Stochastic Structural Dynamics

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Module No. # 04
Lecture No. # 13
Random Vibrations of Mdof Systems-1

In today's lecture, we will begin discussing dynamics of multi-degree freedom systems under random excitations. So, we will start with discussion on discrete multi-degree freedom systems, which have finite degrees of freedom and they are typically governed by a set of ordinary differential equations.

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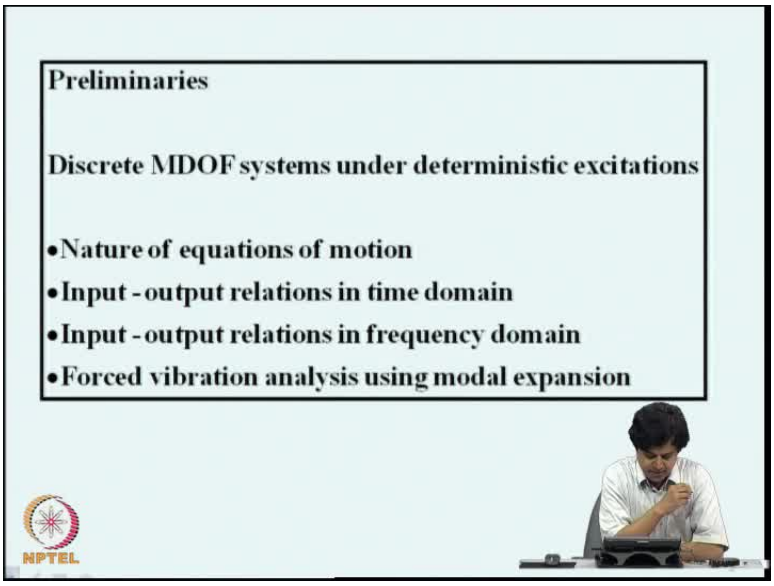

In today's lecture, we will go through some preliminaries. We will consider dynamics of multi-degree freedom systems under deterministic excitations and quickly review the nature of equations of motion and input-output relations in time domain, input-output relations in frequency domain, and the question of forced vibrations analysis using model expansion. This would be the launching pad for discussing input-output relation for systems with random excitations.

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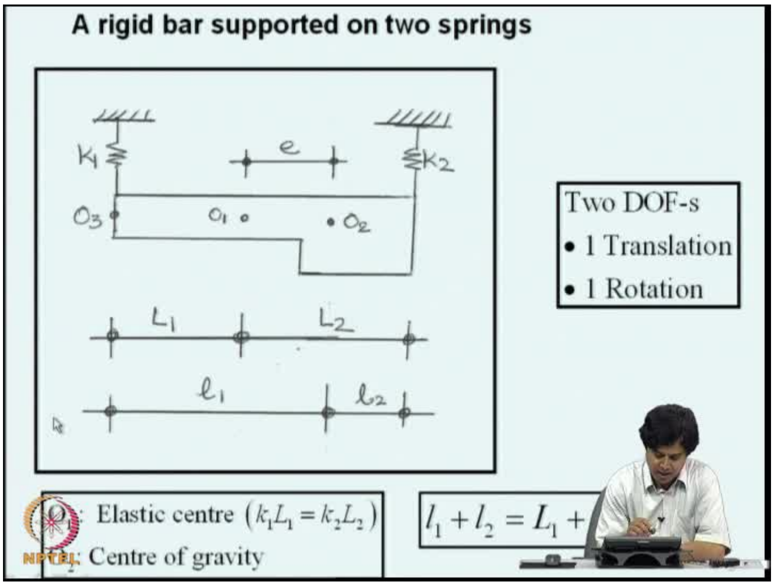

We will begin by discussing the dynamics of a rigid bar. This is a rigid bar; it is not a point mass. It is supported at two ends by springs K 1 and K 2; and, the point O 2 is the center of gravity; and, O 1 – we define it as elastic center; O 1 is defined such that K 1 into L 1 is K 2 into L 2. That is how O 1 is determined. O 2 is the center of gravity with respect to the mass of the system. This system has two degrees of freedom: it can translate and it can rotate.

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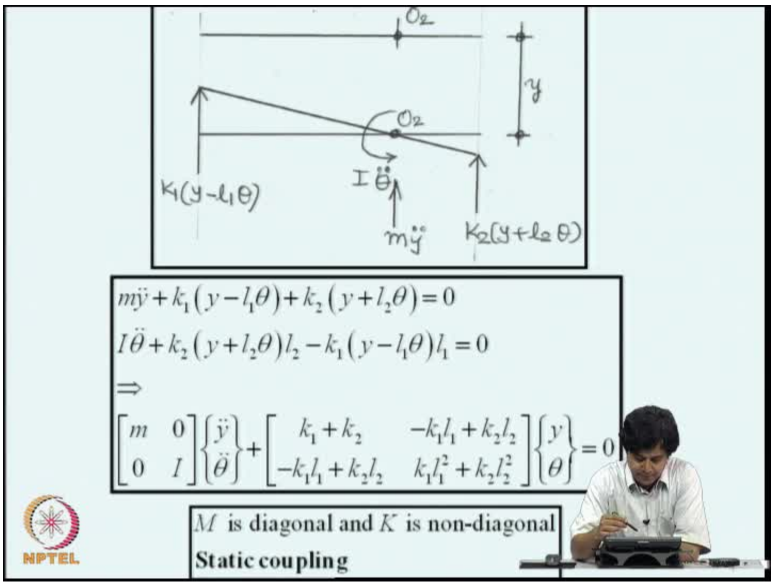

Now, we will try to set up the equations of motion for this system and see what are the basic issues in doing that and what will be the generic nature of equation of motion. We will begin by considering the displacement of the center of gravity as one of the degree of freedom, that is, y and the rotation with respect to the horizontal plane, theta. Now, the m y double dot is an inertial force. I theta double dot is the inertial force due to which opposes rotation. And, m y double dot is inertial force, which opposes translation. And, this force in the left spring is k 1 in to y minus l 1 theta. This distance is y and this distance is l 1 theta. So, the force in the spring will be product of k 1 into y minus l 1 theta. Force on the right-hand spring will be y plus this distance, (Refer Slide Time: 03:01) that is, k 2 in to y plus l 2 theta.

Now, if we sum the forces in vertical direction, we get one equation; that is, m y double dot plus k 1 into y minus l 1 theta plus k 2 into y plus l 2 theta is 0. Similarly, if you take moments of the forces about point O 2, I get I theta double dot plus k 2 – this into this distance (Refer Slide Time: 03:26) – k 2 y plus l 2 theta into l 2 and k 1 y minus l 1 theta into l 1. So, this is the equation of force equilibrium; this is moment equilibrium (Refer Slide Time: 03:37). We can recast this equation of motion into a matrix form, where I see that from the first equation, I get inertia is associated with y double dot. So, I get m and 0 here. And similarly, inertia is associated with I here – 0 and I; and, this I here. This is the mass matrix. Similarly, this is the stiffness matrix. So, from this equation of motion, we can observe that the mass matrix is diagonal; whereas, the stiffness matrix is non-diagonal.

You write now equation for y; you see this equation (Refer Slide Time: 04:21). It has terms involving theta. And therefore, we cannot solve this equation for y unless we know theta. Similarly, if I look for equation for theta, it involves terms containing y. Therefore, I cannot solve for theta using this equation unless I am able to solve for y; or in other words, the equation for y and theta are coupled. How does this coupling manifest in the matrix form of equation of motion? The coupling manifest in terms of this stiffness matrix being non-diagonal; that means the equation for y – this will be m y double dot plus 0 into theta double dot; that means in terms of inertia, there is no coupling between y and theta. But, if you look at forces in the springs, there is a coupling between y and theta, because the first term will be k 1 plus k 2 into y; and, second term will be this term into theta.

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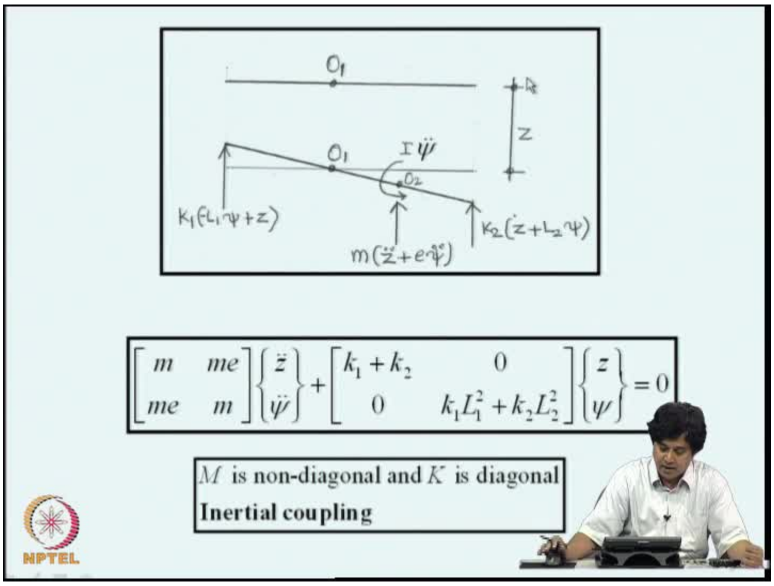

So, here we say that the equation of motion has static coupling, because stiffness matrix is non-diagonal. The choice of defining translation as y, that is, that amount by which the center of gravity translates is what we have taken as degree of freedom here. This choice itself is not unique. I can take the translation of the elastic center as a degree of freedom. This is another choice of coordinate system that I can make. And, if I do that, the inertial force would be now against translation – will be m z double dot plus the acceleration e psi double dot; e is the distance between O 1 and O 2. So, e psi double dot is the acceleration here. There is of course inertial force against rotation, which is I in to psi double dot; where psi is the angle that the bar makes with the horizontal.

I can now write the forces in the springs following the same argument that we used previously. The force here will be k 1 z minus L 1 psi. This (Refer Slide Time: 06:34) is k 2 into z plus L 2 psi. Again, if we sum forces in vertical direction and moments about this point O 2, we get this equilibrium equation. Here the mass matrix now becomes non-diagonal; the stiffness matrix is diagonal. This is for the same system. The difference is now arising essentially because I am selecting a different coordinate system. Earlier I chose y and theta; now, I am selecting z and psi. Therefore, in this coordinate system, the coupling between z and psi coordinate is through inertial terms and not through the stiffness terms. So, we say that this equation of motion has inertial coupling.

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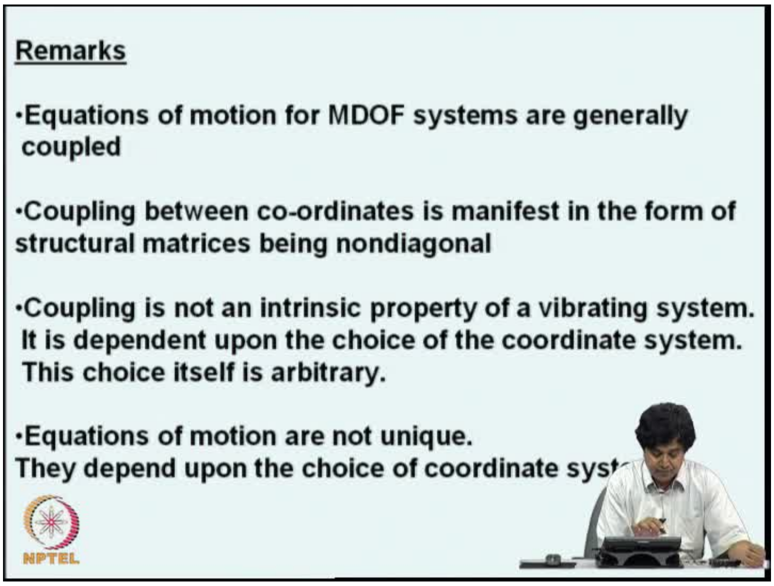

Now, I can make yet another choice, where instead of measuring translation at center of gravity or elastic center, I will simply measure the translation at the free edge. It is a left-hand edge. I call this as x and this rotation as phi. If I do that and write the equations again following the same logic, I get now an equation of motion in which both mass and stiffness matrices are non-diagonal. Here the coupling is through both inertia and stiffness terms. So, here we say that in this form of equation of motion, there exist static and inertial coupling. Based on this, we can make some observations now that equations of motion for MDOF systems are generally coupled. In this system, it is true in any case. Coupling between coordinates is manifest in the form of structural matrices being non-diagonal. Coupling is not an intrinsic property of a vibrating system. It is dependent on the choice of the coordinate system. This choice itself is arbitrary. So, we made three choices. In one of the choice, mass matrix was diagonal; stiffness was non-diagonal. In the second choice, mass was non-diagonal and the stiffness was diagonal. In the third choice, both m and k were non-diagonal. And, all these equations are written for the same system, physically the same system. Thus, we can say that equations of motion are not unique. They depend on the choice of coordinate system.

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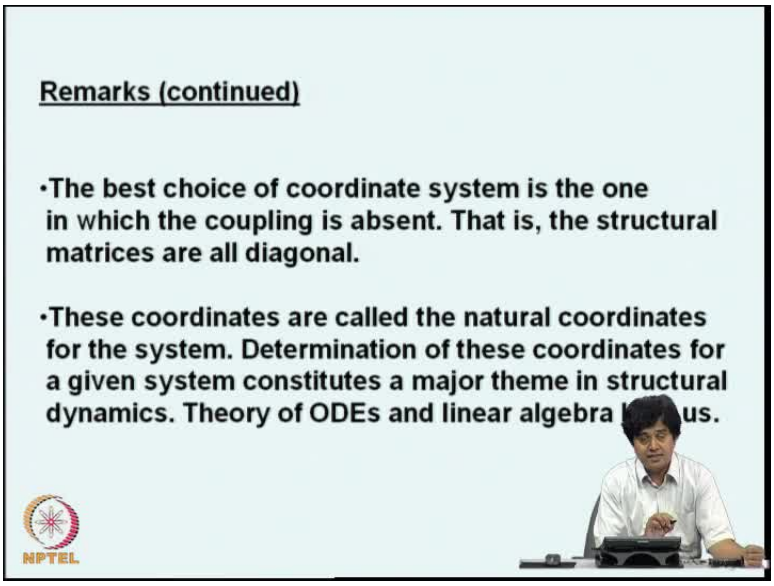

Now, the question would arise, what is the best choice that one can make on coordinate systems? So, the best choice of coordinate system is the one in which the coupling is absent. That means all the structural matrices are diagonal matrices. If that happens, then the two degree freedom system that we are talking just now can be viewed as a set of two single degree freedom systems. So, we can solve them in a straightforward manner using the theory that we already learnt. Now, if such coordinate systems can be found, it simplifies our solution strategy substantially. In fact, the coordinates in which the structural matrices are diagonal are called natural coordinates for the system. Determination of these coordinates for a given system constitutes a major theme is structural dynamic analysis. Here theory of ordinary differential equations and linear algebra are going to help us. So, will see how that happens shortly.

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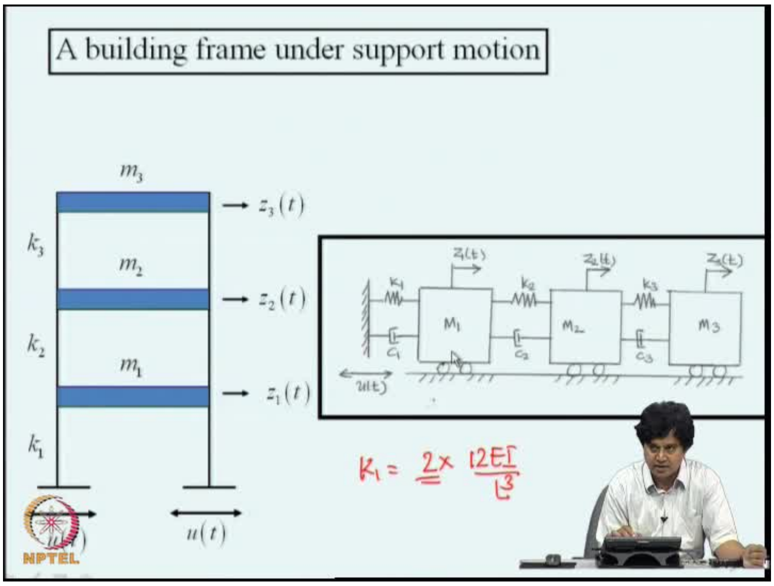

Before we do that, we can consider a few more examples and see the general structure of equation of motion for multi-degree freedom systems. In this case, we have a building frame under support displacement. It is like a three-storey planer frame under earth quake like support motion. Both supports are receiving same input u of t. And, if we assume that slabs are infinitely rigid in their own plane and columns are light in mass in comparison with the mass of the slab, the mass of the columns can be ignored. Then, we can approximate this three-storey frame by a three degree freedom system as repeated here. This m 1 is mass of this slab; this m 2 is mass of this slab; and, m 3 is mass of this slab. We could include in m 1 a part of mass of these columns, which can participate in vibration. That is a refinement of the model.

This k 1 is the stiffness of the columns in the first floor. So, if we assume that this is fixed (Refer Slide Time: 11:24) and this is fixed and is undergoing a support displacement here, the force required to produce unit displacement here would be for instance, k 1 would be 2 into 12EI by L cube. 2 because there are two columns and 12EI by L cube is the force that you should apply to produce unit deformation at the slab level here. Now, this u of t, which is a support displacement appears here. So, m 2, m 3 are here; k 2, k 3 are stiffness of this and c 1, c 2, c 3 are the dampers, which will represent notionally through these entities. We can draw now free-body diagram of each of these masses and represent all the forces. Thus, for example, if you consider mass m 1, the forces that act on these are the inertial force. If motion is taking place in this direction, the inertial would oppose that. And, if motion is taking place in the positive direction, this spring will try to pull it back; this damper will try to pull it back; this spring will force it back; this damper will force it back; and so on and so forth.

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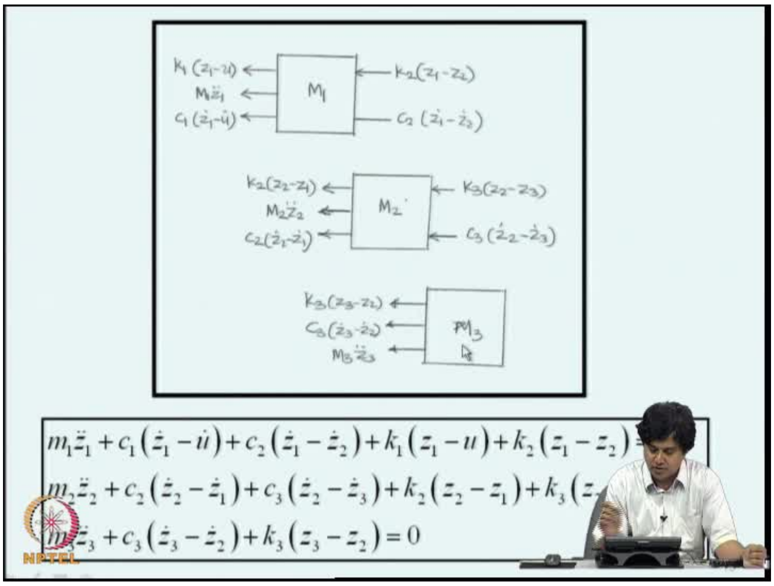

So, if you write all those forces, we get here the inertial force, the force in spring k 1; this is k 1 into z 1 minus u, because support is undergoing a displacement u. And, here k 2 is the… z 1 minus z 2 is a relative displacement between mass m 1 and m 2. Therefore, the force in k 2 is k 2 into z 1 minus z 2. So, using similar logic, we can write the free-body diagram for m 1, m 2, m 3. And, based on this, I can write the three equations of motion by summing the forces in the horizontal direction. So, the support displacement here if you notice, support displacement appears here and here (Refer Slide Time: 13:16). We can now recast this equation by considering relative displacement of the floor with respect to the ground.

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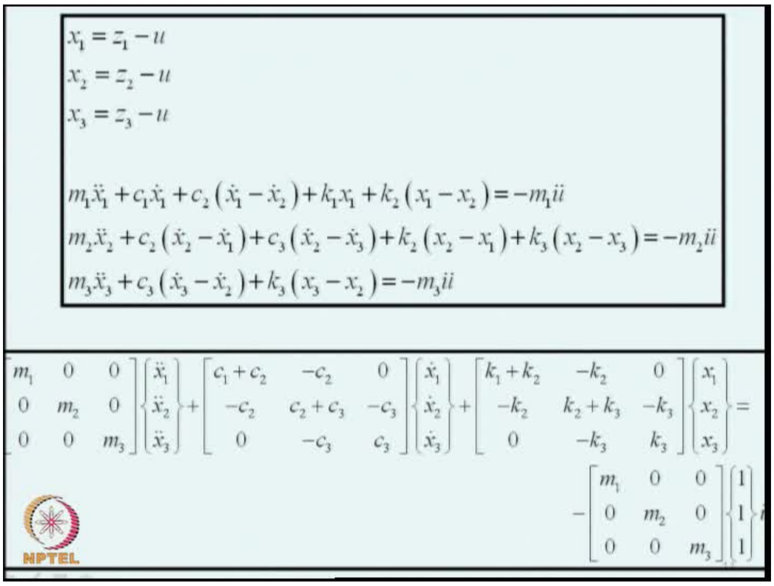

So, if I define x 1 as z 1 minus u, x 2 as z 2 minus u, and x 3 as z 3 minus u, I can rewrite these equations. And, on the right-hand side, now, I will get m 1 into u double dot, m 2 into u double dot, m 3 into u double dot. That means, the effect of support motion is equivalent to the application of lateral forces given by m 1 u double dot, m 2 u double dot and m 3 u double dot at the slab level. This is equation for the relative displacement. So, I can recast this in matrix form and you can see here that the mass matrix in this case is diagonal, but stiffness and damping matrices are non-diagonal indicating coupling between the coordinates x 1, x 2 and x 3. If all these three matrices were to be diagonal, then the three coordinates would be uncoupled from each other.

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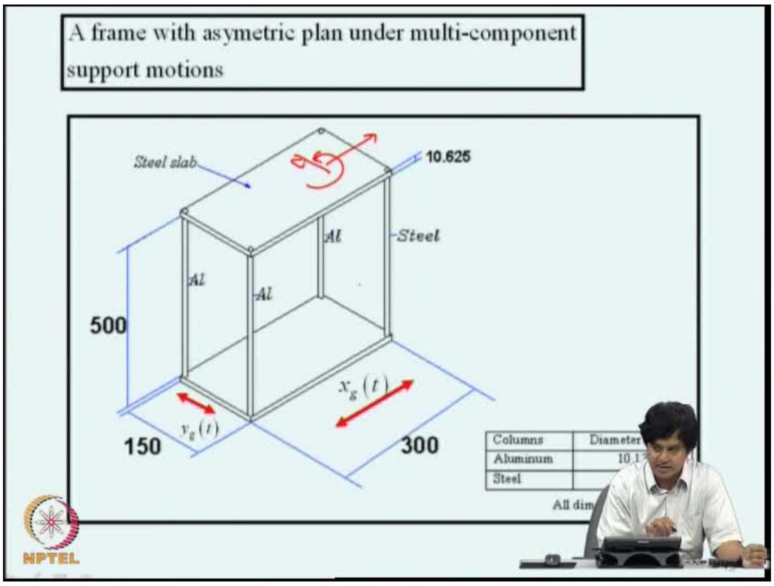

We will consider one more example now. This is a model that is being studied in laboratory. So, this consists of a steel slab, which is supported on four columns as shown here. And, the bottom slab is fixed on a shake table; and, on this table, we can apply horizontal displacement x g of t and y g of t – support displacement. The three columns here are made up of aluminum and one of these columns is made up of steel. So, if you view this structure from the top, you can see that there is an asymmetry in distribution of stiffness, because the Young's modulus of steel is different from Young's modulus of aluminum. So, this member is going to be stiffer than these three members. Therefore, there is an asymmetry in plan. So, if you were to apply at the support say a displacement x g of t, you can expect that the structure would displace in this direction and in this direction. And also, it would twist, because there is a coupling between bending and twisting because of asymmetric distribution of the stiffness.

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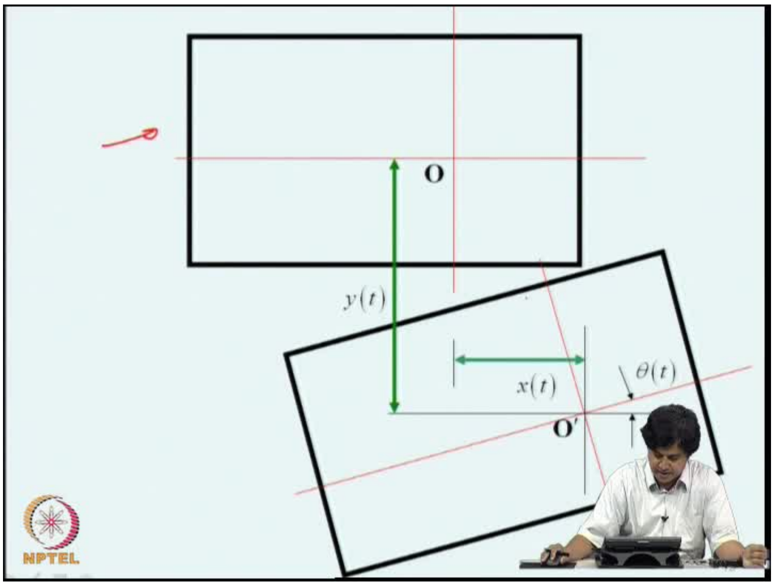

So, this is schematically shown here. The motions here are exaggerated. So, this is the undeformed configuration; this is deformed configuration. And, the three degrees of freedom are translation y, translation x and the rotation theta.

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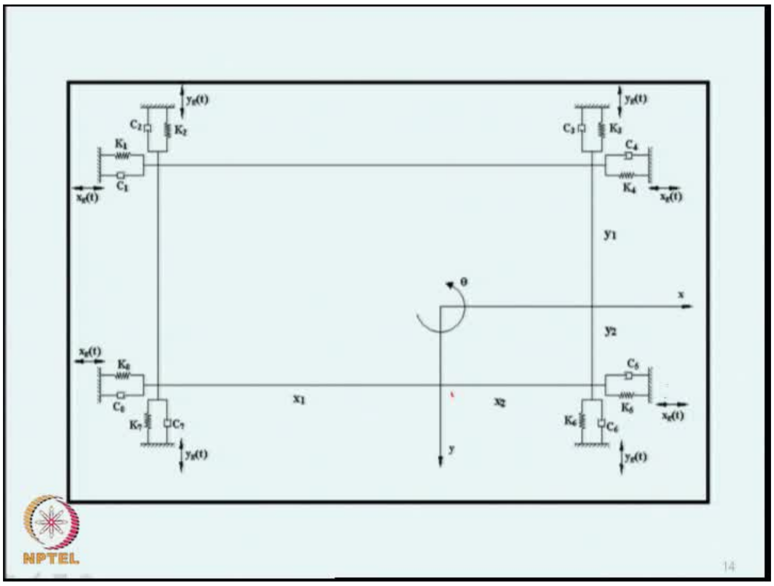

So, we need to write now the equation of motion by considering forces along x, y and this rotation. We assume that the steel slab is infinitely rigid in its own plane and it behaves as a point mass. And, the columns contribute to stiffness in directions x and y and these are shown here. And, these are the three degrees of freedom. And, we can write now the forces in these dampers and springs by considering these displacements and these support motions. These support motions are appearing here; x g is here; y g is here.

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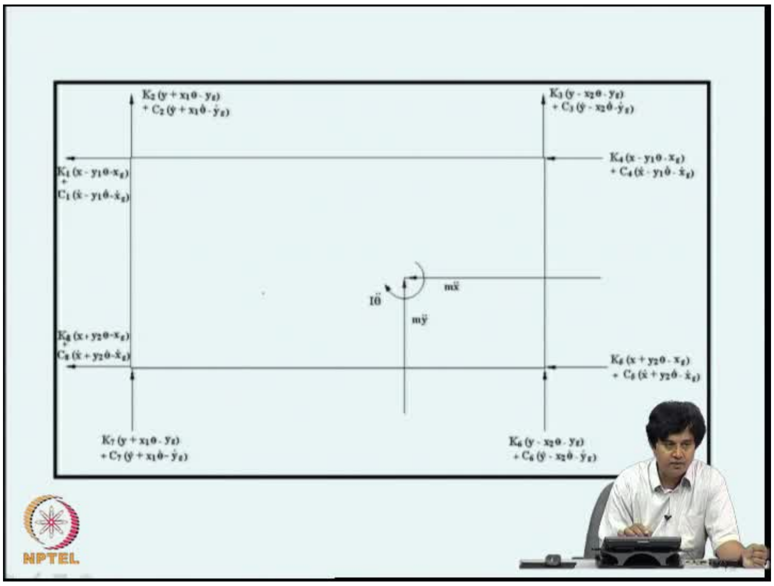

We can represent all the forces here. These are the inertial forces passing through the center of gravity of the system. And, these are the various forces at the springs and dampers introduced at the ends, which represent the action due to the columns. Now, we can consider the equilibrium of forces in horizontal x direction, y direction and this theta direction.

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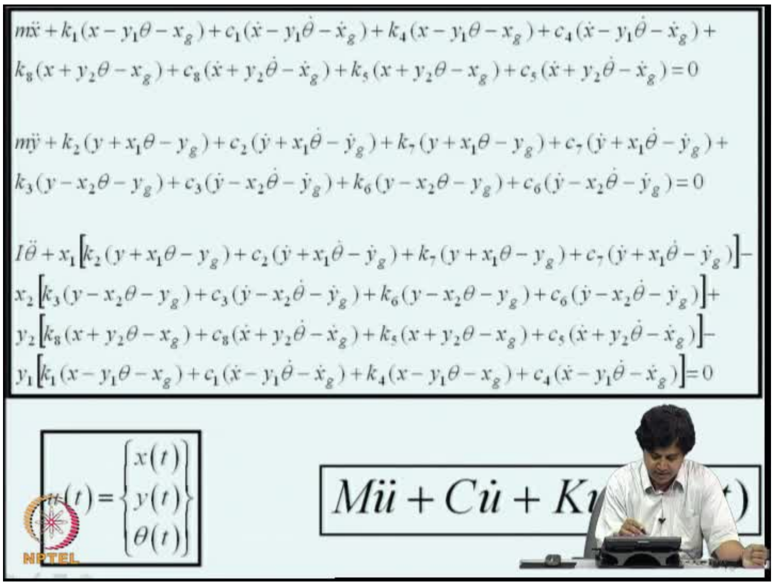

And, if you do that, we get the equation of motion, which is fairly complicated looking equation, but the summary is that this equation of motion can be written in the matrix form – M u double dot plus C u dot plus K u equal to f of t, where the matrix m is diagonal, because we have selected the reference point to pass through center of gravity; but, C and K are non-diagonal. So, the equation for x, y and theta will be mutually coupled.

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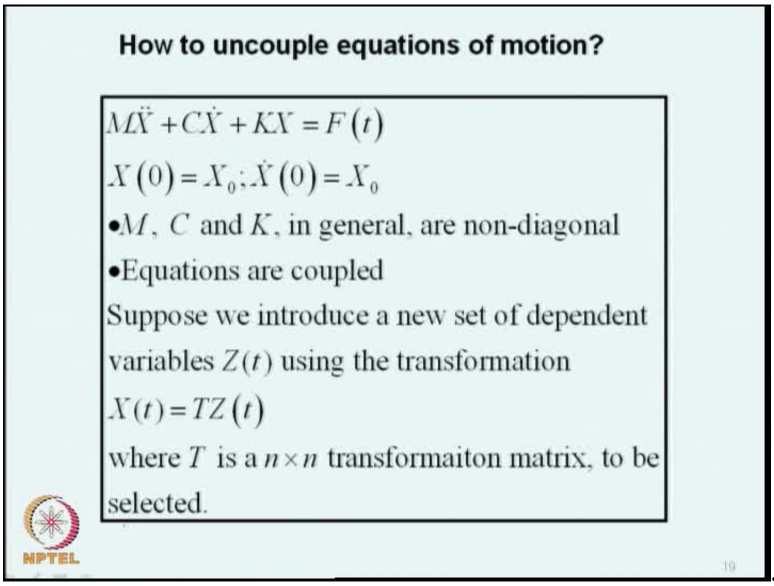

So, what we have seen till now is a generic form of equations of motion for multi-degree freedom systems. Linear multi-degree freedom system has this form, namely, M X double dot plus C X dot plus K X equal to F of t. These represent a set of coupled ordinary differential equations, second order linear ordinary differential equations. And, these are the initial conditions – t equal to 0, the displacement vector x naught is specified and t equal to 0, the velocity vector x naught is specified. These matrices M, C, K are square matrices; n cross n, where n is the degree of freedom. And, they are typically non-diagonal. At least one of them will be non-diagonal. And therefore, the equations for x 1, x 2, x 3, x n would be coupled.

Now, our interest would be to see if we can formulate the equation of motion in a coordinate system in which M, C, K are all diagonal. If that happens, we would have uncoupled the equations of motion. Now, we have already seen in the example of rigid bar that by making different choices of coordinate system, in one instance, we got masses diagonal, but stiffness was non-diagonal. This next instance we got masses non-diagonal, but stiffness was diagonal. In the third instance, both were non-diagonal. So, one could think of searching for a coordinate system, where M and K both are diagonal. But, how do you conduct that search? We should have a systematic recipe for that. So, the strategy that we follow is we will formulate the equation of motion in a coordinate system that appeals to the analyst, which is physically appealing to person, who is making the model. Then, we will try to transform the coordinate system into a new coordinate system in which the equations could become uncoupled. So, what we do is, we introduce a set of new dependent variable Z of t; using the transformation, X of t is T into Z of t; where, T is a transformation matrix, which is n by n, which is to be selected; we do not know. It has to be selected in such a way that after I make this transformation, the equation in z coordinate system would be uncoupled; that means the mass stiffness and damping matrices in z coordinate system would be all diagonal. How do we do that?

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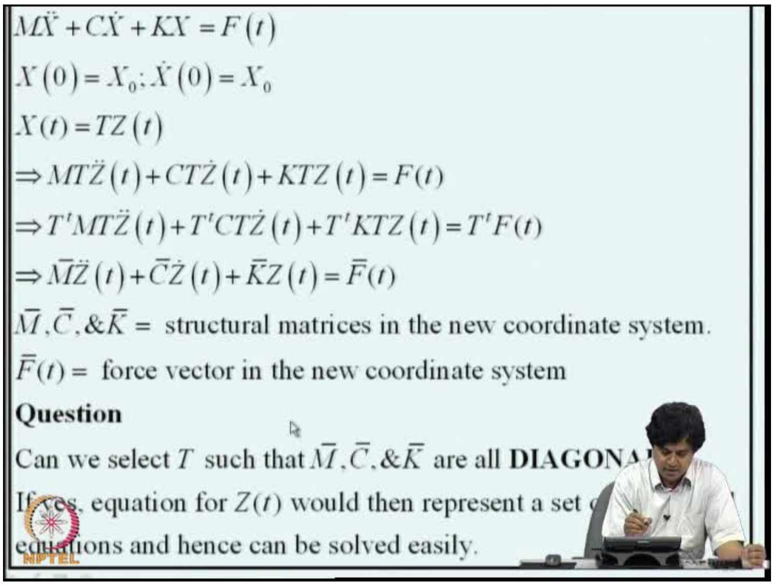

So, let us consider the effect of this transformation X of t is T Z of t. So, I will substitute this in to the given equation; I get for M X double dot, MT Z double dot; for C X dot, I get CT Z dot; K X, I get KT Z equal to F of t. If I now pre-multiply this equation by T transpose, I get T transpose MT Z double dot plus T transpose CT Z dot plus T transpose KTZ is equal to T transpose into F of t. I will now call these matrices, T transpose MT as M bar; T transpose CT as C bar; T transpose KT as K bar; and, T transpose F of t as F bar. These M bar, C bar and K bar matrices are the structural matrices in the new coordinate system. M bar is the mass matrix; K bar is the stiffness matrix; C bar is the damping matrix. Similarly, F bar of t is the force vector in the new coordinate system. What is that we are looking for? We are looking for the transformation matrix T, such that the three matrices M bar, C bar and K bar are all diagonal; that means we are looking for a linear transformation, which simultaneously diagonalizes three matrices M, C and K.

If we can find that then the equation for Z of t would then represent a set of uncoupled equations and hence can be easily solved easily. If in this equation, (Refer Slide Time: 21:51) if M bar, C bar and K bar are all diagonal, then equation for Z 1 will not contain terms involving Z 2, Z 3, Z n, etcetera. They are all uncoupled. So, I can take one by one these equations and solve them using theory of single degree freedom system, which we have already studied. Therefore, now the fundamental question here is how do we select this transformation matrix T.

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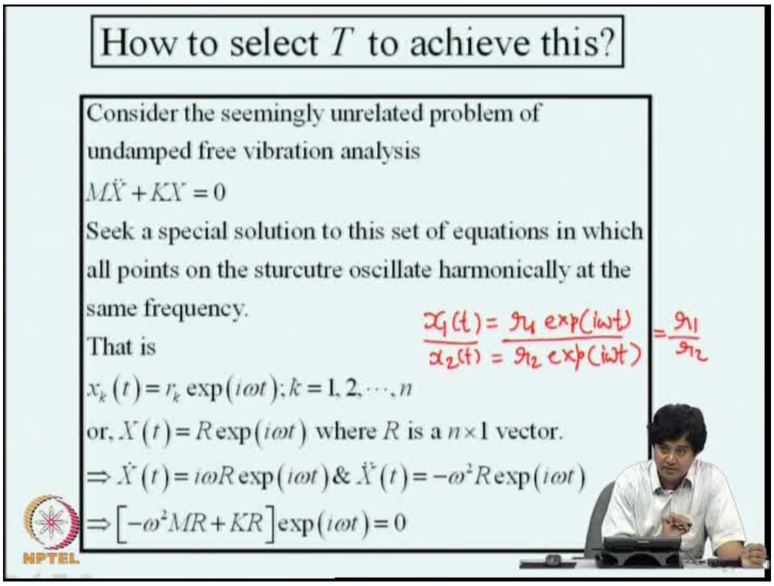

To answer that question, we start by considering a seemingly unrelated problem of undamped free vibration analysis, that is, M X double dot plus K X equal to 0. This is nothing to do with the problem that I actually wish to solve; where in the problem that I wish to solve, there is damping and there is an external forcing and there are specified initial conditions. So, now, I have sacrificed the damping term, excitation term and I am not even considering the initial conditions that are specified. So, I am considering in a way is totally unrelated problem. Not only that; we seek a special solution to this set of equations in which all points on the structure oscillate harmonically at the same frequency, that is, x k of t; I want a solution to this where x k of t is r k in to exponential i omega t. So, if you consider now say for example, x 1 of t, this will be r 1 exponential i omega t and x 2 of t would be r 2 exponential i omega t; that means both x 1 and x 2 are oscillating at the same frequency omega. So, if I were to divide that, the ratio of this displacement would simplify r 1 divided by r 2, which is not a function of time.

So, the solution I am looking for is that all points on the structure vibrate harmonically at the same frequency. And, if you take ratio of displacements of any two coordinates or any two displacements, that ratio would be independent of time; that means we are looking for synchronous harmonic motions. Why are we looking for that? That is not what we are intending to; I mean that is not our original problem anyway, but there is a profound reason why we are looking for that. So, let us see that. So, if I now write this as x of t is r exponential i omega t, where r is n cross 1 vector. So, X dot of t is i omega R exponential i omega t; X double dot of t is i square omega square R exponential i omega t; i square is minus 1. So, I can substitute this into (Refer Slide Time: 24:33) the governing equation and I get this equation.

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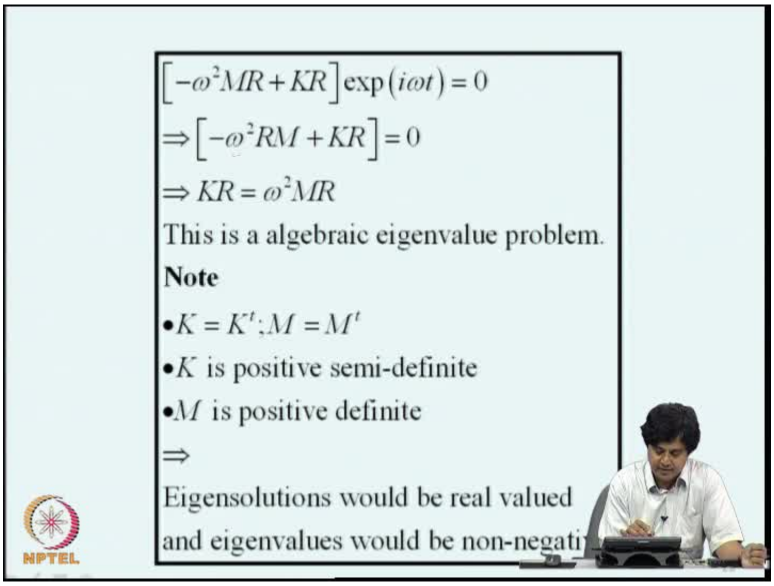

Now, exponential i omega t cannot be 0. Therefore, I get this equation minus omega square RM plus KR equal to 0 or KR equal to omega square MR. Now, what are unknowns here? Omega is unknown; R is unknown. So, we need to find now a solution for this equation, which consists of determining omega and R. You can quickly see that if R equal to 0, that immediately satisfies this equation. But, R equal to 0 is a trivial solution in which we are not interested. So, the question we ask is, are there any specific values of omega for which R is not equal to 0 is also a solution? R equal to 0 is a solution for any value of omega, but are there any values of omega for which R not equal to 0 is also a solution? Or, in other words, we are looking at the algebraic eigenvalue problem.

Here (Refer Slide Time: 25:33) we should notice that this matrices k and m are symmetric; K is equal to K transpose and M is M transpose; and, K is positive semi-definite and M is positive definite. K is derived from the potential energy of the system. So, the energy can be 0 because we permit rigid body motions. Therefore, we allow for K becoming semi-definite; whereas, kinetic energy is strictly positive and mass matrix originates from kinetic energy. Therefore, M is positive definite.

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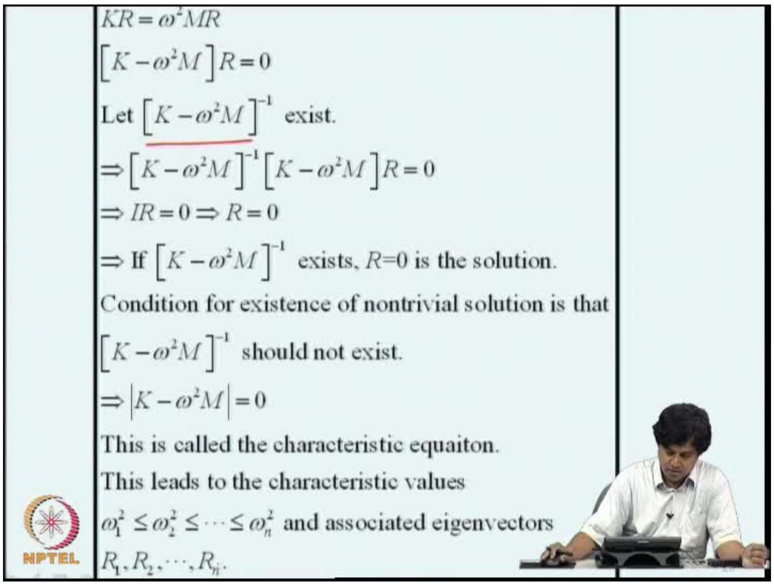

As a consequence of this, the eigensolutions, that means, omega comma R would be real valued; and, not only that the eigenvalues of omega square would be non-negative. So, how do we solve this? We can recast this as KR equal to omega square MR as K minus omega square M into R equal to 0. If we assume that inverse of this matrix K minus omega square M exist, then I can premultiply this by this K minus omega square M inverse and I get immediately a solution that R must be equal to 0; that means if K minus omega square M inverse exists, then R equal to 0 is the solution, which is the trivial solution in which we are not interested. So, the condition for non-trivial solution, that means, R not equal to 0 being a solution, is that the inverse of this matrix; K minus omega squared M must not exist. That would happen when that matrix is singular or the determinant of that is 0. This gives an equation for omega, which are unknown; omega square, which are unknown. And, this is called characteristic equation and the associated omega squares are known as characteristic values. As we have already seen, omega squares are real valued. Therefore, they can be arranged in an increasing order as shown here; and, there will be n eigenvalues, where n is a degree of freedom. And, associated with each of this n eigenvalues, there will be eigenvectors R 1, R 2, R n; each one is n cross 1 vector.

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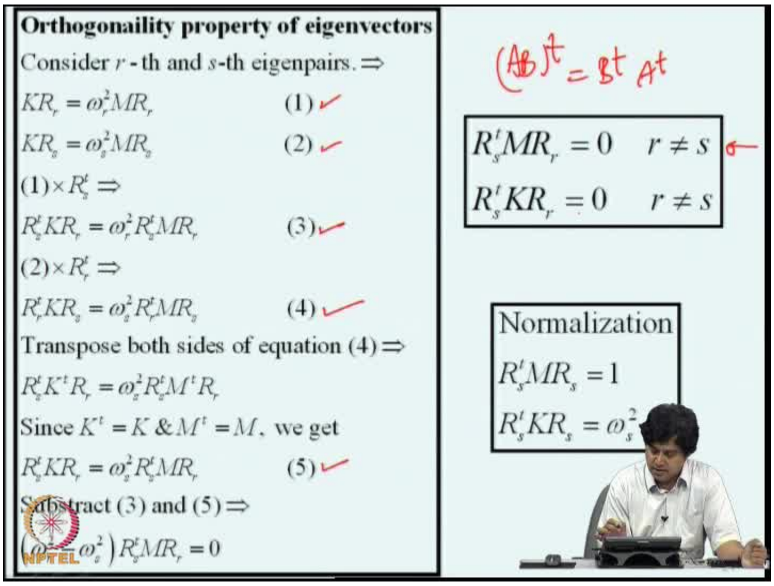

Now, again, we will ask this question – why are we doing all this? Originally, we are looking for a transformation matrix, which will diagonalize KM and C. So, you should bear that in mind. Now, these eigenvectors have an important property known as orthogonaility property. So, what it means can be illustrated by considering two eigenpairs say Rth eigenpair and Sth eigenpair. Eigenpair means eigenvalue and associated eigenvector. So, the Rth eigenpair: omega R and R r satisfy this – equation 1 and S th pair satisfies equation 2.

Now, we carry out some mathematical manipulations; we premultiply equation one by R s transpose; I get this equation (Refer Slide Time: 28:36). And, I premultiply 2 by R r transpose; I get this equation. Now, I transpose both sides of equation 4. So, if you use the result AB transpose is B transpose A transpose, the transpose of R transpose K R s will be R s transpose K transpose R r; R r transpose transpose is R r itself. And, on the right-hand side, similarly, I get this term. Now, since K is symmetric, K transpose will be K; and similarly, since M is symmetric, M transpose will be M. Consequently, I get R s transpose K R r is equal to this. This is (Refer Slide Time: 29:14) equation 5.

Now, you can see here; if you compare equation 3 and 5, the left-hand sides of these equations are identical. Therefore, if I subtract these two equations, (Refer Slide Time: 29:24) I get the equation omega r square minus omega s square into R s transpose M R r is 0. So, if r is not equal to s, that is, omega r is not equal to omega s, then I get the relation that R s transpose M R r is 0 for r not equal to s. Now, since this is true, you can use any of these equations; say equation 5 if you use, you will use R s transpose K R r equal to 0 for r not equal to s. This set of relations are known as orthogonaility relations.

Now, if you look at the eigenvalue problem, (Refer Slide Time: 30:05) K R equal to M omega square MR; if R is an eigenvector, a scalar multiple of R is also an eigenvector. Therefore, eigenvector is not unique; it is non-unique to the extent of a multiplying constant. Now, that multiplying constant I can select, so that (Refer Slide Time: 30:27) if I consider R s transpose M R s, this right-hand side would not be 0. But, if I make it as 1, we say that the eigenvectors have been normalized with respect to mass matrix. This is normalization, is removing that non-uniqueness associated with eigenvector by making a specific choice of that constant. So, if we do that, we get R s transpose M R s equal to 1. And, if I now go to equation 5, since R s transpose M R r is 1, it means R s transpose K R s is omega s square.

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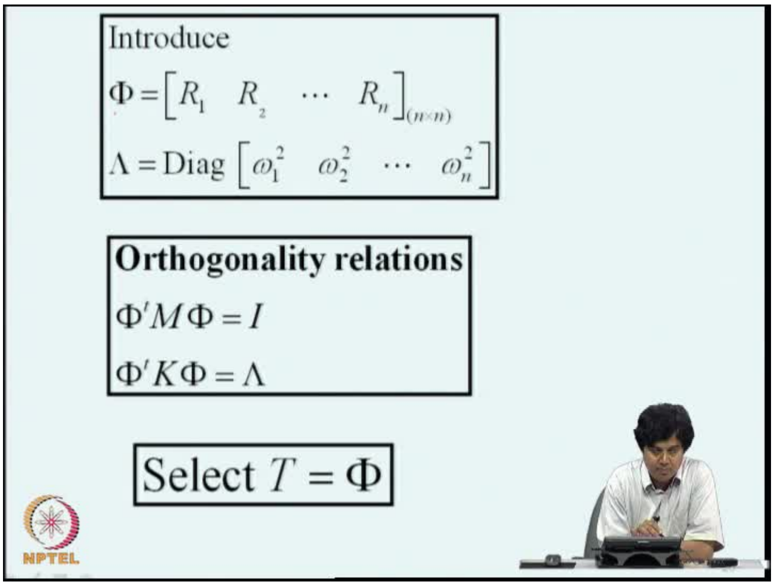

If I now assemble all the eigenvectors into a single matrix phi – R 1, R 2, R n, which is square matrix n cross n and all the eigenvalues – omega i squares into a diagonal matrix capital lambda, then the orthogonaility relation actually means phi transpose M phi equals I and phi transpose K phi is capital lambda. So, phi transpose M phi is a diagonal matrix; phi transpose K phi is also a diagonal matrix. So, the proposition is now that the transformation matrix T that we are looking for can be selected, such that this T is this modal matrix phi. This is the importance of doing this (( )). It helps us to find the transformation matrix, which uncouples equations of motion. So, we have been able to now uncouple M and K matrices.

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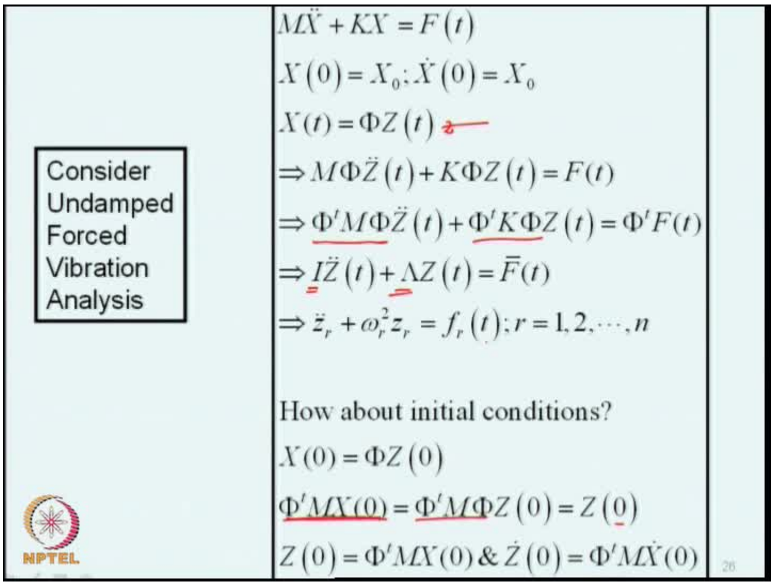

Suppose if we now consider undamped forced vibration problem, M X double dot plus K X equal to F of t with certain specified initial conditions, I will first find natural frequencies and the matrix of more shapes and make this transformation, X of t is equal to phi into Z of t. You substitute into the equation of motion; I get M phi Z double dot of t plus K phi Z of t equal to F of t. Now, premultiply by phi transpose; I get phi transpose, M phi into Z double dot of t plus phi transpose, K phi Z of t is equal to phi transpose F of t. Using orthogonaility relation, this phi transpose M phi is the identity matrix; phi transpose K phi is the diagonal matrix with diagonal entries beings square of the natural frequencies. So, I get now a set of uncoupled equations – z r double dot plus omega r square z r is equal to f r of t. So, n number of single degree freedom systems I have got as the equations of motions. This can easily be solved.

There is one small problem still that we need to address – how do we get initial conditions on z r? To see that, I can take X of 0 as phi of Z of 0 based on this equation (Refer Slide Time: 33:15). Clearly you can say Z of 0 is phi inverse X of 0. That is one we have to doing. But, we can use orthogonaility relations to actually find Z of 0. To do that, what we do is we premultiply this equation by phi transpose M. So, I get phi transpose M into X of 0 is phi transpose M phi into Z of 0. But, phi transpose M phi is identity matrix. So, I get simply Z of 0. So, Z of 0 is obtained as this (Refer Slide Time: 33:46). If we differentiate this, I will get Z dot of 0 as phi transpose M of X dot of 0. So, the initial conditions on z r are known and f r of t is obtained by multiplying this phi of transpose into F of t.

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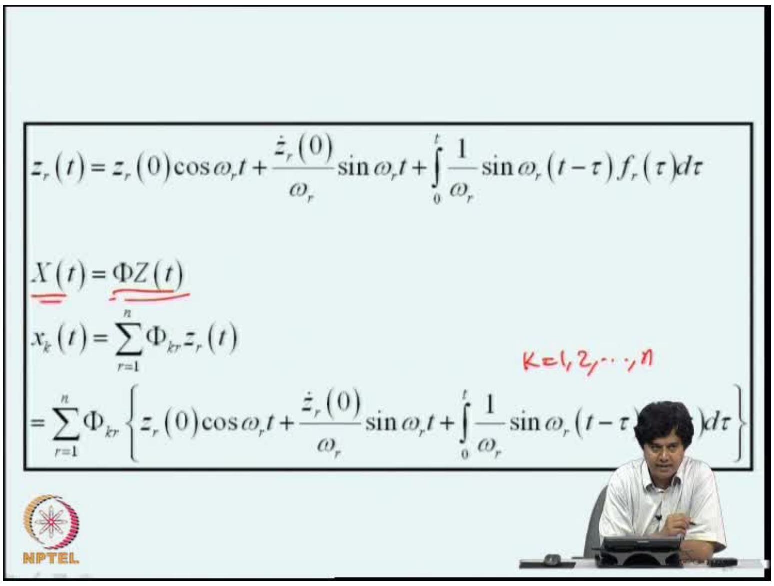

So, I have this set of single degree freedom equations. I can solve them using the theory of single degree freedom system, is an input-output relation in time domain. And, once I solve this, I can substitute back into the physically appealing coordinate system. Here I need to take decisions in x coordinate system. Z is a linear transformation on X; it may not have immediate physical meaning; no engineering; probably engineering design decisions cannot be made in z coordinate system. So, we have to return to x coordinate system. But, that transformation is ready. Therefore, if you are interested in x k of t, it is summation of phi k r z r of t; and, z r of t is already found out. So, I find out k and this can be done for k equal to 1, 2, n. And, I am ready with the solution. Once I find a displacement, I can find out the forces by multiplying by stiffness matrix; I can find out reactions; I can do any other calculation that we are typically interested in. So, the effort here is in finding the modal matrix and uncoupling the equation of motion, and then, integrating the set of single degree freedom equations.

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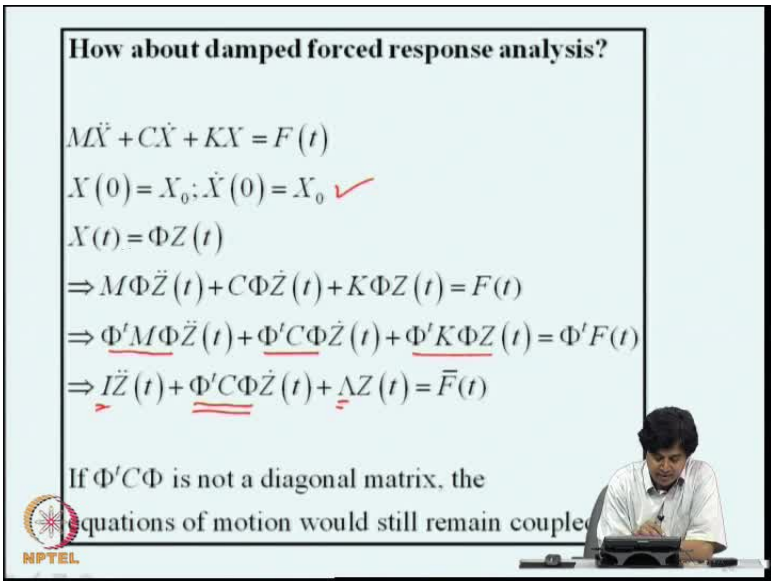

We consider till now undamped forced vibration; what happens if damping is also present? So, if I do that, again, let us start with the M X double dot plus C X dot plus K X is equal to F of t. So, again is specified initial condition. So, I make the transformation X of t is phi into Z of t. Substitute into the original equation and premultiply by phi transpose; I get this equation. By virtue of orthogonaility of phi matrix with respect to M, this is identity matrix; phi transpose K phi is another diagonal matrix. But, we are stuck with this matrix – phi transpose C phi; phi transpose C phi need not be a diagonal matrix. There is no theorem, which ensures that phi transpose C phi is diagonal for any choice of C. This is clear, because while finding this phi, we did not take into account C matrix. If however, phi transpose C phi – I mean, if it is not a diagonal matrix, even after transformation, although we are able to uncouple the inertial and stiffness terms, the equations in Z coordinate system continues to be coupled through damping terms. So, all our effort involved in implementing this transformation would not pay any dividend.

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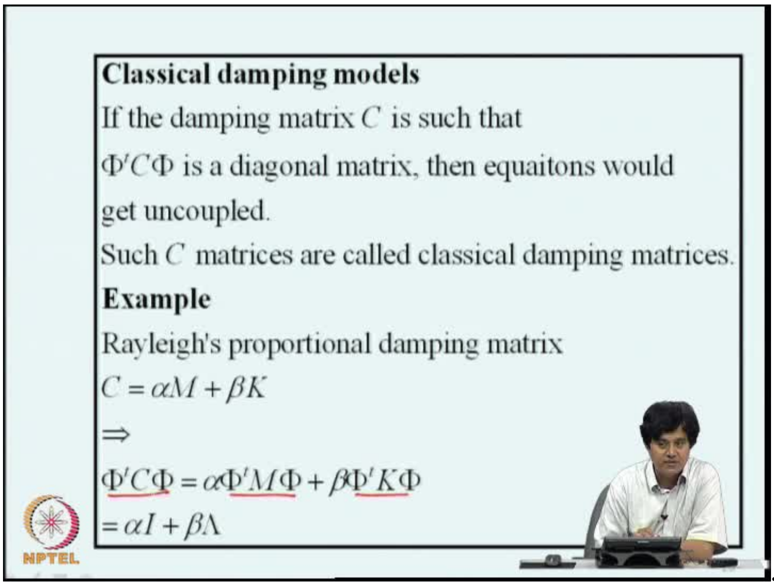

But, what we do is we assume that C matrix is such that phi transpose C phi is a diagonal matrix. Such C matrices for which phi transpose C phi is diagonal are known as classical damping matrices. Many of the engineering applications, it is assumed that C matrix is classical. One of the examples of a classical C matrix is the so-called Rayleigh's proportional damping matrix; where C is taken to be a linear combination of mass and stiffness matrices. So, if C is such that it is alpha M plus beta K, then if you were to look at phi transpose C phi, you will get alpha into phi transpose M phi plus beta into phi transpose K phi. And, this is diagonal and this is diagonal. Therefore, phi transpose C phi also would become diagonal.

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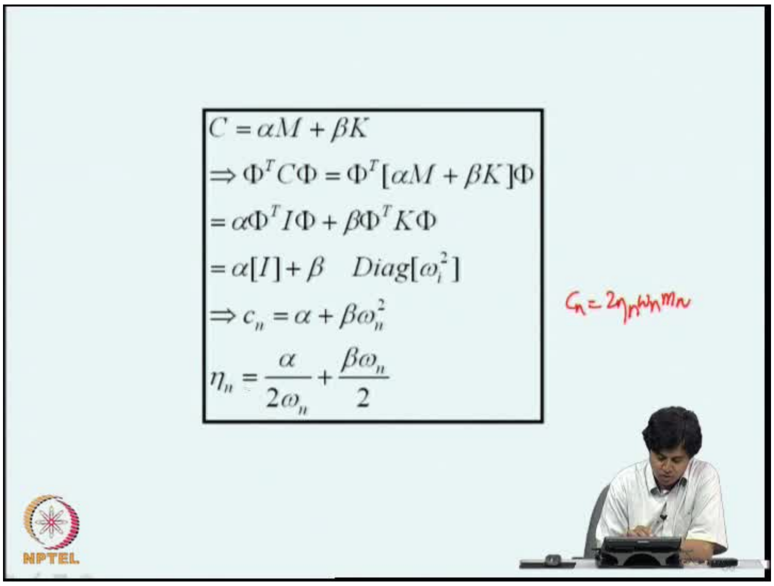

Now, we can explore this a bit further. If C is alpha I plus beta into diagonal of omega i square, then C n will be alpha plus beta omega n square. And, if I write c n as 2 eta n omega n into m n – m n is 1 in this case, I can solve for eta n and I get this equation alpha by 2 omega n plus beta omega n into 2.

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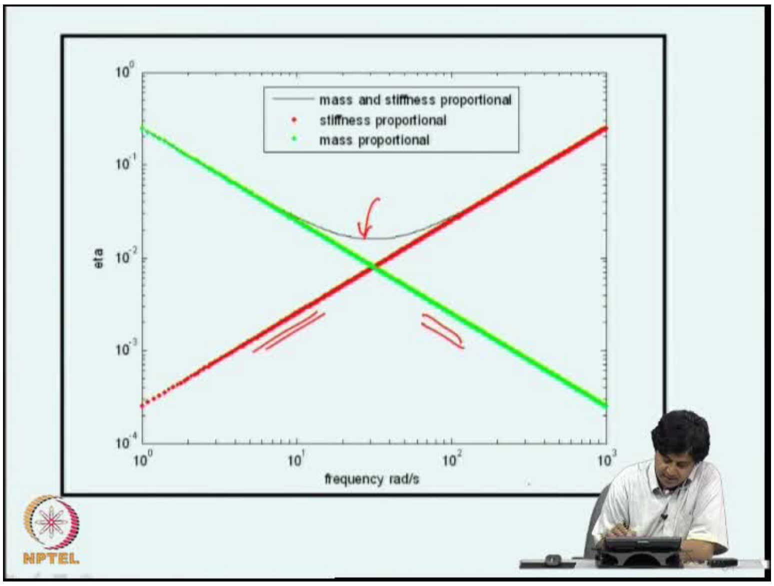

So, how does damping vary with respect to omega n? We can plot that on a log-log scale. If you plot, if C matrix is proportional to only stiffness term, I get this red line. If it is proportional to only mass matrix, I get this green line. If it is proportional to both mass and stiffness matrix, I get this black line. So, according to Rayleigh's damping model, the damping varies as a function of frequency in certain prescribed manners; that we have to take into account.

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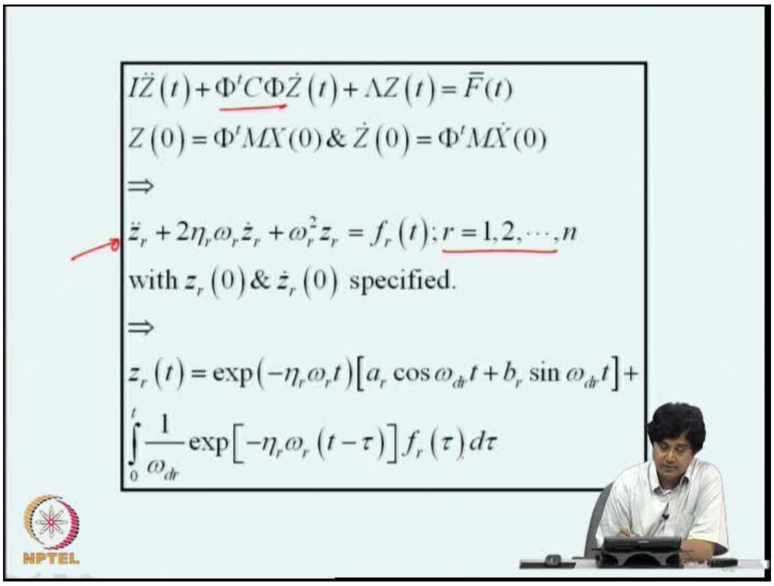

In any case, we have been able to now diagonalize C matrix also. If I restrict now C to only classical damping matrices, I get phi transpose C phi also diagonal. Therefore, this set of equations can be written as a set of second order ordinary differential equations, which are mutually coupled. So, this again can be solved using the theory that we already picked up. This is the complementary function. This is the particular integral in terms of the Duhamel's integral. So, this again completes the response analysis using normal modes even for a damped system.

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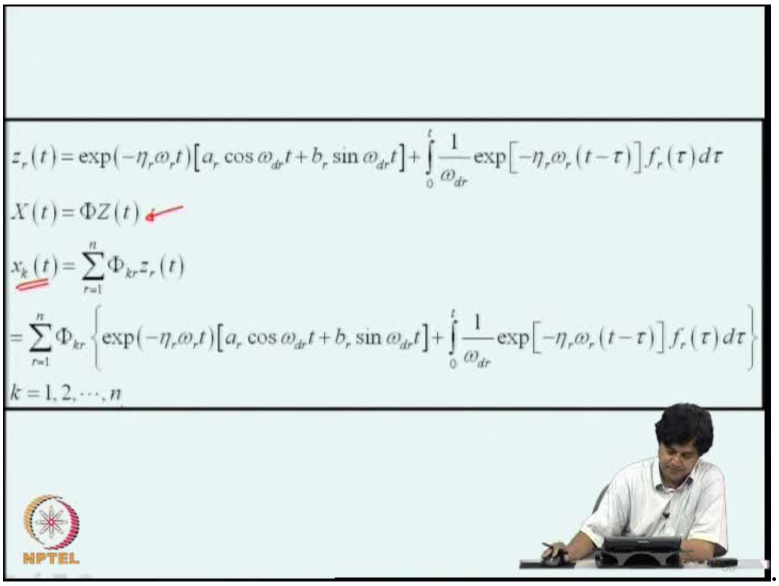

Once I find z r of t, I can get again x k of t, any k th coordinate by returning to this transformation. And, that is given here. So, I am able to now find x k of t for all values of k.

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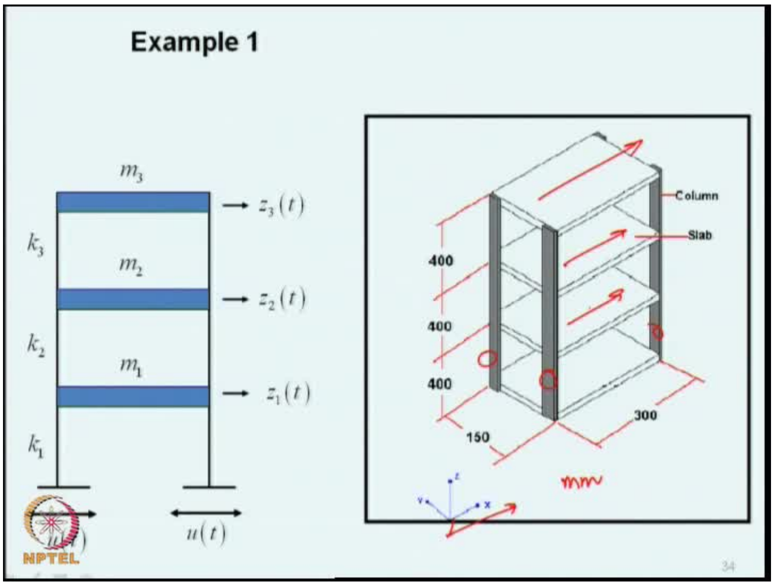

Now, let us consider a couple of numerical examples. These examples are fashioned after certain laboratory models and they are not real engineering structures. This is the model in which there are three slabs. The dimensions are shown here in mm. And, columns are made up of rectangular flats and slabs are rectangular plates. If I assume planar motion along x-axis, I can make a simple model consisting of three masses, point masses. And, subjected to horizontal displacement support motions, this can be applied on a shake table typically in a laboratory. And, the degrees of freedom that I select are z 1, z 2, z 3, which are the horizontal displacements of these three slabs. k 1, k 2, k 3 are the stiffnesses at individual floor levels. For example, k 1 is made up of contribution to stiffness from this column, this column, this column and the column that is hidden there.

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Part	Dimensions in mm		
	Depth (D)	Width (B)	Length (L)
Column	D_A =3.00	B_A =25.11	L_A = **400.00**
Slab	D_B =12.70	B_B =150.00	L_B =300.00

Sl. No.	Part	Material	Mass kg	Material Properties	
				Young's Modulus (E) N/m^2	Mass density (ρ) kg/m^3
1	Column	Aluminum	M_c **=0.0814**	69.0E+009	2700
2	Slab	Aluminum	M_s= 1.5430	69.0E+009	2700
3	Allen screw, M8	Steel	M_{sc}=0.0035	-	-

So, if we do that… I have given some numerical values; you can run through these calculations and verify if you indeed get the solution that I am going to show just now.

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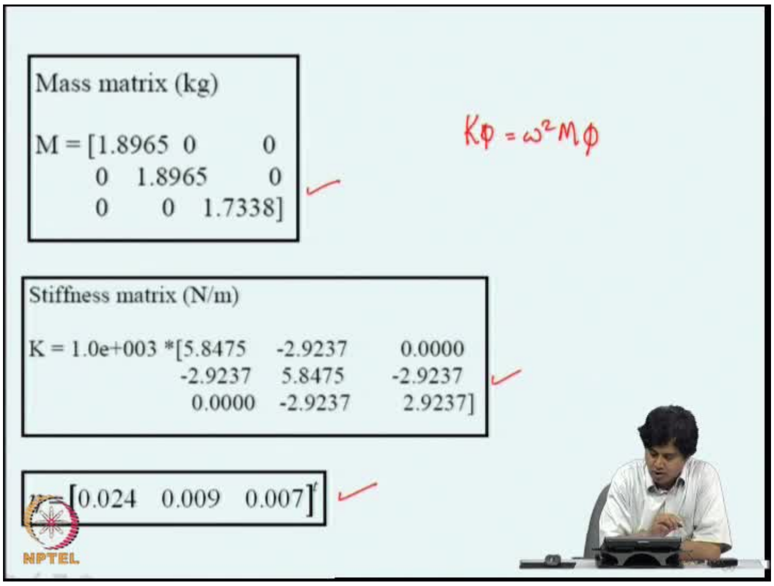

So, the details of column cross section and the material, that is, the column and plates are made up of etcetera are provided here. If we do that, I get the mass matrix, which is a diagonal matrix as shown here and the stiffness matrix, which is a non-diagonal matrix, which is shown here. And, we have experimentally measured the damping in that frame in the laboratory and these are the values of the damping ratios for the three modes as shown here.

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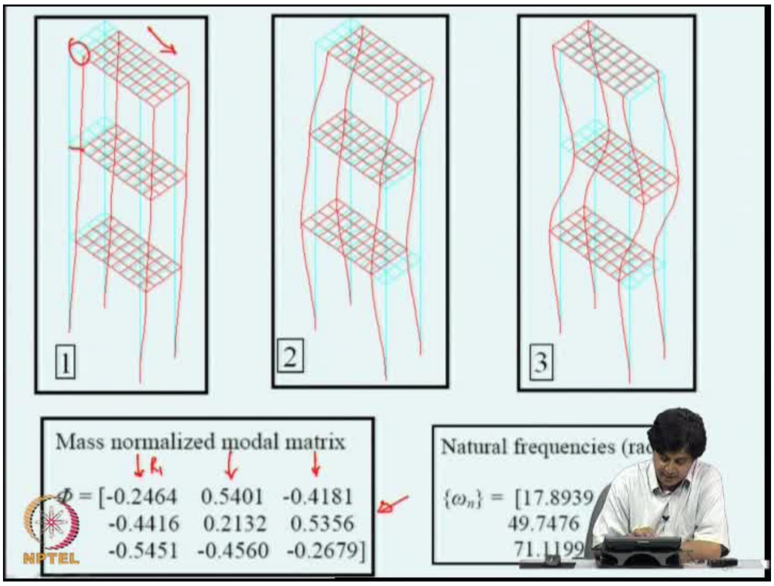

If we now perform the eigenvalue analysis and obtain the eigenvectors and the natural frequencies, we can display the eigenvectors pictorially as shown here. This is so-called the first mode shape. Here for example, if this is the first eigenvector, this quantity – 0.2464 represents the displacement of the first slab, this is the second slab, third slab. All of them have same sign; that means all of them are in phase. So, you see here this cyan color is the undeformed geometry and red is the deformed geometry. So, all the three slabs have moved in phase in this direction. This is the first mode shape. So, these numbers – 0.2464 appear here; 0.4416 appears here; and, 0.5451 appear here.

The second mode is in this column. So, here you can see that the first two masses are in phase; (Refer Slide Time: 43:14) whereas, the third one is out of phase. That means when the frame is vibrating in its second mode, the first two masses are in phase, the other one is out of phase. Therefore, the phase difference between the third and first two masses is pi; whereas, the phase difference between these two will be 0. For the third mode, this is the eigenvector. So, we see that m 1 and m 2 are out of phase, m 2 and m 3 are out of phase. So, we get this kind of displacement pattern and this we called as mode shape. So, associated with each mode shape, there are natural frequencies the first natural frequency is about 17.9 radians per second; second one is about 49.7 radians per second; and, this is 71.1 radians per second. So, these are the normal modes. So, this is the matrix, which will uncouple the equation of motion. This is the transformation matrix that we have to use. And, the elements of this transformation matrix are displayed graphically in this and they are known as mode shapes.

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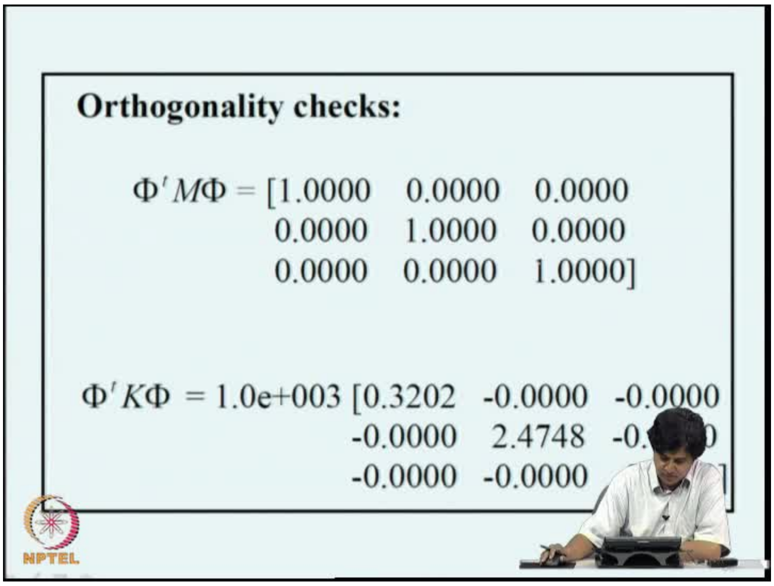

We can check if the orthogonaility properties are satisfied are not. So, if we compute phi transpose M phi, we get the identity matrix as we are expecting. And, if we check phi transpose K phi, I get this matrix. And, if we compare, this is nothing but omega 1 square, this is omega 2 square and this is omega 3 square. That can be verified.

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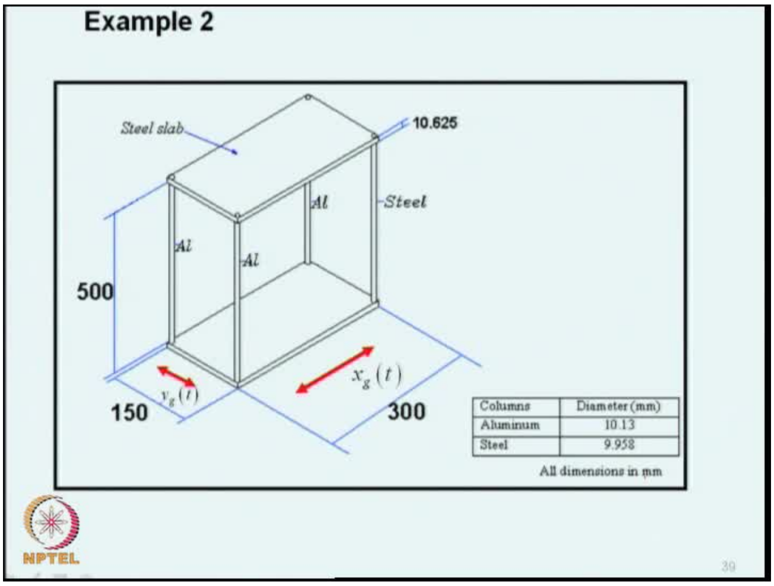

Similarly we can consider the second example, where a steel slab is supported on three aluminums and the one steel column. As I mentioned already, this is asymmetric in plan.

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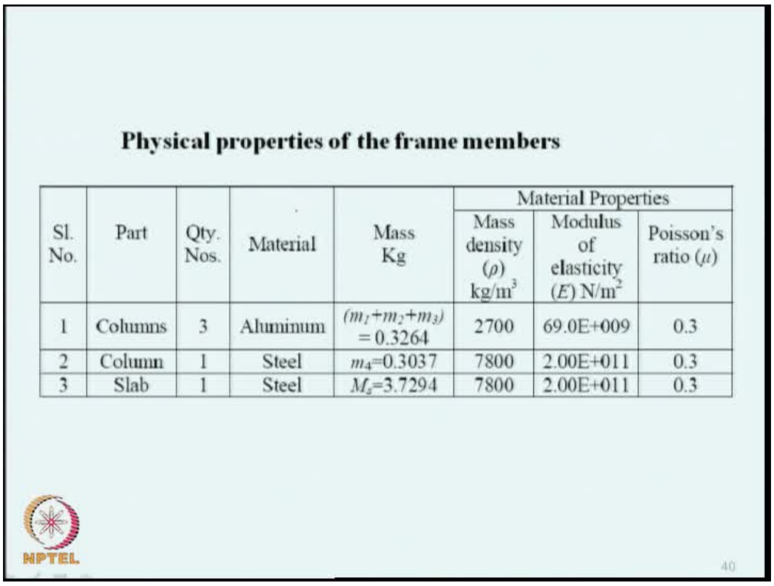
Physical properties of the frame members

Sl. No.	Part	Qty. Nos.	Material	Mass Kg	Material Properties: Mass density (ρ) kg/m^3	Modulus of elasticity (E) N/m^2	Poisson's ratio (μ)
1	Columns	3	Aluminum	$(m_1+m_2+m_3)$ = 0.3264	2700	69.0E+009	0.3
2	Column	1	Steel	m_4=0.3037	7800	2.00E+011	0.3
3	Slab	1	Steel	M_s=3.7294	7800	2.00E+011	0.3

Again, the properties of different columns and slabs are tabulated here. The masses and the mass density and the Young's modulus, Poisson's ratio – all these details are provided here. This would enable you to make the models for stiffness and mass matrices.

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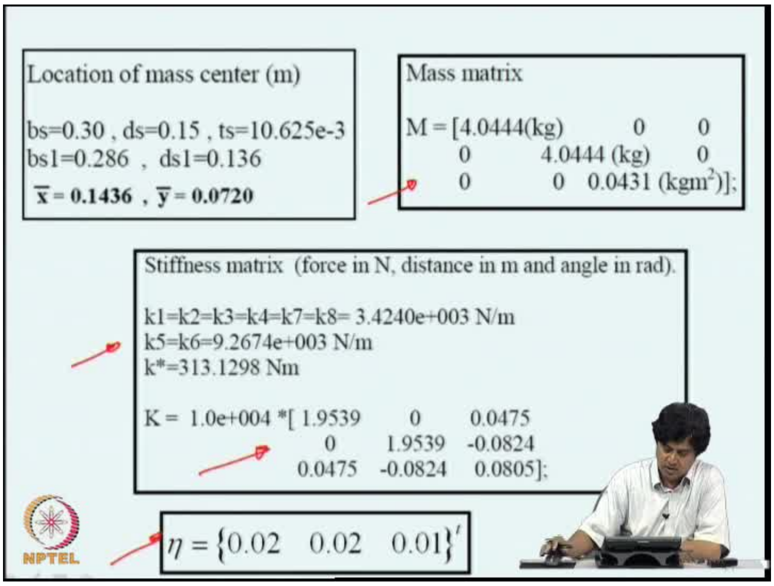

And, the geometry of the structure is specified. Based on which you can first locate the center of gravity, that is, the mass center; and then, compute the mass matrix; this is the mass matrix we get. And, for computing stiffness matrix, we need certain details of computation of those k 1, k 2, k 3 and k 8. If we do that, this is what we get as stiffness matrix. And, in the experiment, we have determined the damping values to be 0.02, 0.02, 0.01 for the 3 modes.

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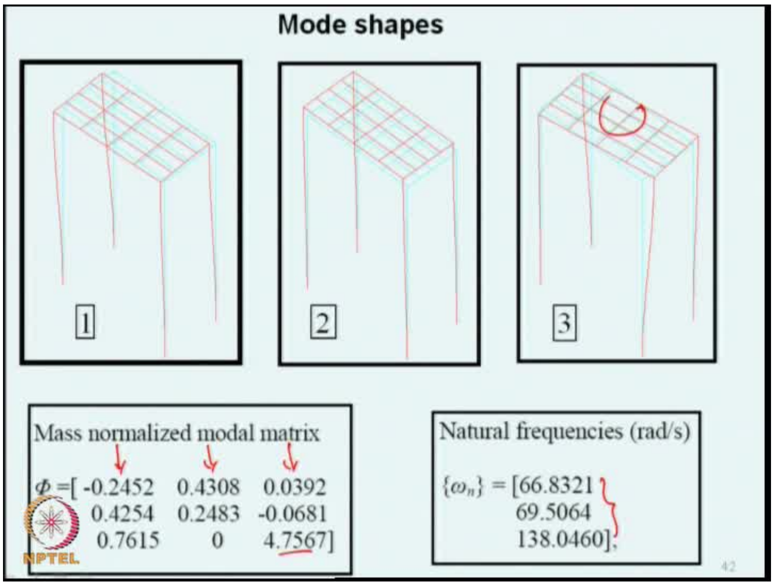

So, the three mode shapes, which are obtained by carrying out eigenvalue analysis on these (Refer Slide Time: 46:08) M and K matrices are shown here. This is the first eigenvector; this is the second eigenvector; and, this is the third eigenvector. And, these three are the natural frequencies. You can see here that the first and the second natural frequencies are closely spaced and third one is quite removed. From the first mode shape, if you carefully see this, again here the red line is a deformed geometry and the cyan line is an undeformed geometry. When the structure is vibrating in its first mode, all the three degrees of freedom, namely, x, y and theta are all coupled. There is nothing like a pure translation mode here or a pure torsion mode. The translational and torsion modes are coupled.

Similarly, in the second mode, we see that the two translations are coupled, but they are uncoupled from rotation for this particular example. In the third mode, you can see that it is predominantly a twisting mode. If you see here, (Refer Slide Time: 47:15) the amplitude of translation in x direction is 0.03; this is 0.06, but there is substantial rotation. So, it is a mode shape that is dominated by torsion or the rotation. And, that induces torsion in these columns leading to shear stresses and so on and so forth. And, these kinds of structures are quite important from a practical point of view. When we consider earthquake response of structures, one of the major sources of complexity in earthquake response analysis of real structures is the torsional behavior of building frames due to asymmetry in plan. So, this example illustrates that. And, we can see here the bending and torsional modes in this particular example are mutually coupled.

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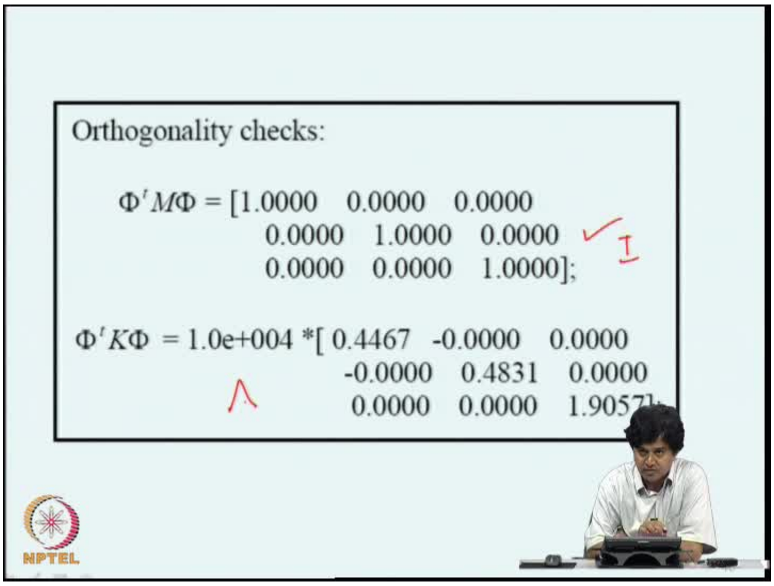

Here again, we can make the checks on orthogonality by computing phi transpose M phi and phi transpose K phi. This transferred to be the I matrix and this transferred to the lambda matrix. This can be verified.

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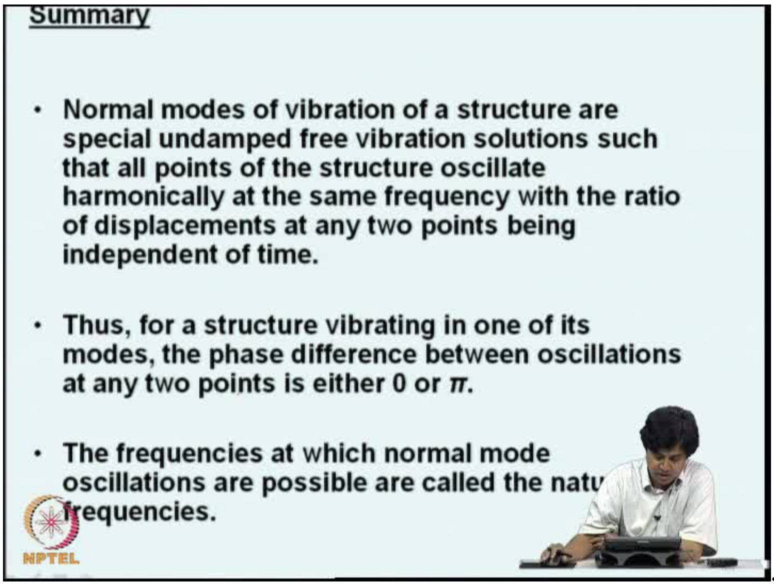

So, we can make a few comments in summary. Normal modes of vibration of a structure are special undamped free vibration solutions, such that all points of the structure oscillate harmonically at the same frequency with the ratio of displacements at any two points being independent of time. That means they are synchronous harmonic motions, undamped free vibrations. Thus, for a structure vibrating in one of its modes, the phase difference between oscillations at any two points is either 0 or pi. The frequencies at which normal mode oscillations are possible are called the natural frequencies. These appear as square root of the eigenvalues when you solve the eigenvalue problem, K r equal to omega squared M r.

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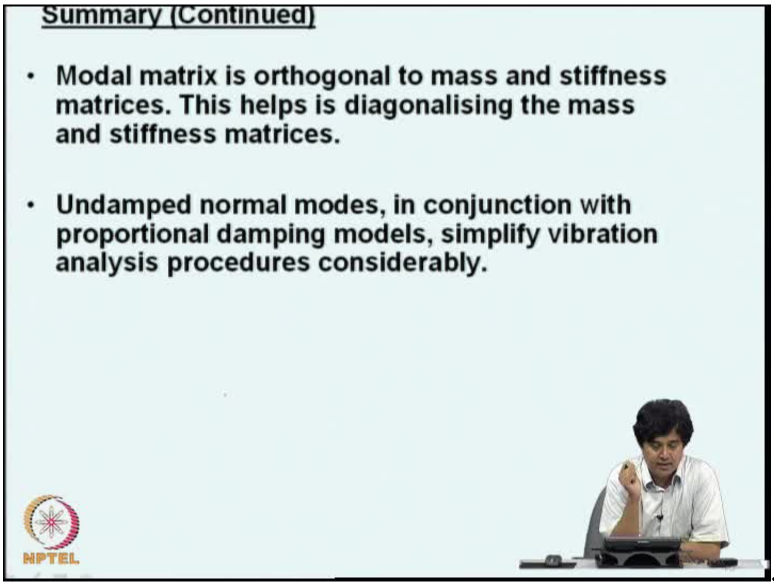

This modal matrix, that is, the matrix of eigenvectors is orthogonal with respect to mass and stiffness matrices. This helps in diagonalizing the mass and stiffness matrices through a transformation, where the transformation matrix is the matrix of eigenvectors. Undamped normal modes in conjunction with proportional damping models, simplify vibration analysis procedures considerably, because the matrix of undamped normal modes would also diagonalize C matrix if C matrix is classical.

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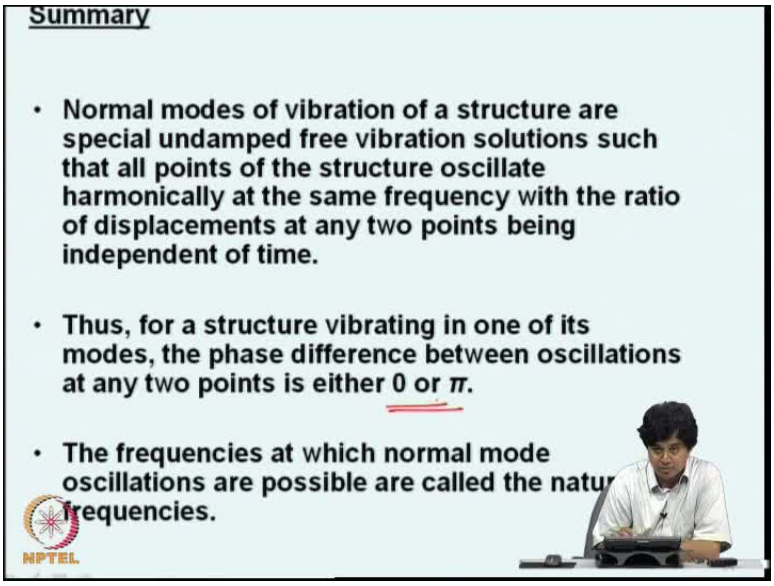

In reality, if you were to actually excite a structure in one of its normal modes, you would see that damping would invariably be present whenever you do such exercise. There is no magic switch to switch off damping in reality. Therefore, the normal modes that one gets in practice in a laboratory are always damped normal modes. So, the phase difference that we are getting here as 0 or pi would not be true; there will be the departure of this phase from 0 or pi, indicates how serious is the effect of damping on normal modes. If it is of the order of plus minus 5 degrees, you can ignore the effect of damping on normal modes; otherwise, you need to consider the effect of damping in evaluating normal mode.

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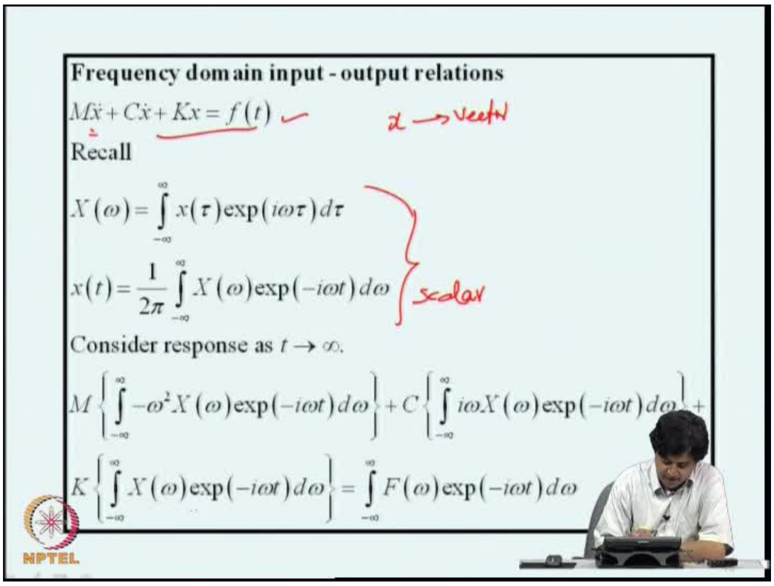

We will now move on to another topic, how to characterize input output relation for multi-degree freedom systems in frequency domain. So, we will start with the equation on motion in time domain, M x double dot plus C x dot plus K x equal to f of t. If you recall, the definition of fourier transform pair for a function x of t is through this pair of integrals. Now, this X is of course a vector. Although in this equation, I use the same notation X, this is scalar here; x here is a vector.

If I now take fourier transform on both sides of this equation, I get M into – for x double dot, I will write its fourier transform – this is minus infinity to infinity omega square X of omega exponential minus i omega t d omega; minus omega square arises because when you differentiate x of t twice, you get i square omega square and that becomes this

(Refer Slide Time: 51:55). The second term is C x dot; that is, i omega into X of omega; you have to differentiate this one; you get this. Third one is this – K into X of omega exponential minus i omega t d omega. This is equal to this. This is the fourier transform of excitation process.

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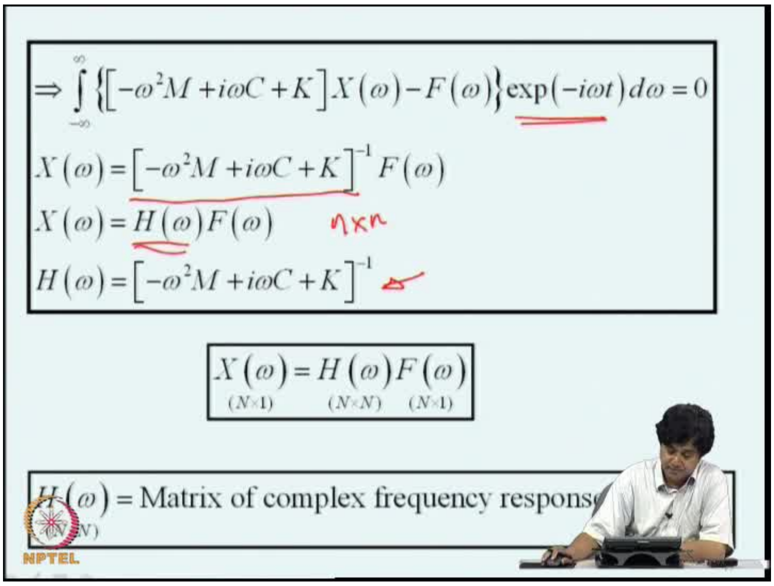

Now, if you are organize these terms, I can bring in all the terms to one side and take the integral side outside; I get this equation; the term inside the braces is multiplied by exponentially minus i omega t d omega is equal to 0. This can be true if we select now X of omega; that means, the term inside the braces taken to be 0, I get X of omega as minus omega squared M plus i omega C plus K in inverse F of omega. This quantity – I denote it by H of omega, which is now a n by n matrix; H of omega is n by n matrix given by this expression; that is, X of omega, which is the fourier transform of X of t is a n cross 1 vector, is related to now n cross 1 vector of excitation signals through n by n matrix H of omega. And, this is known as matrix of complex frequency response functions. This is complex valued; is not symmetric; is not hermitian. So, we need to take care of some of these complexities when we compute this.

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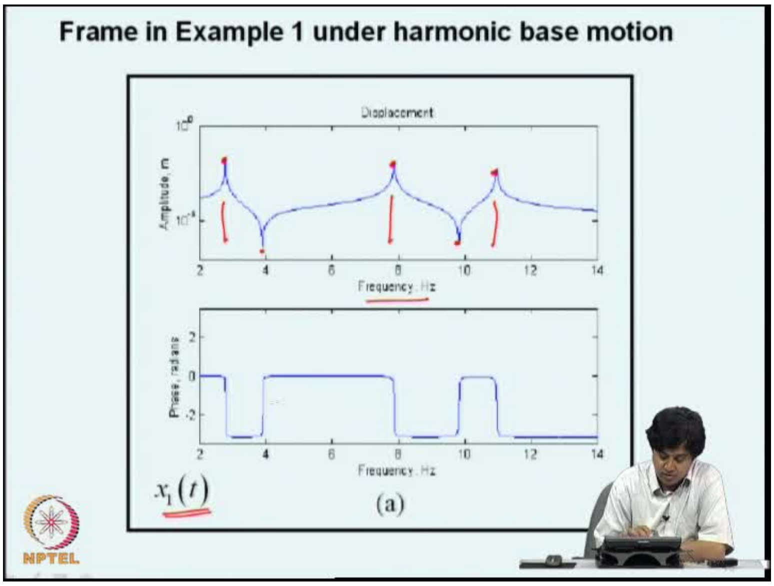

Now, a further example of a three-storey frame that I showed you earlier. If I were to plot X of omega – x 1, x 2, x 3 using this frequency domain representation, on the x-axis, I have plotted frequency in hertz. And, the amplitude of the displacement at degree of freedom, x 1 is shown here. And, this curve peak set the three natural frequencies of the system and it is characterized by minima at these two places and these peaks at these two places. Mind you, this is plotted on y-axis is on log scale and this is the plot of phase angle. So, phase undergoes rapid changes near the resonance, so that I have already seen is single degree freedom system. So, this can be viewed as the manifestation of resonance in multi-degree freedom systems; where the propensity for occurrence of resonance increases now, because there are more natural frequencies. In a three degree freedom system, there are three natural frequencies. Therefore, there are three situations under which resonance can occur.

Similarly, you can plot for other degrees of freedom; this is for x 2 and this is for x 3. Again, each one of them peak at the three natural frequencies; there are certain interesting observations that between these two peaks there is a minima. And, the nature of this minima and this minima (Refer Slide Time: 55:10) are different. This is known as anti-resonance; this is known as a minimum. So, there is a theory to explain why such things happen, but probably, this is not of immediate concern in this course. With this, we will conclude today's lecture.