Stochastic Structural Dynamics

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Lecture No. # 12

Random Vibrations of Sdof Systems-4

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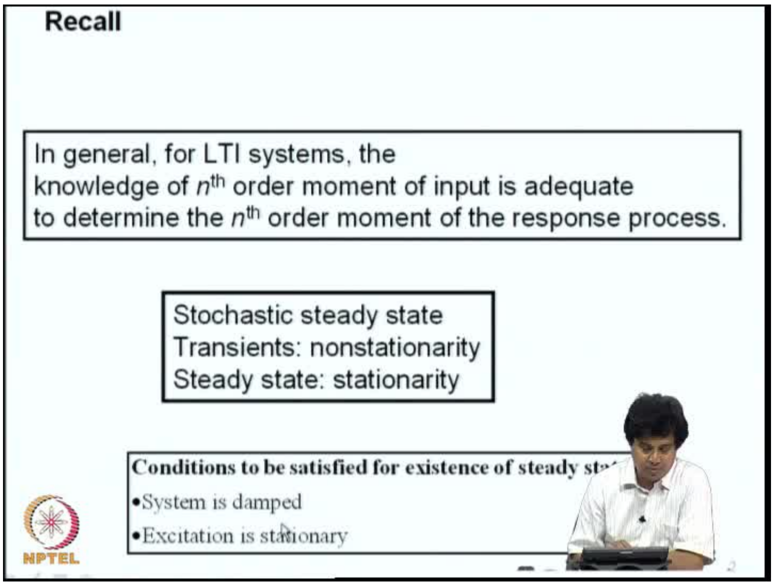

In the previous lecture, we have been considering input-output relation for randomly driven linear time in variance systems, and we have shown, that for such systems the knowledge of nth order moment of the input is adequate to determine the nth order moment of the response process; that means, if you know the auto covariance of the input, you can find the auto covariance of the output; if you know mean of the input, you can find the mean of the output.

We also discuss the notion of stochastic study state; so, if a linear time variance system is driven by stationary random excitation, the response can be, the response can exit neither in a transients state or in a study state, in the phase, in which the response is in the transients state, the response process will be a non-stationary random process and in the study state, the response becomes a stationary random process and in transients state, the system response will be affected by the initial conditions, whereas in study state the initial effect of the initial conditions vanish. For the system enter into a study state, that means, conditions to be satisfied for existence of study state or that the system should be damped and excitation is stationary. If the system is un-damped, no study state is possible, and similarly, if we apply an excitation which itself is not stationary, there is no possibility of a stationary response.

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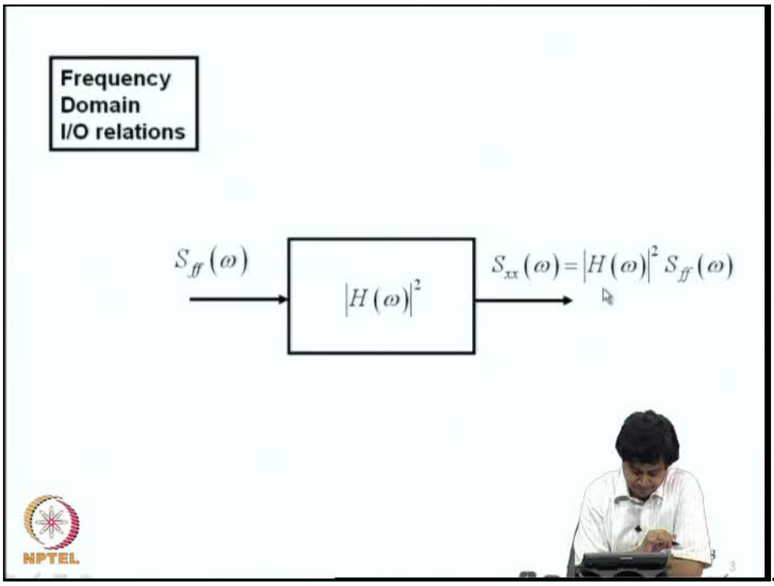

The input-output relation in frequency domain related the prospective density function of response to prospective density function of the input, through a transfer function, which is square of the frequency response function; this is valid only in the study state.

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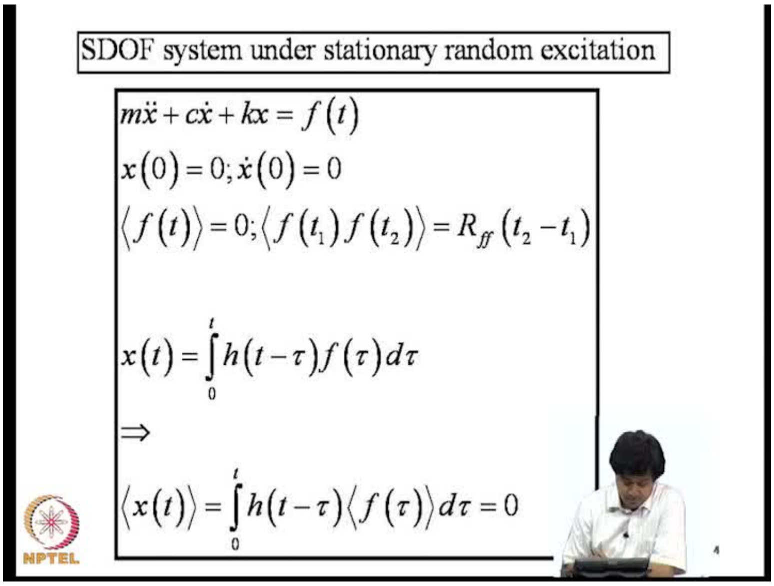

We will continue this discussion. Now, we will consider the response of a single degree freedom system to a stationary random excitation; still now, we have considered f of t to be a white noise process. Now, we will consider more general forms of stationary random inputs, we will consider the system start from rest and mean of the excitation is 0 and the auto covariance of the excitation process is given by this, where the auto covariance is function of the time difference t 2 and t 1. We will start with the Duhamel's integral representation for the response; since the system is starting from rest the Duhamel's integral representation provides the complete solution h of t is impulses response function f of t is the excitation. Now, if you take the expectation of the response, so you take expectation on both sides, you get expected value of x of t is expected value of this integral and that itself is this integral 0 to t h of t minus tau expected value of f of tau t tau and since expected value of f of t is 0, it follows that expected value of x of t is 0, in this case.

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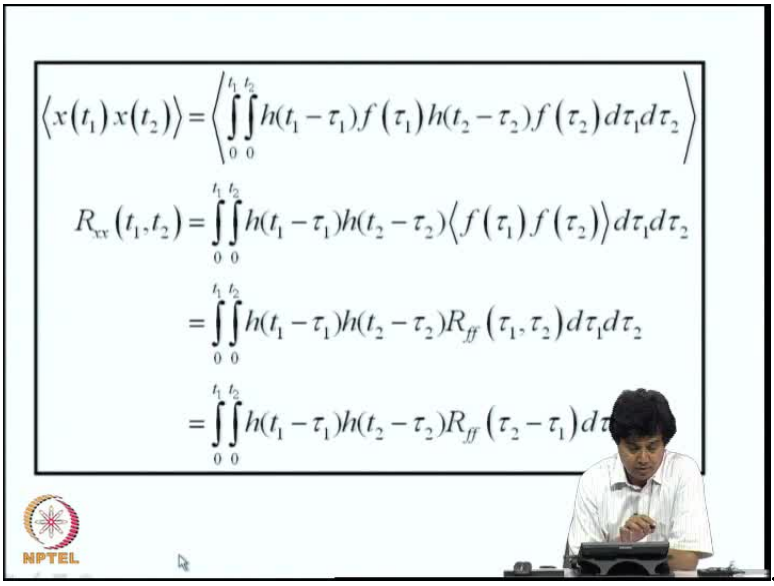

How about what a covariance of the response processes; so, you consider the expectation of x of t 1 into x of t 2, so this is given by expected value of h of t 1 minus tau 1 into f of tau 1 h of t 2 minus tau 2 into f of tau 2 d tau 1 d tau 2. So, interchanging the expectation operator with these integrals, we get, we can take the expectation operators in inside and we get the auto covariance of the response to be given by this integral h of t 1 minus tau 1 h of t 2 minus tau 2 R ff (tau 1, tau 2) which is a auto covariance of the excitation process d tau 1 d tau 2. Since we have assume f of t 2 is to be stationary, R ff (tau 1, tau 2) will be the function of R ff of tau 2 minus tau 1, so I replace this function by R ff of tau 2 minus tau 1.

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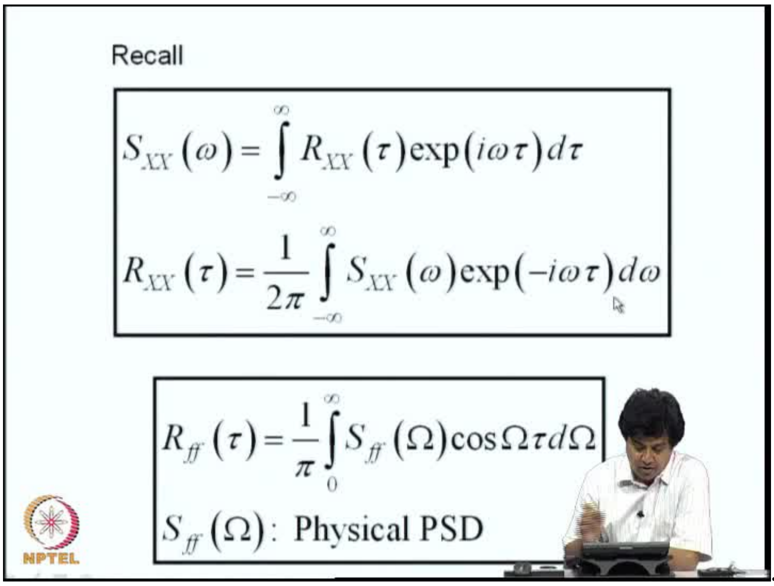

Now, if you recall the prospective density function of a random process and auto covariance function of a random process are related through these equations. So, in terms of a physical power spectral density function, the auto covariance functions of a random process f of t can be expressed in this form; physical power spectral density means the prospective density function is define for only positive values of the frequency parameter; the frequency parameter here in this particular derivations I am using capital omega, because small omega we often used to represent natural of frequency of the system, so to make that clear and I am using upper case omega.

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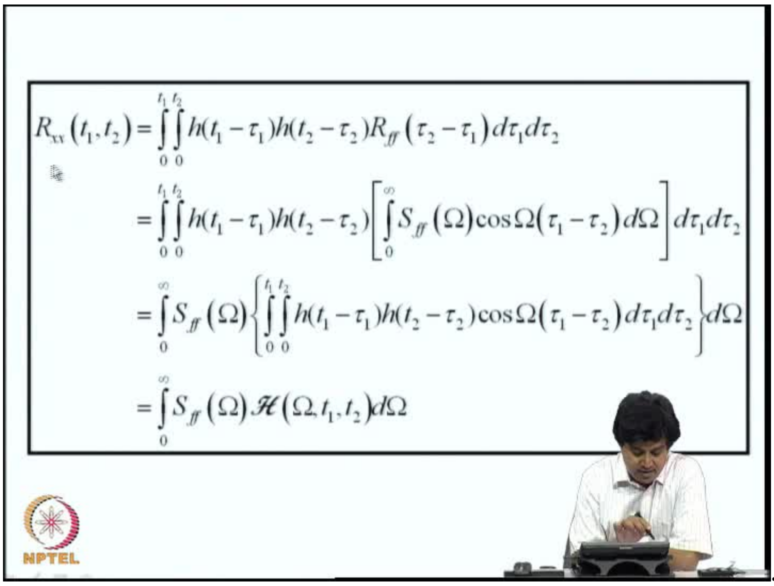

So, we had this auto covariance of the response to be given, in terms of auto covariance of the input through this relation. Now, for this auto covariance of the input, now I use the frequency domain representation and write this R ff of tau 2 minus tau 1, in terms of power spectral density function of the response process and that is written here. And next, we interchange the order of integration with respect to omega and time and I will pull out this S ff of omega here and inside the braces, I will have integration with respect to the time; time appears not only in this h tau 1 tau 2 but also in this cosine of tau 1 minus tau 2 also has to tau 1 and tau 2, so that gets into the braces here. So, we call the term inside this brace as script H (omega, t 1, t 2); this H can be viewed as a time dependent system transfer function.

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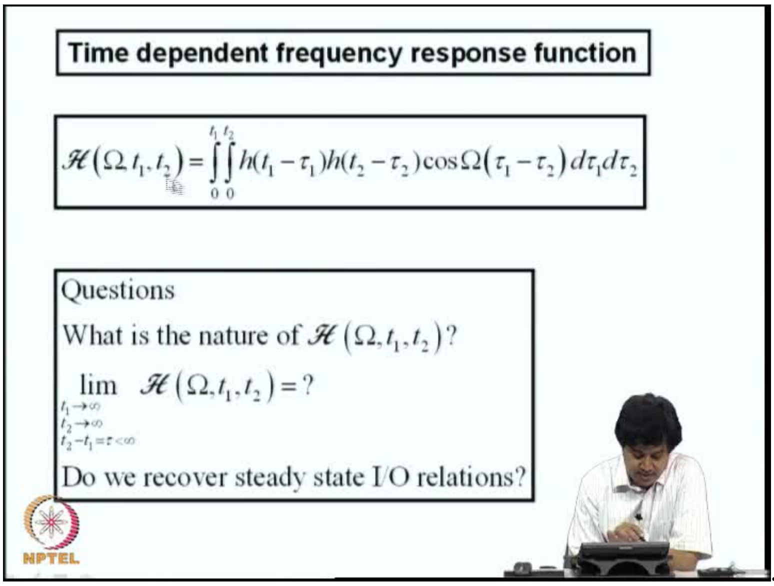

So, this is given by 0 to t 1 0 to t 2 h of t 1 minus tau 1 and this, this particular integral. The question would be interested in answering is, what is nature of this frequency response function as time becomes large, that means, t 1 tends to infinity and t 2 tends to infinity and t 2 minus tau 1 becomes tau which is finite, what happens to this transfer function, under this limiting operation the auto covariance of the input is stationary, I mean it is always stationary, so the question would be, therefore to consider what would happened to this transfer function as time becomes large and time difference is finite. We have already seen through a direct analysis of response in frequency domain, that the power spectral density function of the response is related to the power spectral density function of the input through the frequency response function.

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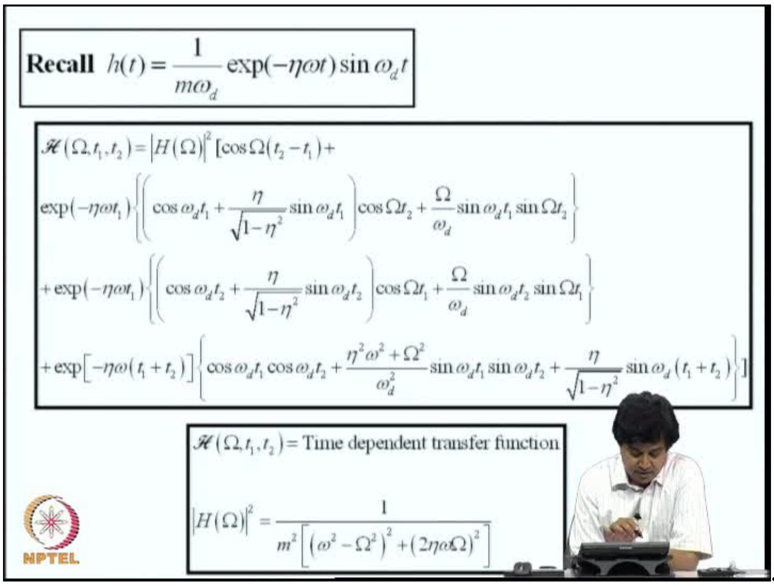

Now, the question would be if you approach this problem through time domain and pass through transients, do you reach the same study state which we should, but it is nice to be able to verify that that indeed happens. Now, the impulse response function is given by this for a damped single degree freedom system; now, if I now substitute this into this integral for a h, this is known function; so, for h of t, I will write this function and carry out this integration, if i indeed do that I get this line the expression; this H of omega whole square that appears here is the modulus square of the frequency response function that we are encountered already. This capital omega is a frequency parameter which appears at various places here.

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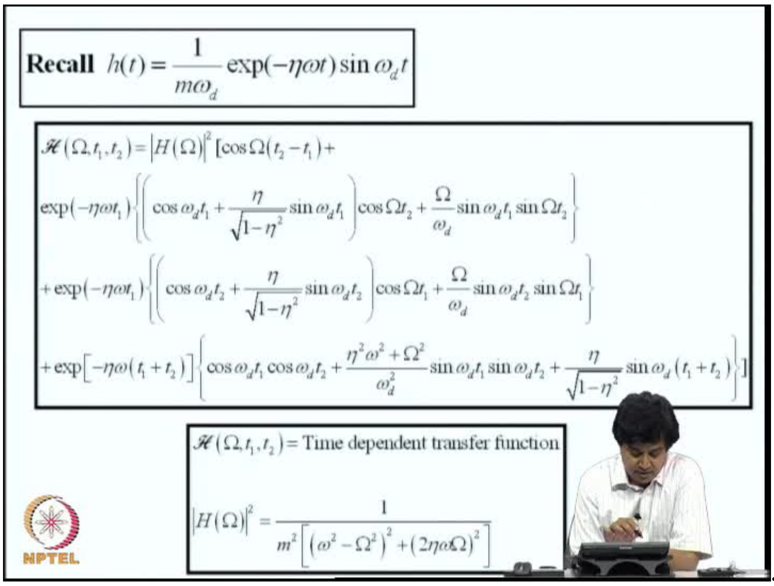

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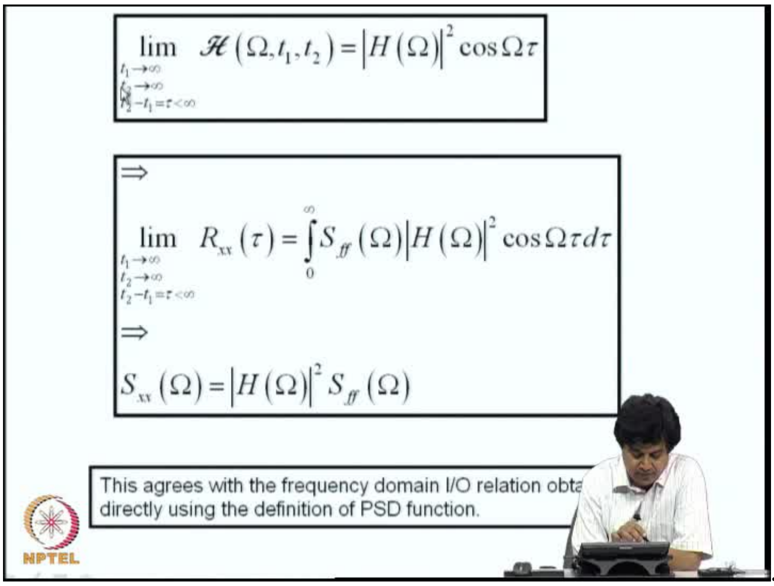

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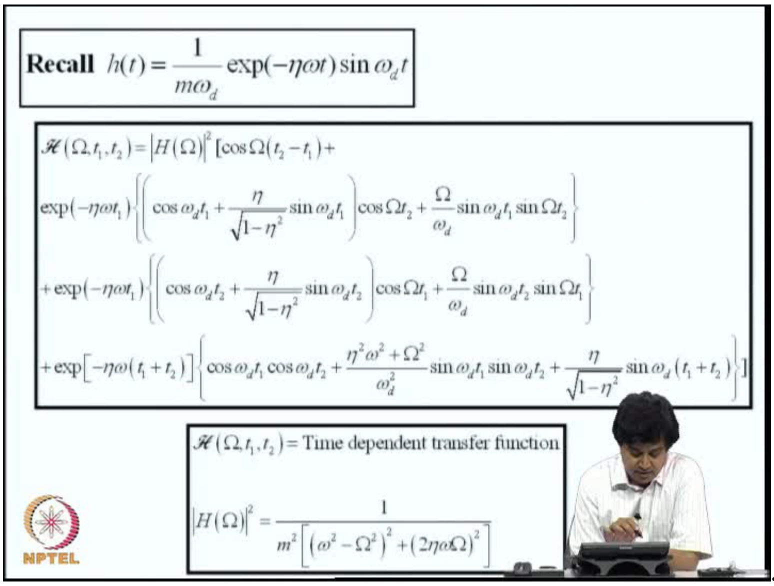

Now, clearly this H is the function of both t 1 and t 2, all the, at few places the time different also appears, but if you look at these multiplies, it is function of t 1 as well as t 2. Now, if I know consider the limit of t 1 going to infinity t 2 going to infinity and t 2 minus t 1 is tau which is finite, what would happen; clearly, this exponential function, since they are multiplied by time and time goes to infinity this go to 0, so this terms inside the brace will vanish, because I am multiplying by a function exponentially decaying function and therefore these two terms will vanish. Similarly, there is another term here inside this brace which is multiplied by minus eta omega t 1 plus t 2; as t 1 and t 2 becomes large, this function also becomes 0; therefore, the only term that will be left with will be this H of omega whole square into this cosine function; please notice that, there is a bracket running across these terms begins here and ends here and the entire time with in the bracket is multiplied by H of omega whole square, so what would remain as t 1 tends to infinity and t 2 tends to infinity t 2 minus t 1 being tau is only this terms h of omega whole square cos of omega tau, so I get this one.

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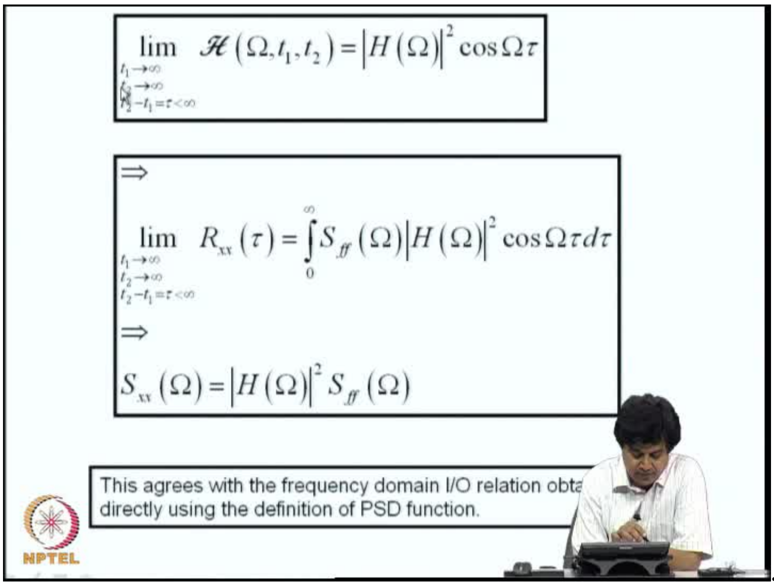

So, therefore under these limiting operation, the output auto covariance function is given by S ff of omega H of omega whole square cos omega tau. Now, seeing that, this is nothing but the Fourier transform of R xx of tau, we can infer that the output power spectral density function is given by H of omega whole square S ff of omega and this is, what we have already derived by using a frequency domain analysis of response in the steady state, so this agrees, this nice to able to verified that.

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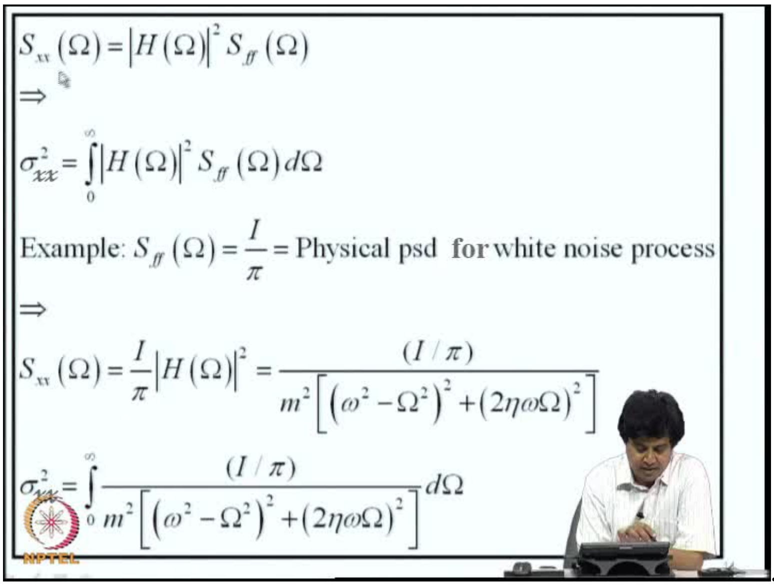

So, we have the output power spectral density function is input PSD multiplied by square of H of omega. So, the study state variance is given by area, under this function and this is given by this. Suppose, if we take the case of white noise, then the output PSD function, if I now substitute into this, I would end up needing to evaluate this particular integral 0 to infinity and this is a rational function, a polynomial in omega.

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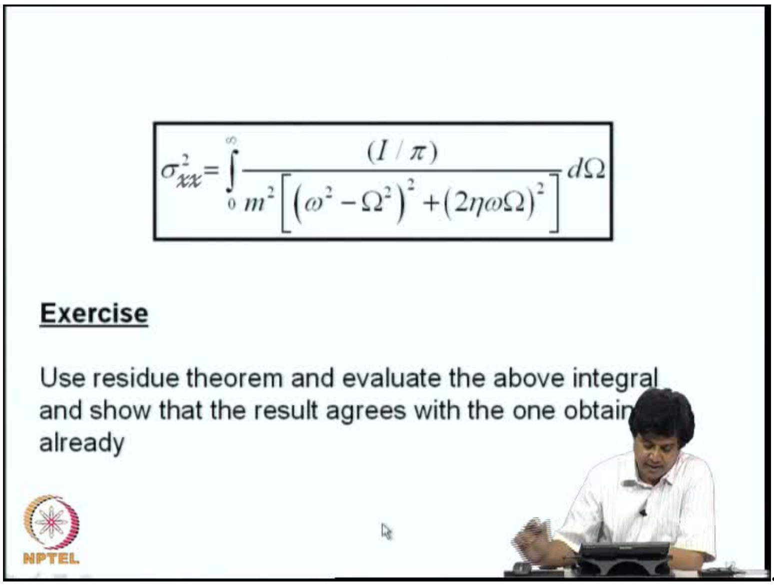

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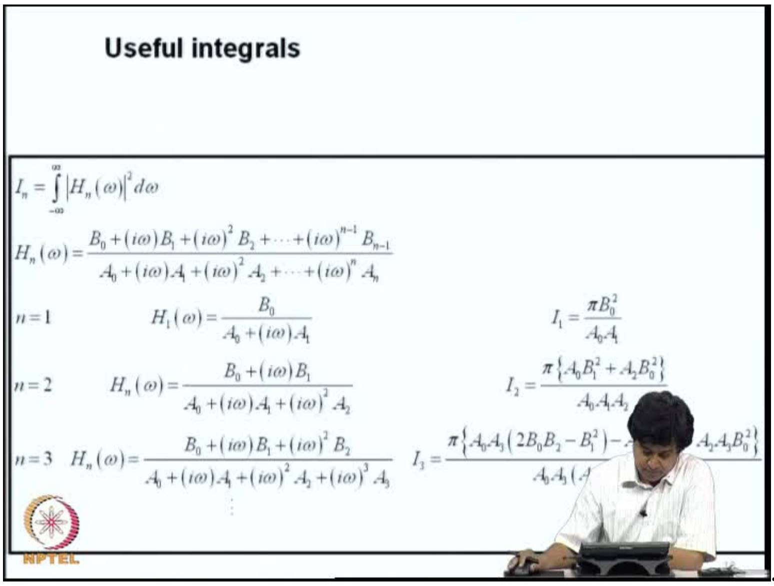

Evaluation of such integrals can be carried out using Cauchy's residue theorem using contour integration in complex plane, I leave that as an exercise and exercise is to evaluate this integral, using the residue theorem and show that, this study state variance is equal to the value, that we have already derived using frequency domain analysis. There are few integrals, that will be useful in frequency domain analysis of random way driven systems, I have provided at here, the structure of this integration integrals are as follows. In the integrant, we have ratio of two polynomials in omega and for different values of n, for example, n equal to 1, H 1 of omega is b naught by a plus something like A plus omega and I 1 is given by this value. So, for n equal to 2, I get a linear function here and quadratic function here and this is the integral; and n equal to 3, I get a quadratic here, cubic here and this is the value, this is further results for n equal to 5, 6, 7, 8, etcetera also available in the existing literature.

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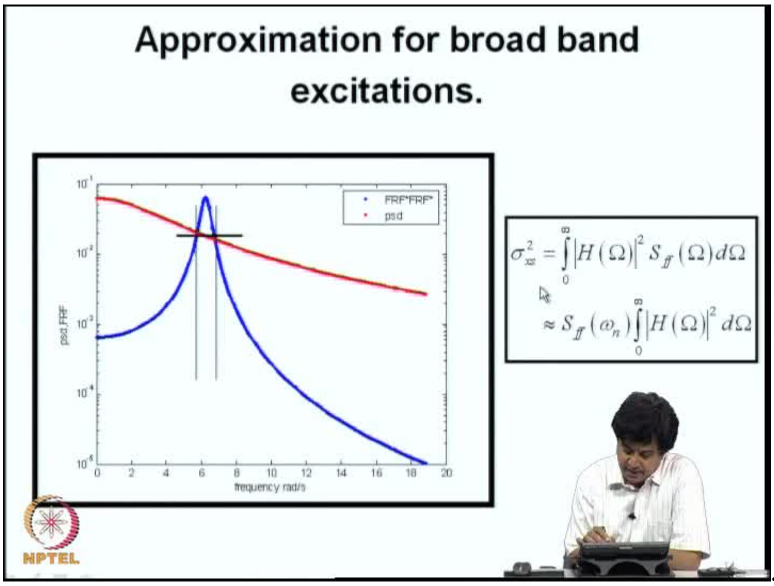

So, one could use such readily available information to carry out the requisite integrations. We have seen that the study state response variance is given by this integral 0 to infinity H of omega whole square S ff of omega d omega. If S ff of omega is white noise process, then it will be a constant and will be left with evaluation of the integral 0 to infinity H of omega whole square d omega, but if S ff of omega is not a white noise, it may be possible to carry evaluate this integral, using an approximation which is explained as shown here. The blue line that you see is the frequency response function of this system, that is, this H of omega whole square.

If red line shown here is the prospective density function of the input, then you could see that the output PSD is clearly product of these two and in the, if you want the variance of the output, you have to find the area under the curve that is obtain by multiplying these two curves. Most of the area in the product will be contributed by the function lying in this region, where the frequency response function as well as excitations has relatively higher values. So, for purpose of an approximation in regions, where the frequency response functions has significant values, we can approximate the prospective density function is the constant and we can ignore this variability.

So, if I now replace the prospective density here by its value, at omega capital omega equal to omega n and pull it outside; that means you are basically applying white noise excitations at pitched, at that, this frequency at this value. If you do that, then the integration of this function can be carried out by integrating the only the term involving the frequency response function and it affords certain simplifications and we can get a quick estimate of the study state variance.

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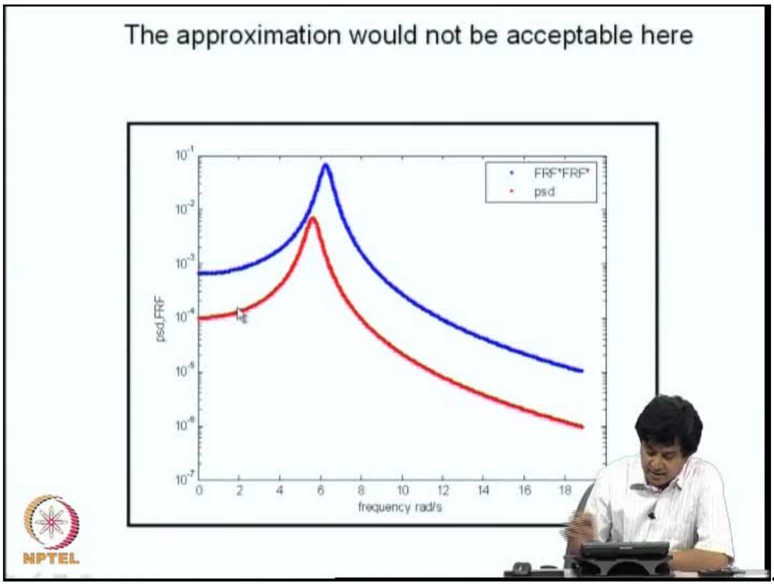

So, this, an approximation that is often used in practice, if you want a quick estimate of the study state variance, for this approximation to be valid the prospective density function of the excitations should not change dramatically in regions, where the frequency response function as significant values. So, this assumption, for this particular view graph shown seems to be expectable, but if you consider this particular case, where the blue line is the frequency response function and red line is prospective density function of the input. Here, we cannot make the assumption that the prospective input can be approximated as a white noise with the prospective density value at the system natural frequency, because when we multiply here, over the regions, where frequency response function is significant; the prospective density function of the excitation also changes dramatically.

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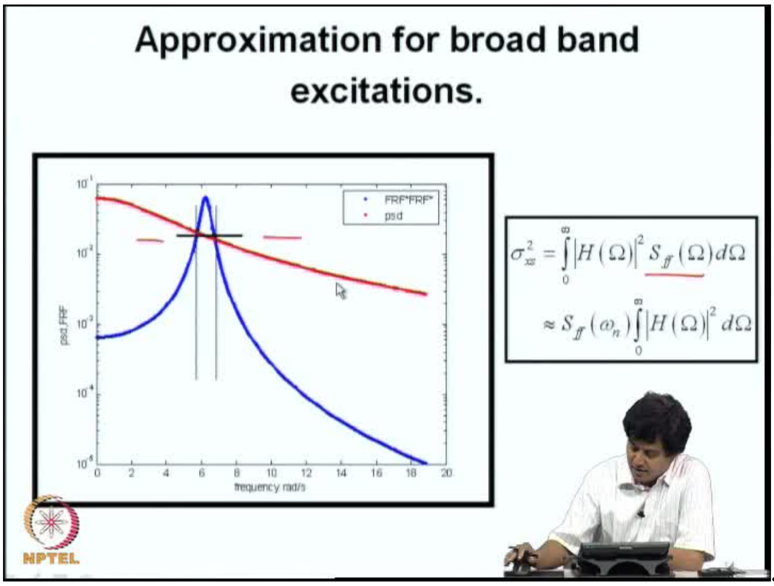

So, the details of this variation need to be allowed, for in evaluating the integral and we will not be able pull out this outside the integral, if you do that, we will not be getting a reasonable approximation. So, this approximation that is shown here, will not clearly work for this particular case; so, when you are using this approximation if you need to you have to sure that we are not making gross errors.

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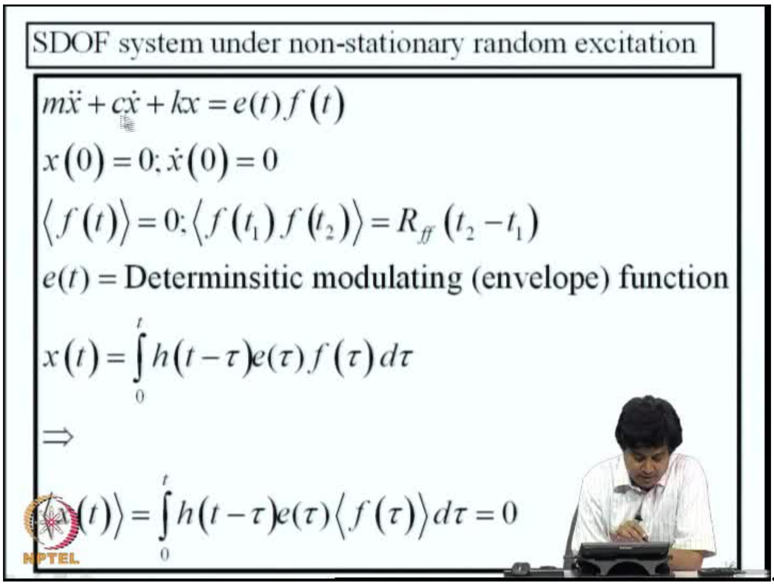

Now, we can continue this discussion and consider now the response of a system to a non-stationary excitation. Here, I consider the excitation to be a product of deterministic envelope and a stationary random processes, so f of t is still stationary system starts from rest mean of f of t is 0 and auto covariance of f of t is still a function of time difference, but it is multiply by an envelope function e of t, it is also called as a modulating function or an envelope function. So, what happens to the response if system is driven by this type of excitation? Again, we start with Duhamel's integral, now on the write integrant, we also have this modulating function, this is deterministic.

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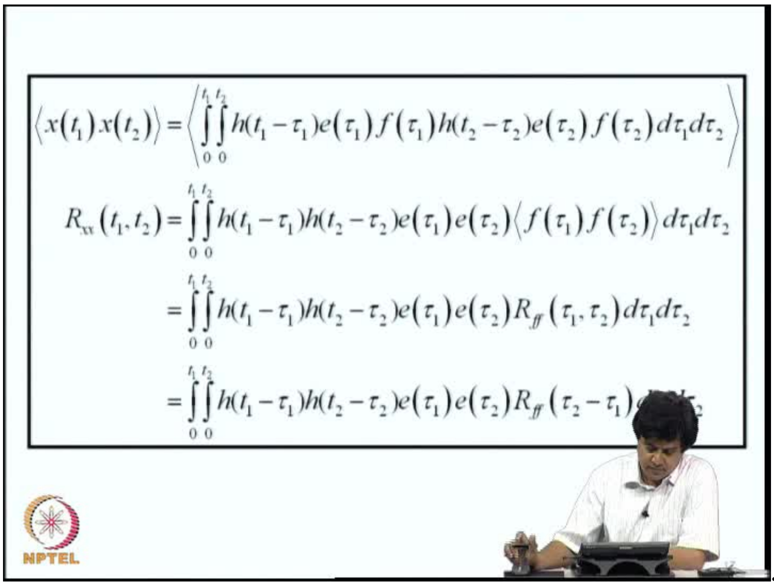

So, if you take the expected value, now since we are assuming that, expectation of f of t is 0, the expected value of f of tau that appears here is 0, therefore this expected value is also 0; so, the expected value of the response process is 0. How about auto covariance of the response; so, we multiply x of t 1 with x of t 2 and take expectation, if you do that you are multiplying two Duhamel’s integrals that becomes a double integral and this is a first integral, part of the first integrant is here, this is the second integrant. And if we interchange, now the expectations operator with these integration, the expectation operator enters inside the integrant and we get expectation of f of tau 1 into f of tau 2 in the integrant and this is replace by R ff (tau 1, tau 2) and since f of t itself is stationary, this becomes tau 2 minus tau 1. So, the auto covariance of the response process is now given by this expression; the new thing is, now that we have the terms involving the analog functions present here.

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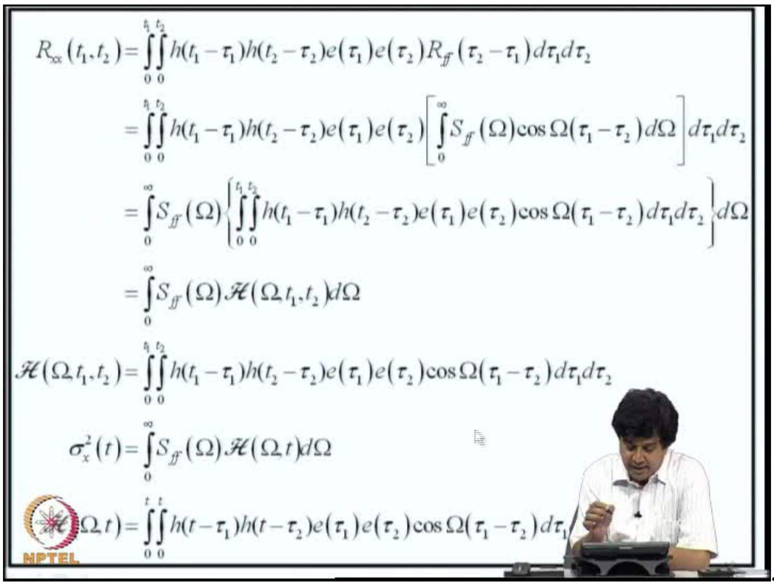

Now, again for the auto covariance of the input, if I now use the frequency domain representation, I will express R ff of tau 2 minus tau 1, in terms of prospective density function and again interchange the integration with respect to frequency and time, I get this expression and the time inside the brace, I again, I will call it as a time varying frequency response function which is function of capital omega as well as t 1 and t 2.

So, if I write that, that is, this. Now, this frequency response function is now a function of the envelope functions; so, that means, it carries certain information with excitation, also is not just a system property, if this where to be not there, h of t is the system property and cos omega tau is the terms that associated the Fourier transform, Fourier transformation; therefore, on the right hand side, there would be no terms involving the excitation process, but in this particular mode of representing the edge, this transfer function is now a property also contains property of the excitation process, so one has to be careful in interpreting this as a transfer function . If you are interested in variance t 1 becomes t 2 and this h (omega, t 1, t 2) can be written as h (omega, t), which is this function, this again a double integral function of envelopes this is the expression.

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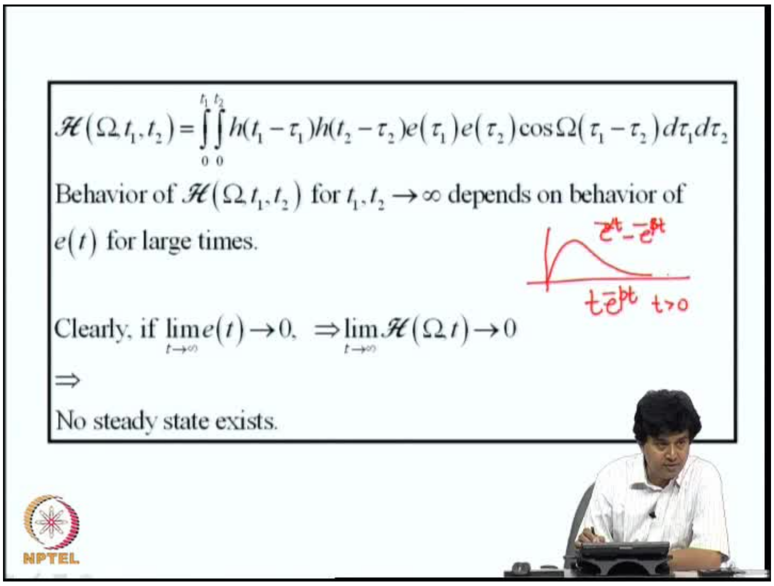

Now, if I now look at this time dependent transfer function and examine what happens as t 1 and t 2 become large, you can easily see that the behavior of this function depends on how would this envelopes should behave for last times. If envelopes itself goes to 0 as time becomes 0, clearly the transfer function also go to 0 as time becomes large and therefore, one can immediately concludes that the response would not reach a study state. Typical envelope function we already seen for earthquakes like loads, it is something like this; this can be of the forum e raise to minus alpha t minus e raise to minus beta t or of the forum t e raise to some minus p t with t greater than 0.

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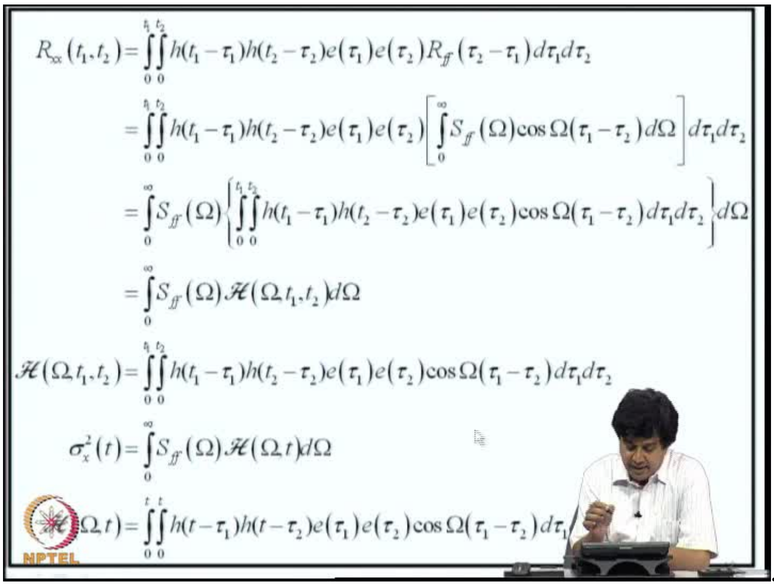

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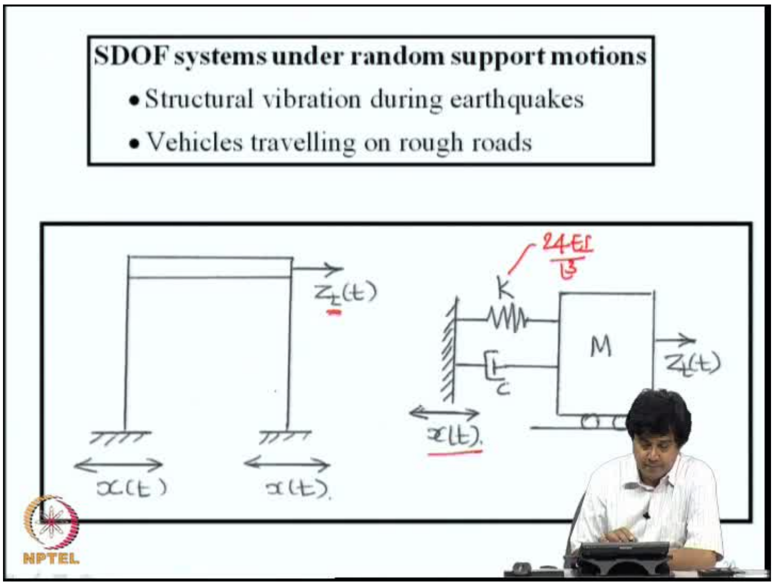

So, clearly as t tends to infinity, these two functions behave like this first, of course, there will be some restriction alpha and beta, if that is satisfied, the functions goes to 0 as t tends to infinity, in which case as t tends to infinity, the response variance goes to 0 and x of t would no longer be a able to reach any study state. We will now try to apply, what we have learned to problems of random vibration of single degree freedom system under support motions; these support motions could be due to vibration during earthquakes or these could also be vibrations of vehicles, when they travels on rough roads.

So, if you consider a simple portal frame, one story one way and if x of t is support motion, a simple model for these as a single degree freedom system is shown here. We assume that slab is infinitely rigid in its own planes and the stiffness of the system in this direction, here depends on stiffness of these two columns; so, this k, for example could be 24 E I by L cube and C represents the damping in the columns and M is the mass of this slab and part maybe a part of column, can also be added to this. So, notionally, we have now a single degree free system in which the support displacement appears here and Z t of t is the total displacement of the slab, this subscribe t here denotes total, total displacement. So, this is a simple physical model for a portal frame undergoing support motions.

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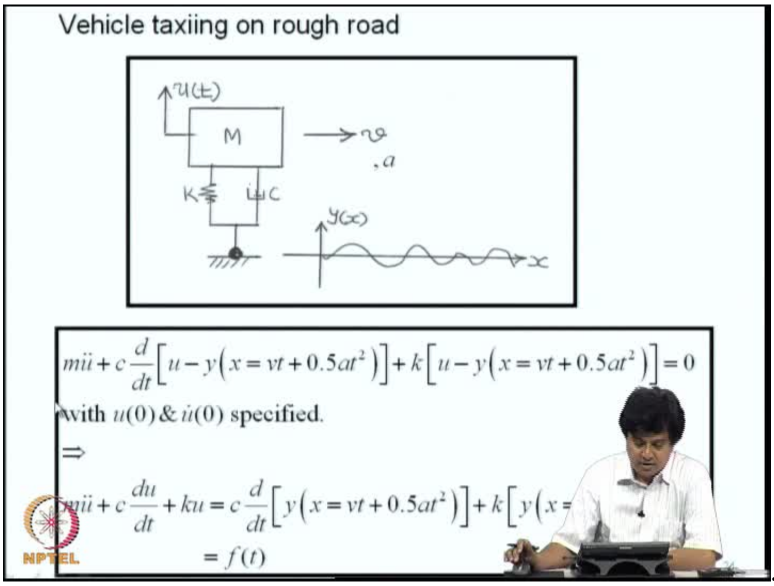

Similarly, if a vehicles x is on a rough road, suppose y of x is the plot of the road roughness and this M C K system is a model for a simple vehicle, which is travelling with velocity v and acceleration a and say t equal to 0, it enters this points and it moves with a velocity v and acceleration a and we are interested in the displacement of the this mass u of t, u as a function of time.

If you now set of the equation of motions for this vehicle, we will get this equation m u double dot is a inertial force and the force in the damped is d by dt of u minus y of x where x is v t plus half at square. Similarly, k the force in the spring is k into u minus y of x, where x is v t plus half at square with the initial condition u of 0 and u dot of 0 specified. So, we can write this equation is m u double dot plus u dot plus k u equal to c d by dt of this plus k into y of this. So, this, the force that acts on the single degree freedom system; clearly if y of x is a random processes, this f of t would also be random process, there can be additional complexity due to any uncertainties in the vehicles velocity and acceleration that also would affect this right hand side; therefore, f of t would be a random process. So, a structure which is subjected to support displacement, where guide way unevenness is a random process, is again equivalent to a single degree freedom system with a random excitation on the right hand side.

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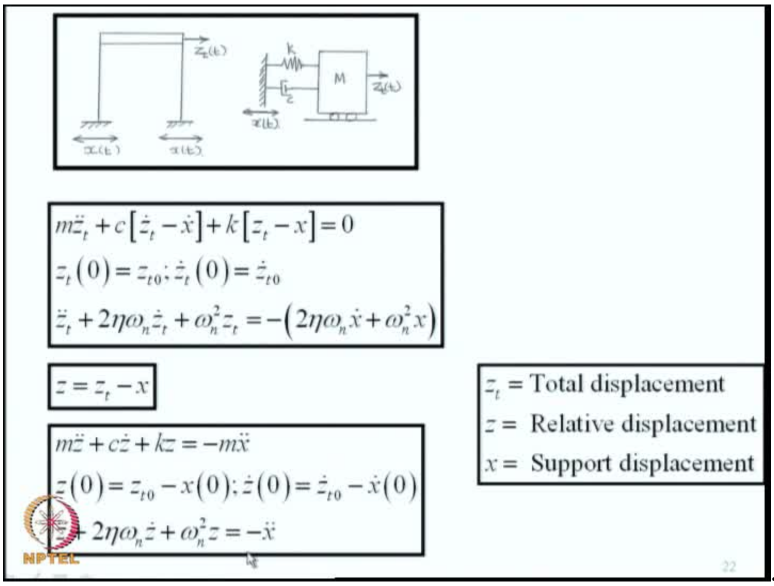

Let us consider this problem of support motions on a portal frame and try to formulate the problem bit further. So, we starts with this model for portal frame, this is the physical model and based on the free bird diagram of the mass, here I can write the equation of motion as shown here, m Z t double dot is a inertial force and C into Z t minus Z t dot minus x dot of t is the force in the damper and k into Z t minus x is the force in the spring and some of these forces must equal to 0 and these are the initial condition.

So, if I divide, now both sides by m I get the equation in the standard form, is the t double dot two eta omega n Z dot t plus omega n square Z t equal to, now, the forcing function, is in terms of support displacement and support velocity. Now, I can also write an equation for the relative displacement of this mass M relative to the support, so if I now introduce the variable Z which is Z t minus x, I can substitute that into this equation Z t minus x already appears here and Z t dot minus x dot appear here; so, these two terms will simply become c z dot and k z, but these m z t double dot will be m into Z t double dot plus x double dot and that m x double dot, I take it to the right hand side, so and this will be now the initial condition. The initial condition for the relative displacement involves the initial condition on Z t as well as the ground displacement at t equal to 0, the displacement and velocity.

So, if I now divide this by m, I get an equation, where right hand side contains the ground acceleration. So, if displacement is the random process, clearly the displacement process and velocity when added in this manner, also lead to a random process and if x of t is a random process x double dot of t would also be random processes; so, neither case we get single degree free system driven by random excitation.

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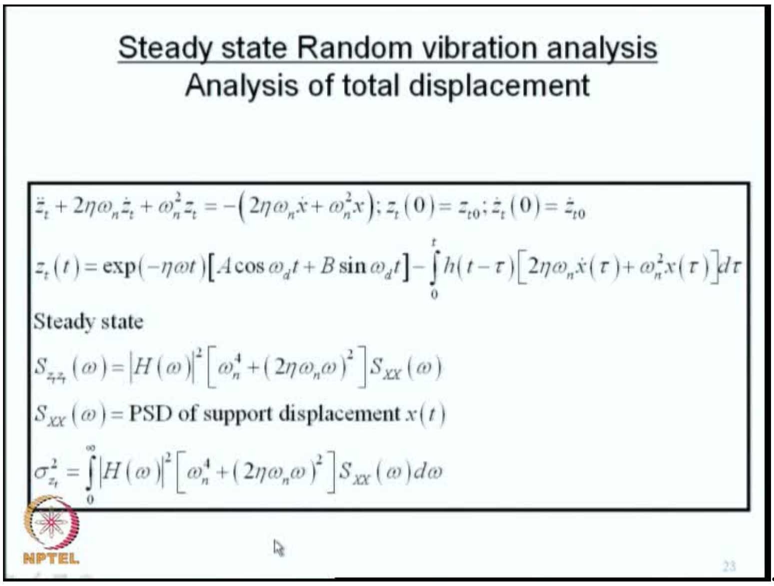

Now, we can, if now, I, model x of t is a as a stationary random process, can we do a study state response analysis and find out power spectral density function, for example, for total displacement, if you want do that we start with this equation; on the right hand side, I have 2 eta omega and x dot plus omega n square x and the solution for Z t of t will be transients part minus plus this Duhamel’s integral h of t minus tau into 2 eta omega n x dot of tau plus omega n square x of tau d tau. in this study state, we have to find now, the PSD of the input can be shown to be given by omega n to the power of 4 2 eta omega n omega whole square into S x x of omega, where S x x of omega in this particular example is the PSD of this support displacement; therefore, the output prospective density function is given by this relation. The variance of the total displacement is area under this curve and that is given by this; so, if you know the prospective density function of this, you can evaluate this and if it is, for example, a Kanai Tajimi type of power spectral density function, we get in the integrant ratios of polynomials in omega and we can evaluate it, using the list of integrals that I showed earlier or use residue Cauchy’s residue theorem on this and evaluate the variance.

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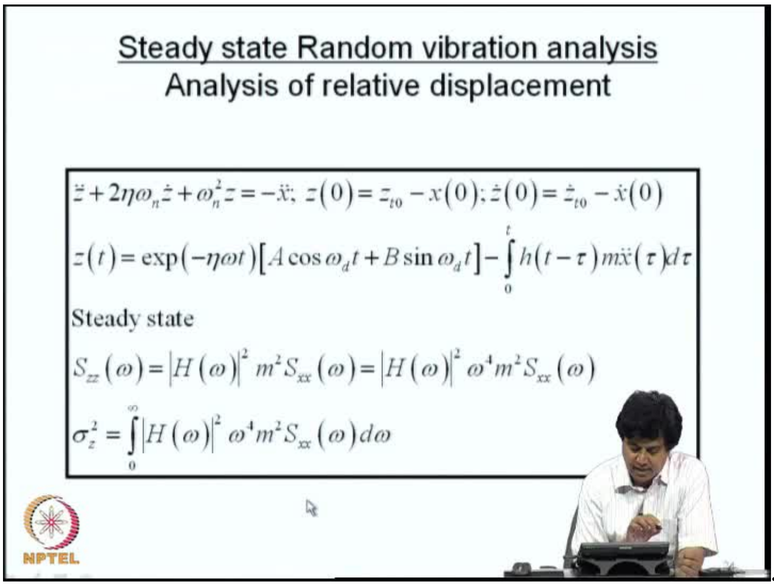

A similar analysis for relative displacement Z of t, if it considers, then equation governing Z of t is this and the solution in terms of initial condition and Duhamel's integral is this. Clearly, the output power spectral density function is H of omega whole square into the power spectral density of the right hand side which is x double dot of t and that prospective density of prospective density function of x dot of t, we already shown is omega square into prospective density of x. So, the prospective density function of x double dot will be omega to the power of 4 into the prospective density functions of the displacement, if you use this, I get the output prospective density; we integrate this function from 0 to infinity, you get the study state variance.

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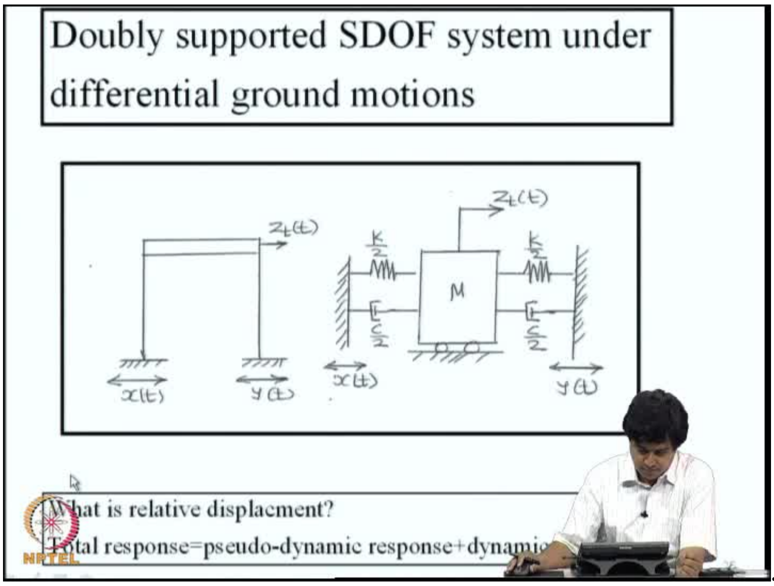

Now, in studies of long span structures, often the two supports of the system may suffer different support motions. So, a simple archetypal model of that would be a portal frame, where one of the leg is subjected to support displacement x of t, the other leg is subject support displacement y of t. The model for these would be of this form,, is a single degree freedom system, it two supports and the left hand support receives the displacement x of t, that is due to this and right hand support receives displacement y of t, which is due to this. This k by 2 is the stiffness of this column and we are assuming that the two columns are identical; therefore, this is k by and 2 k by 2, this c by 2 and c by 2. Now, in this particular case, the notion of a relative displacement is not appears, because the two supports move differently, so there is no unique definition for a relative displacement, it could be with respect to this, this support or this support, so that is not how will talk about relative displacement in this case. Instead, what we do is, we consider the total response Z t of t to be made up of two components, namely a pseudo dynamic response and a dynamic response, what are these things?

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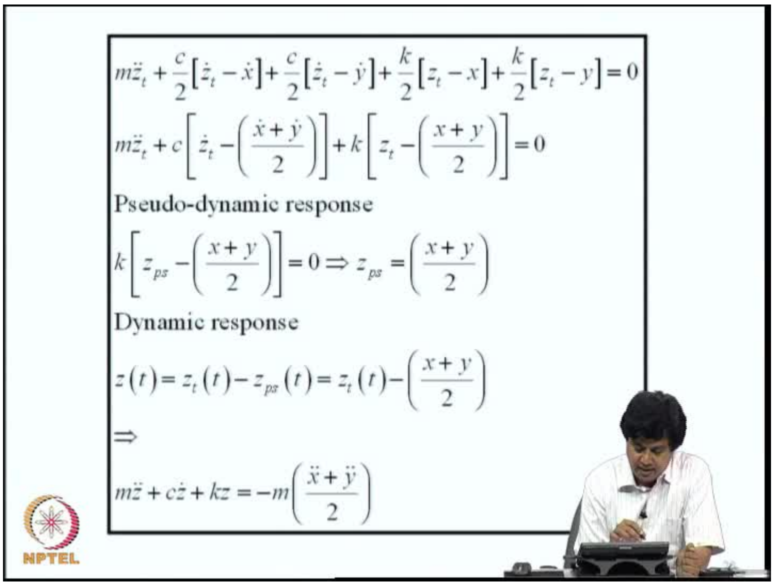

To see that, we will, let us write the equation of motion, so if I draw the free birded diagram for the mass and write the forces in damped and the strings and the inertial force we get this equation of motion. This is the force in the left spring, which is left damper, which is C by 2 into Z t dot minus x dot, whereas force in this damper would be Z t dot minus y dot.

Similarly, force in the spring would be k by 2 into Z t minus x; for this spring, it will be Z t minus y. So, if you do all, the, we get this equations and rearrangement would lead to this equations and we define the pseudo dynamics response, as solution of this system, when there is new dynamic action, that means, Z t double dot is 0 and we consider only the equilibrium due to the static action and we define Z p s as solution of this equation. We call this as pseudo dynamic or pseudo static response, we have denoted here as Z p s, so this will be x plus y by 2. The dynamic response, now we define as Z t of t minus Z p s of t, so this becomes Z t of t minus x plus y by 2. If I now substitute for Z t of t into this equation, simple rearrangement of terms will give me the equilibrium equation for the dynamic component as shown here. Clearly, this Z t minus x plus y by 2 here; therefore, this term will be simply Z, this is Z dot that will be this, but the acceleration term will contribute to the double derivative of this multiply by mass which appears on the right hand side.

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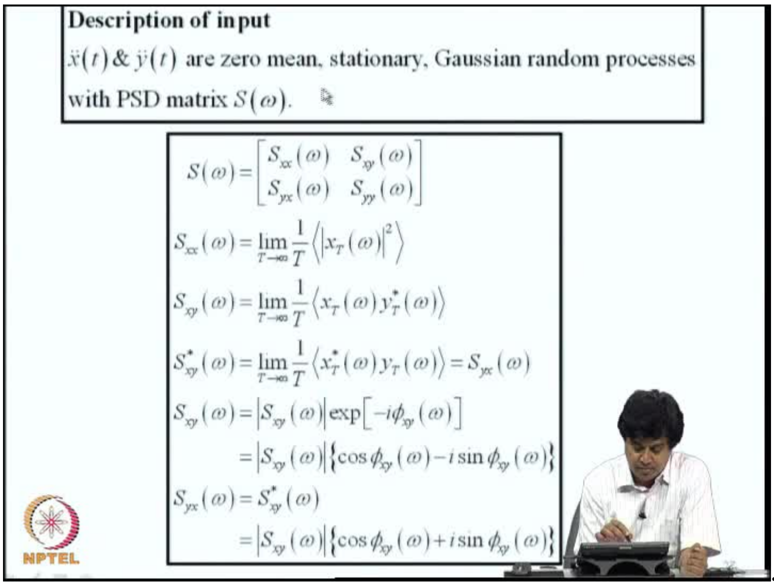

Now, we will try to now proceeded with this analysis will describe input as follows. If x double dot of t and y double of t are 0 means stationary Gaussian random processes with PSD matrix, S of omega given by this S x x of omega is auto prospective density of x double dot of t, this is mind you, this prospective density function is for acceleration in the derivation. S x y of omega is the cross power spectral density function between x of t x double dot of t y double dot of t; S y y of omega is auto covariance of y double dot of t. These are the definitions for this quantity S x y of omega is this limit T tend to infinity 1 by T x T of omega y T star of omega and we already define the meaning of this is already given in the earlier lectures.

Now, if I take conjugation of this, it will be x T star of omega into y T of omega, which is nothing but S y x of omega, that means, this is not symmetric function S x y star of omega S y x of omega. Now, if I write the cross prospective density function, in terms of an amplitude and phase angle; S x y of omega can be written in this form, phi x y of omega is phase spectrum, this is the coherence function. And S y x of omega will be the conjugated of this, we should make this minus i sin phi x y of omega plus i sin phi x y of omega.

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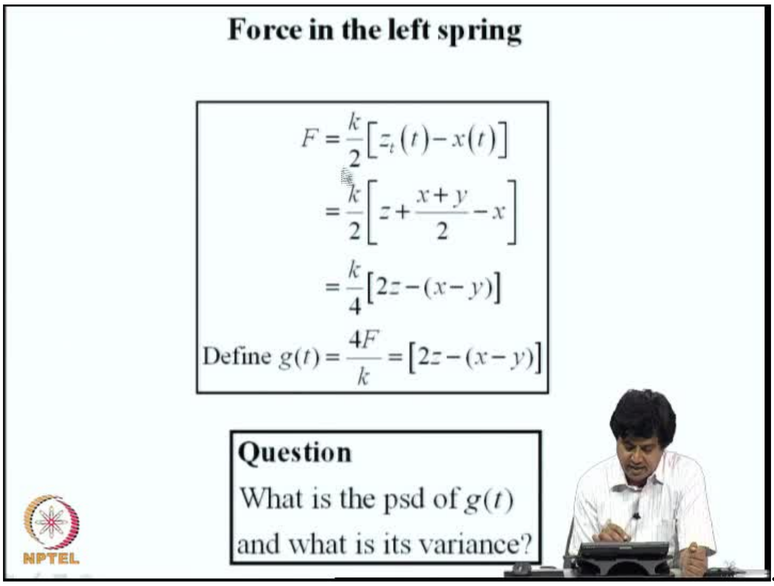

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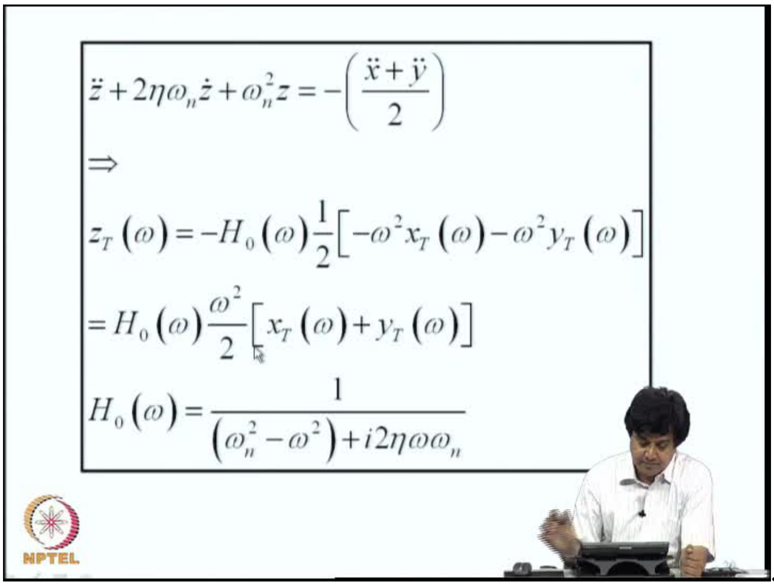

Suppose, for purpose of discussion, we take the force in the left spring; that means, this spring, what, whatever force exit in that as a response variable of interest; so, that is k by 2 into Z t of t minus x of t. Now, Z t of t is written in terms of dynamic component and pseudo dynamic component, if you rearrange this terms, I get this force to be 2 z minus x minus y; I can define another quantity g of t which is 4 F divided by k, so that, I can focus on the quantity 2 z minus x minus y, this is related to the force in the left spring. Suppose, now, I ask what is the variance of g of t? What is its power spectral density function? We could go ahead and do this analysis Z is this and Z T of omega will be this and H naught of omega is the frequency response function which is given by this. Here x T of omega is written a Fourier transform of x double dot is written in terms of Fourier transforms of displacement, that is truncated x of t, that is why we are seen omega square, that omega square is pulled out here and I get this.

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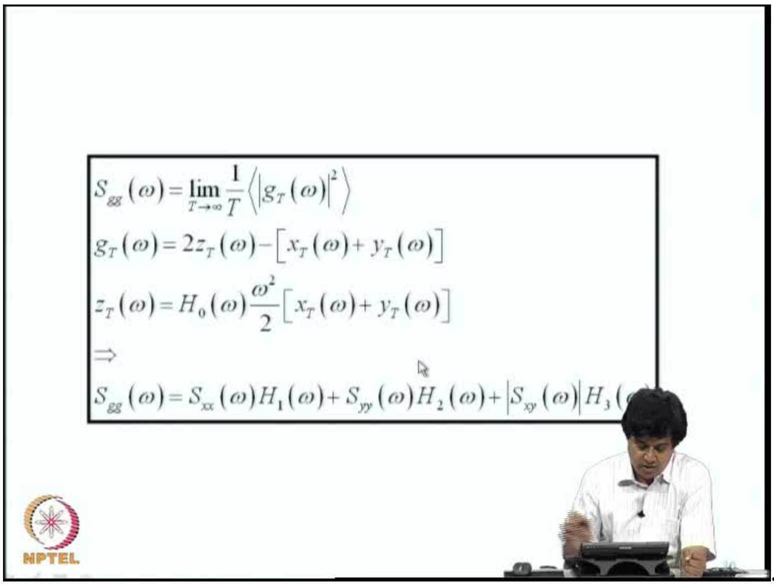

So, S g g of omega, if I use this definition of the power spectral density function, this is given here and g T of omega is, g of t is 2 z of T minus x plus y; so, if I do this, I get this and S g g of omega can be shown to be given by. The auto power spectral density functions of the left hand side, support excitation multiplied by one transfer functions. The auto prospective density function of the excitation, there the right hand side multiply by another transfer function and another transfer function is H 3 of omega which multiplies the amplitude of the cross power spectral density function between x of t and y of t x double dot of t and y double dot of t.

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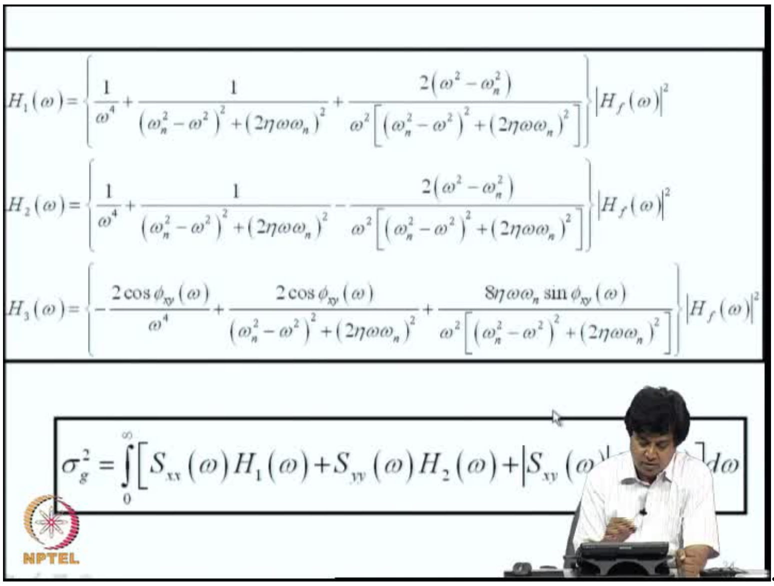

We can show that, these H 1, H 2, H 3 are given by this function, here I have introduce the slight variation, I have multiply this transfer function by a filter H f of omega whole square, if you look at the nature of H 1 of omega, you see that there is a term 1 by omega to the power of 4 and this will contribute a singularities in the response, which is due to improper specification of the power spectral density functions to second one, that we multiply by a transfer function which actually skills this singularity and leads to meaningful result; this is an artifact, I will explain this function shortly. H 3 of H 1 and H 2, we can show that there actually positive, all the, that is known manufacture; here I will show it in the next slide, that H 1 and H 2 are strictly positive, where H 3 depends on this phase spectral and this could take value which could be neither negative or positive; in many case, the variance of the quantity g of t is given by the area under this function.

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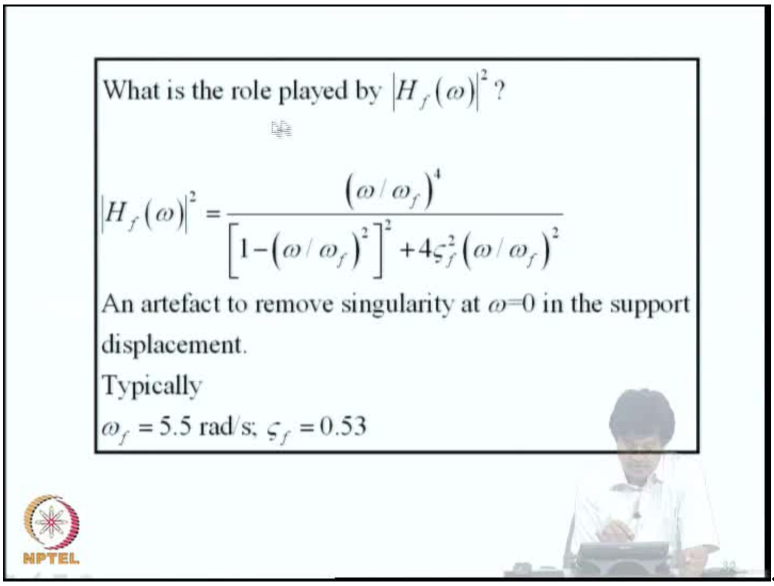

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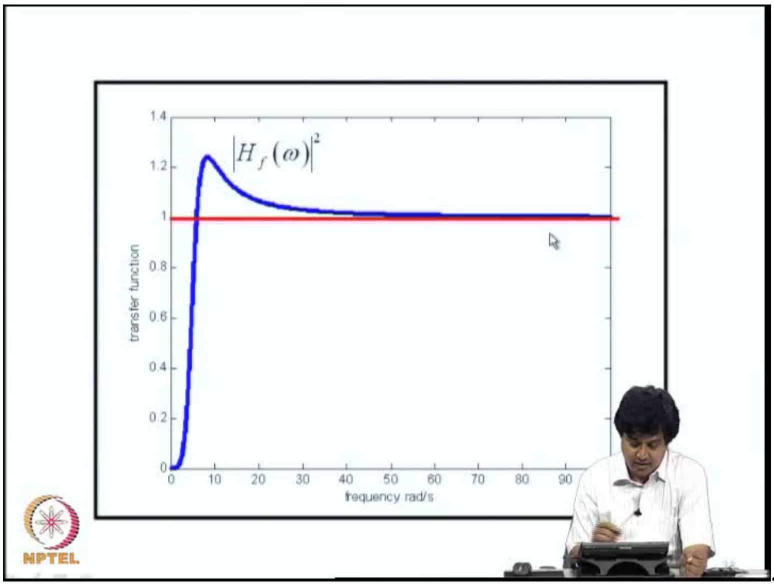

So, again there are three transfer functions H 1, H 2, H 3 which multiply respectively the out of PSD of S of x double dot of t auto PSD of y double dot of t and the cross power spectral density amplitude of cross power spectral density function between x double dot and y double dot. So, to clarify the role played by H f of omega whole square, we can propose a propose form of this transfer functions is given here and if you actually plot it, that is, how it looks like this blue line is this square of this H f of omega whole square; this function is 0 at omega equal to 0 and this enables us to when this is multiplied by function is singularity, that omega equal to 0, a few of them can get eliminate. As omega becomes large, of course, this function becomes one and therefore multiplication of this function does not affect the product; once we cross a frequency of say, 30 radiant per second also.

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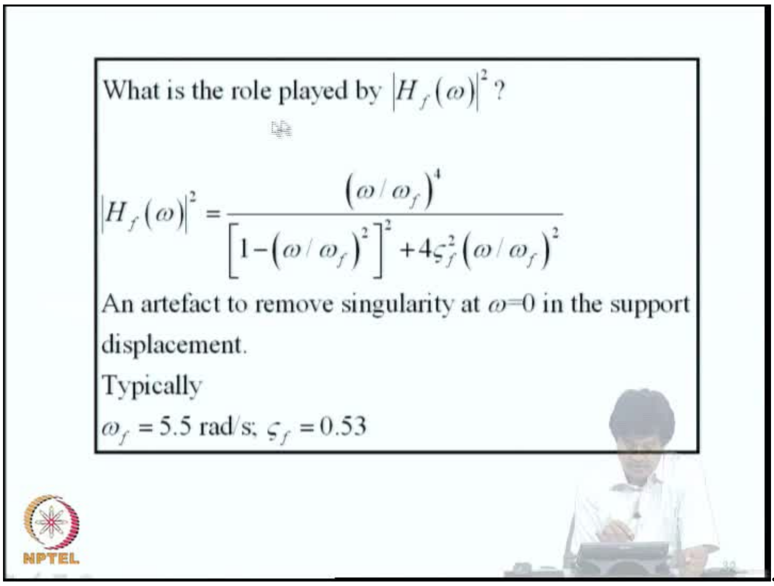

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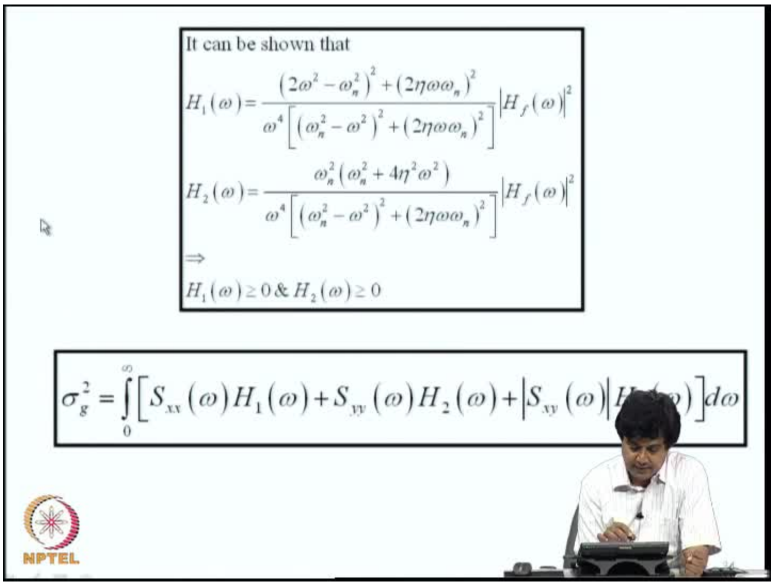

So, the location of this peak can be manipulated by adjusting this filter, frequency omega f and also we can the details of the decade to 1, can be control by varying this damping factor zeta. Typically, the plot that I have shown is for the central frequency of the filter is 5.5 radians per second and damping is fairly high. This is basically an artifact to remove singularity at omega equal to 0 in the response. By rearranging the terms in H 1 and H 2, we can also show that H 1 and H 2 have this representation and if you carefully observe this, you can see that H 1 and H 2 are non-negative, there actually positive.

So, if you look at this expression for variance, we see that the first two terms essentially make positive contribution and third time can make neither a negative nor a positive contribution. So, if you now ask the question, what should be the natural of cross power spectral density function, which produces highest or the lowest variance, that will be control by this product; basically that will be control by the nature of this H 3 of omega. If S x y is 0, that is when x n x of n omega t are independent, then we get, of course, the variance to be simply the sum of contribution from first two terms; if it is fully coherent, that means, if this is equal to product of S x x into S y y, I get another value of this. One may the tempted to thing, that these two represent limiting values of the variance of g, but it terms out that there are not the limiting forms, which leads to highest and lowest variance.

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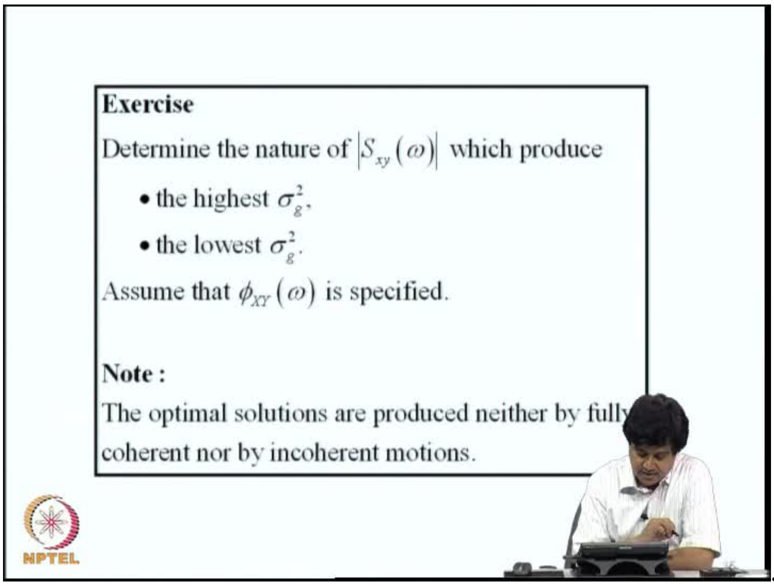

So, I leave this is an exercise, if you to determine the nature of this cross power spectral density function, which produces the highest variance and the lowest variance. We could assume that phi x of phi x y of omega is specified and hint to this solution, is that the optimal solutions are produced neither by fully coherent nor by incoherent motions, that means, there exits special forms of cross prospective density functions, which produce the highest response.

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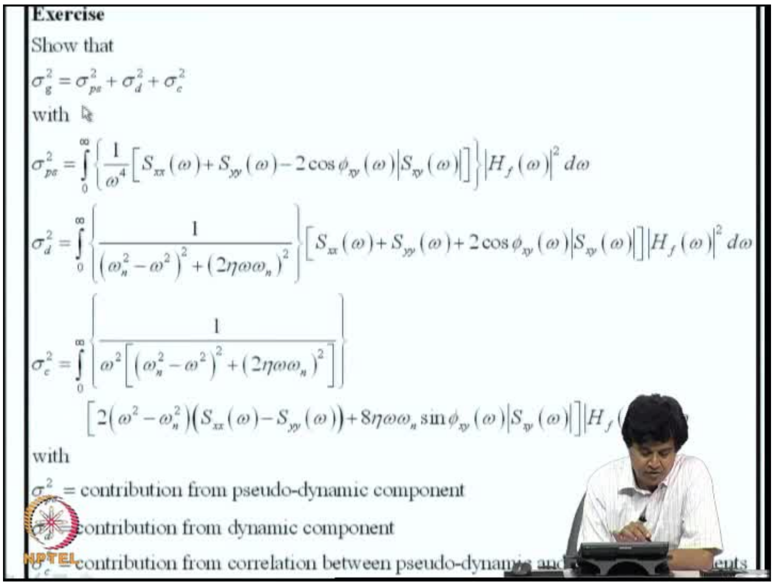

So, in case you do not know the properties of cross PSD function and we want to get the worst response, we can use this optimal cross power spectral density function and get an idea of what could be the highest variance, there is another exercise associated with this problem, we have shown the variance of the response, in terms of contributions from auto PSD of x auto PSD of y and cross PSD of x and y. Another way of looking at this is, if you look at the way, we have written the response; we had a pseudo static component and dynamic component Z of t as written a total response total response is written as a dynamic component plus pseudo static component. So, if we look at variance of the total any of the total response quantities, it will be having contribution from variance to pseudo static component and variance due to dynamic component and another contributions which comes because of code relation between dynamic and pseudo static component. So, we can also write the variance of the response, if you all you have to do really rearrange this terms in a slightly different manner, you can show that the variance of the quantity g of t consists of a contribution due to pseudo static component contribution, due to dynamic component and contribution, due to correlation between dynamic and pseudo static component; if you do that, you can show that the pseudo static component is given by this, the dynamic component by this and the correlation component by this, these I leave it as an exercise; this essentially requires rearrangement of these terms.

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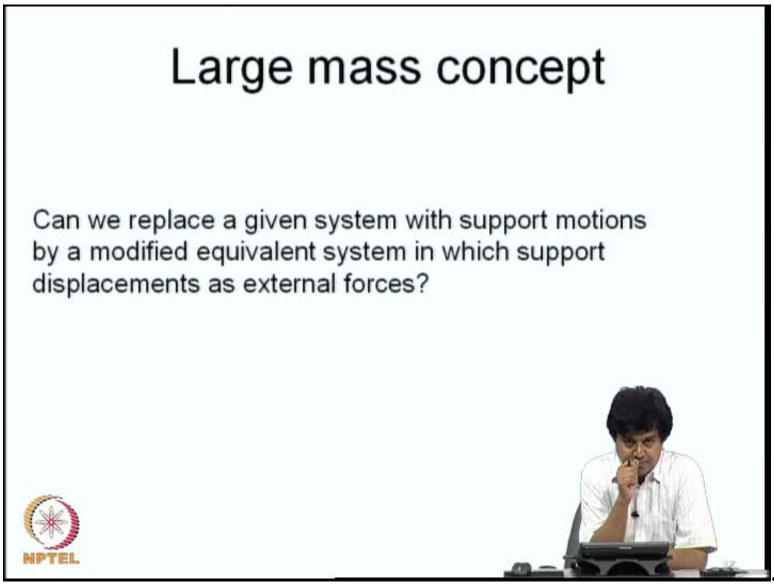

In problems involving support motions, in computational modeling, many finite elements software do not have capability to apply support displacement, instead they convert the problem of a support displacement into an equivalent problem of an external force. How it is done. So, there, we use what is known as a large mass concept, the basic issue here is can be replace a given system, which support motions by a modified equivalent system in which support displacements or replaced by external forces.

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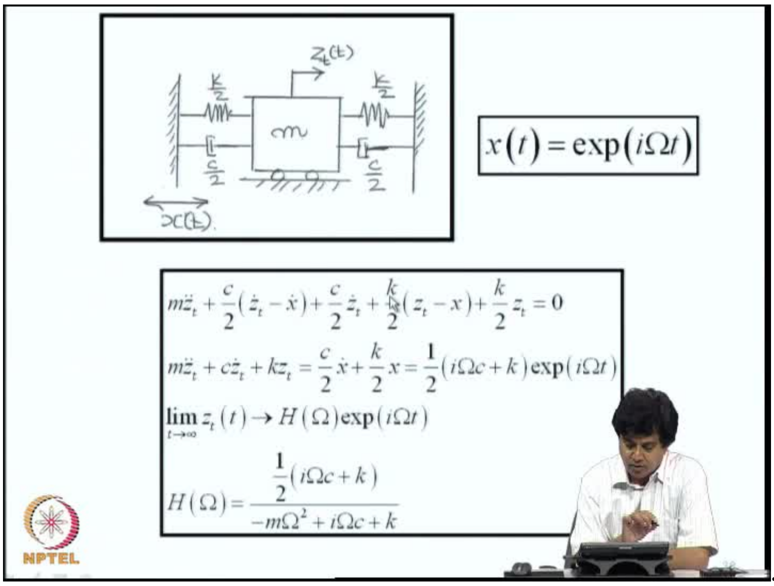

So, to consider that problems, we will consider a simple example, a problem of a differential support motion, where one of the support does not actually move other one moves by x of t and let us assume that x of t actually a harmonic displacement, we can easily write the equation for this, in terms of the displacement and I can get the study state value of Z t of t given by H of omega exponential, I omega t and H of omega is the transfer function, you can easily do that, because this is an equilibrium equation and the right hand side I have harmonic driving. In the study state, the response also would be a harmonic at the driving frequency with an unknown amplitude H of omega, if substitute this into this equation and rearrange the terms you can easily drive this.

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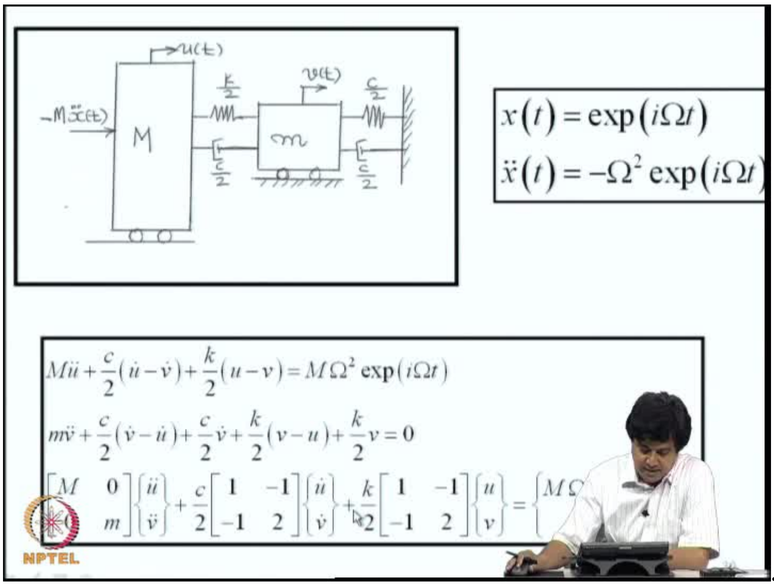

Now, what is done in using large mass concept is, this support is replace by a huge mass, these are point masses, the size of the rectangle is not so important, but the main just to pictorially represent, this is the large mass, I shown a larger rectangle here and the support is a replace by large mass and the degree of freedom is released and this mass we apply a force minus M x double dot of t. What is x of t, x of t is a support displacement here what I have done, I removed this support and added a large mass there, on the large mass, I am applying the force M x double dot right, but this support is now removed.

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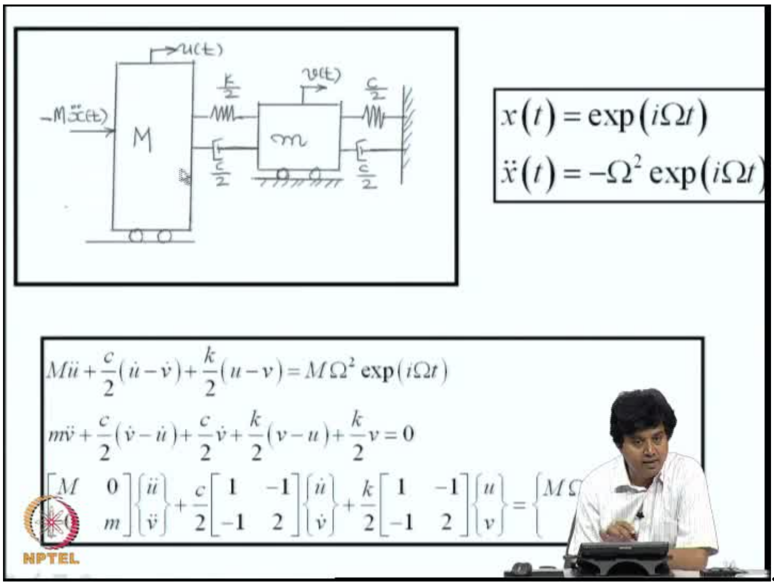

So, this is now become a two degree freedom system, we have not at encountered multi degree freedom system, in this course, but we can proceed a bit, if I write now the equilibrium equation for this system, there will one equation governing u of t which is given by this. There is an external force m omega square exponential i omega t and this mass, where degree of freedom is v of t I get this. I can put it in the matrix form and an, the right hand side, I get a forcing function for this degree of freedom u of t know for function on this. Again, if now I consider study state response, the system here is again linearly time in variant driven harmonically; therefore, response is also harmonic at the driving frequency, but which certain unknown amplitude.

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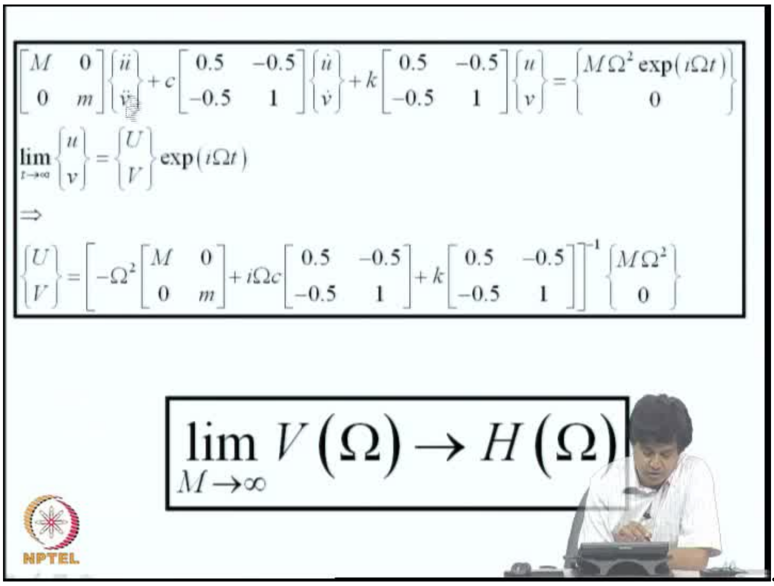

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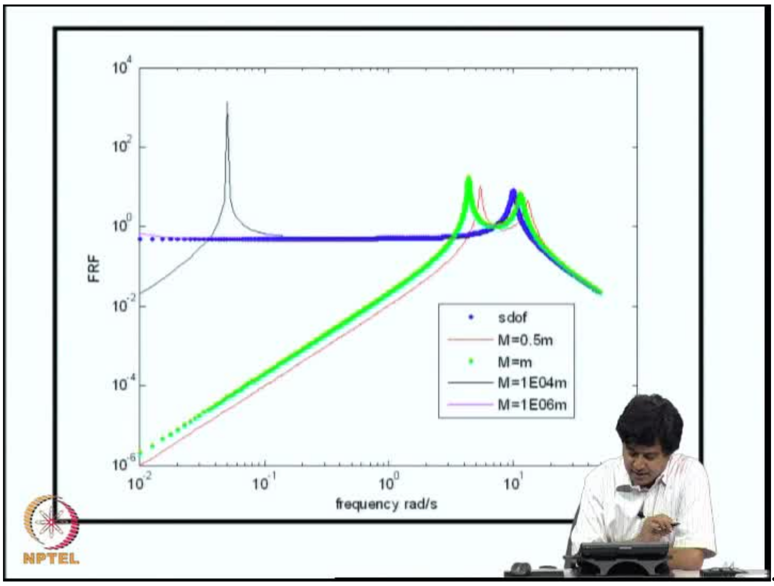

So, if I now assume in the study state, this u and v is given by some capital u and v exponential i omega t, I do not know what are these u and v. If I substitute this expression into this and solve for this vector U V, I get the expression in this form; this is something that you can easily verify. So, the claim that is being made here is, as these mass becomes large, the amplitude of response of this mass for this two degree freedom system approaches the amplitude of response of this system for large times, that means, what we are assuming the amplitude of response here is H, whereas amplitude of this mass here is capital V, according to this notation. So, the claim is limit capital M tend to infinity v of omega goes to H of omega, this can be verified, I have shown here as an illustration here the blue line is response of a single degree freedom system, in which one of the support is moving harmonically. The other lines are response of the two degree freedom system, where the left hand support is replaced is removed and mass is inserted there and that mass carries the 4 M S double dot. The various colors that you see here are for different values of this M; so, if M is very small half of the mean mass, then the system has two degrees of freedom and there will be two peaks, but as this capital M becomes larger what happens is, there lower frequency starts moving towards 0 frequency and in the case, where capital M is 10 to the power of 6 times, the mass of the main system; you see here this magenta line as a small peak almost at omega is equal to 0 and this actually now lies on the blue line, which validate the claim, that this claim, that M tends to infinity V of omega is H of omega. So, you could analyze, therefore a single degree freedom system, under one support motion as by an equivalent; you can study on equivalent problem of a two degree freedom system, where there is no support motion, but there is an external excitation. So, much software is geared to handle external excitation then support motions.

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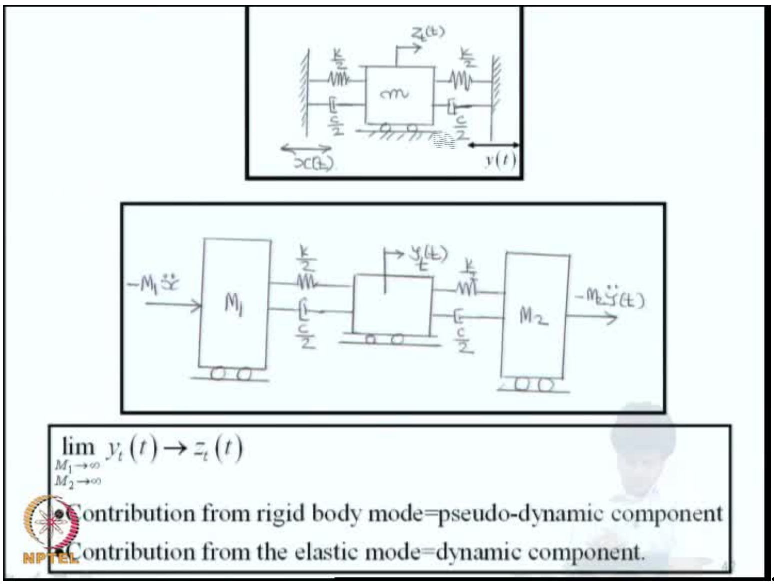

What happens, if the system has now two supports and each one receives a differential support displacement; so, I replace this support by one mass, this support by a another mass and on this mass I am apply minus M 1 x double dot and this I am apply minus M 2 y double dot and if our to analyze this, the times M 1 and M 2 becomes large; these two becomes large, the response of this mass becomes the response of this mass, but here you should see that, this has a rigid body motion possible, because omega equal to 0 would be one of the natural frequencies here. In consequently, you can also show that the contribution from that rigid body mode, in this problem would be actually the pseudo static response for this system and contribution from elastic mode will be the dynamic component of the response here.

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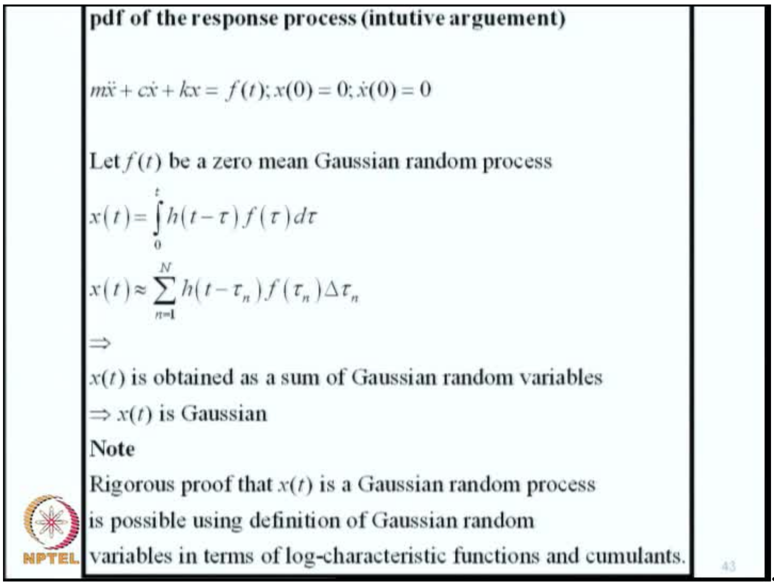

So, this is one useful trick that is done, if you are modeling vibration of large system in using safe a natal method. So far, we have being taking about response covariance power spectral density, mean, variance so and so far, how about the probability density function of the response that question we not get addressed. We start with the issue of the case of f of t being a 0 mean, the Gaussian random process, the output processes that is response processes is given by this Duhamel’s integral and this is a linear transformation on the input process, but this an integral, this a differential equation right, the second order differential equation here, this an integral convolution integral here. How do you prove that if f of t Gaussian, x of t is also Gaussian, that what we are going to use a simple intuitive proof for, that is, you replace this integral by a summation and we see here that x of t is obtain a by linear super pollution difference Gaussian random variable f of tau for tau n, for different values of n actually Gaussian or Gaussian random variables and you are adding them linearly. So, linear transformation of this kind, a Gaussian random variable, we are already seen that it retains Gaussian property. So, based on this argument, we can conclude that x of t is Gaussian random processes, but more rigorous proof of this can be obtain by considering the characteristic functional of f of t and study what are known as is cumulates, I leave that as an exercise that for purpose of preliminarily understanding this argument should suffice.

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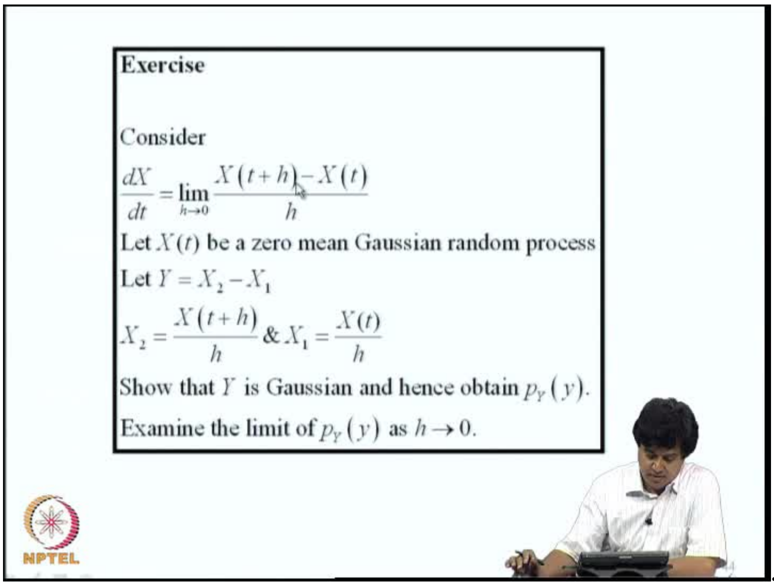

Another a small exercise, in the same, in this context would be, if you consider now the derivate d x by dt, we already defined the derivate of a random process in the means square sense, that is given here; if with this limit is interpreted, as limit in mean square sense, this will be mean square derivate of random process x of t.

Now, let x of t be a 0 mean Gaussian random process and suppose if I define y X 2 minus X 1, where X 2 is X of t plus h by h and X 1 means X of t by h, where h is we are we are not still imposing the limit h is going to 0, h is some finite non-zero quantity. We can easily show that, y is Gaussian and we can write the probability density function of y. Now, the exercise I want you to tackle is you have to examine the limit of this probability density function as h goes to 0. The limiting operation here is a density function, it is quite different from mean square the definition of limit is mean square sense. So, the discussion of this should be with reference to the two different modes of convergence that we are using.

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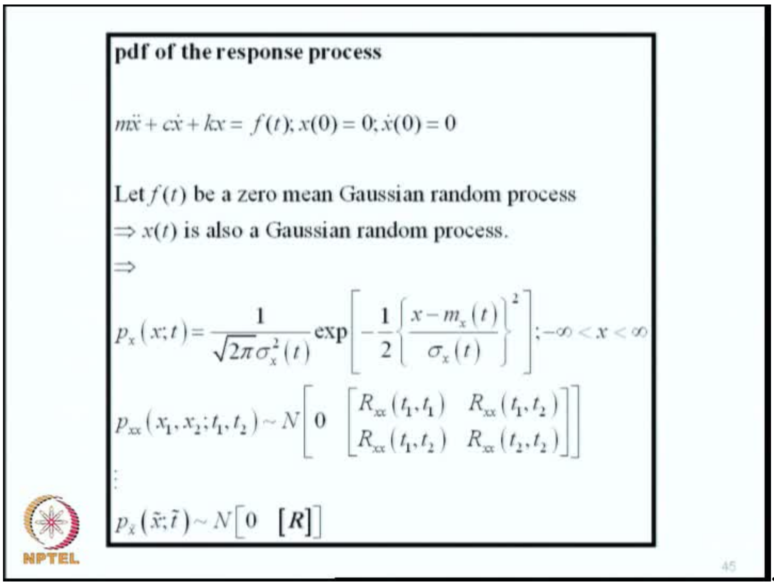

If we now consider the pdf of the response process, we are now taken that x of t also a Gaussian random process; what it means, the first order density function is given by this second order density function, I am now not writing the expression, but I am notionally representing a normal random variable is 0, 0 mean and covariance given by this. If you consider n time instance, this will be a, n fold n-dimensional Gaussian random variable. The means continues to be 0; R is n by n covariance matrix.

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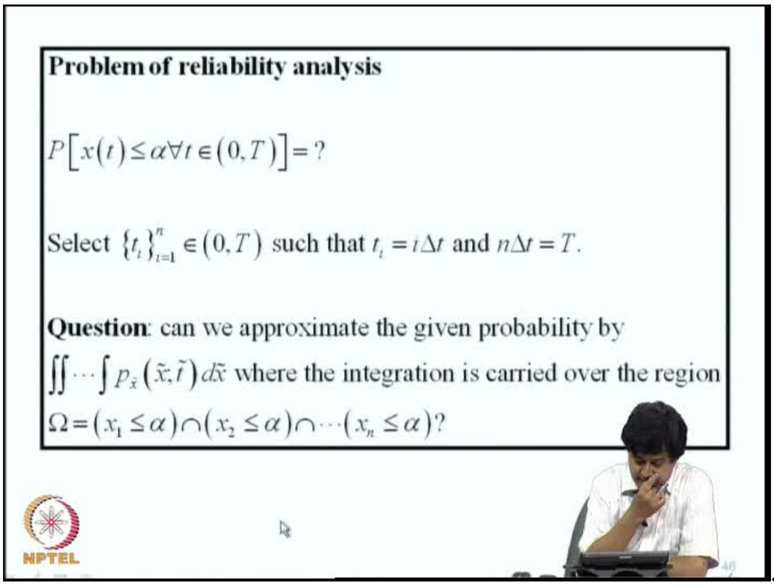

Now, typically we are interested in answering the question, what is the probability that the response, these below a safe limit alpha for a time duration 0 to T. When you say x of t is completely specified, when I know it is n th order probability density function with such complete description of x of t would enable you to answer this question or determine this probability, I am not sure, why, suppose, if you select t 1, t 2, t 3, t n which all belong to 0 to T, such that t i is i delta t and n delta t is T.

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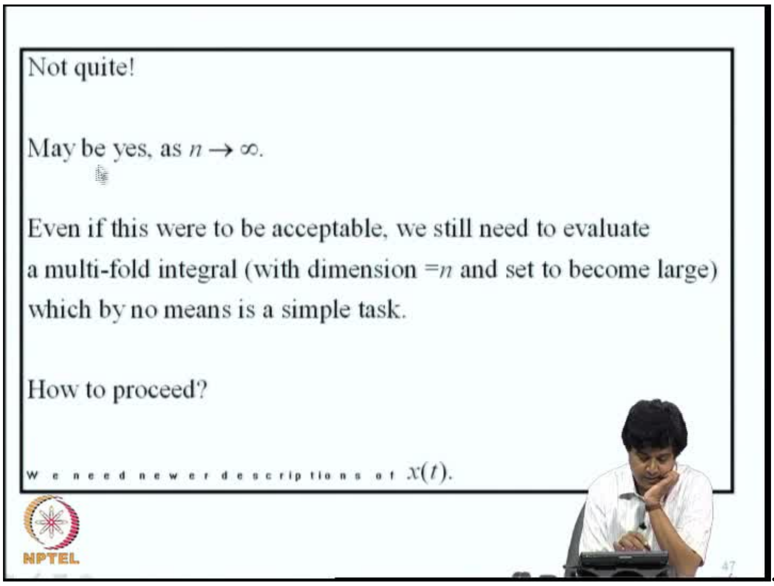

Now, can we approximate this probability can be find this probability by approximating by this integral, that means, a n fold integration on n th order density function, where domain of integration is the intersection of all these x is being less than or equal to alpha; so, it is a n fold integration. Even if you are able to evaluated this, I do not think will be able to approximate, this I means, this would serve as an approximation to this, because this integral will be equal to this, provide at this n goes to infinity; as n goes to infinity, the dimensional of the integral is becoming increasing large, that means, it appears as though that the answer to the question, we posed is yes maybe yes as n tends to infinity; even if this were to be acceptable, we still need to evaluate a multi fold integral with dimensional n and that n set to become very large. I think this is by no means, is a simple task. So, we seems to reach a dead end here, where we seems to have manage a complete description of x of t, but still were unable to answer a very important question on whether the response stays below a critical value over a given duration. So, this point was a fact that, we need different kind of descriptions of x of t, a join density function of any order still does not same to be adequate. So, we will address this question in the next coming lectures; so, we conclude this lecture here.

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