Stochastic Structural Dynamics

Prof. Dr. C. S. Manohar
Department of Civil Engineering
Indian Institute of Science, Bangalore

Lecture No. # 16

Random Vibrations of MDOF Systems-4

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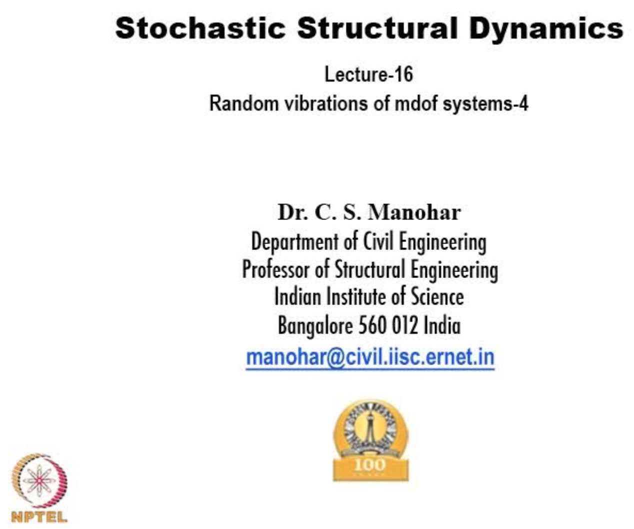

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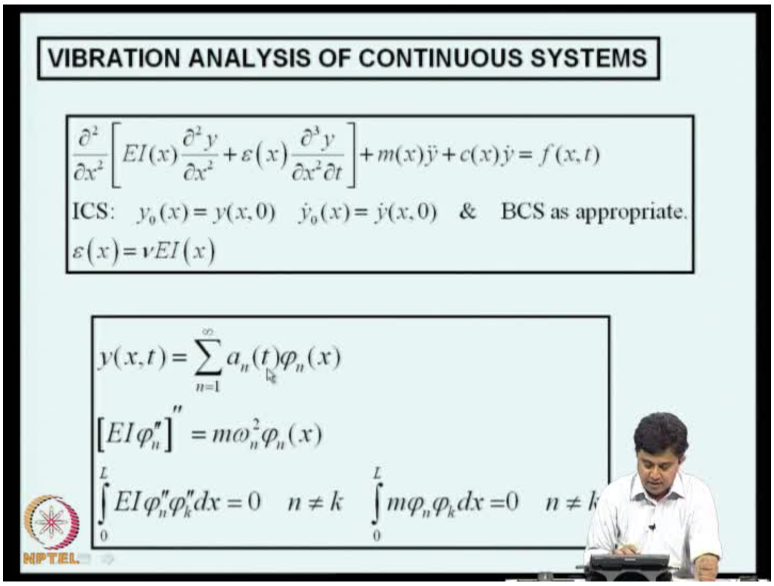

We have been discussing dynamics of continuous systems; towards the end of the last lecture, we reviewed the analysis of continuous systems under deterministic load. So, we considered as an example the dynamics of an Euler Bernoulli beam, whose equation is displayed here, the first term here is the stiffness term, this term is the inertial term, this is the, there are two damping terms, one is c of x into y dot, this is velocity dependent damping, this is strain rate dependent damping and f of x, t is the external force, these are the initial conditions and boundary conditions has appropriate have to be specified and we assume that this epsilon of x is proportional to the flexural rigidity; so that, this damping term will be proportional to the stiffness of the system.

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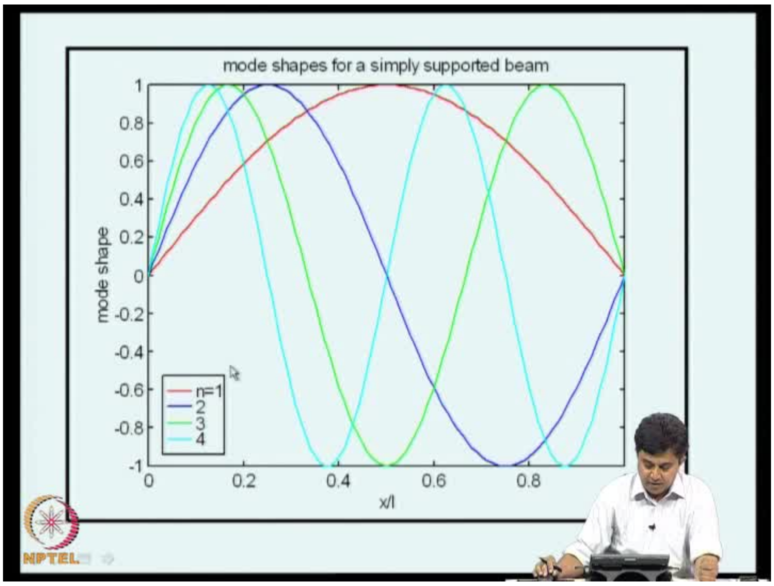

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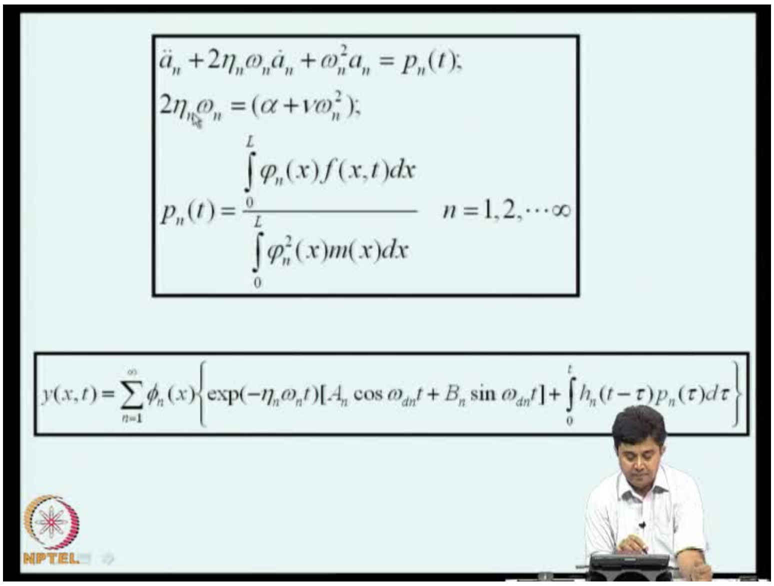

We first carry out an un-damped free vibration analysis and determine the normal modes, here these are the Eigen functions phi n of x and they are governed by this equation; this is the Eigen value problem and the Eigen functions satisfy, this pair of orthogonality relations one with respect flexural rigidity, where the second derivative of the Eigen functions are involved and the other with respect to the mass function, where phi n of x and phi k of x are involved and we expand the solution to the forced vibration problem, in terms of the generalized coordinates a n of t and phi n of x, this n runs here from 1 to infinity. And the governing equations for a n of t turns out to be of this form a n double dot plus two eta n omega n a n dot plus omega n square a n is p n of t. P n of t is the generalized force given here and is damping term 2 eta omega n is expressed, in terms of the, for the proportional damping model, that we are assuming it will be in terms of two constants alpha and nu and omega n square.

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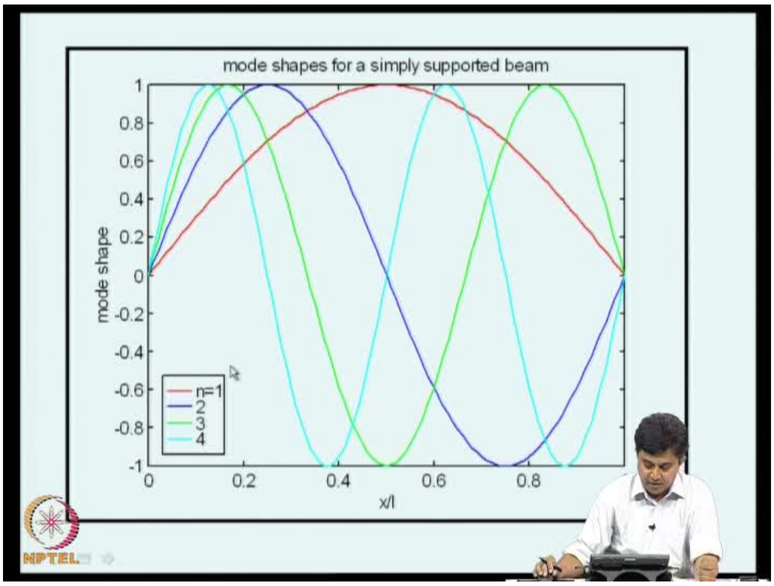

The mode shapes for the Eigen functions, for a simply supported beam, we saw to be sine functions; the first mode is a half sine wave, second one is one complete sine wave and so on. So, alternate Eigen functions are symmetric with respect to mid span and anti symmetric with respect to mid span.

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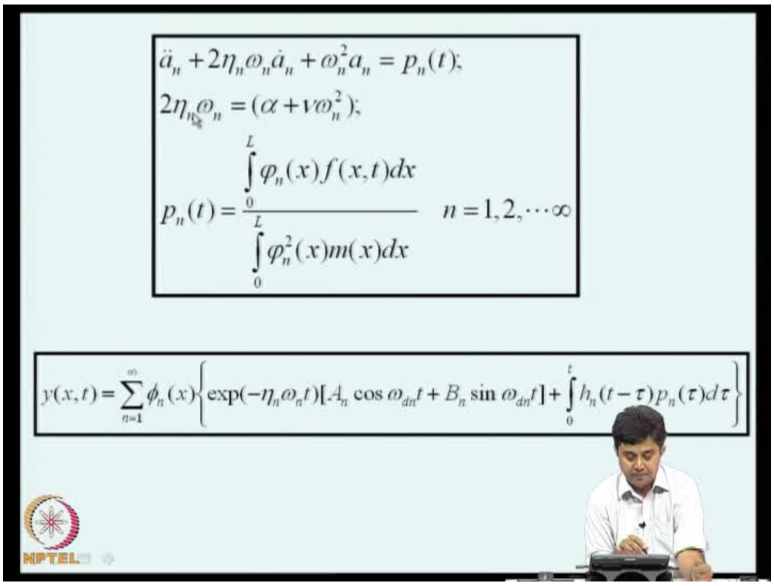

So, we have reduced the given partial differential equation into an infinite set of single degree freedom system, each one of which can be solved using the methods that we have already learned. And based on the solution for a n of t the expression for the displacement function can be expressed, in this series form. So, the first terms here shown in the bracket here are the contributions from initial conditions and this is the particular integral, in terms of the Duhamel's integral associated with each one of these single degree freedom systems.

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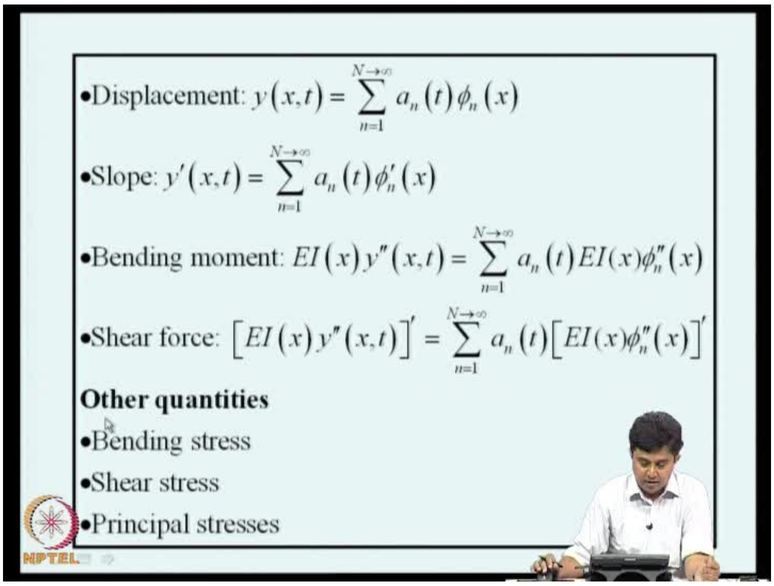

Once we find displacement, we could go ahead and evaluate the slope which is the derivative of y x, t, here we need phi n prime of x and bending moment, which is E I of x into y double prime x, t as shown here. And shear force, which is the derivative of the bending moment shown here and other quantities like bending stress, shear stress and principal stresses at any point, can be eventually evaluated.

Here, we are summing the summation should run from n equal to 1 to infinity, but typically in numerical work, we stop at some finite capital N and the rate at which these series converges is different for displacement slope bending moment and shear force. Typically, lesser number of terms is needed for convergence of displacement and higher number of terms is needed for shear force.

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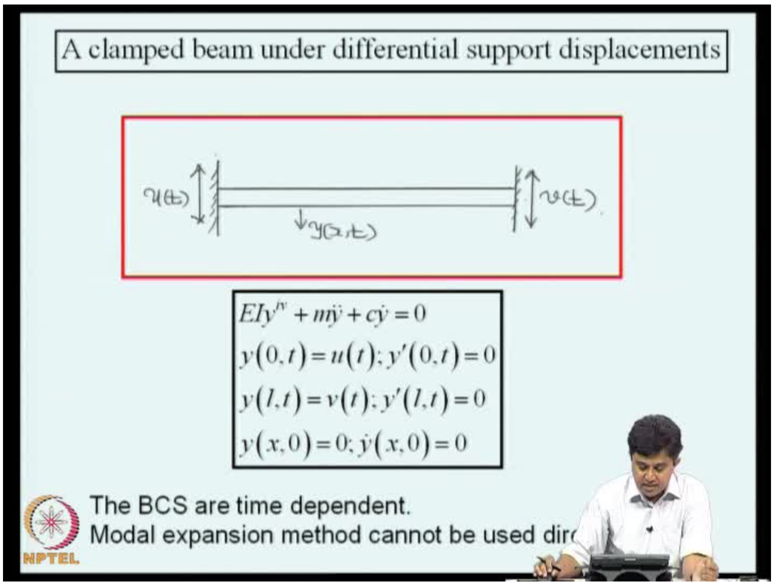

We will continue with this discussion now and we will see how to analyze problems of single span beams, if excitations appear as support motions; this, a typically what would happen in an earthquake engineering problems. So, to illustrate that, we consider a single span beam which is clamped at the two ends and the two supports are subjected to displacement u of t and v of t.

The field equation here we are assuming a simpler model for damping and we are also assuming that beam is homogeneous; so, we get E I y 4 plus m y double dot plus c y dot equal to 0. The bound, the excitations u of t and v of t appear as boundary conditions, that is y (0, t) is u of t and y (l, t) is v of t. Since, the beam is clamped at the two ends, we get y prime (0, t) is 0, y prime (l, t) is 0. We also assume for shake of simplicity that this beam starts from rest. Now, a complicating future here, is that, the boundary conditions here are time varying. So, consequently, we will not able to use the Eigen function expansion method to solve this problem. So, we overcome this difficulty by implementing a transformation of the dependent variable.

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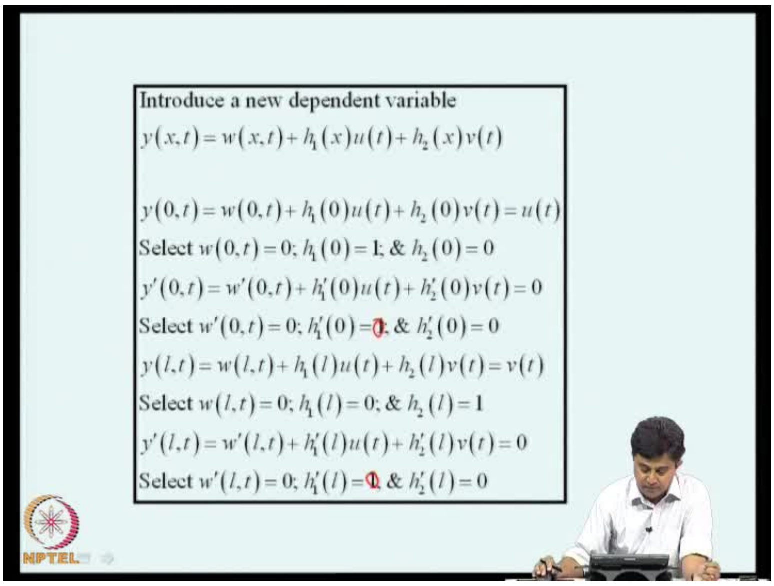

And we introduce a new dependent variable w and related to y (x, t) through this relation. Here, w is the new dependent variable, h 1of x and h 2 of x are unknown functions, which we need to determine and u of t and v of t are the support displacements. So, now you look at boundary condition at x equal to 0, we get y (0 , t) is w (0, t) plus h 1 of 0 u of t plus h 2 of 0 v of t and this must be equal to u of t.

We can satisfy this requirement by taking w (0, t) as 0 and h 1 of 0 as 1 and h 2 of 0 as 0. Similarly, the condition y prime (0, t) can be satisfied by taking w prime (0, t) equal to 0, h 1 prime of 0 is, this should be 0 and h 2, h 2 prime of 0 is 0. Similarly, we satisfy boundary condition at x equal to l and put condition on displacement and condition on slope, we will get, now there are two functions h 1 and h 2 for each one of these functions, we get four conditions at x equal to 0 and at x equal to l.

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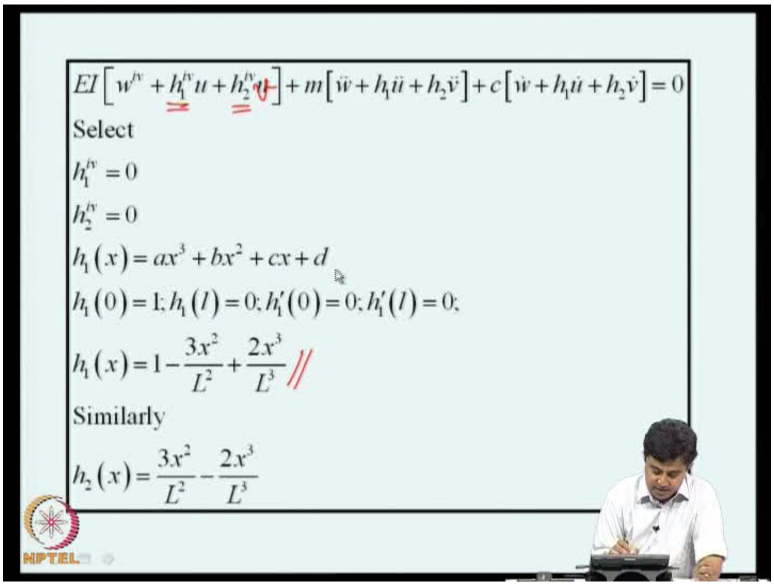

Now, if we substitute this assume solution form of the solution into the governing equation, I get E I w 4 h 1 4 u plus h 2 4 u and m w dot, w double dot and so on equal to 0, now what we do is, we select h 1 4 and h 2 4 to be equal to 0. And consequently, for example, for h 1 of x I get a cubic polynomial with four constants and I have four conditions specified on h 1, using that, I can determine these four constants a, b, c, d and if you do that, we get this function h 1 of x is 1 minus 3 x square by l square plus 2 x cube by l cube as shown here. Similarly, we can use h 2 of fourth derivative of h 2 equal to 0, would mean, again h 2 of x, again a similar cubic polynomial, similar to this; and then, again on h 2, I have 4 conditions, using that, we can show that h 2 of x is another cubic polynomial as shown here.

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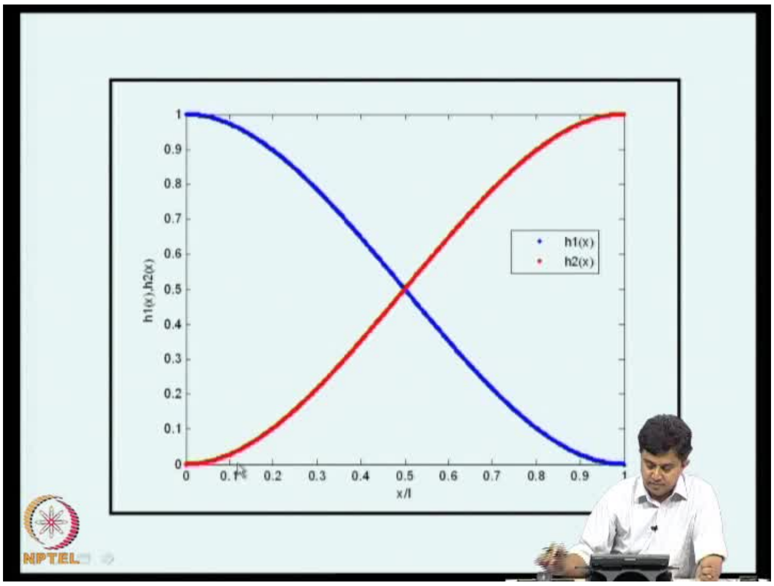

If you have to plot them, they look like this h 1 of x is this and h 2 of x is this. So, if you are familiar with finite element modeling of beam elements, you will recognize that these are nothing but two of the shape functions that we use in discretizing beam element.

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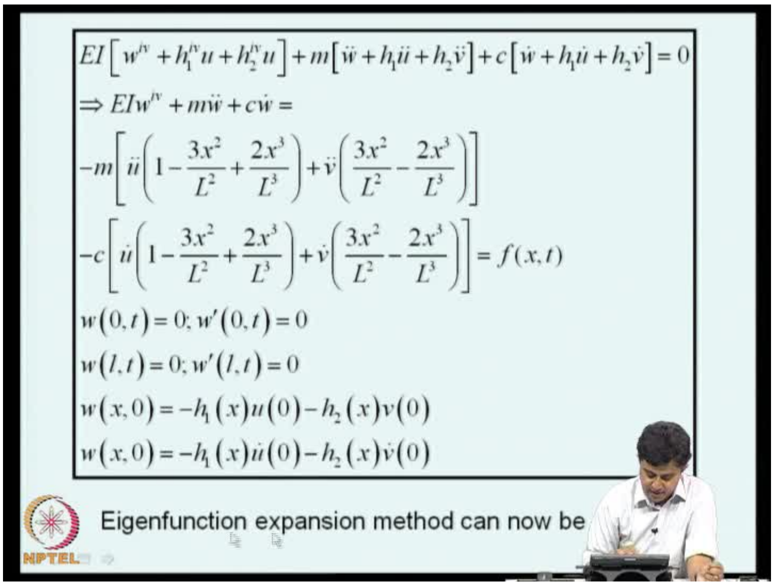

So, with these, we now go back to the original field equation h 1 of 4 is 0, h 2 4 of 0, is 0; so, I get E I w 4 m w dot c w dot is equal to the remaining non-zero terms, are put on the right hand side. And this, the quantities on the right hand side are given m is known u double dot is known, v double dot is known, c is known, u dot is known, v dot is known so on and I call this right hand side as f (x, t).

In this analysis, now if you look at the boundary conditions on w, we see that the boundary conditions here become time independent; so, w (0, t) is 0, w prime (0, t) is 0 and w (l, t) is 0 and w prime (l, t) is 0. We could, of course derive the initial conditions as well and this problem is now amenable for Eigen function expansion method and we can proceed and solve this problem.

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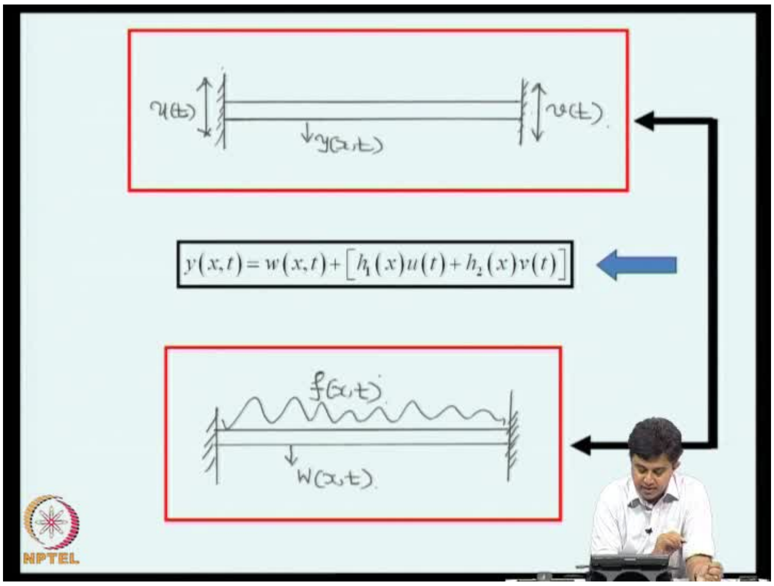

So, what we are doing is, the given problem where the excitation appear as time dependent boundary conditions, through this transformation we are converting, then this problem into an equivalent problem, where the beam now carries a hypothetical load which varies space, in space and time, and the displacement of this is w not y, and this w is related to y through this equation, where w is solution of this u and v are given support displacement, and h 1 and h 2, we have selected suitably.

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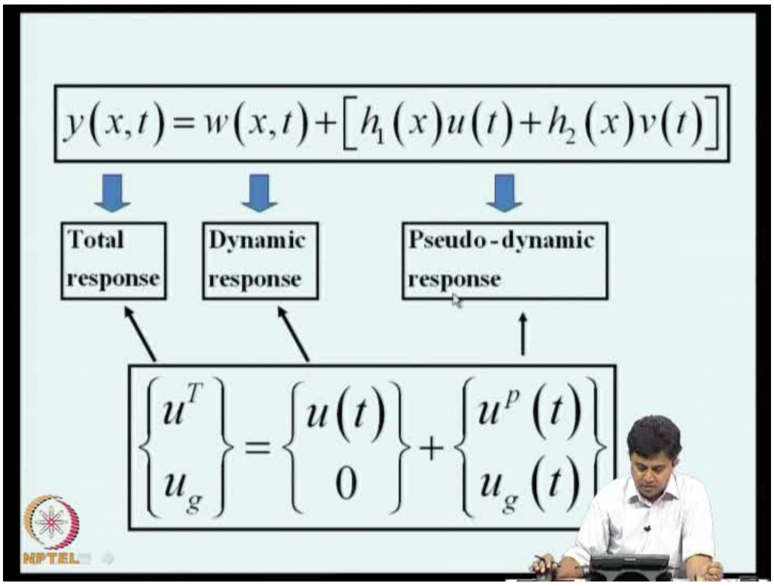

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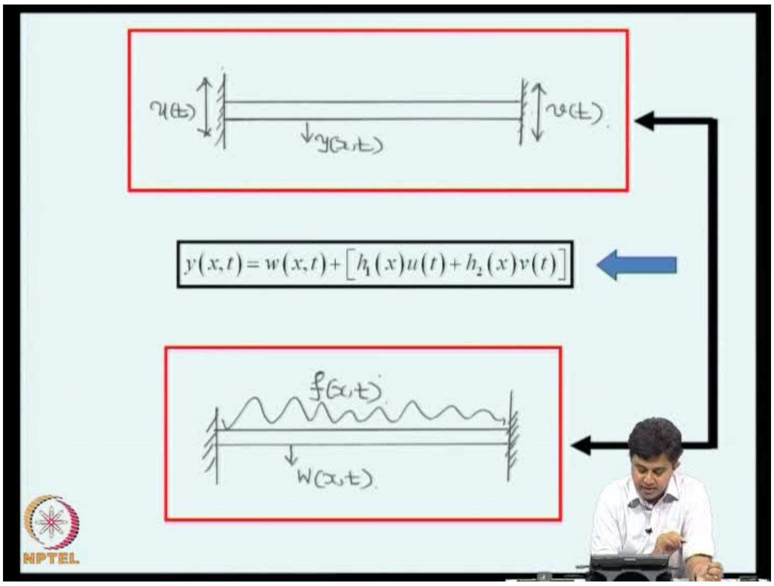

So, we can look at the transformation that we have done; this is amenable for an interpretation. this y (x, t) is a, can be viewed as a total response and w as a dynamic response and the term inside this bracket is the pseudo dynamic response, this is analogous to what we did for discrete multi-degree freedom systems, that is, in this structure, without the inertial terms participating in the solution, we still get certain displacement, and stresses in the beam due to differential support motions, and that is the pseudo dynamic response, where inertial effects and damping effects are not included.

This is a dynamic component, where inertial components are also included, but the supports are now not moving. So, this is as I already said is analogous to what we already saw, when we studied discrete multi-degree freedom systems which undergoing differential support motions.

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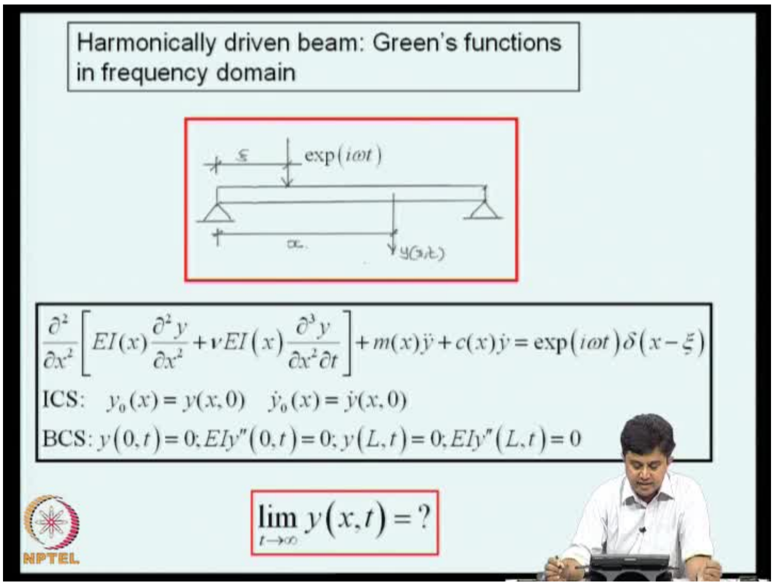

We are eventually interested in analyzing the response of the beam to random excitations; so, to prepare the basic formulations to achieve that, we need to consider a few deterministic problems, one is the problem of a harmonically driven beam, we consider this beam, the boundary condition here is shown to be hinged at the two ends, the beam is hinged at the two ends, but this is just for illustration, it can be any other boundary condition as well. So, we assume that, this beam is driven harmonically by a force e rise to i omega t applied at distance psi from this end and we are interested in knowing displacement at a point x, at this point which is at a distance x from this left hand. So, the governing equation here is displaced, here the left hand side remains the same.

Now, on the right hand side, we apply the force e rise to i omega t and this is a concentrated load applied at x is equal to psi; therefore, I represent that by using direct delta function delta of x minus psi. We assume certain initial conditions and the boundary condition appropriate for this, and conditions shown here is displacement is 0, bending moment is 0 here and similarly displacement is 0 and bending moment is 0. We are interested in solution of this problem as t becomes t goes to infinity, that means, we are basically interested in investigating the harmonic steady state of the Euler Bernoulli beam.

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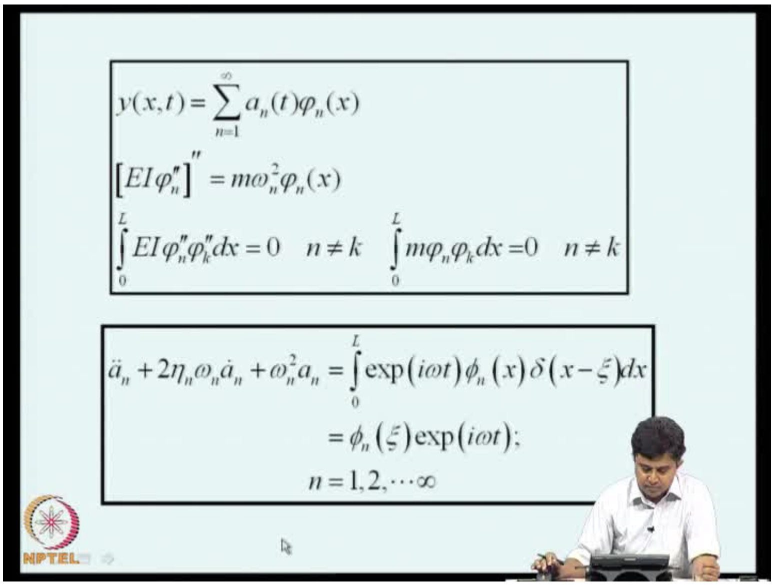

So, we can expand the solution in terms of Eigen functions phi n of x and generalize coordinates a n of t and this phi n of x satisfies this equation and this orthogonality relations with respect to E I and m; using this we can derive the equations for the generalize coordinates, and on the right hand side, the generalize force is now given by 0 to l e rise to i omega t phi n of x delta of x minus psi d x.

So, this integration can be carried out in a straight forward manner, so we replace x by psi and I get phi n of psi exponential i omega t and this is true for n equal to 1 2 infinity. So, the given partial differential equation is thus equivalent to a family of single degree freedom systems, each one of which is driven by e rise to i omega t and the magnitude is given by the value of the mode shape at the drive point. This can easily we solved, we know how to get a, n of as t tends to infinity.

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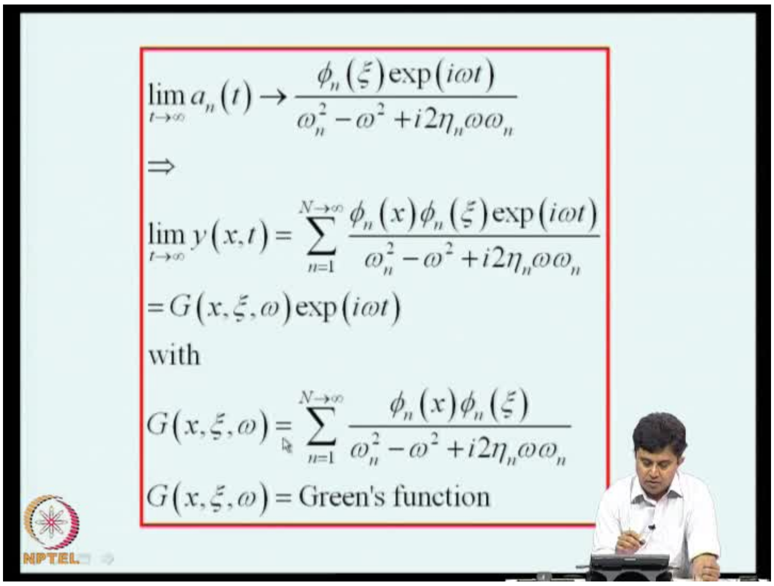

So, we get t tend to infinity a n of t is phi n of psi e rise to i omega t omega n square minus omega square plus i two eta n omega n. Now, if we now substitute this solution into the expression for y (x, t), I get phi n of x into a n of t, phi n of x is here, the remaining terms are the a n of t which I had derived just now.

This summation, I write this solution as G of x comma psi comma omega e rise to i omega t, where G is given by this summation; this function is known as the Green's function for the beam. It is function of the point, where you are majoring the response the point where you are driving and the driving frequency; it is a complex valued quantity, it has an amplitude and a phase.

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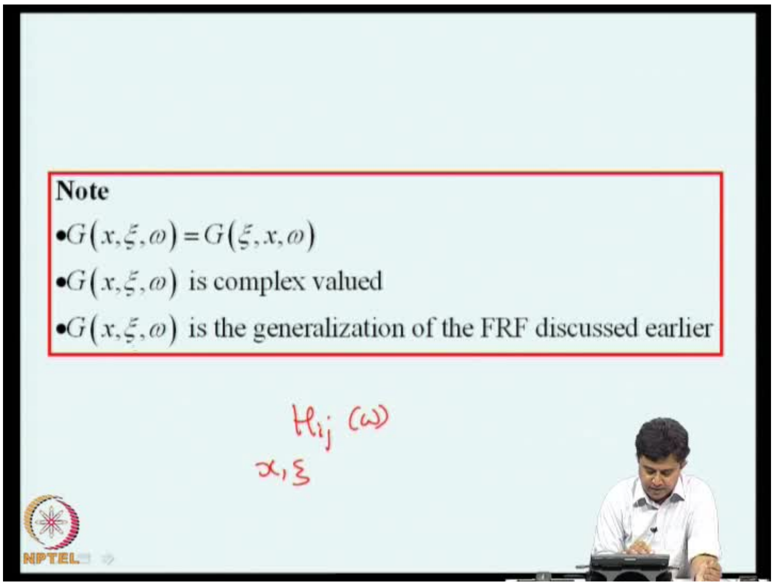

And it is actually dissymmetric in x and psi, this is actually manifestation of reciprocity theorem, the Green’s function is symmetric in x and psi, it is complex valued and this is actually the generalization of the frequency response function matrix that we considered for discrete multi-degree freedom systems. We had there a n by n matrix, now it is a function in x and psi; so, the H i j of omega that we used in discrete multi-degree freedom system, now this role of i and j is played by x and psi here.

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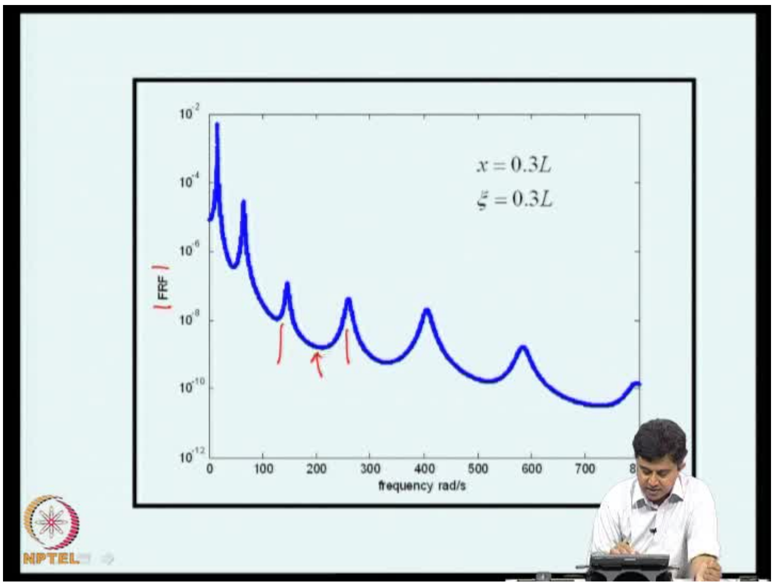

A typical Green's function when measurement point is 0.3 l and drive point is 0.3 l is shown here, and we see here, that the amplitude of the Green's function, this is actually the amplitude peaks at the system natural frequency, these are actually the system natural frequencies, and between the two modes, there is a characteristic minimum.

So, here what happens is, the contribution suppose you are driving here, the contribution to response is pre-dominantly this mode and this mode, and they add up produce a small quantity and that appears as a minimum.

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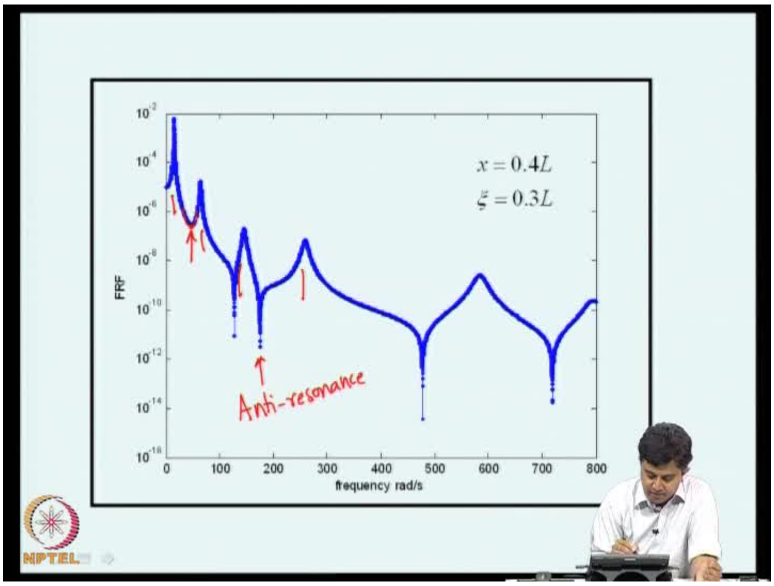

Now, if you drive at a different point, then where measurements are made, that means, now you measure at point 4 l but drive at 0.3 l. Here, again the frequency response function peaks at the natural frequencies, but between the two modes there, is now a different kind of behavior possible. Here, what happens is, again if you drive at this frequency, the response is mainly made up of contribution from this mode and this mode, but here what happens is the contribution from this mode cancels with the contribution from this mode and we get an anti-resonance point.

Whereas, here if you see the contribution from this mode and this mode, add to produce this value and they are of the same sign; therefore, we get this characteristic minimum, whereas this is a sharp anti resonance point, this actually does not go to 0, because there will be contribution, from other higher modes which will be small and yet but not 0 anyway.

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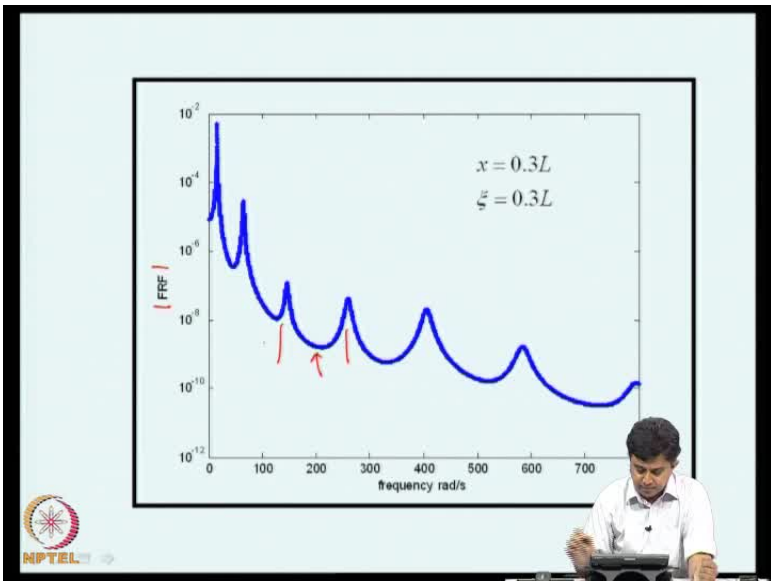

So, another thing that you should notice here is that, the beam natural frequencies as you go higher up in the frequency, the spacing between frequency increases; in fact, this increases linearly, this is property of a Euler Bernoulli beam, if it were to be an actually vibrating rod, these peaks will be uniformly is best. And if it is a membrane or a shell, these peaks comes closer as we go higher up in the frequency and this frequency response function tend to become smooth. We will write to that point later, when we briefly discuss what are known as statistical energy analysis methods.

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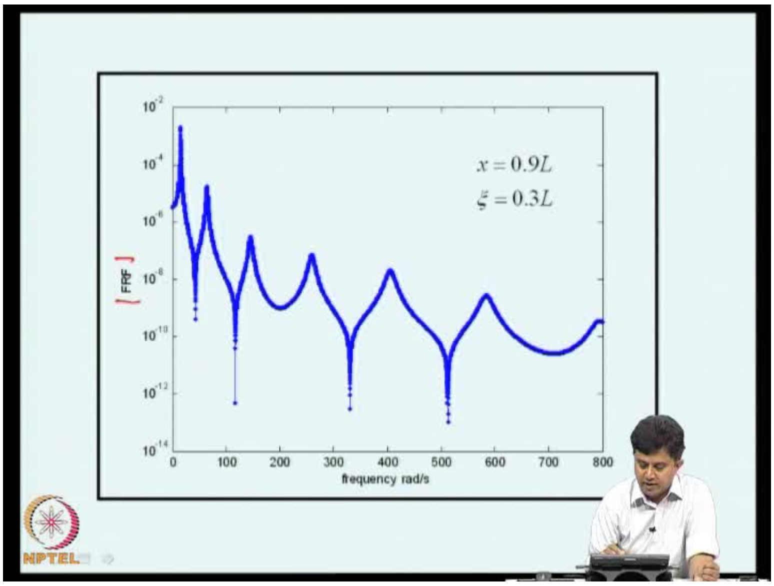

So, this is another plot of amplitude of frequency response function; here drive point is quit removed from the excitation point, and we see that, here there are more anti-resonance points, this is again one of the properties of Green's function, if drive point and measurement point move away, there will be more anti-resonances in the response.

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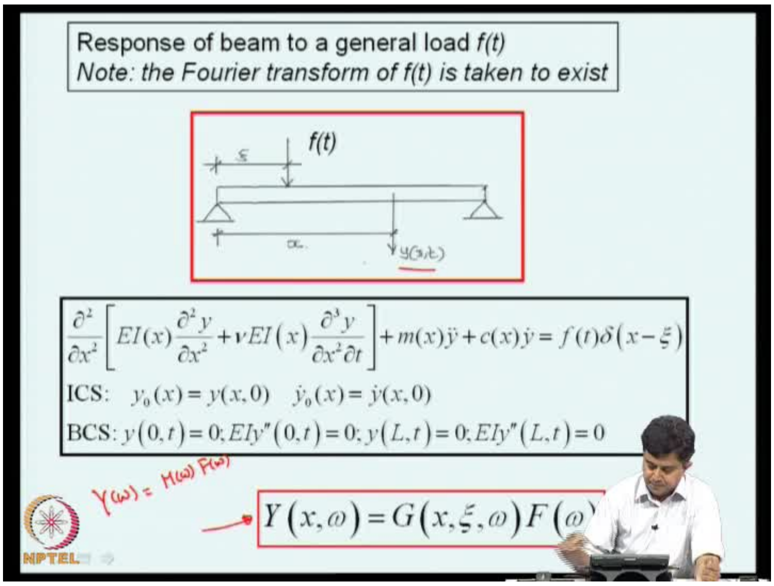

Now, equipped with now the definition of Green's function, we will now consider the response of the beam to a general force f of t and will perform the analysis in frequency domain. So, the field equation is the left hand side, is the same the right hand side f of t into delta of x minus psi, in the example, that we just now discussed f of t was e rise to i omega t; now, it is more general function f of t, we will assume that f of t admits a Fourier representation, that means, that Fourier transformation exists and we are interested in the Fourier transform of the response quantity y (x, t). It can easily be shown that, the Fourier transform of y (x, t) is given in terms of the Green's function G of x, psi, omega into the Fourier transform of f of t, which I had denoted as f of omega.

This is quite similar to our input output relation in frequency domain, which we had derived for single-degree freedom systems and multi-degree freedom system; so, these are generalization for a continuous system. This g here just to emphasis again is obtained as a series made up infinite number of terms, each term coming from one of the normal modes.

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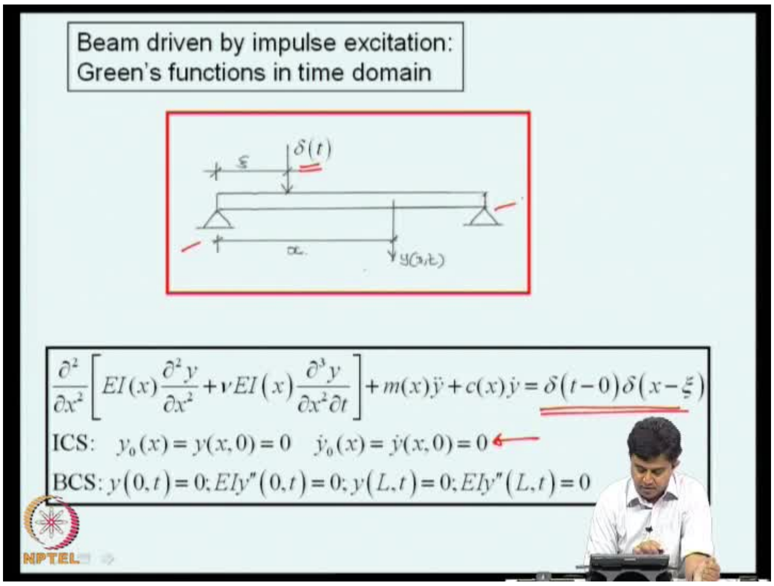

Now, we look at this problems in time domain, instead of applying a unit-harmonic load, if I were to apply a unit impulse load at x equal to psi, and again ask the question what is the response at this point x at a distance x from this end; so, on the right hand side, here I get an impulse applied at t equal to 0 and at x equal to psi; so, that is represented as product of two direct delta functions. So, we assume that system start from rest, so initial displacement and initial velocity are 0 and again the boundary conditions are displacement and bending moment are 0 here and here.

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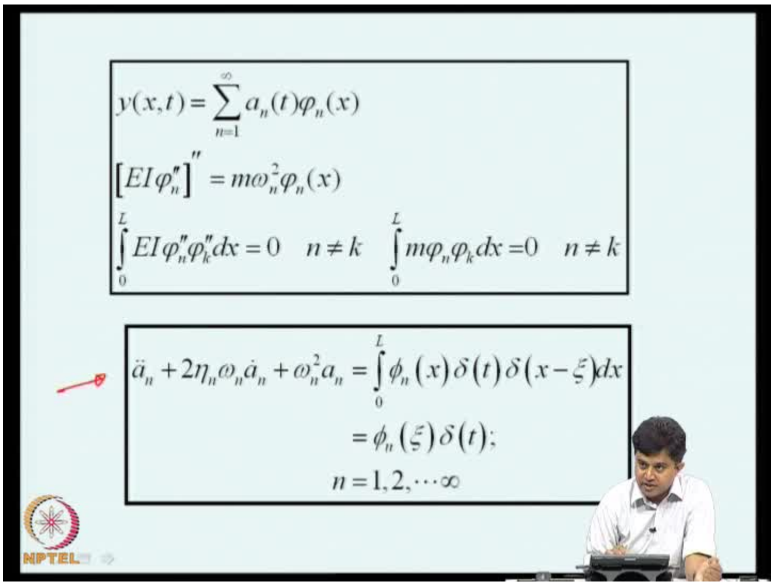

we can again use the Eigen function expansion representation for the solution a n of t are the generalized coordinates phi n of x are the Eigen functions, which satisfies these orthogonality relations and the equation for the generalized coordinates is displayed here; and on the right hand side, the generalized force is obtained as 0 to l phi n of x delta of t delta of x minus psi d x.

So, one of the integration with the integration with respect to x, now can easily be carried out, we replace x by psi and write phi n of psi into delta of t. So, now, each of these single degree freedom systems is now driven by unit impulse and impulse applied at t equal to 0 and its amplitude is given by the value of the mode shape, the n th mode shape evaluated at a drive point.

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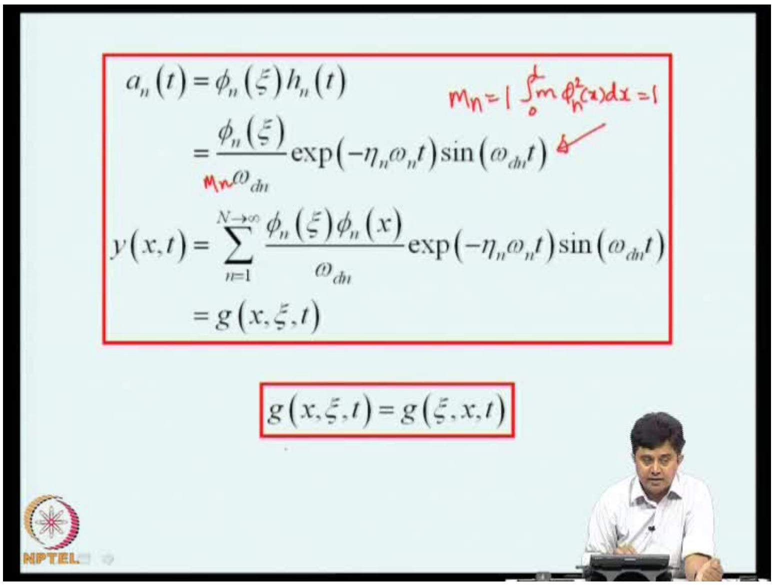

So, a n of t consequently will be phi n of psi into h n of t, where h n of t is a response to a unit impulse response function, this is a unit impulse response function; so, a n of t is given by this, we are assuming system to be all the modes to be under damped. So, we write the impulse response function for the n th generalize coordinate as 1 by omega d n e rise to minus eta omega n t sin omega d n t.

The generalized mass here m n is 1, because that is how we have normalized the Eigen functions, so we are taking 0 to l m phi n square of x d x as 1; therefore, this m n does not appear here. We will now substitute a n of t into the series expansion y (x, t) and we get this function; this function is again known as Green’s function, but it is Green’s function in time domain, is again a function of drive point and measurement point, now it is a function of time. It is again symmetric in x and psi, that is G (x, psi, t) is G (psi, x, t) that means, if you interchange the drive point and measurement point, the Green's function will not change.

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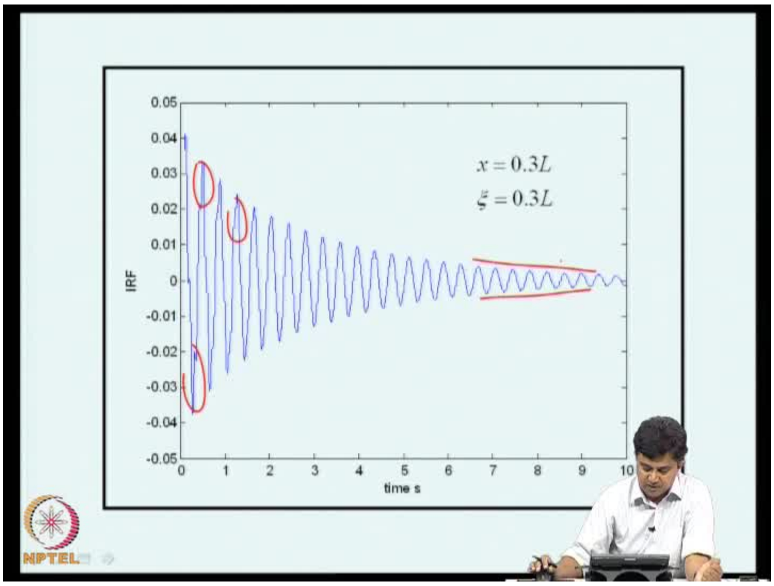

This is a plot of Green's function, when the drive point and measurement points coincide and both are equal to 0.3 l; you can see here, that initially the response is made up of different frequencies, and eventually, it is used to be decaying at the same frequency, the single frequency; that means, in free vibration, the beam oscillates, the beam oscillation is made up of contributions from several modes typically. This is drive point is 0.3 l and measurement point is 0.4 l, again the characteristic exponential decay is seen.

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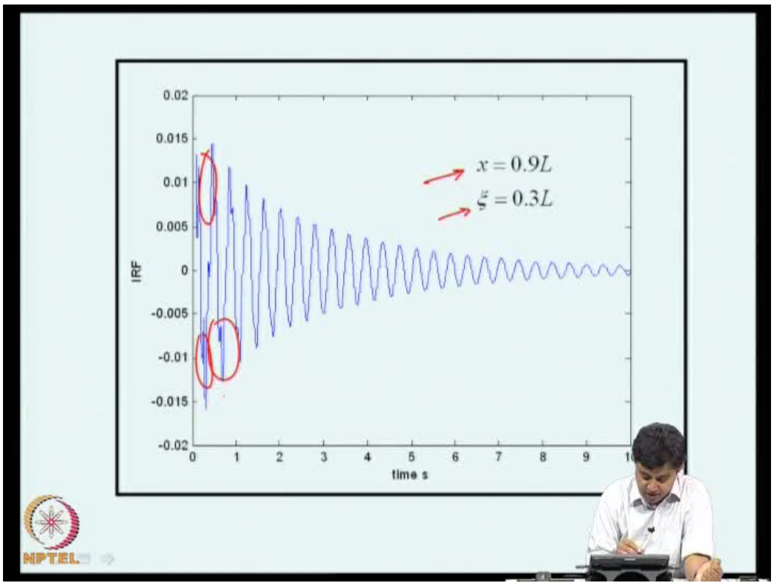

This is another variation, the drive point is 0.3 l and measurement point is 0.9 l, and we can see here the contribution from different, more than one mode is evident, especially for response near t equal to 0, where we are applying the impulse.

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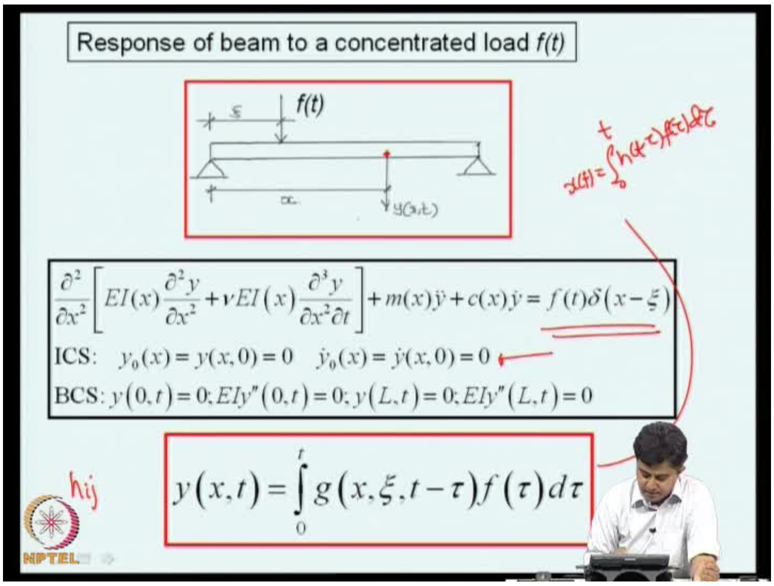

Now, if we now apply a force f of t at x equal to psi, and I want to now characterize the time history of response, say the displacement at this point; just now we analyze this problem in frequency domain, we are now considering problem in time domain.

So, the beam equation the right hand side changes, now it is f of t into delta of x minus psi, and we are assuming the system starts from rest, and we can show that, the displacement y of x, t is actually the convolution of Green's functions appropriate for the psi and x, and this convolves with the time function f of t, to give me the displacement y (x, t).

So this analogous to our x of t is equal to 0 to t h of t minus tau f of tau d tau, that we have seen that single degree freedom system. And for multi degree freedom system, we had impulse response function with indices H i j right; now, the role of i and j are now played by the independent variables x and psi, which take values between 0 and L now.

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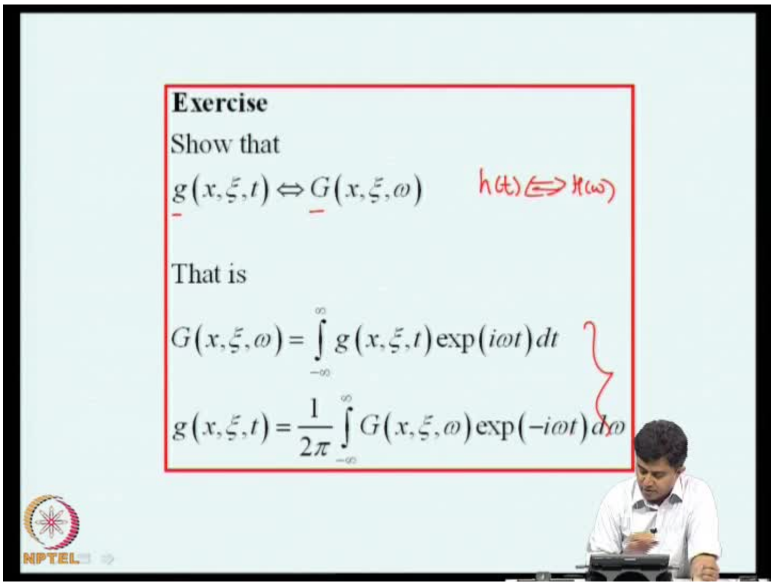

Now, I suggest an exercise, I can show that the Green's function in time domain and Green's function in frequency domain form a Fourier transform there; so, this is again analogous to the result, that we are shown that impulse response function and frequency response function for discrete multi-discrete, single degree as well as multi-degree freedom system form a Fourier transform here, that means, what we need to show is, this G and this G satisfied this pair of equations.

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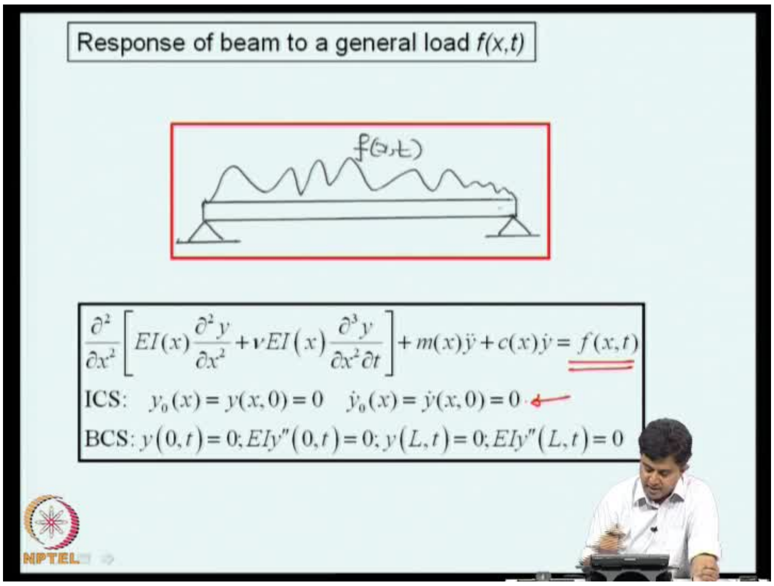

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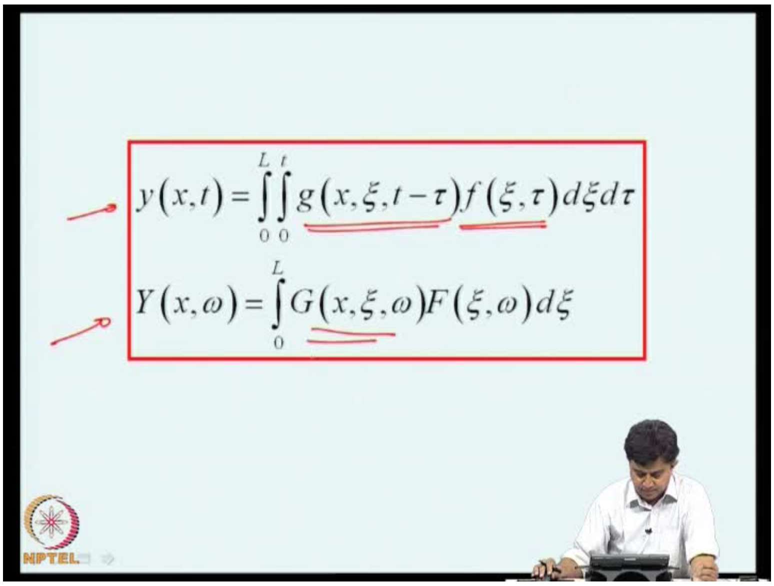

We will now consider what happens if the beam carries the general load f (x, t); so, on the right hand side, I have f (x, t) and we are assuming the system starts from rest. So, in time domain, we can derive the response in terms of the Green's functions as shown here, and in frequency domain, we can derive the Fourier transform the response in terms of Green's function in frequency domain.

So, this is basically the input output relation in time and frequency domains for a deterministic system, subjected to a general load f (x, t); this again requires few steps of derivations, I leave this as an exercise.

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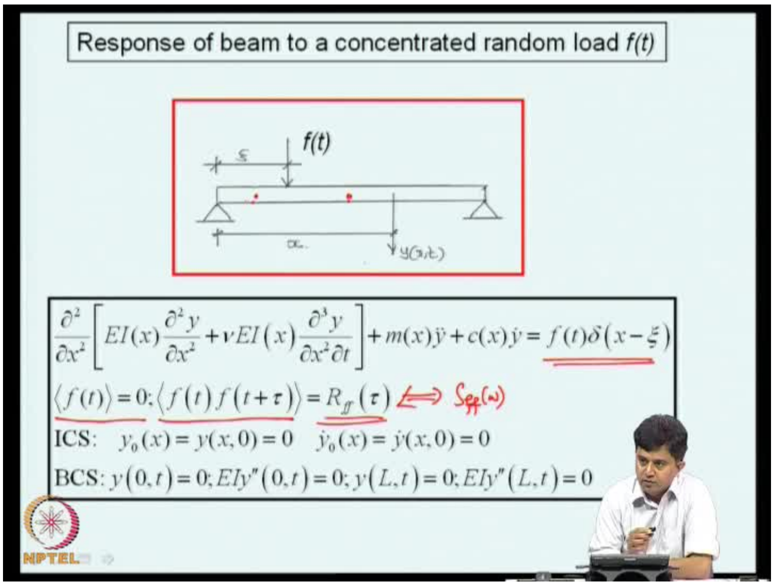

Now, we will consider the problem of beam subject, beam subjected to random excitations; to start with we will consider the problem, where the beam is driven by a random process f of t, at a point x equal to psi and we are interested in response y (x , t). So, the governing equation, this remains the same, this is the sample you know equation for sample of f of t, and this f of t itself, we assume that its mean is 0, and it is a stationary random process with auto covariance function R f f of tau, and associated with this, we have the power spectral density function S f f of omega.

We assume that the system start from rest and the boundary condition as appropriate are mentioned here. So, we are interested in finding out mean response, auto covariance of the response and cross covariance of the response at different points, here I have y 1 and y 2, what is the covariance between the two? What is the power spectral density function of y (x, t)? What is the cross power spectral density function between response here and response here? So, this, such question can be posed and we need to find answers to these questions and we need to formulate the problem in time domain and frequency domain; so, we will see how I will show a few of this formulations.

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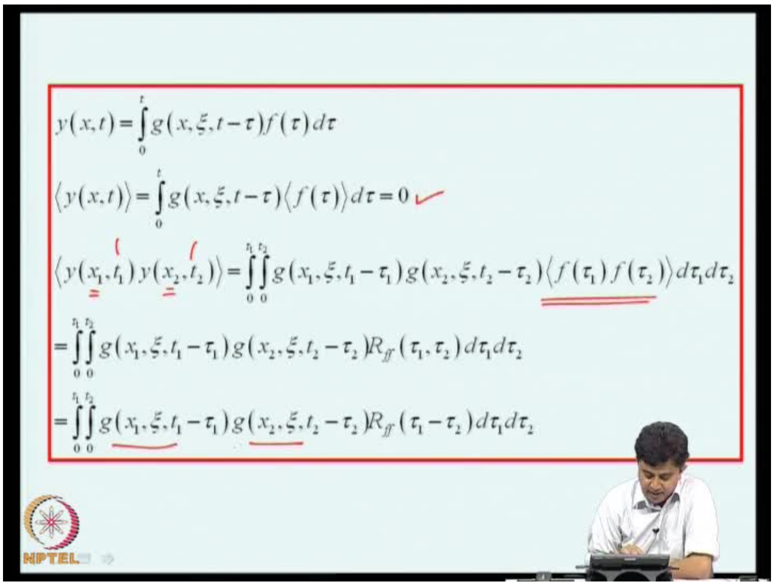

Now, we start by writing y (x, t) in terms of the Green's function, convolution of Green's functions with the applied f of t, and if you take now the expected value of this, is the expected value of f of tau is 0; therefore, this is 0. What happens to the covariance between response at x equal to x 1 and x equal to x 2 and time t equal to t 1 and time t equal to t 2?

So, we need to find the expectation of, I mean, we need to evaluate this integral, where the expectation of applied force appears here. This is the stationary random process, so R f f of tau 1, tau 2 can be written as R f f of tau 1 minus tau 2. We have already derived the Green's functions, in terms of the natural frequencies, and mode shapes of the system, modeled damping natural frequencies and mode shapes; so, they can that can be plugged here and these integrals can be evaluated to determine this requisite moments of the response.

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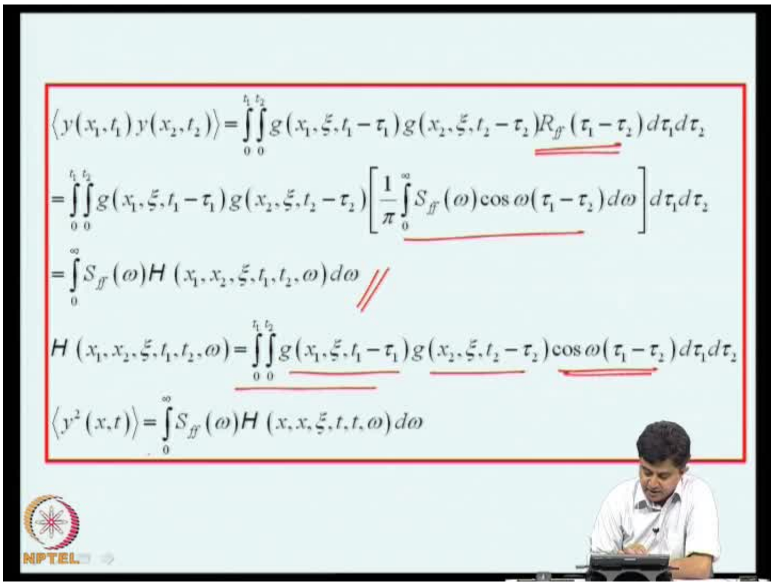

We can of course write the response also in terms of power spectral density function of the input, to see that what we do is, I represent R f f (tau 1, tau 2) in terms of its Fourier transform, once I had Fourier transform, which is shown here and we interchange the order of integration. So, we first integrate with respect to tau 1 and tau 2, and define a function capital H, which is this double integral, in terms of system Green’s functions and this cosine function; this can be viewed as the generalized transfer function, which is function of space as well as time.

So, I get the cross covariance between y at x 1 and y x 2 to be given by this expression. If you want mean square value what I have to do is, I have to put x 1 equal to x 2 to x and t 1 equal to t 2 to t, and if you do that, in this expression I get x psi t, t omega d omega; so, this gives me the variance of displacement. So, the key to the evaluation is actually the evaluation of this double integral and this contains the Eigen function expansions for Green’s functions, in terms of natural frequencies, model damping and mode shapes; so, this in principle can be evaluated.

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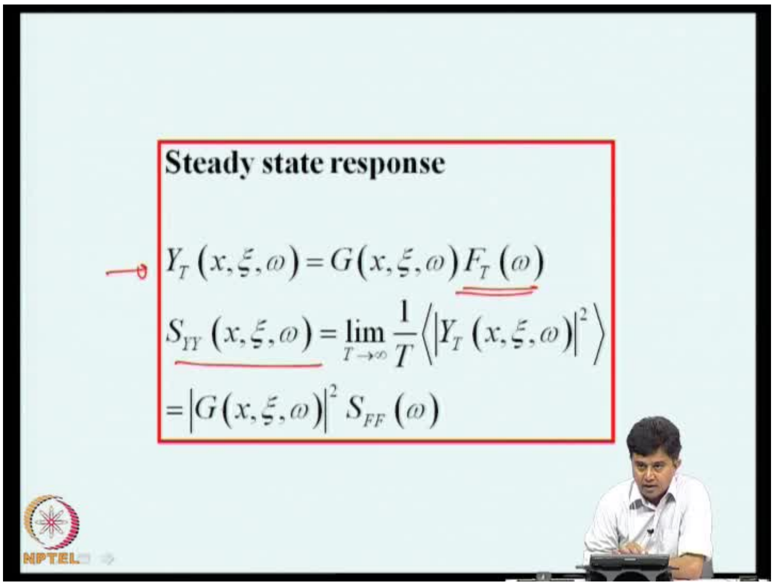

If you are interested in steady state response, we can look at the response in frequency domain and we consider the Fourier transform of a truncated sample of y (x, t) and that is given in terms of a truncated Fourier transform of a truncated damper of excitation through this relation. And we know that, power spectral density function is define through this expectations as t tends to infinity and if we apply this, we get the input output relation, that is the response power spectral density function at x due to driving at psi, is given by the amplitude of Green's functions squared into S f f of omega. This analysis, of course, we can repeat for different values of x, that means, you can find out the cross power spectral density function between response at x equal to x 1 and response at x equal to x 2.

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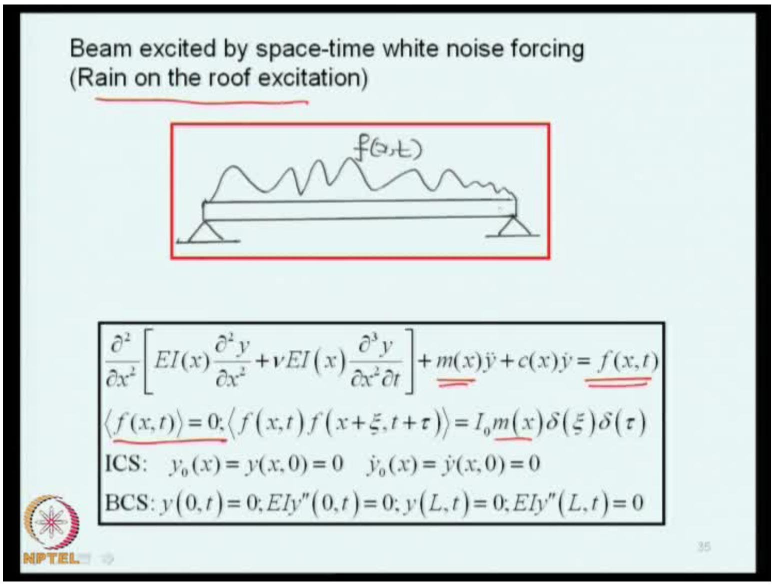

We now consider another example; we consider the beam excited by what is known as rain on the roof excitation. This is a space time white noise, having certain special properties; the equation is here on the right hand side is f (x, t), left hand side remains the same. Now, the property of f (x, t) is, that the mean is 0 expected value of f f (x, t) is 0, but if you find the covariance between f (x, t) and f of x plus psi and t plus tau, this is given by an intensity parameter i naught of x i naught and this m of x which is this mass here and delta of psi into delta of tau.

This type of excitation is known as rain on the roof excitations. This is used in response of structures to say boundary layer turbulence and in problems of high frequency evaporation analysis, at some point we will return to this use usefulness of these models, but right now we will analyze this problem.

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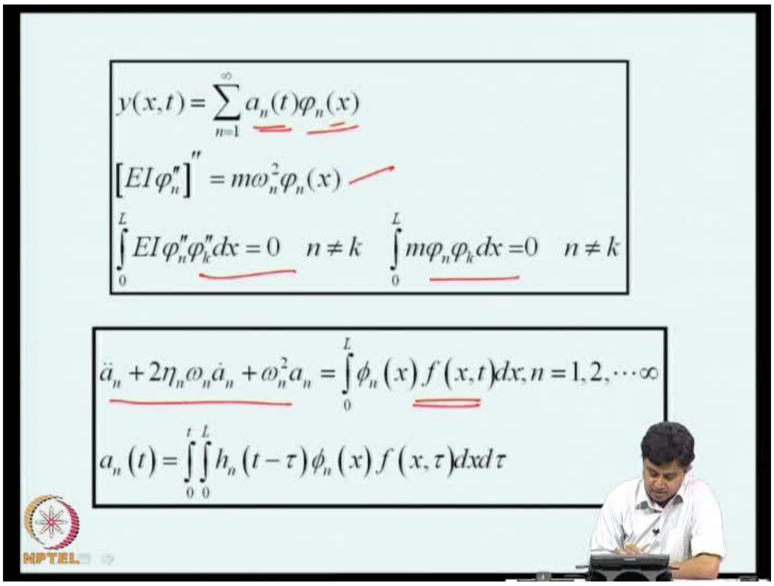

So, the first few steps are similar to what we have been doing till now, we assumed displacement in terms of generalized coordinates and the Eigen functions and these Eigen functions satisfy, this pair of orthogonality relations; and this Eigen value problem, this is the Eigen value problem and this leads to the equation for the generalized coordinates; and a generalized force on the right hand side is now double integral 0 to l phi n of x f of x, t d x and this n runs for 1 to infinity a n of t; therefore, in terms of the n th impulse response, for this system is given by this double integral.

One of these integral is with respect to space, that arises in definition of the load and other one is the integral that appears in the Duhamel integral or the convolution integral; this is in time, this is in space.

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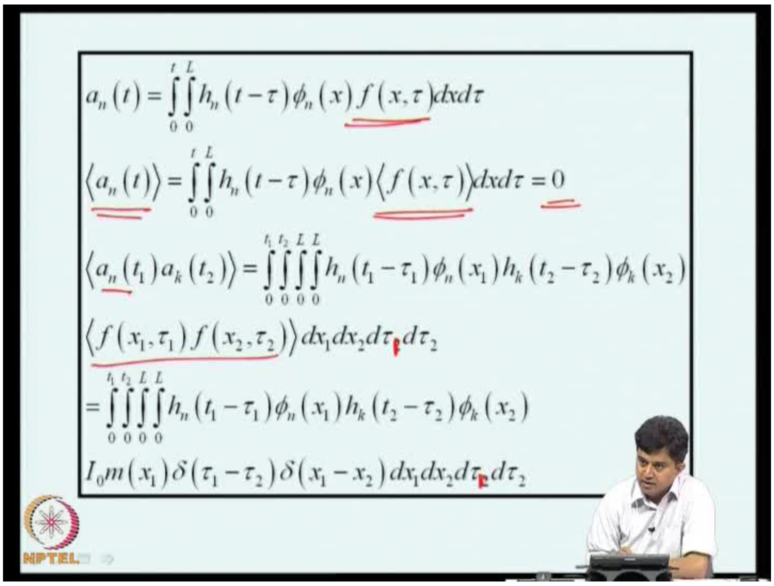

Now, let us look at the moments of a n of t, so if you look at mean of a n of t, this is expected value of a n of t, is expected value of this right hand side, and if you take expectation, inside the expectation, now operates on a f (x, tau) and it is specified that the mean of f (x, tau) is 0; therefore, mean of n of tau is 0.

Now, you consider two generalized coordinates a n of t and a k of t and you consider a n of t 1 and a k of t 2, and I want now the cross covariance between these two quantities. So, here each one a n of t itself is express as double integral; therefore, when you multiply these two, we get four integrals here and the integrant will contain the expectation of f (x 1, tau 1), f (x 2, tau 2); for this, we have the model, that is, this is given by i naught m of x1 delta of tau 1 minus tau 2 delta of x 1 minus x 2. So, the integration on a x 1, x 2 tau 1, tau 1 and tau 2.

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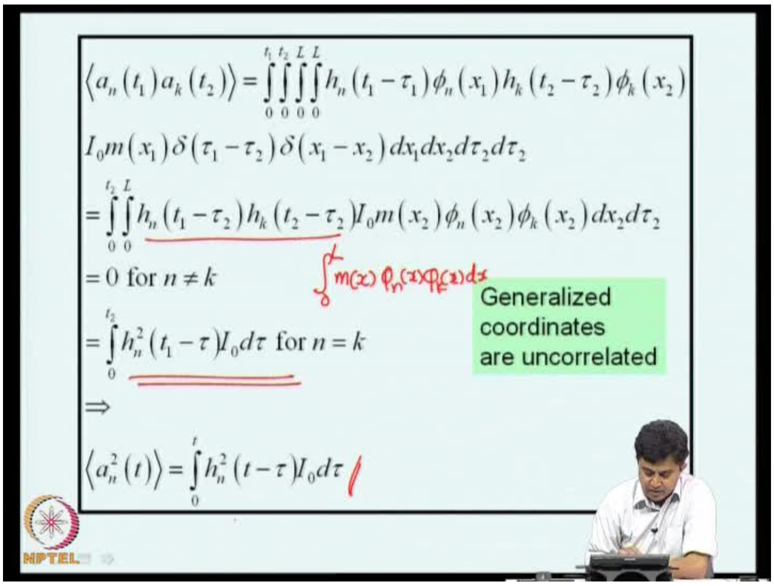

Now two of these integration can easily be done, because there are two direct delta functions; one in time 1 in space, so if you do that, I get this double integral 0 to t 2 0 to l this product and i naught m of x 2 phi n of x 2 phi k of x 2 d x 2 and d tau 2. Now, if you now look at the integration in space, what is this, this actually 0 to l, it can be written as m of x phi n of x phi k of x d x; by virtual of orthogonality property of phi n of x, this is 0, when n is not equal to k. So, what happens is, this will contribute, only when n is equal to k, otherwise it is 0, that would mean, the cross covariance between a n and a k is 0, for n not equal to k and for n equal to k; the auto covariance now of a n of t 1, a n of t 2 is given by this quantity. And of course, the variance itself given by you put t 1 equal to t 2, you get this variance.

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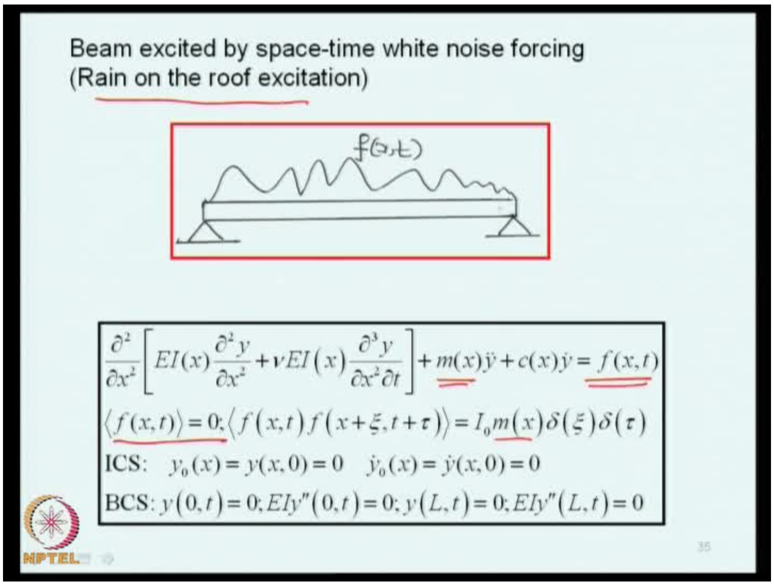

So, for this excitation, that is this, so-called rain on the roof excitation, the generalize coordinates become uncorrelated; if excitation is Gaussian that would mean generalize coordinates as stochastically independent. So, the coupling, uncoupling of equations is now total in the sense the equation for a n of t does not contain any terms involving a k of t, in their mechanical sense, it is already uncouple, but in stochastic sense the generalize coordinates a n of t is independent of generalize coordinate a k of t.

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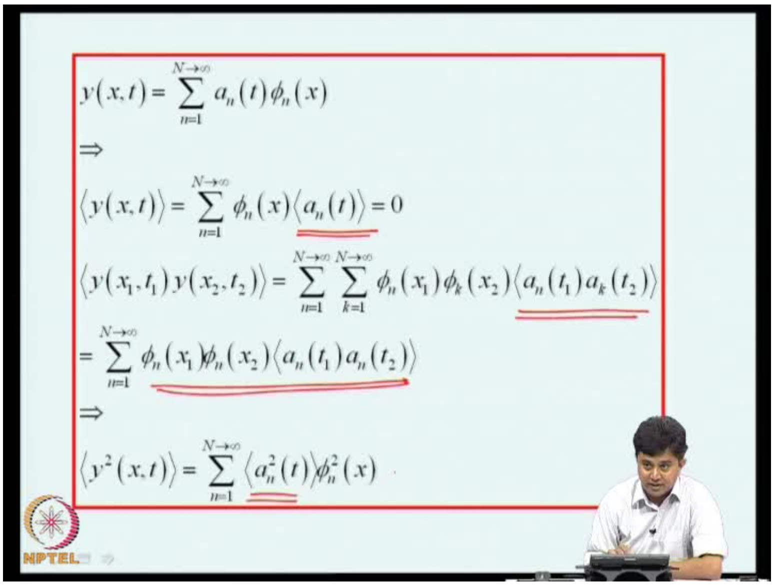

So, this type of model affords significant simplification in modeling, and that is some time exploited, in representing distributed loads like, turbulence and things like that. Now, if you written to the moments of displacement y (x, t), again we have to do, the, apply, the expectation operator on these expressions. So, expected value of y (x, t) is the expected value of this right hand side and that is expected value of a n of t which is 0.

You now want covariance between response at x 1 and x 2, we get this double summation, but we already seen a n of t 1 and a k of t 2 are uncorrelated; so, this collapse it to a single integral. And if you are finding variance of the response, it simply, it consist of summation of some kind, of sum of variance of individual generalize coordinates; that means, in computing variance of the total the beam displacement y (x, t), there is no cross terms between contribution from cross terms involving product of a n of t and a k of t. This is only true for the case of this rain on the roof type of excitation, it is not generally true but many people use that assumption, even when the excitation is not of the type, that is the of the type of rain on the roof type of excitation, but that modeling is in appropriate.

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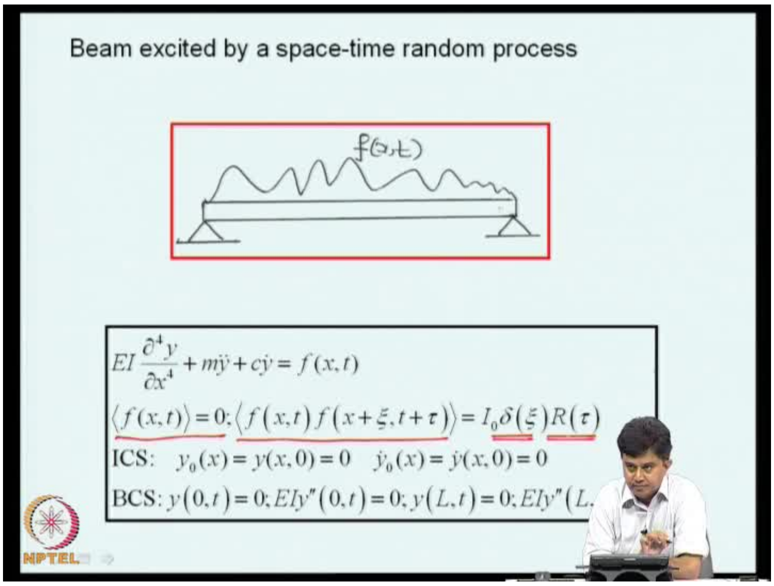

So, we will now consider a slightly more general random excitation, where f (x, t) is a random field evolving in x as well as time; it is not a rain on the roof type of excitation, but instead we assume that the mean is 0, but the second order moment f (x, t) into f of x plus psi t, tau is now uncorrelated in space, but in time it is a stationary random process.

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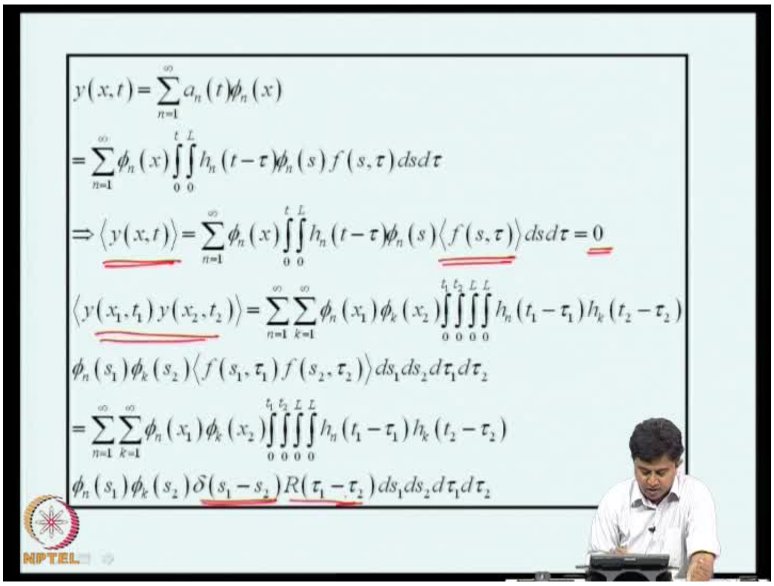

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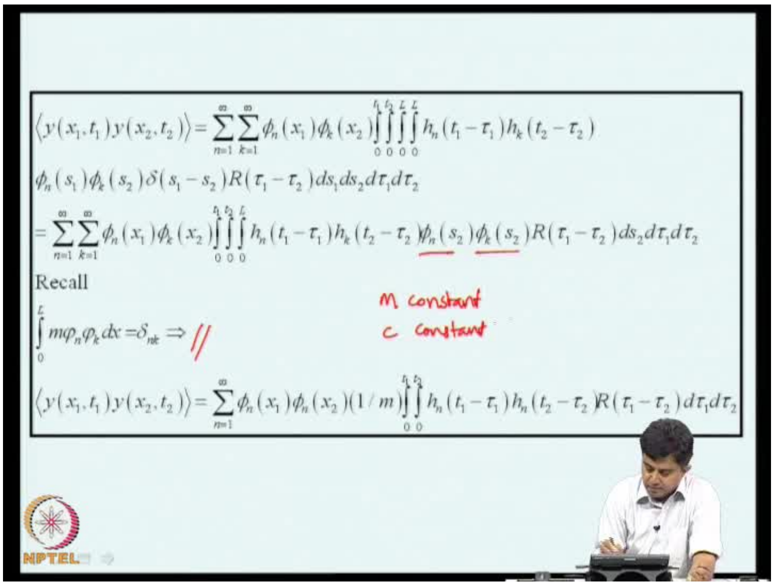

Now, the analysis here is reasonably straight forward, we assume y (x, t), again in the terms are generalize coordinates and Eigen functions. So, this is given in terms of the double integral h n of t minus tau phi n of f (s, tau) d s d tau; so, from this I can compute the expected value of the response, and since this is given to be 0, this expected value becomes 0. And I can consider the cross covariance, and if we run through this integration two of the integration, we have this expected value of f (s 1, tau), f (s 2, tau 2), and that appears here as a direct delta function in space, but in time, it is r of tau 1 minus tau 2, earlier we had a direct delta function here again, but now, right now it is not. So, one of the integration can be done easily and we from four integrals, we come down to three integrals; again by using orthogonality relation of this phi n of s 2, phi k of s 2, we can show that, this double summation reduced it to a single summation and this triple integral become as a double integral.

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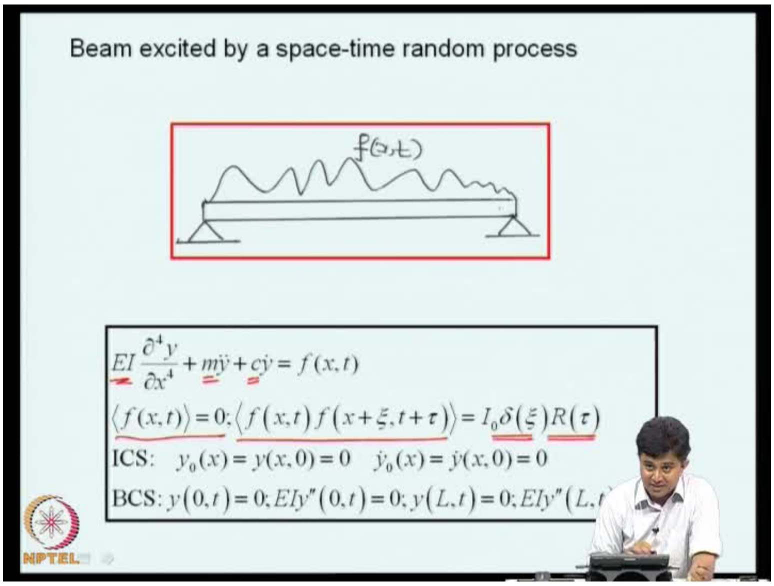

So, some simplification is possible, but here mind you we are assuming m to be constant and c to be constant, that means, the beam problem that I am considering here E I m and c are not functions of x, earlier I was including that, when I consider rain on the roof type of excitation m was the function of the x and E I was function of a x and c was a function of a x.

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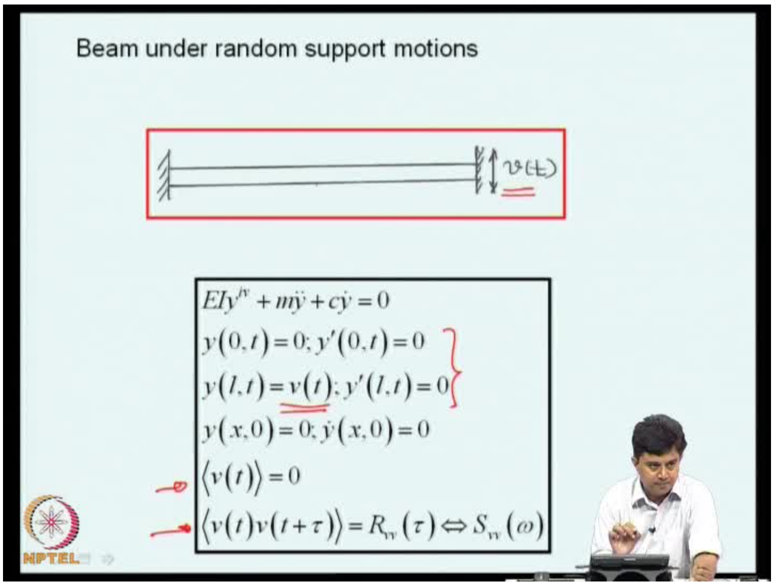

So, with this simplification, again we can get this expression with slightly less amount of effort. Now, we already outlined the formulation for response of a beam under time dependent boundary conditions. We will now consider the time dependent boundary condition to be a random process; so, we will consider now a clamp beam with one of the support being subjected to a time function v of t, which is modeled as a stationary random process. So, the governing equation is E I y 4 m y double dot plus c y dot equal to 0, I am assuming E I m s c etcetera to be independent of x and these are the boundary conditions; the excitation v of t appears as a boundary condition and we are assuming that system starts from rest; v of t we take it to be such that its mean is 0 and its auto covariance is the function of time difference, which is R v v of tau, and associated with that, there is a power spectral density S v v of omega.

Now, given this question is, what is the statistical properties of beam displacement, what is the mean, what is the covariance, what is cross covariance, what is auto power spectral density, what is cross power spectral density and so on and so forth.

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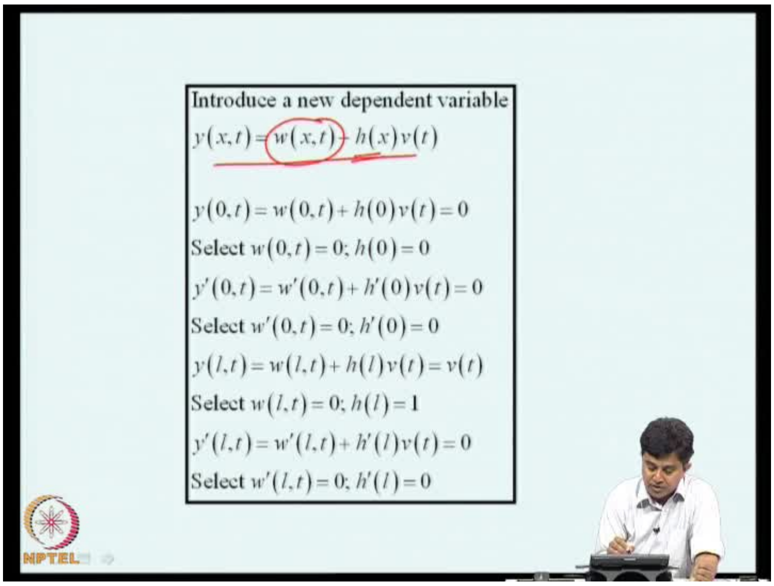

So, since the boundary conditions are time dependent, we implement this transformation, this we have discussed just a while before. So, I need not have to go through this, but the logic is displayed here, if we do that, we get an equation for the new dependent variable w, introduce a new dependent variable w and an unknown, an unknown function of h of x, w is unknown, h is unknown, h is selected in a certain manner, such that w becomes the governing equation for w, we will have time in variant boundary conditions.

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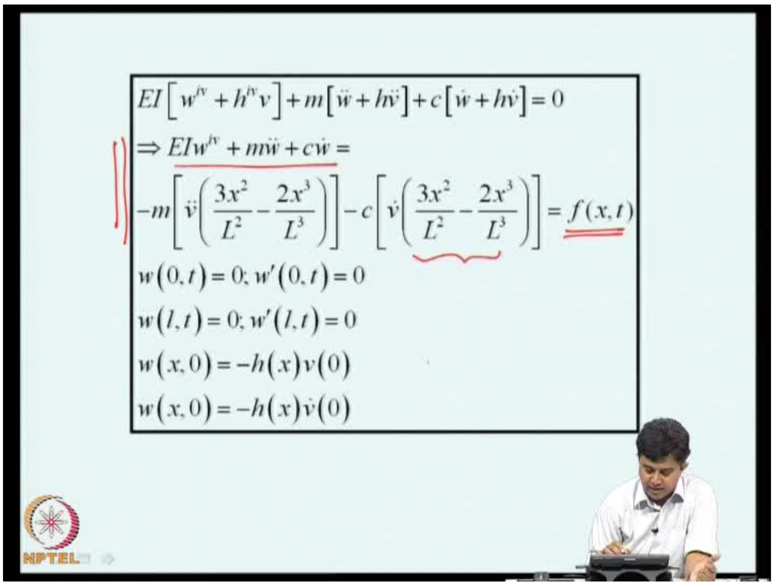

So, if you do that, we get the equation for w to be this, E I w 4 m w dot c w dot equal to a function of x and t, where h of x is this function, which we derive just a while before. At this stage, we have now reduced the problem to a form, which we have just now seen how to handle. So, this one, this is one approach of, one approach for handling time varying boundary conditions, and on this, we can do now statistics find out mean and variance etcetera.

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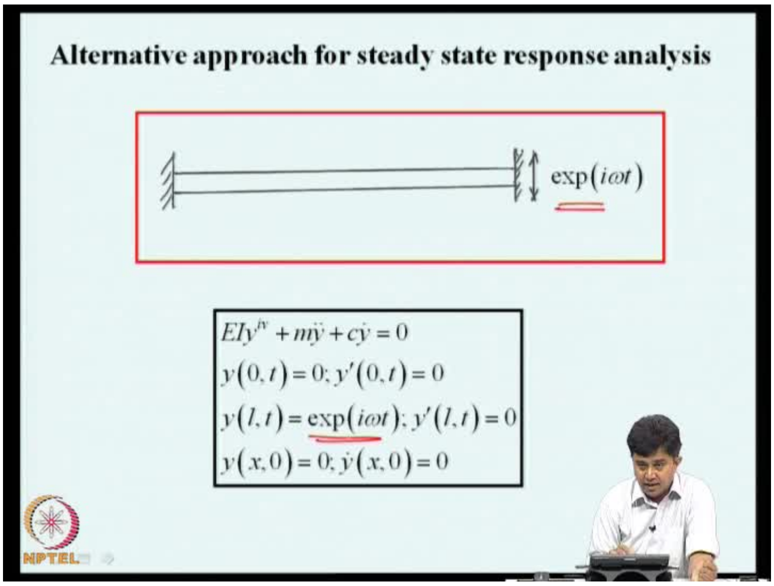

Now, there is yet another approach which is valid when support motions are stationary random processes, here we do not take the root of using Green's function, in terms of system natural frequency, is normal modes and modeled damping, but we directly solve the field equation. To illustrate that, we will consider this problem; again to start with, we will assume that, the support displacement is a harmonic function. So, the support displacement appears as a boundary condition and this is a linear partial differential equation and the excitation source is harmonic; so, as time becomes large, the response of the structure also is harmonic, at the driving frequency but with an unknown amplitude.

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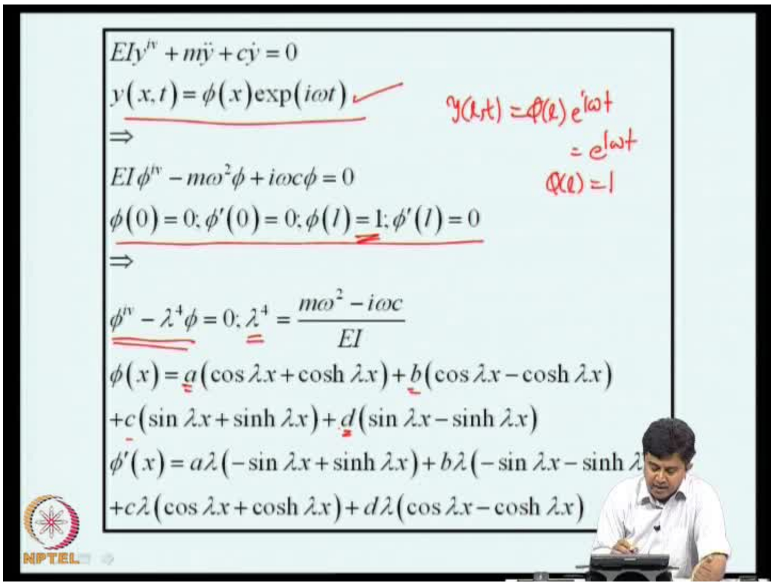

So, with that logic in mind I represent y (x, t) is phi of x into e rise to i omega t. Now, phi of x is an unknown function, is not in an Eigen function; we substitute this into this equation, we get EI phi 4 minus m omega square phi plus i omega c phi equal to 0 and the prescribed boundary condition on y. Now, translates to boundary conditions on phi and phi of 0 is 0, phi prime of 0 is 0, phi of l is1, because y (l, t) is phi of l e rise to i omega t and this is given to be e rise to i omega t; therefore, phi of l is 1, phi prime l is 0.

Now, I introduce a parameter lambda to the power of 4 as m omega square minus i omega c by E I and this is now an ordinary differential equation with x as the independent variable; so, it has this simple solution in terms of sin and cosine functions, sin, cosine, sin h and cos h function, this we have seen, when we derived the beam Eigen functions, the same form of the solution is relevant here, but except that now we have a inhomogeneous boundary conditions; this is not an Eigen value problem.

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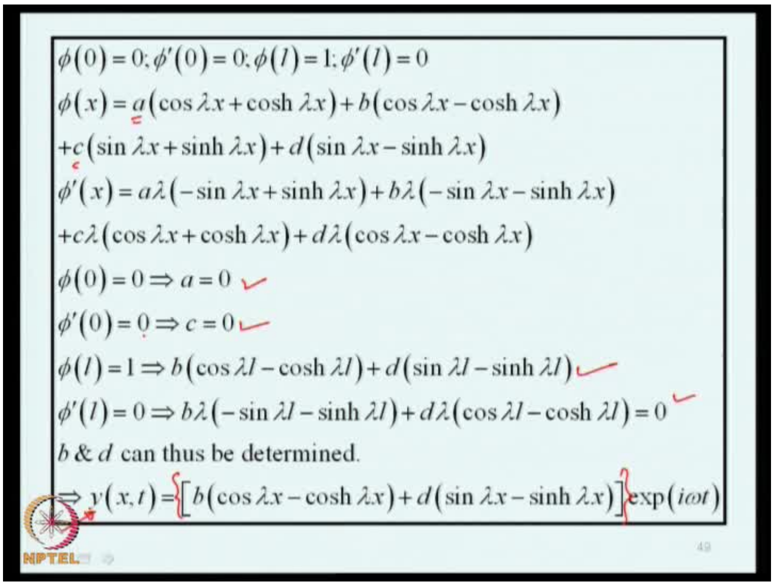

So, a, b, c, d are the arbitrary constant to be determine using this four boundary condition; if we do that, phi of 0 is 0, phi prime of 0 is 0, phi of l is 1, phi prime of l is 0; so, if we put phi of 0 is 0, a become 0; phi prime of 0 is 0, c become 0; so, a is gone, c is gone and I am left to b and d. And I have two more equations, at x equal to l phi of l equal to 1, leads to this equation; phi prime of l equal to 0, leads to this equation; and I can solve for b and d from this and get an expression for y (x, t), then this form.

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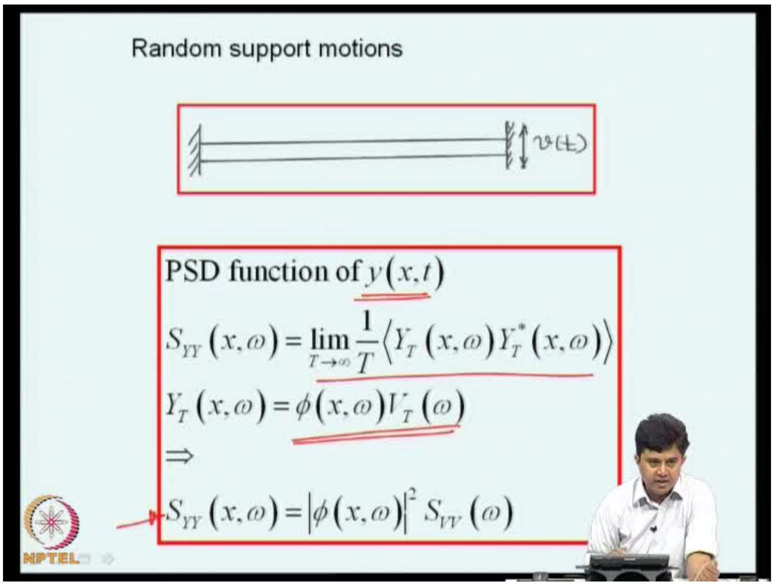

So, this quantity which multiplies e x p i omega t is nothing but the transfer function, right. And this can directly, we use in our random vibration analysis, to find out power spectral density of the response and so on and s forth. How we do that we are interested in the case, where v of t is the stationary random process, with a power spectral density S v v of omega and I am interested in power spectral density function say y (x, t); so, S y y (x, omega) is given by this is a basic definition of power spectral density function and for y t (x, omega) I use this relation, this phi (x, omega) is nothing but a transfer function which we have derived just now. So, now substituting this into the definition of power spectral density function, I get the response power spectral density function to be given in terms of this transfer function into the power spectral density function.

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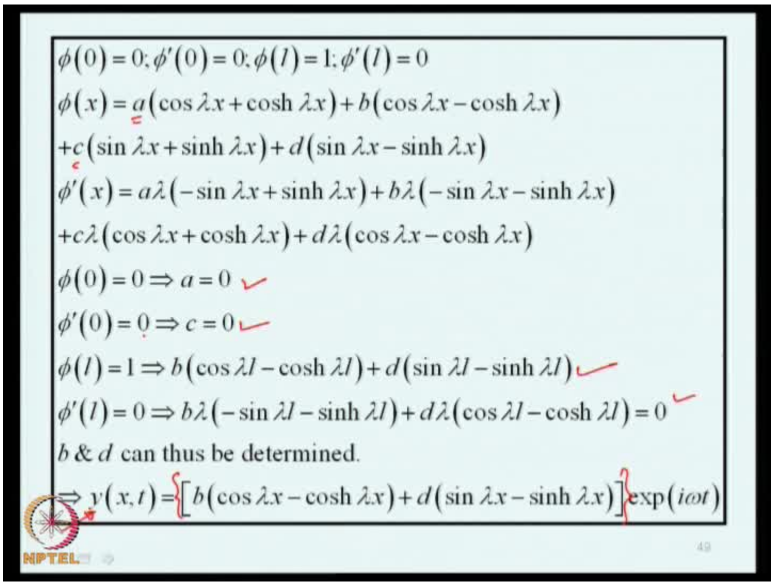

So, this solution in fact does not involve any model expansion, so more general models for damping can be used, but the penalty that you have to pay is you have to deal with trigonometric and hyperbolic sin, cosine and sin h, cos h functions.

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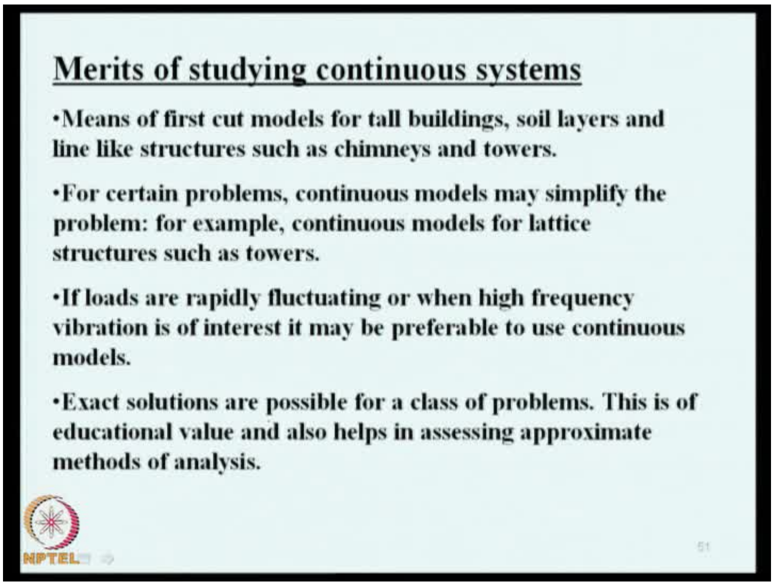

So, we can make some concluding remarks on vibration of continuous systems. Why do we study continuous systems? They save us means of first cut models for a class of structures like tall buildings, soil layers and line like structures, such as chimneys and towers; so, in their own standing, they have certain worth. For certain problems continuous models may simplify the problem, for example, continuous models for lattice structures such as towers; tower is essentially a lattice structure, but we can represent the behavior of a tower by a say a ventilation curve beam and then we can study of a beam like a ventilation curve beam, cantilever ventilation curve beam is not easier than studying a huge lattice structure.

Another thing is, if loads are rapidly fluctuating or when high frequency vibration is of interest, it may be preferable to use continuous parameter models than discrete models. The most important application of continuous system is that, certain problems can be solved exactly. This is of educational value and also helps in assessing approximate methods of analysis. So, these are some of the reasons, why we study continuous systems in practical application, there seldom used, but they serve as very useful models for various reasons mentioned here.

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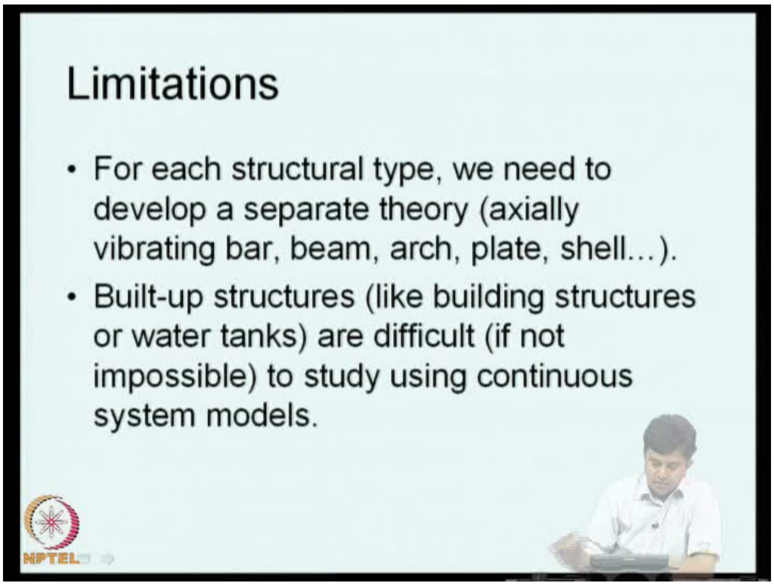

The limitations of using continuous system models, is that, for each structural type we have to develop a model, I have been to taking about Euler Bernoulli beam; so, you have to study an axial vibration problem, a beam problem, where Euler Bernoulli beam we discussed and if you include shared information and rotary inertia, you have to develop a theory for ventilation curve beam. And you want to study an arch, you have to combine theory of Euler Bernoulli beam with actual vibration models or if you are interested in departures, you have to develop another theory in two-dimensions, you have to deal with plane, stress models, plane, strain models, plates, shells and so on and so forth; so, each one you have to develop separately.

Even if you achieve all that built up structures, like building structures, a building structure has beams, slaps, may be trusses part of the roofs etcetera; so, it is hard to imagine how continuous system models can be used in studying such systems. We will end this lecture here; in the next lecture, we will consider problems of reliability, analysis of randomly vibrating systems.