 6 here it is 12; we are able to count it if you are given a sample of X of t, but how do you develop a counter in mathematical language. We cannot be doing this visual counting, so how do we actually develop a formula that actually when applied on this sample it gives 6, when applied on this sample it gives 12 and so on and so forth or how to develop that counter, is the first problem that we have to tackle.

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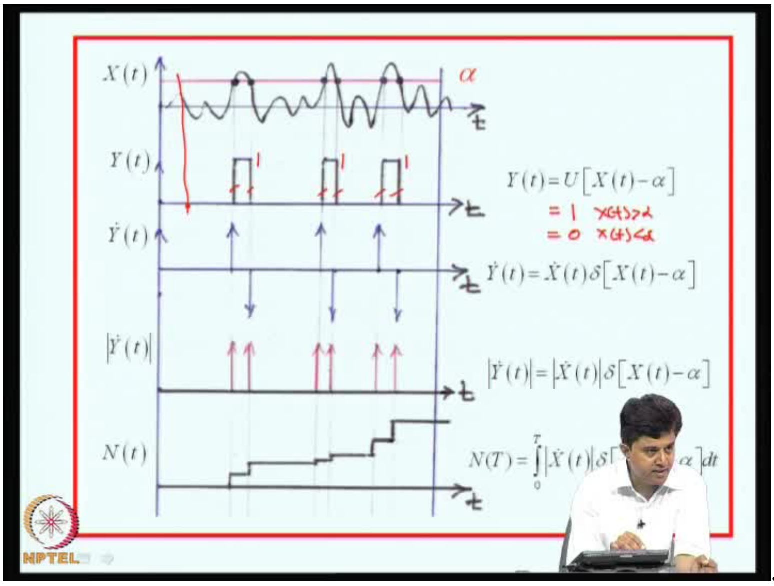

So, to do that we carefully consider one realization of X of t as shown it here; this is one realization of X of t and this is the level alpha we are interested in. As you can see here, in this sample, there are six times that X of t crosses level alpha; now, I have to get a counter, to do that what I do is, I define a random process Y of t as shown here, U is a step function, so this is equal to 1, if X of t is greater than alpha; equal to 0 if X of t is less than alpha.

So, when I start this at every time t, I will look at this Y of t now, here for instance at this time instant X of t is less than alpha; therefore, this is 0. And at this, when we enter this region take Y of t becomes 1 right, this is 1; this is 1 and all other places it is 0.Now, you can see here the 6 that I am looking for a is already here 1, 2, 3, 4, 5, 6 I have to somehow now find out how to count these walls of this small pillars.

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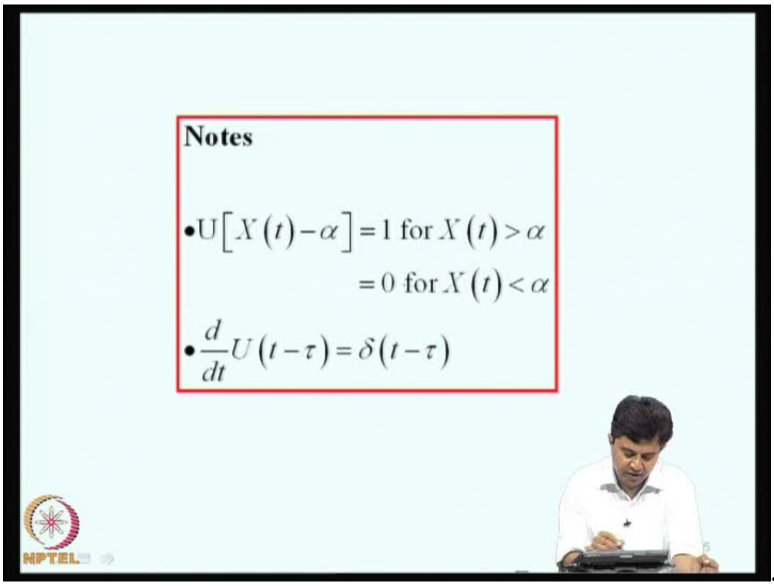

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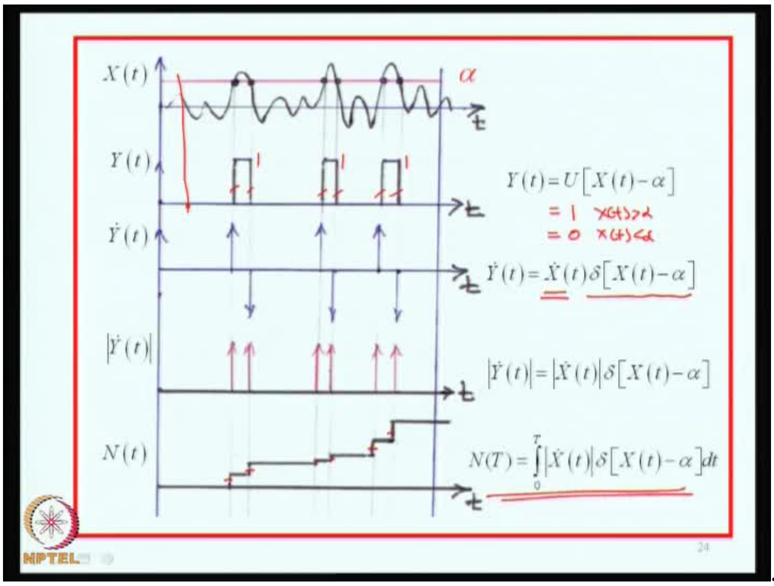

So, what I do I differentiate Y of t, so differentiation of… You can quickly recall this is the definition of step function U of X of t minus alpha is 1 for X of t greater than alpha it is 0 for X of t less than alpha. The formal derivative of a step function is a direct delta function right. So, if you apply that now if I differentiate this I get a direct delta here, I get another direct delta here; so, wherever there is a spike, I get a direct delta.

So, what is this function I had to differentiate Y of t with respect to t, so differentiation U of X of t minus alpha gives me direct delta of X of t minus alpha and that has to be multiplied by the derivative of this argument which is X dot of t. Now, I have this 6 year 1, 2, 3, 4, 5, 6 but if I simplify, add them I will get o, because these two will get cancel, so I will take now absolute values; so, if I take absolute values, these direct delta functions, all of them now surface above 0 and now I have 1, 2, 3, 4, 5, 6.

So, what is absolute value of Y dot of t, it is absolute value of X dot of t in to direct delta function. Now, I have to know count, suppose I start counting now number of times level alpha is crossed it is 0, 0, 0, 0, 0 here as soon as the entire here it becomes 1, I continue 1, 1, 1 and I cross this it becomes 2 so and so.

So, I have to define an integral of this, I call it as N of T which is 0 to t X dot of t delta of X of t minus alpha d t; so, here you see, here there is 1, 2, 3, 4, 5, 6 the value of this N of T will be 6; so, this is the actually the counter that we are looking for. So, it involves a very complicated transformation on the parent process X of t, it has a modulus operation and there is an indirect delta function here.

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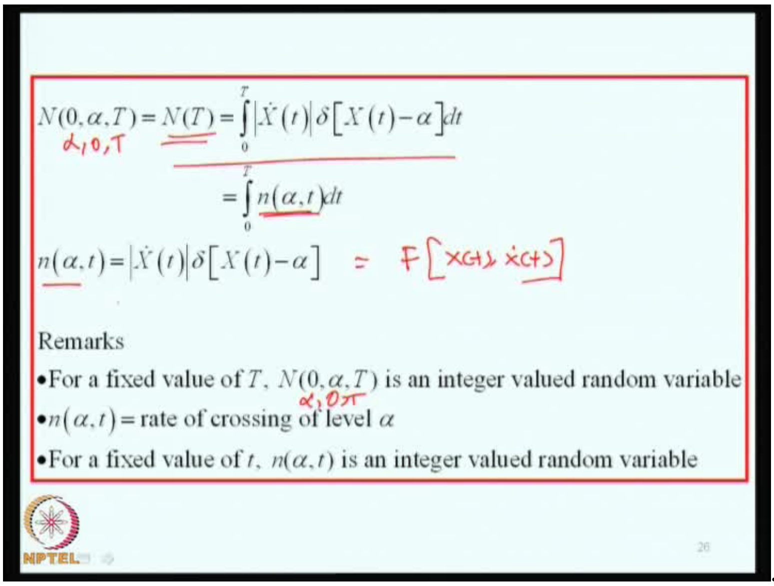

So, I have now the counter ready N (0, alpha, T), I will, now onwards write it as N of T; it is understood that it is level crossing of level alpha in the interval 0 to T, I call it as N of T, so this is integral 0 to T modulus of X dot of t direct delta of X of t minus alpha dt.

This integrant, I call it as N (alpha, t) this is actually the rate at which the level alpha is getting crossed, so total will be the integrals 0 to T n (alpha, t) dt, because this quantity multiplied by dt integrated towards 0 to capital T gives the number; therefore, this is the rate.

Now, for a fixed value of capital T N (alpha, 0, T) is the integer valued random variable; this rate is again an integer valued random variable; for a fixed value of T, this is again a random variable. Now, you look at this transformation, since X of t is a random process which is completely specified, so in the integrant, for example, on the right hand side of this, there are two random variables, modulus of X dot of t and direct delta of X of t minus alpha; so, the two random variables here are X of t and X dot of t.

So, this can be viewed as some function of X of t X dot of t. We have considered the problem of transformation of random variables, earlier a one function of two random variables, we deduce the strategy to find the probability distribution of the resulting random variable, but this would be difficult, that idea would be difficult to apply here, because the nature of the transformation here are quite complicated; so, we need to device some other strategies.

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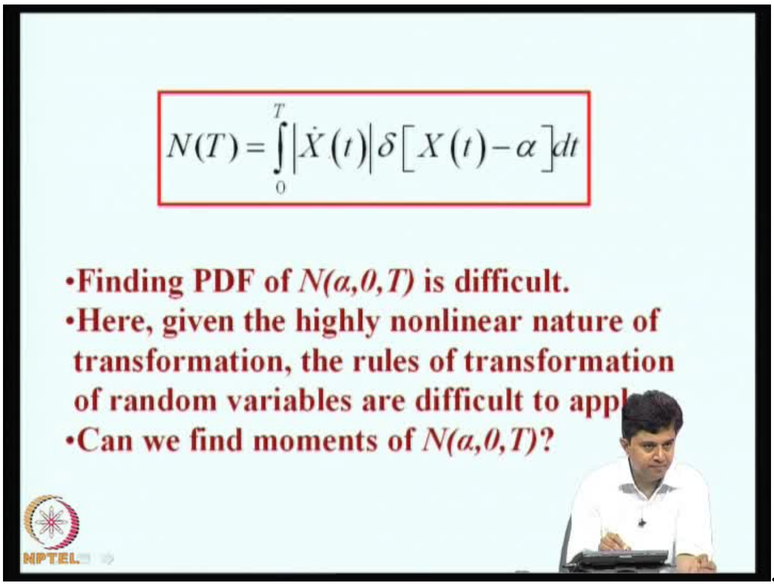

So, finding probability distribution function of this quantity is difficult. Here, given the highly non-linear nature of transformation, the rules of transformation of random variables are difficult to apply. So, we concede defeat here, the in fact, this is an unsolved problem even as of now.

So, what we do is, we settle to find the moments of this random variable; suppose, now, I ask what is mean of N of T, can we determine, that right, what is the data we have, it is complete description of X of t; if X of t is completely described, I can get the joint density function between X of t and X dot of t; so, it appears possible to compute mean of N of T.

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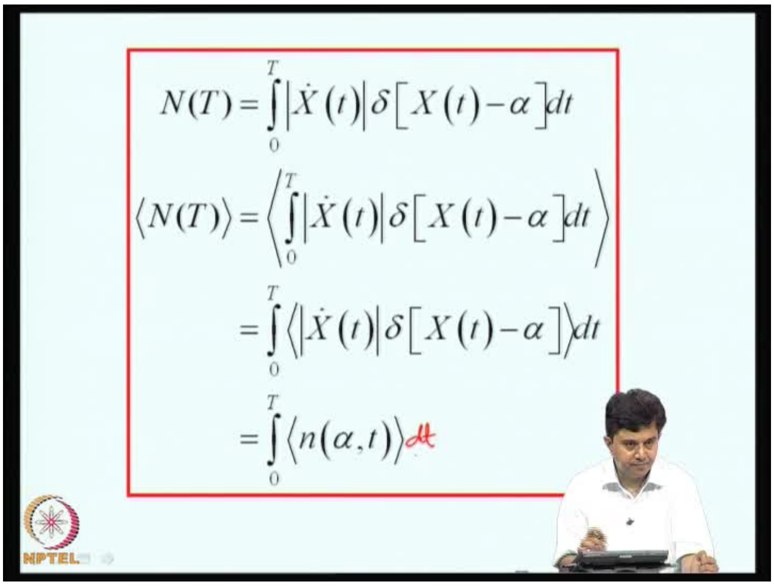

So, how do we do that? So, N of T this, so expected value of N of T is expected value of this integral and if we interchange the order of integration and expectation operator, the we get 0 to t expected value of this integrant and this is nothing but the expected value of this rate of crossing of level alpha into dt.

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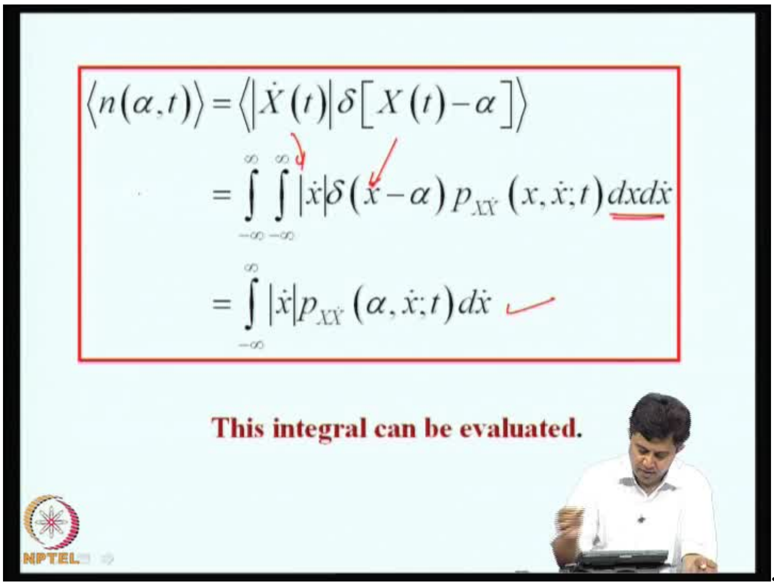

So, can we determine this, expected value of N (alpha, T), this is actually expectation of modulus of X dot into direct delta of X of t minus alpha, so this is nothing but double integral over X dx x and x dot minus infinity to plus infinity; mod X dot, that is this term and delta of x minus alpha, that is that term and the joint density function of these two random variables p X X dot x, x dot evaluated the same time dt.

So, this is the expected value of the rate, this integral can be evaluated; so, that would mean, I can find the average rate of crossing of level alpha and once this is known I can integrate from 0 to capital T and get the average number of crossings of level alpha.

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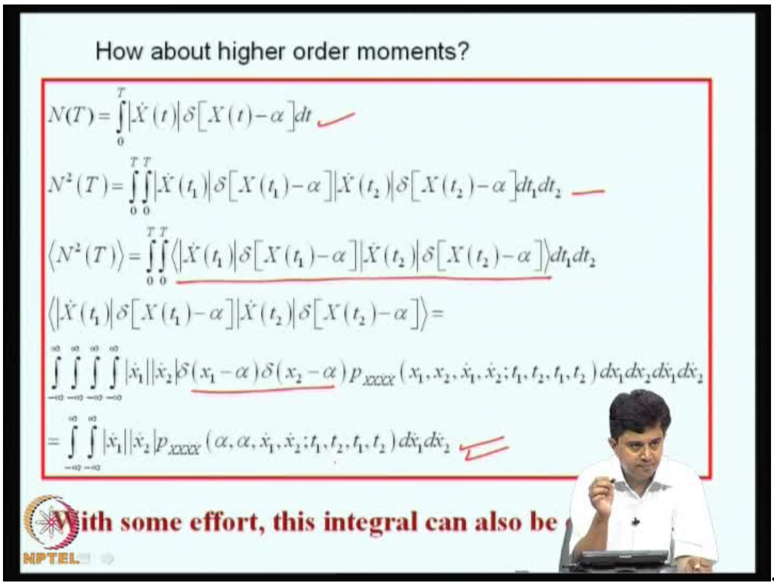

How about higher order moments? So, N of T is this, N square of T becomes a double integral that can be written as shown here. Now, if you are interested in expected value of N square of T, which would eventually facilitate you to find standard deviation of N of T, because mean has been already evaluated, it becomes an expectation of these expectation of function of now four random variables namely X dot of t 1, X of t 1, X dot of t 2, X of t 2, is again is a highly non-linear transformation of the four random variables involved, and if you look at the expected value of the integrant, it becomes a 4 4 integral on X 1 dot, X 2 dot, X 1, X 2 and the joint density of X evaluated at t 1, X evaluated at t 2; similarly, X dot evaluated t 1 and X dot evaluated t 2, there are four random variables here.

Since, there are two direct delta functions here, two of these integrals can easily be evaluated and we end up with a double integral as shown here. This again is not beyond, you know, its value within the reach, we could evaluate this if you choose to do it right; this is this, can be done; so, that would mean, if you know the properties of parent process, all though it is difficult to find the probability distribution function of the number of times level alpha is crossed, we can evaluate its means and its standard deviation. And if we choose, we can also evaluate higher order moments, but that task would become increasingly more complicated.

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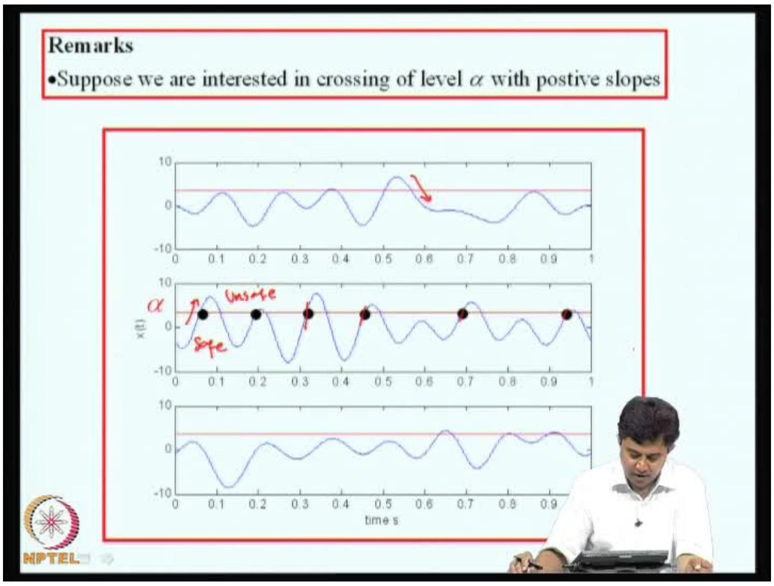

Now, suppose we are interested in crossing of level alpha with positive slopes, for example, if you think that X of t less than alpha is the safe region and this is unsafe. We are typically interested in studying crossing from safe to unsafe; so, this crossing from unsafe to safe may not be of interest. So, if that is the logic you accept, then I may be interested in simply counting the number of times level alpha is crossed from with the positive slope; so, you can see here this slopes are positive.

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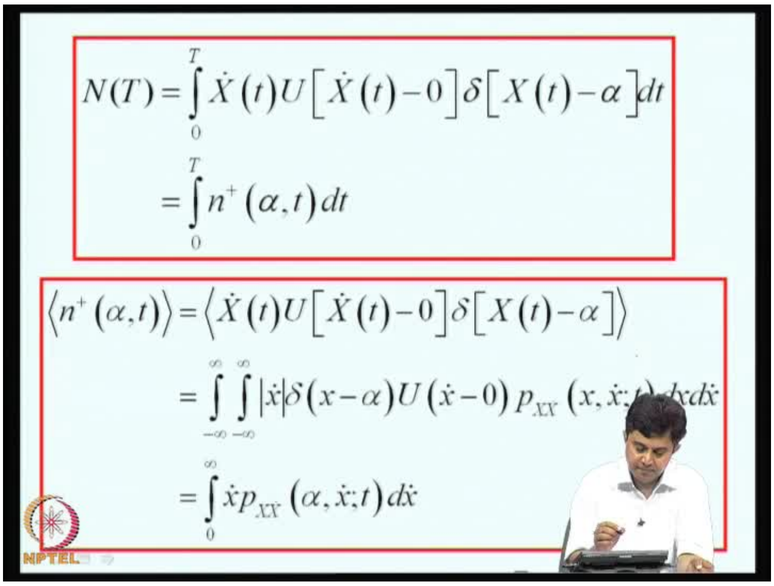

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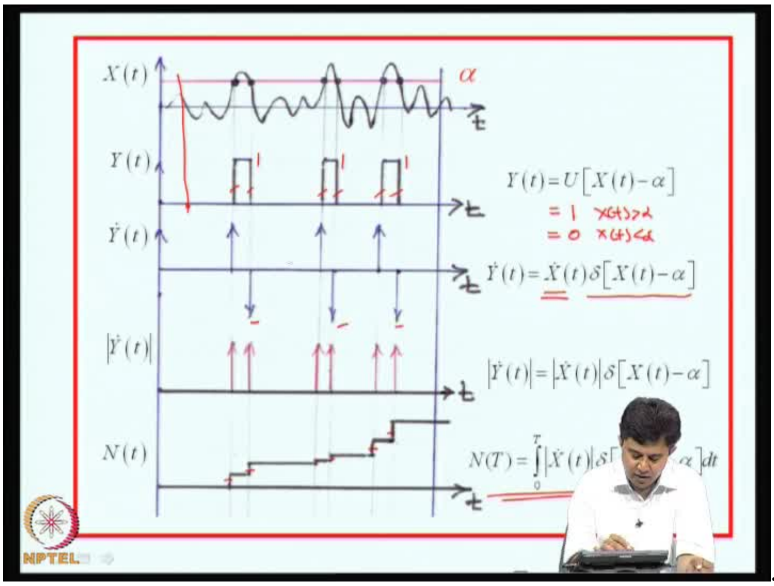

So, the counter has to be slightly modified, this again is the random variable, the counter becomes slightly different the counter would be now X dot of t into… What you have to do in that sketch that assured, here if you want only positive slopes, these three direct delta functions should be excluded. So, what I will do is, I will multiply this by a step function, where this Y dot of t is positive; so, that would eliminate this.

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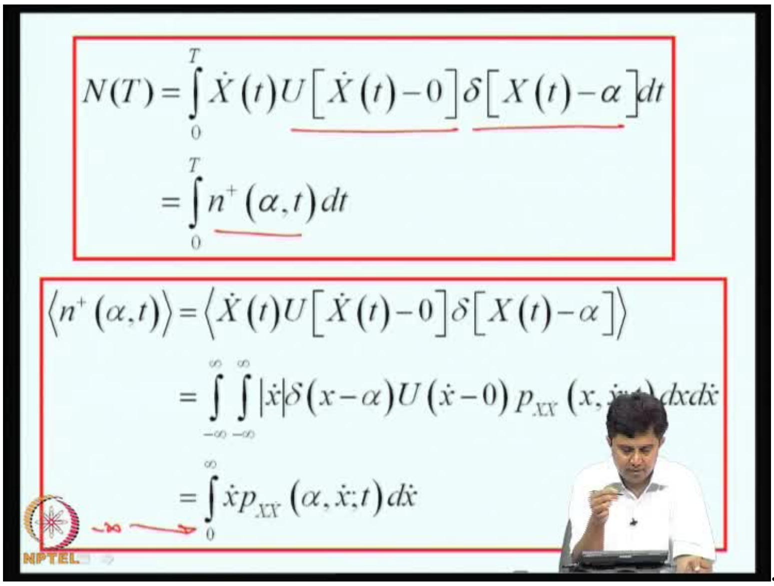

So, if you do that the counter would be N of T would be 0 to t X dot of t and we want only positive slopes and direct delta function remains, and we call, we denote this quantity as N of N plus alpha, t, that means, rate of crossing of level alpha with positive slopes. And we can evaluate, of course, it is expected value and this would be in the earlier case, we got the limit was minus infinity, now this becomes 0, 0 to infinity X dot p X X dot alpha X dot comma t dx dot.

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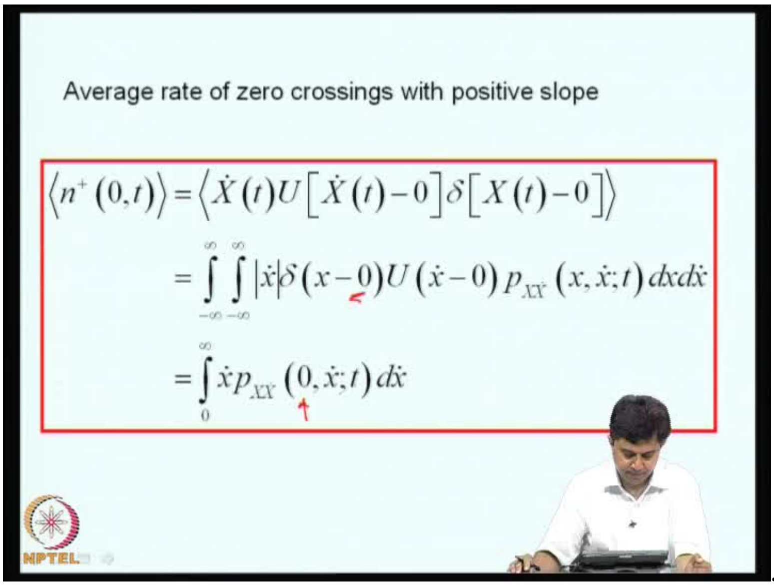

So, this logic can be extended, we can find out higher moments if you choose. Now, if alpha is 0, we call that as 0 crossings; so, we are interested, for example, in knowing the average rate of 0 crossings with positive slope, I will shortly, so that, this is the measure of the dominant frequency in the signal.

So, this n plus 0,t is expected value of this, where alpha is 0; so, that is the modification one, the other terms remain the same; here we had alpha and alpha is 0 here and this integral can be again be evaluated.

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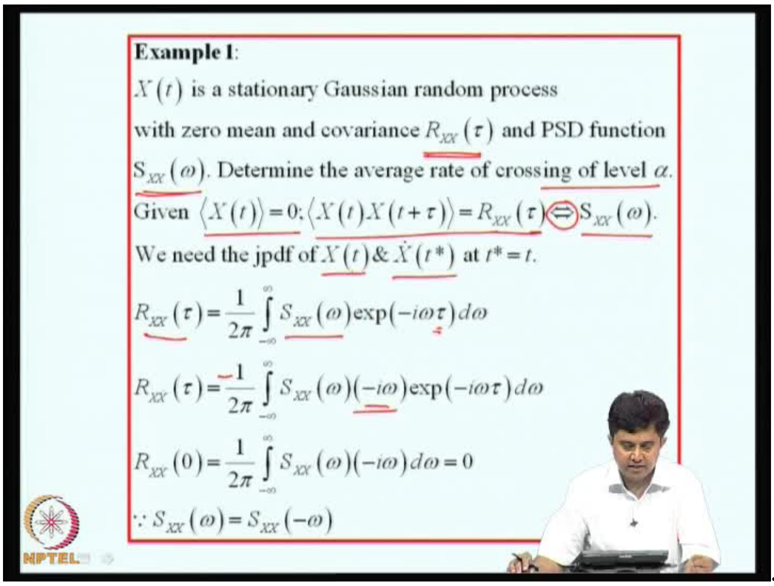

We will consider few examples, so that, this description becomes clear. So, we will start with random processes X of t, let it be stationary Gaussian random process with 0 mean and it is covariance be R x x of tau and PSD function be S X X of omega. The problem is to determine the average rate of crossing of level alpha. So, what we are given mean is 0 and covariance is this and the it is Fourier transform, this is S X X omega, that means, R x x of tau and of omega form a Fourier transform of pair, that is what this notations means.

Now, to find out the average rate of crossing of level alpha, we need the joint probability density function of X of t and X dot of t star, t star is equal to t, that means, the same time that process and its derivative, the two random variables, we need to find out the joint probability function of these two random variables. Now, if process is stationary, we can show that X of t and X dot of t would be un correlated; so, I have shown this earlier, but we can quickly recall, this is the relationship between auto covariance and power spectral density function, if you now differentiate with respect to tau, we get the auto covariance or cross covariance between X of t and X dot of t and that is the differentiation tau is here; so, it comes out here.

Now, if you evaluate the cross covariance is the same time, then this time difference will be 0; so, that means, this exponential i omega tau, tau will be 0, therefore this becomes one and I am left with this integral minus infinity to plus infinity minus i omega S xx of omega d omega, but we know that S xx of omega is the symmetric function, therefore this integral if we look is an odd function under symmetric limits; so, it must be 0.

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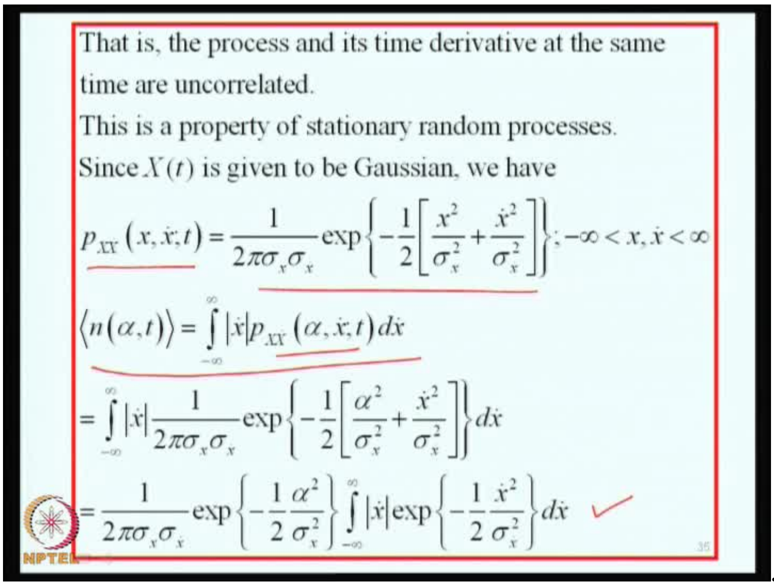

So, this is a simple way of showing that X of t and X dot of t are uncorrelated, if X of t is stationary. This is actually a property of stationary random processes; it is true for even when process is not Gaussian. Since, X of t is now given to be Gaussian, we have now we can write now the expression for the joint density function between X of t and X dot of t.

So, since there uncorrelated, there will, there will not be any times involving correlation coefficients and this will be the form of the joint probability density function. Sigma X is the standard deviation of x of t, sigma x dot is the standard deviation of X dot of t and x x naught takes value from minus infinity to plus infinity.

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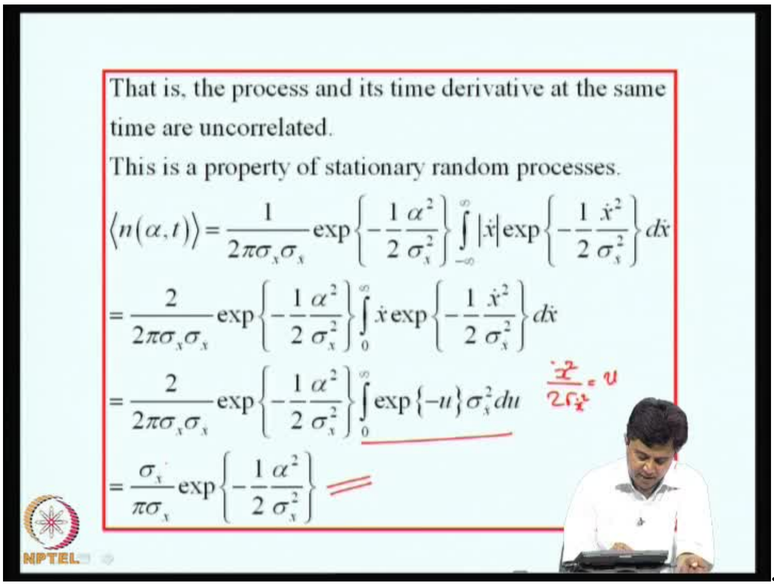

So, what is our interest? Our interest is to evaluate this integral, expected value of n (alpha, t), which were just know shown is given by this; so, you substitute now the expression for p X X, we get this and we can pull out, we can retain inside the integral, only those terms which are function of X dot the remaining can be pulled out; if you do that, we get this expression and this is now even function in symmetric limits; therefore, I can write it as 2 into 0 to infinity, 2 into, this is 2 into 0 to infinity and if we make now a substitution x dot square by 2 sigma x dot square equal to u, and simplify this, we get this integral and 0 to infinite e raise to minus u d u is 1 and thus we get; this is the answer that we are looking for.

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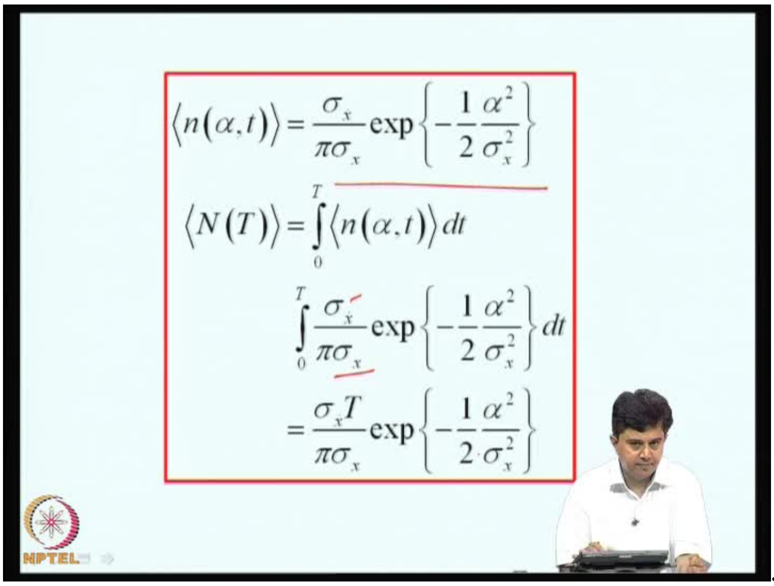

So, this is the rate at which level alpha is crossed; it is the function of sigma x and sigma x dot sigma x dot is here, sigma x appears at two places. Now, the expected value of number of times level alpha is crossed is now integral of this 0 to t n (alpha, t) dt and this, for this, I will substitute this expression and since process is stationary, the variance and standard deviations are independent of time; therefore, they can be pulled out and this integral will be simply capital T into this remaining terms and this is an average number of crossings of level alpha in 0 to capital T.

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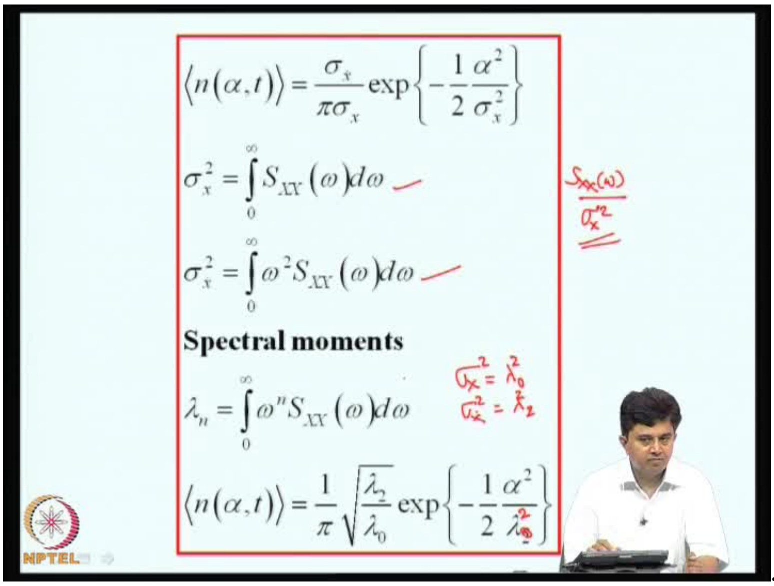

Now, we know that the variance is given by the area under the power spectral density function; so, assuming that, this is the physical power spectral density function, this is the expression. Now, the variance of the derivate processes is given by this omega square S X X of omega d omega; so, we see here, if I right now, lambda n s omega to the power of n S X X of omega d omega, sigma x square can be viewed as lambda naught square; sigma x dot square can be viewed as lambda 2 square, where this lambda's are known as spectral moments, because their moments of this power spectral density function.

So, the rate of crossings can now be expressed in terms of spectral moments lambda to lambda naught as shown here; this terms, spectral moments we will be using again, so this is the first time we are encountering, you can see that the power spectral density function, if you normalize with respect to the variance, the area under the curve of this function is 1 and it is always positive; so, it has all the properties of a probability density function.

So, this can be viewed as some kind of expectation function spectral moments can be thought of as some kind of expectation defined on this power spectral density function. So, unless we normalize with respect to sigma x square, it would not have the property of probability density function; so, we call lambda n as spectral moments.

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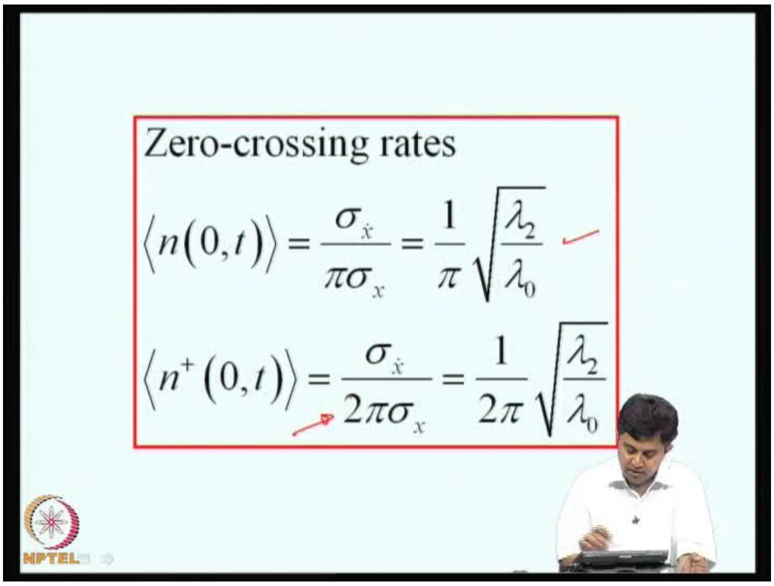

Now, if alpha is 0, we can compute the rate at which the 0 level is crossed and the average rate of 0 crossings is given by this and the average rate of 0 crossing with positive slope is given by this.

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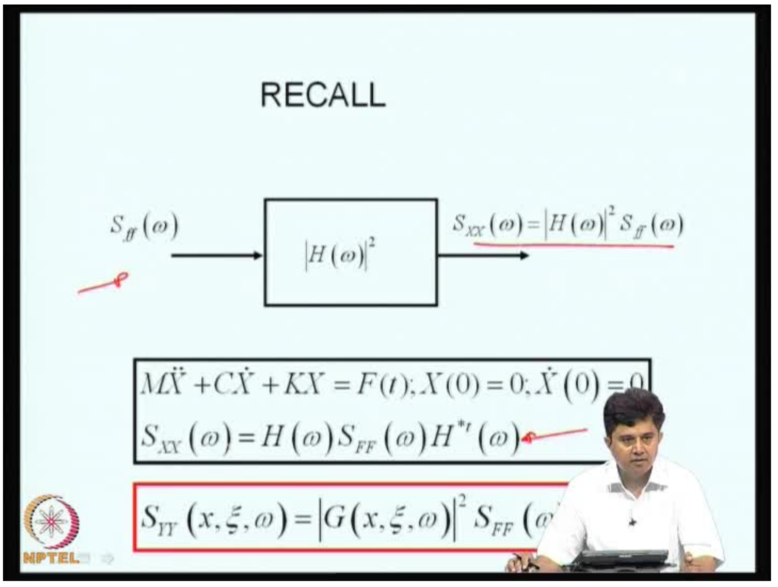

Now, we will recall, now x of t till now, in the example, that I consider was a random process, which I did not refer it to any specific response of a structure, but we have already developed the requisite theory for response of linear systems to random excitation; so, X of t could as well be the response of a vibrating system to an applied random excitation.

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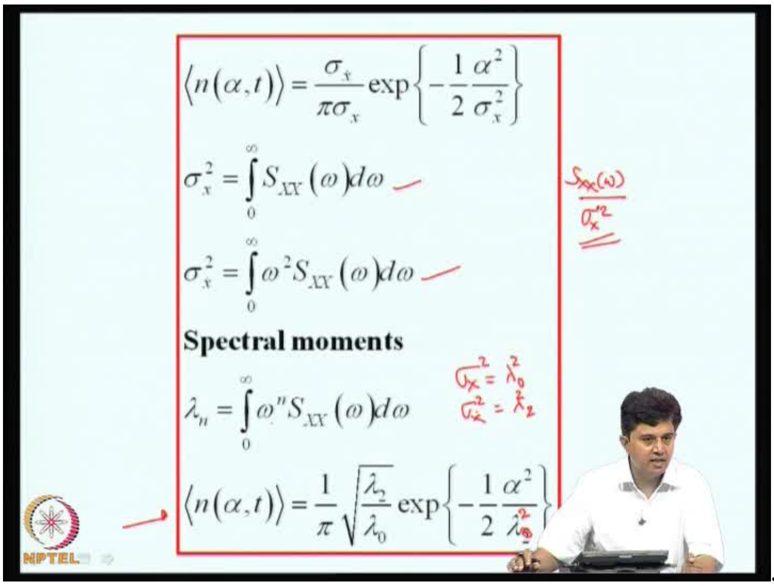

So, for a single degree freedom systems, we have derived this input-output relation; for multi degree freedom systems we have derived this and for continue systems, we have derived this; that would mean, if you are interested is stationary response of vibrating system to stationary random excitations. We, if we know how to compute the power spectral density of the response, then the problem of level crossing is solved, because we are getting our description of rates of crossings, in terms of power spectral density function.

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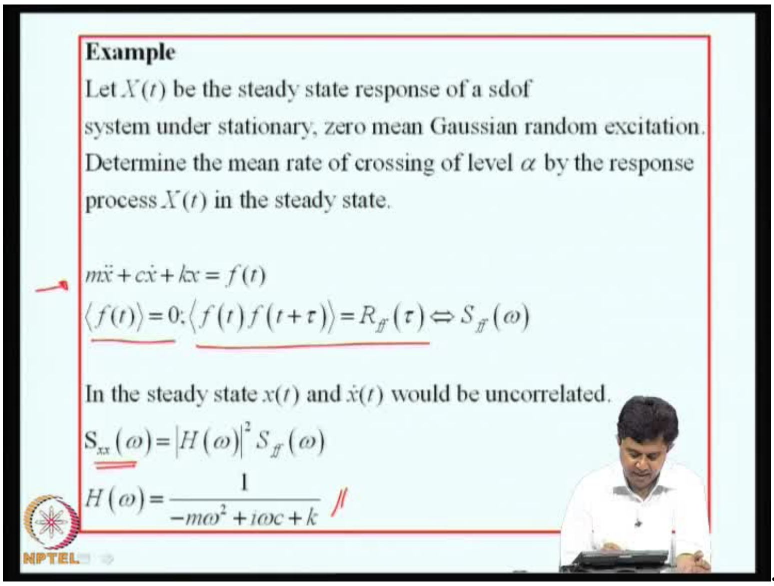

So, moment we will find the power spectral density function, the rates of crossings of level alpha etcetera can easily be solved, can easily be determined. Now, to illustrate that let me consider X of t to be the steady state response of a single degree freedom system under stationary Zero mean Gaussian random excitation; the problem is determine the mean rate of crossing of level alpha by the response process X of t in the steady state.

So, the problem is that we have this system m x double dot plus c x dot plus k x equal to f of t and f of t 0 mean stationary random process; so, in the steady state, we know the power spectral density of the response is given by H of omega whole square S f f of omega and H of omega itself is expressed in terms of m c k using this expression.

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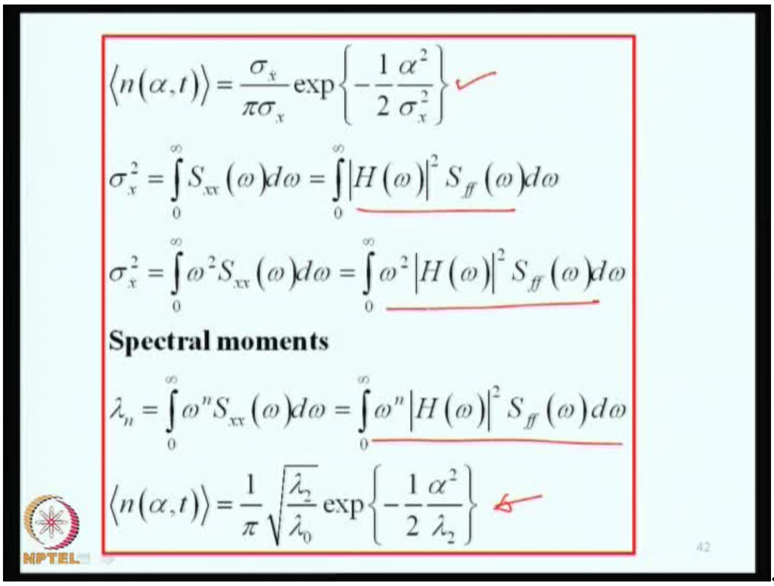

So, I have the average rate of crossing of level alpha, we have evaluated for a Gaussian random processes and it is given by this; now, the sigma x square and sigma x dot square are the quantities, that I need to derive, sigma x square is area under power spectral density, therefore it is area under H of omega whole square into S f f of omega; this is the output power spectral density sigma x dot square, is area under the power spectral density function of the velocity process, so that is given by this. So, the spectral moments now are in terms of system transfer function and the input power spectral density. So, determining the rate of crossings is a straight forward exercise, once you evaluate this spectral moments.

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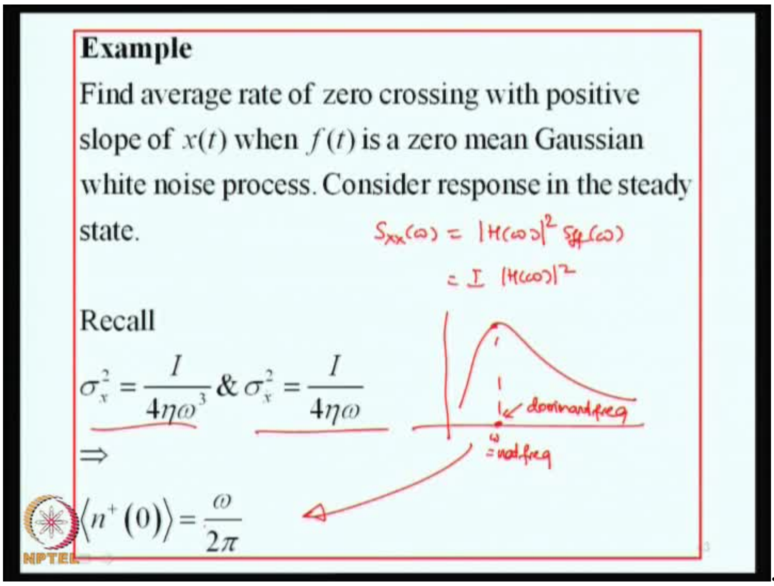

Now, we can consider a special case when f of t is a white noise process and suppose if we ask the question, what is the average rate of 0 crossing with positive slope of x of t, when f of t is a zero mean Gaussian white noise process. Consider the response in steady state. This problem we have already solved, I will leave it as an exercise, but I have given some hints here, this is the variance of the displacement process, this is the variance of velocity process and if you substitute into the formulary that we have develop, we get the result that the rate at which level 0, that is 0 crossing with positive slopes occur is nothing but the system natural frequency in hertz.

So, that shows that the 0 crossings with positive slopes essentially provide the dominant frequency. So, this can be, you know what we are doing is, we have S X X of omega is H of omega whole square s f f of omega; for a white noise excitation, this S ff of omega is constant and if you plot this, this typically peaks around omega which is a natural frequency.

So, this is the dominant frequency, it is quite satisfying to see that the rate at which 0 crossings with positive slope occur coincides with the peak of the power spectral density function the location of the peak; so, this is the dominant frequency and this is the nice measure of the dominant frequency.

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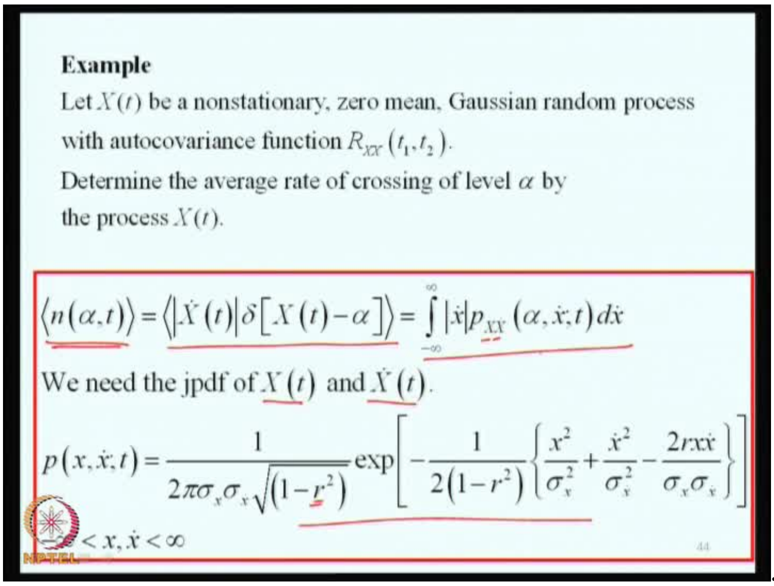

Now, we have considered till now stationary random processes, treatment of non-stationary processes will be more difficult, but it is in principal possible, suppose X of t is a non-stationary Zero mean Gaussian random process with auto covariance R X X (t 1, t 2); now, the problem will determine the average rate of crossing of level alpha by process X of t. This crossings we are just now shown is given by the expected value of this quantity.

So, we need to evaluate this integral and as prerequisite we have to evaluate the joint density between x of t and x dot of t at the same time instant. So, we need a joint p d f of x of t and X dot of t the process is non-stationary; therefore, they will now be correlated. So, the general form of p (x, x dot, t) is displayed here, where r is the coefficient of correlation coefficient which involves the cross covariance and standard deviations of X and X dot.

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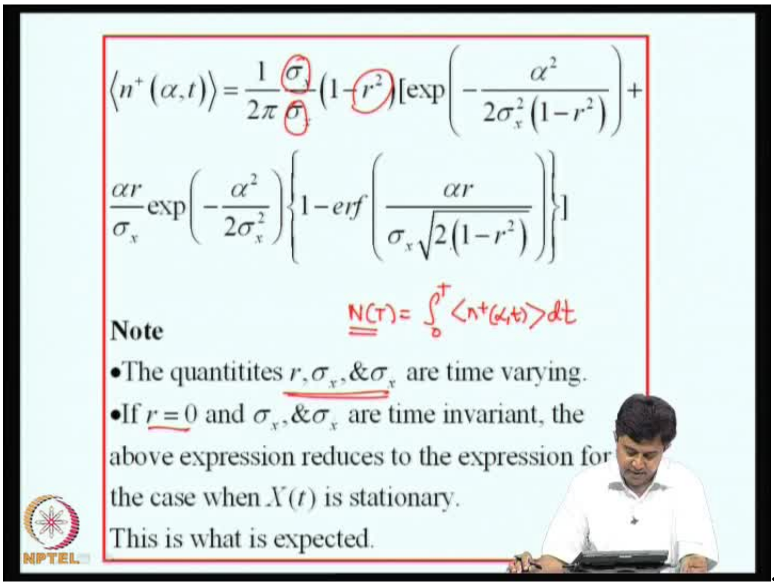

Now, we have to substitute this, this expression into this integral and if you do that we get a fairly complicated expression for the rate of crossings with positive slope that is what I have displayed here; I leave it as an exercise for you to actually verify, that this expression is correct.

Now, on the right hand side, we have sigma x dot sigma x and r appearing at various places and these quantities are time varying, because process is non-stationary. So, when we evaluate N of T, so I have to use n plus alpha comma t dt; so, to evaluate this number, I will have to carry out a fairly complicated integral integration in time.

So, the analysis will be lot more complicated, in this case. If we now take r equal to 0 and sigma x and sigma x dot to be independent of time, then we recover the result for a stationary random process which is as it should be, because for when x and x dot are when X of t is stationary x and x dot will be uncorrelated; therefore, r will be 0 and the standard deviation of the displacement process our X of t and it is derivate will be independent of time; therefore, if we use those facts here, we get we recover the result that we derived just a while before.

Now, with this, I will conclude this lecture. In the next lecture, we will consider development of approximate models for number of times, the level alpha is crossed and we will propose a poisson model for this counting process and let us see the amplifications of such models.