Stochastic Structural Dynamics

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Lecture No. # 11

Random Vibrations of SDOF Systems-3

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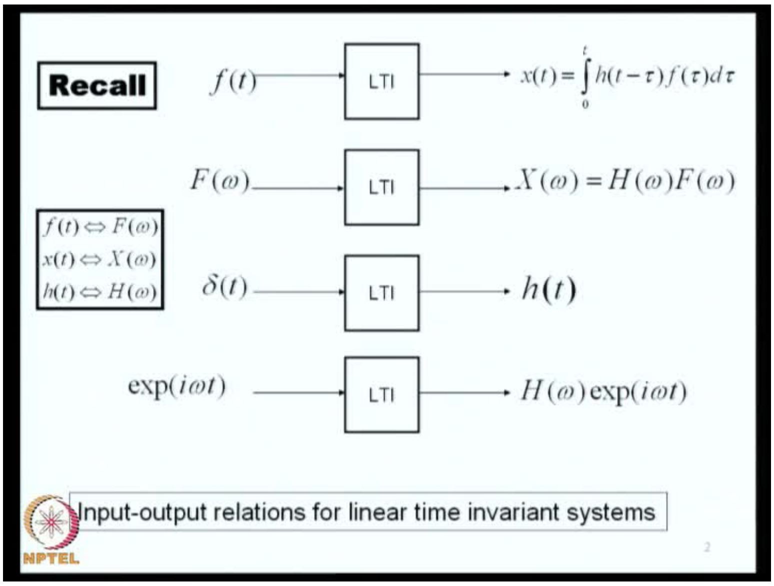

In the previous lecture, we considered the input output relations for linear time invariant systems. We show that, if f of t is the input x of t is the output, it is obtained as convolution between f of t and impulse response function of the system and that is representatives with this; this is time domain input output relation. If input is specified in terms of its Fourier transform, the output Fourier transform will became by multiplication of the frequency response function H of omega and the Fourier transform of the input; so, convolution operation in time domain becomes multiplication in frequency convolution. The definition of impulse response function itself is shown here; h of t is the response of the system to a unit impulse at applied at t equal to 0. In similarly, the frequency response function is the amplitude of the steady state response of the system to a harmonic excitation; this is H of omega. This f of t and F of omega form Fourier transform pair x of t and X of omega form Fourier transform pair; similarly, h of t and H of omega form Fourier transform pair. So, this in a nutshell, the summary of input output relations for linear time invariant systems.

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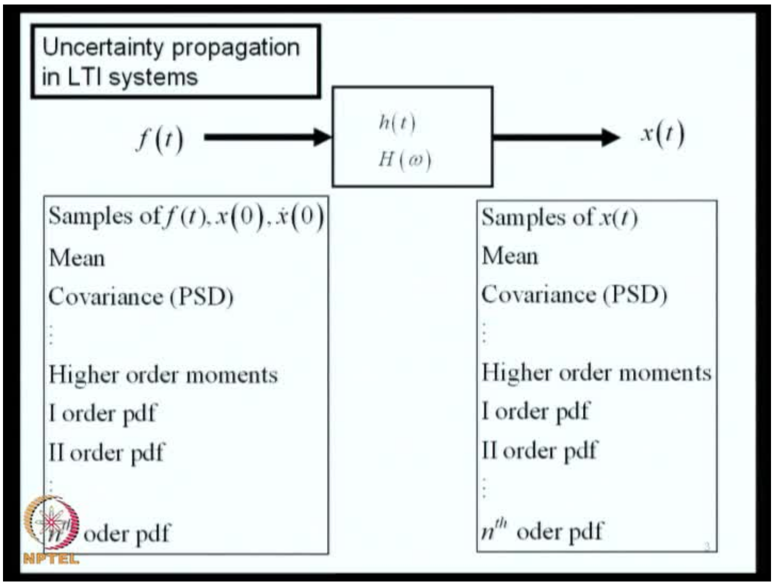

We also considered the problem of uncertainty propagation, if f of t is a input and if f of t is a models as random process, then for every realization of f of t, there will be a corresponding realization of x of t, that can be obtained for instance via convolution between impulse response function and f of t. So, if we are given samples of f of t and then the initial conditions, the problem of uncertainty propagation consist of finding corresponding samples of x of t; this would mean that, we may have to solve a family of deterministic problems, each of the this problems corresponds to one realization of f of t to produce one realization for x of t, that if f of t is specified in terms of, it is new, how one derives a mean of the response, in terms of mean of f of t, is it possible, if so, how to do it.

Similarly, if one, we has description of covariance or the power spectral density function in the frequency domain, then correspondingly how to define the covariance of the output, can we define the power spectral density for x of t, if we can define how to find it? Similarly, such questions can also be as with respect to higher order moments or for the probability density functions, for example, if we are given the first order probability density function of f of t, what is the first order probability density function of x of t? So, the similarly, you can repeat this question for nth order p d f for f of t and corresponding nth order p d f of x of t. So, basic fact that has to be emphasized here, is that, the propagation of uncertainty in such systems follows allows of mechanics, that has to be understood, because the output is obtained as a transformation of f of t; therefore, essentially we are making transformations on random quantities, and as we already seen, there are rules for handling such transformations of random quantities and they come into play while solving this problem.

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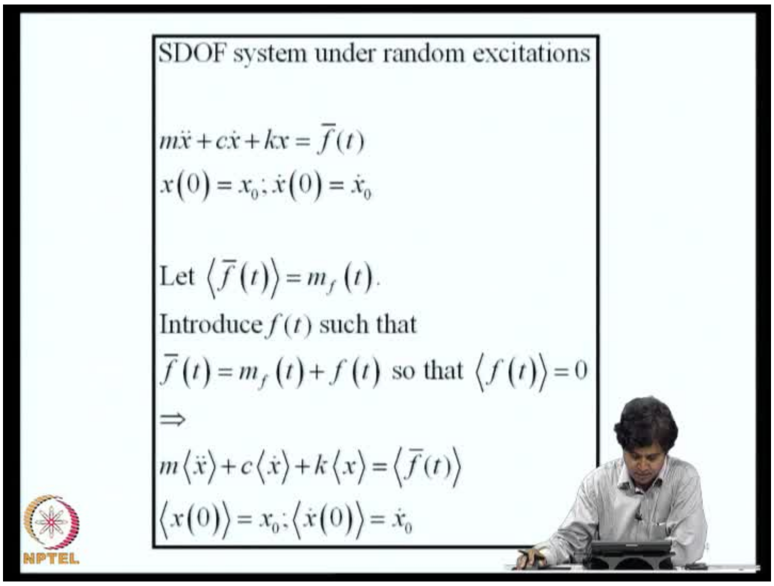

Now, we reconsider the problem of a single degree freedom system under random excitations. Let f bar of t be a random excitation and let x naught and x naught dot to the initial conditions; let mean of f bar of to be m f of t, the what we do, introduce the f of t, such that f bar of t can be written as some of the mean component and fluctuating component, so that, this mean of this fluctuating component can be 0. Now, before take expectations on both sides on this equation, we get m of x double dot plus c of expectation of x dot plus k into expectation of x, this must be equal to expectation of f bar of t; similarly, the expected value of initial conditions will be x naught and x naught dot. So, never analysis we are assuming that x naught and x naught dot or deterministic, they can also the random for the time during were not including that possibility.

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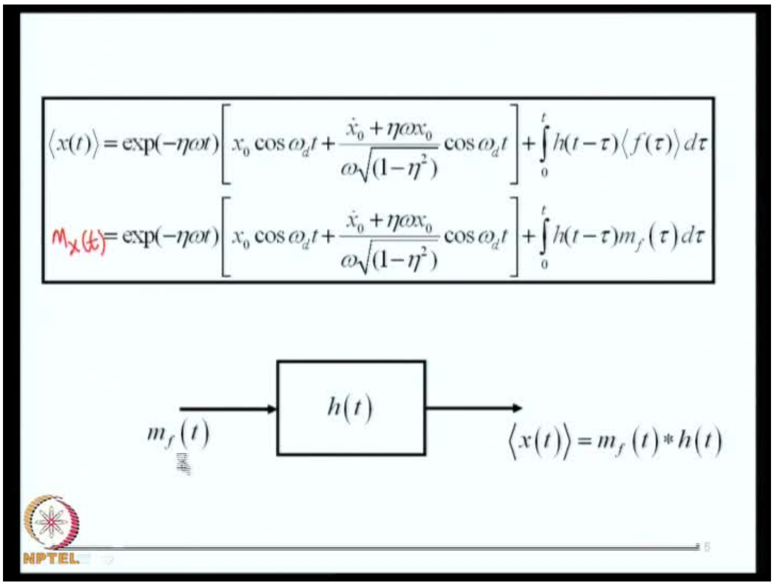

Now, this is the equation that girths the mean and this is deterministic equation. So, we can use the standard formulation, and write the expression for time history of this mean, in terms of initial condition x naught x naught dot, and this Duhamel's integral in terms of mean of f of t; so, this expected value of f of tau is m f of tau and this is nothing but m x of t. Thus we can consider this a equation exclamatically as shown here, what goes into the system, is the expected value of f of t and what comes out is the convolution between h of t and m f of t to produce the expected value of x of t. So, this is the input output relation for the expected value of their response, in terms of expected value of the input.

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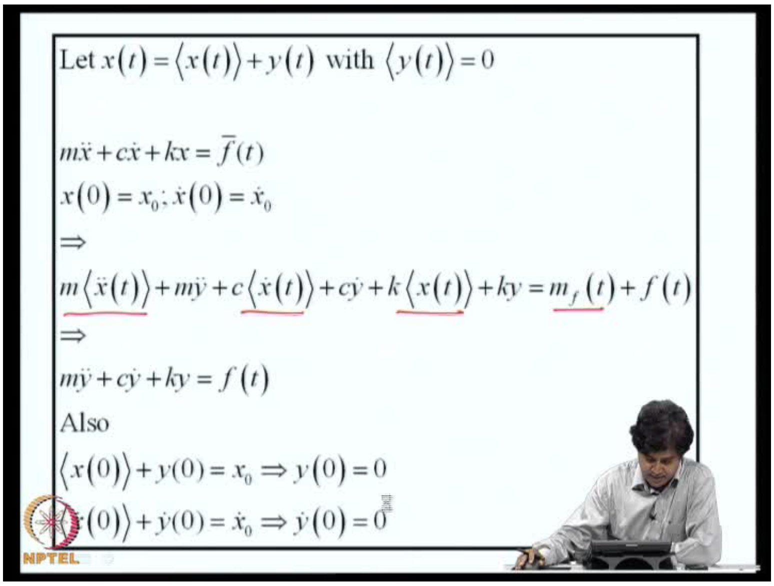

Now, just we wrote f bar of t to be f of t plus its means, we can also right the mean the response process itself to be some of its mean and then random fluctuating component whose mean is zero; suppose, now I substitute this into the only equation, I will have for x of t expected value of x double dot of t plus y dot of t, that is what written here, similarly for c x dot, I will have c into expected value for x dot of t plus y dot of t, so that is here, similarly for k y k x, I will have k into expected value of x of t into k y, this is this, and for f bar of t I will write m f of t plus f of t. We already seen that the some of these terms, terms is equal to this, because that is the governing equation for expected value of x of t; therefore, we can cancel out these three terms and what were left is, left worth is m y double dot plus c y dot plus k y equal to f of t. Similarly, the initial condition, if now I consider expected value of x of 0 plus y of 0 is x naught to be already seen that expected value of x of 0 is x naught, this should mean the expected value of y of 0 will be 0 and by the same argument expected value of y dot of 0 is also 0. So, for this random component y of t, we have now in equation a right hand side is a random process will 0 mean and this system starts from rest.

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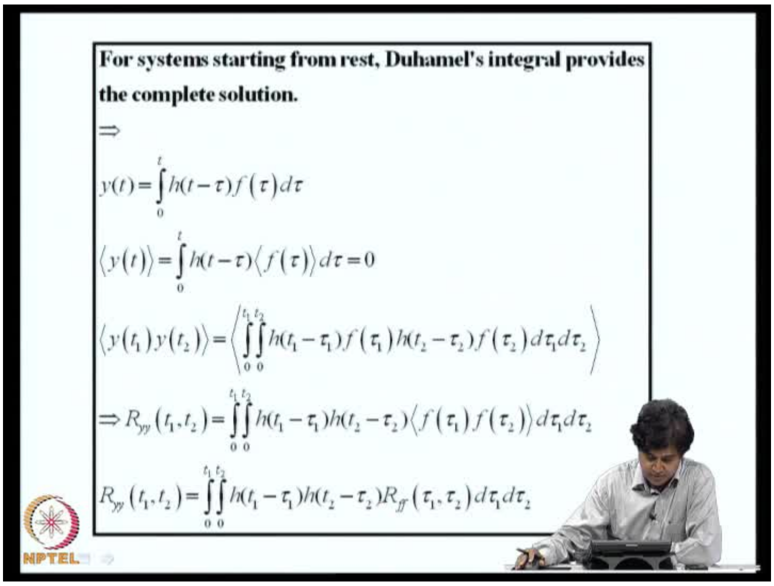

So, for such systems, that is for systems starting from rest, the Duhamel's integral provides the complete solution. So, y of t can be written as 0 to t h of t minus tau f of tau d tau; just verify, you can take expected value of y of t, that is expected value of f of tau this is 0; therefore, this integral is 0. How about second level of properties, now you consider expected value of y of t 1 into y of 2, so we have to multiply the Duhamel integral corresponding into y of t 2 with another Duhamel integral corresponding to y of t 2; so, if you do that, we get double integral, were now the integration is on tau 1 to tau 2 and this t 1 and t 2 appear in the integrant here, in the integrant here, as well as in the arguments here. So, the expectation of operator assuming that, this integration and expectation operator can be interchange, if we do that we will be left the expected value f of tau 1 into f of tau 2; this is deterministic so this can be pulled out. So, we get now here the expected value of the response process, the covariance of the response process in terms of covariance of the input process. So, this can be viewed as the input output relation for covariance of the output in terms of covariance of the input.

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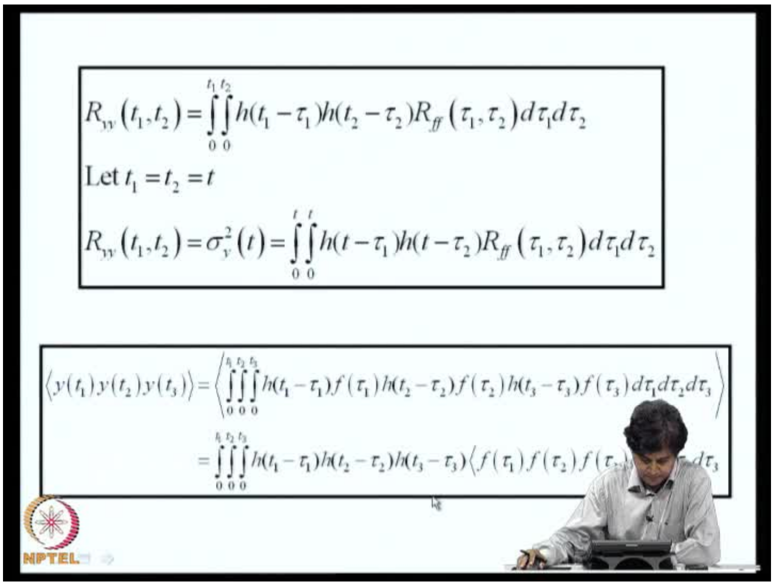

Now, in the expression for r y y (t 1, t 2), if I take t 1 equal to t 2, I get r y (t 1, t 2) is variance, mind you, mean of y is 0; therefore, autocorrelation function and auto covariance function are the same. Therefore, t 1 equal to t 2, the mean square value and variation will be the same, so that integral right hand side here now becomes wherever t 1 and t 2 is replace by t and I get this expression. So, one thing that we can notice immediately is, if the auto covariance of the input, you can find the auto covariance of the output, that means, to find the auto covariance of the output, you need to know the auto covariance of the input; if only the variance of the input you will not be able to find variance of the output, so to find variance of the output you need to know the covariance of the input, that has to be understood. Now, the autocorrelation function is the second order property, we can consider higher order moments, so we can take expected value of y of t 1 y of t 2 y of t 3, this will be a triple integral and now expectation will be taken on f of tau 1 f of tau 2 and f of tau 3, and this, this integral can be rearrange in this form, so I have in the integrant expected value of f of tau 1 f of f of tau 2 into f of tau 3, so this is the third order moment of f of t, if the third order moment of f of t you can derive the third order moment of the response.

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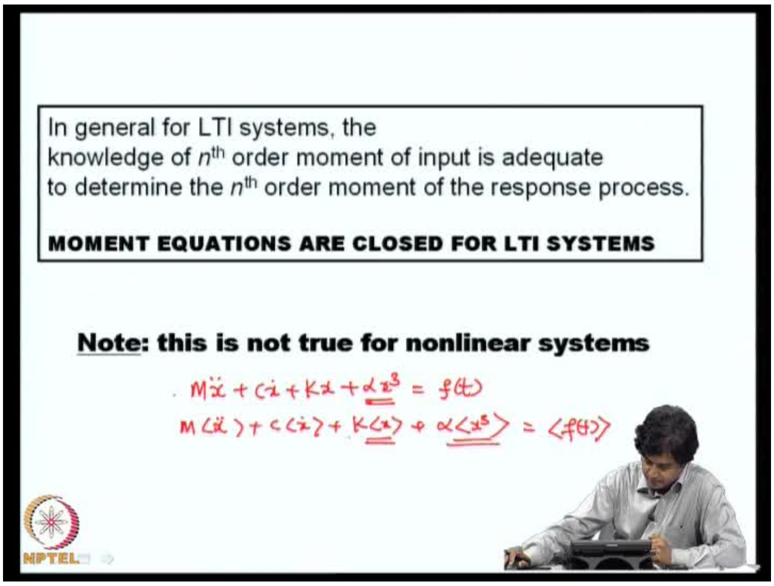

So, we can generalize this now, and say, in general for linear time invariant systems, the knowledge of nth order moment of input is adequate to determine the nth order moment of the response process; this what mean, the moment equations are closed for linear time invariant systems; this is not in general true for dynamical systems, for instance, if you have a non-linear system, this is not true, this can be see if you consider a simple non-linear system, where I have now a cubic non-linear term, if I know, the consider the expected value of the response, this will be expected value of x double dot plus c into expected value of x dot plus k into expected value of x plus alpha into expected value of x cube is equal to expected value of f of t. Now, you please notice here the equation for expected value of x contains the expected value of x cube, if you now try to write a another equation for expected value of x cube, for instance, you can multiply this equation by x square of t, x square of tau, for example, and take expectation on both sides, you will immediately see that, you will need, if do that you will need expected value of x cube of t into x square of tau again which will be unknown. So, the moment equations here you will never get closed, they will form in infinite hierarchy of equations which at most is in expectable set of equations, which can result in resolution for the moments. So, this is known as a closer problem, later when we discuss some issues about the non-linear random vibration problems, we will briefly touch upon this again.

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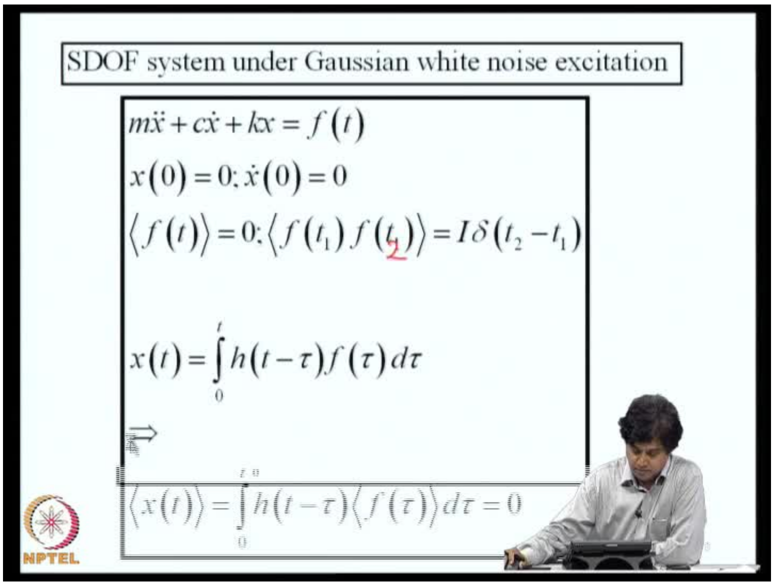

Now, to illustrate the idea that we just know describe, we will consider now the response of the single degree of freedom system, under Gaussian white noise excitation. The governing equation here is again m x double dot plus c x dot plus k x equal to f of t, we will assume that system starts from rest and the expected value of f of t 0. Without lose of generality, these assumptions can now be made, that mean of the applied force is 0 and initial condition have 0, that means, the system from rest and excitation has 0 mean. If they, if any of these conditions are violated, we can solve a separated deterministic problem and find out the contribution from non-zero mean for excitation and non-zero initial conditions, that, that is reasonably straightforward. So, we focus only on issues related to randomness in f of t, so the expected value of f of t is 0, since we are saying that f of t is the Gaussian white noise and the expected value of f of t 1 into f of t 2 is i into direct delta of t 2 minus t 1. Now, x of t can be written in terms of the Duhamel integral, again as shown here, now if you consider the expected value of x of t expected value of x of t is expected value of f of tau, will come here Duhamel integral, this is the given to be 0; therefore, this is 0. This system starts from rest; therefore there are no other terms in your expression for x of t.

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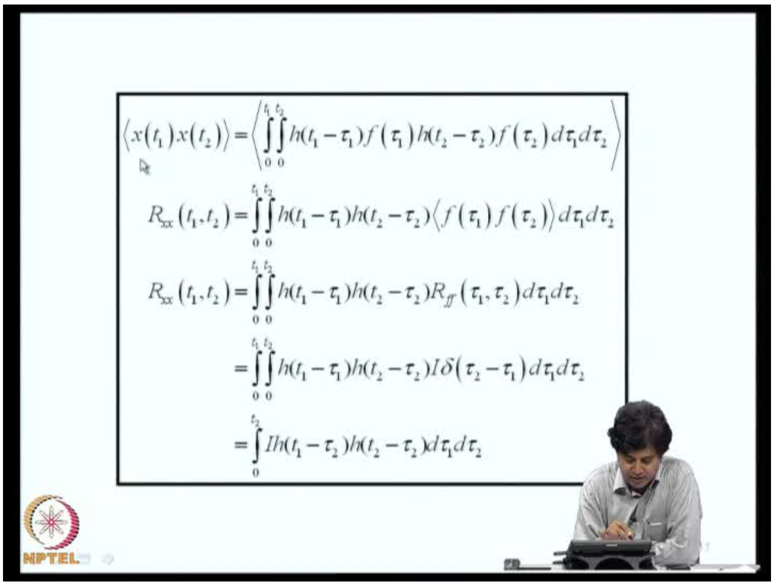

How about covariance of x of t, so you now consider expected value for x of t 1 into x of t 2, so this is the expected value of this Duhamel integral; here we can get the expected auto covariance of the output, in terms of auto covariance of the input, in this form. And for r f of tau 1, tau 2, now I will write the expression for auto covariance of a white noise, which is i into direct delta of tau 2 minus tau 1. So, one integration can easily be perform, I can replace tau 2 by tau 1 and I get h of t one minus tau 2 h of t 2 minus tau 2 and then 1 and only integral remains that is the integration with respect to tau 2; so, this is 0 to t 2 and 1 integral with respect to tau 2.

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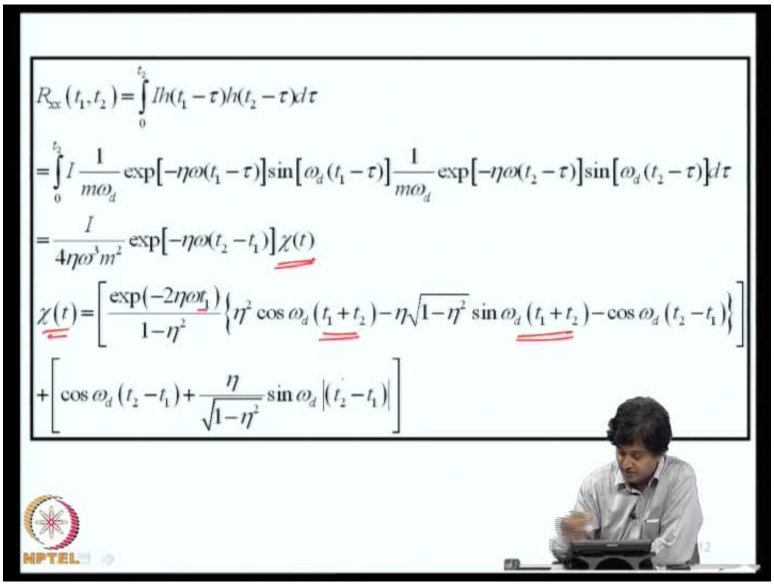

Now, h of 2 is given by this damped exponential a sinusoidal with exponentially decaying multiplier, so the 1 by m omega d exponential minus eta omega t 1 minus tau etcetera. So, this can, in fact be integrated within straightforward application of rules integral calculus, and we get this expression, then this says I have written as slightly form that can help us to analyze the nature of this function. So, there is function chi of t sitting here and this chi of t given by this expression, and then we can multiplier which is exponentially minus eta omega t 2 minus t 1. If I now carefully look at this expression, we can see that r x x (t 1, t 2) this is actually the function of both t 1 and t 2, because there is t 1 plus t 2, where the t 1 here and the sign omega t 1 plus t 2 etcetera. So, from this we can conclude that x of t is a non-stationary random process, because 0 mean constant mean, that it has time varying auto correlation function which is not a function of time difference.

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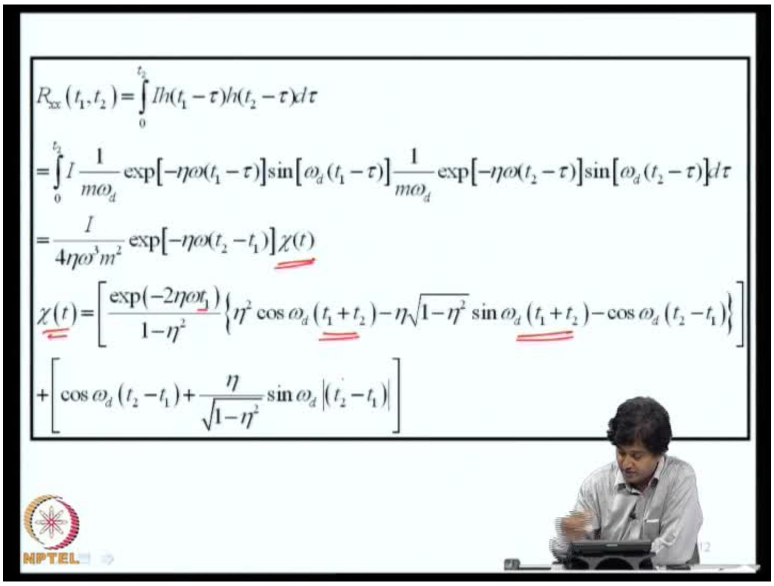

We can consider the variance by replacing t 1 equal to t 2 equal to t, and if I perform this integration, we get this expression. Here again you can see that variance is the function of time, therefore again it points towards fact that response is non-stationary. What would happened now, if for example, in the expression for the variance time becomes large, if we take look here, the term inside these braces is sinusoidal and constant here cosine and sine terms there bounded between plus and minus 1 for all values of t, but this multiplier function exponential minus 2 eta omega t ensures that as time become very large and the product of this function and the terms inside the braces go to 0 and we are left with this one constant term; so, the variance that is here still becomes large, becomes a constant; how about auto covariance and you now take t 1 to infinity, t 2 to infinity, but t 2 minus t 1 to be tau.

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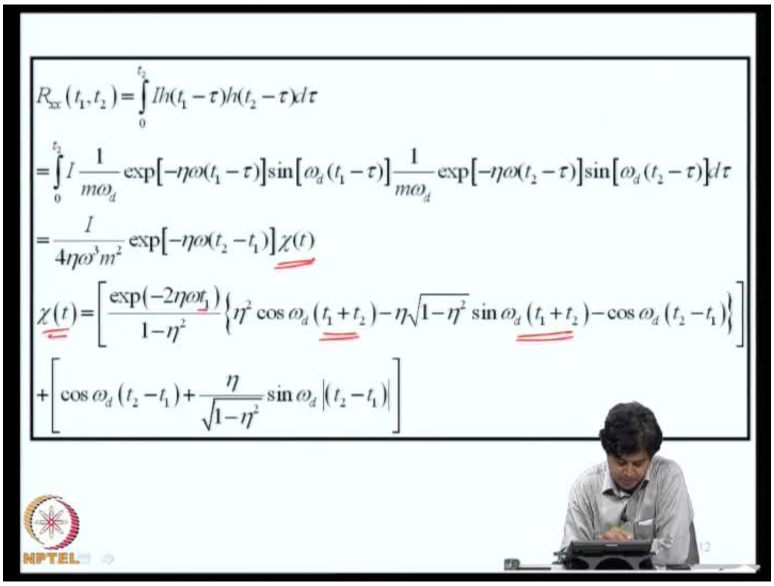

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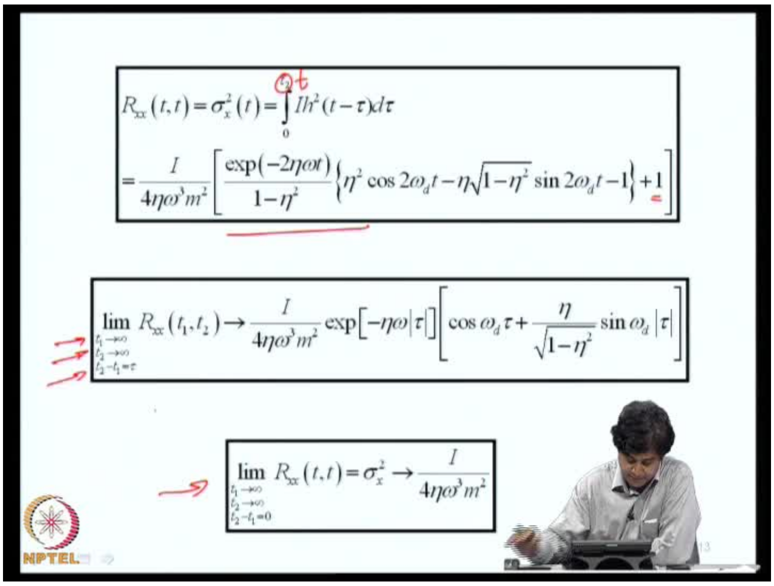

If we put the limit, what happens, here as t 1 goes to infinity the multiplier to these terms inside this brace goes to 0, where as the terms inside the second bracket, I have sin in cosine and also they have their functions of t 2 minus tau 1, therefore t 2 minus tau 1 is going to tau; therefore, I get the auto covariance function to be function of only tau. And therefore, we can conclude that as time becomes large, the process becomes weakly stationary.

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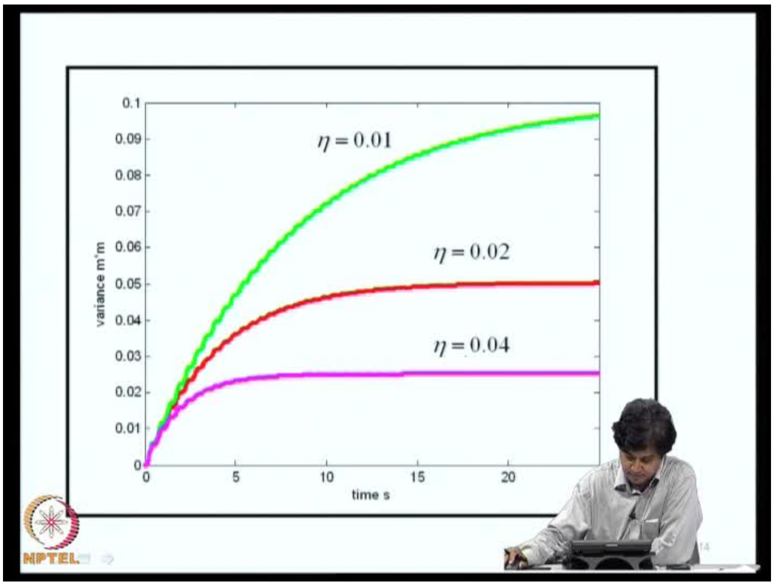

So, I have shown a few plots of auto covariance and variance, is a plot of variance system starts from rest and this is for different values of dumping parameter, so as you see here time becomes large, after some time there is a initial growth and after some time the response which is a constant value, the variance becomes constant. The steady state value, that is the steady state value of the variance different on damping; so, lesser than damping higher will be the response variance and not only that, the time taken for this system to reach steady state increases with reduction in damping, because the function exponential minus 2 eta 2 omega t, this case relatively more slowly for lesser value of the eta. So, any case as long as eta is not equal to 0, the steady state exist and variance becomes constant; if damping is 0, this system will never reach steady state.

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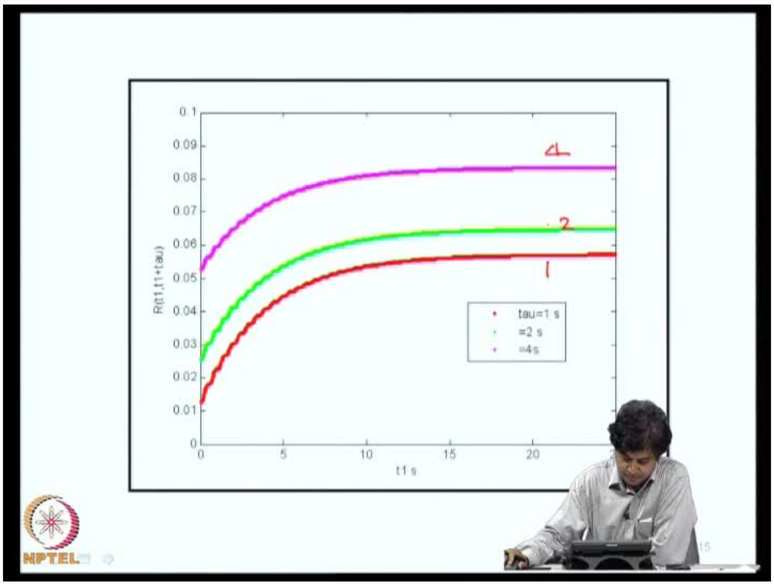

Here, I have shown plot of auto covariance function t 1, t 1 plus tau as the function of t 1 and for different values of time lags. So, here again what happens for small values of t 1, the function is varying with respect to time, but for large values of t 1, this function is also becomes constant. So, let this for large values of t 1, the auto covariance function simply, because the function of tau, this is tau equal to 1 second and this is 2 seconds, this is 1 second, this is 2 second, this is 4 second, so it reaches the different values.

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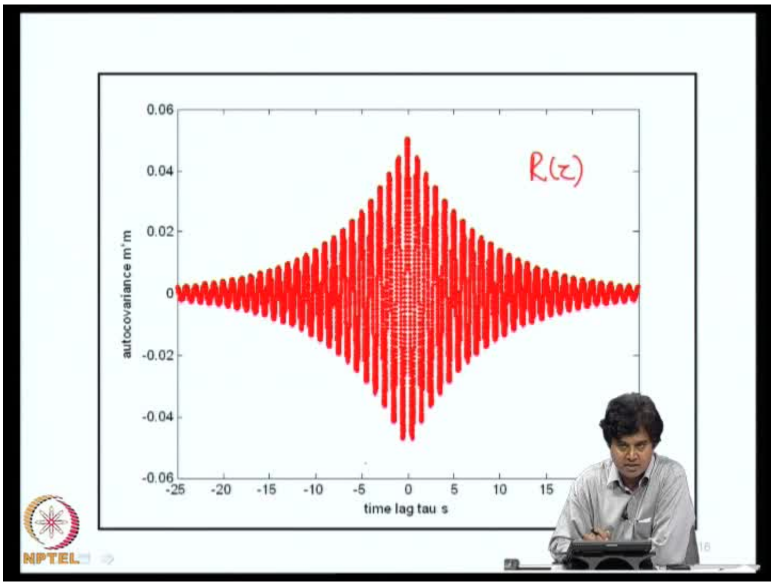

This is the plot of R of tau, as a function of tau, you can see that, this is the shown for both the positive and negative time lags, the you can easily see that, this function is symmetric about i equal to 0 and it represents a decaying sinusoidal function. So, this is the auto covariance of the output in steady state; so, the process is stationary in steady state.

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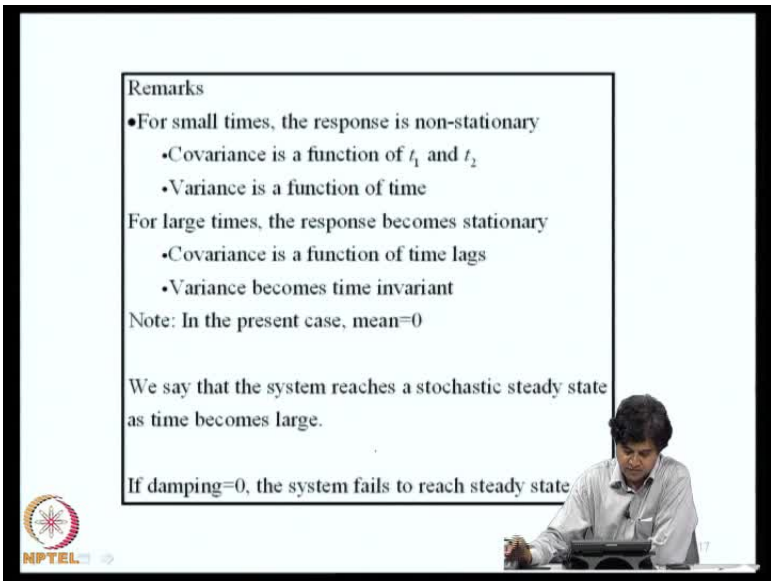

So, we can summarize now, for small times the response is non-stationary, the covariance is the function of both t 1 and t 2 and variance is function of time. For large times the response becomes stationary, that could be the variance of time lags and variance becomes the in variant; in present case, mean always 0. When the covariance becomes function of time lags and variance becomes constant, we say that, this is the time to reach the stochastic steady state. If damping is 0, this system fails to reach the steady state. This motion of transient state and stochastic steady state is synonyms or in analogs to the motion of periodic steady state and transient state and harmonically driven the single freedom systems, if you recall for small times, single degree of freedom systems and harmonically, there response will be periodic, thus the time becomes large the response will becomes harmonics, with this amplitude and phase becoming independent of time. So, there is an analogy between stochastic steady state and harmonics steady state.

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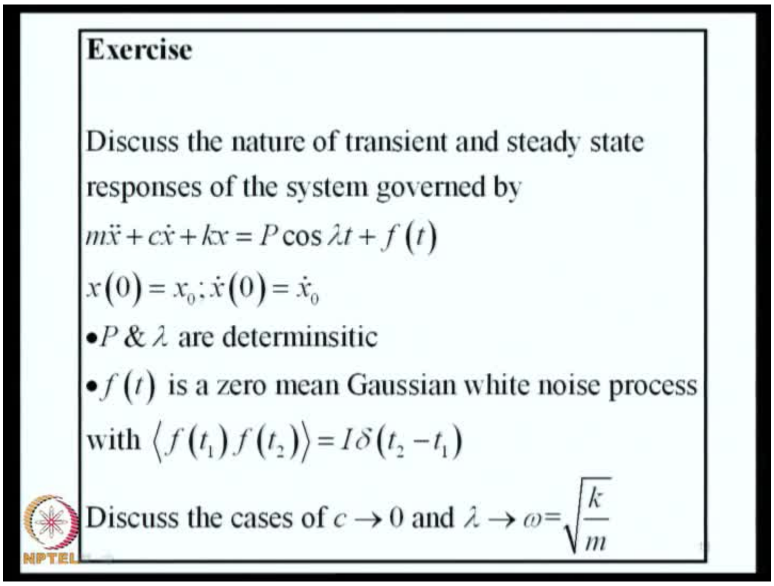

So, I leave this is an exercise, you now consider the single degree of freedom systems, which is driven by a combination of a harmonic load and then random process f of t with non-zero initial conditions. So, this P and lambda deterministic; therefore, P cos lambda t is a mean of the excitation process, but f of t is 0 means Gaussian white noise process. So, with this covariance, now the question is you need to discuss the nature of steady state response of x of t; so, it has both components half harmonic steady state and stochastic steady state, so you have to see if there exist any interaction between them and what exactly happens as damping becomes 0 and the driving frequency goes to the nature frequency of this system. So, what are all the issues if you discuss, you would appreciate the points being made in the context of excitation under system, under a stationary excitation.

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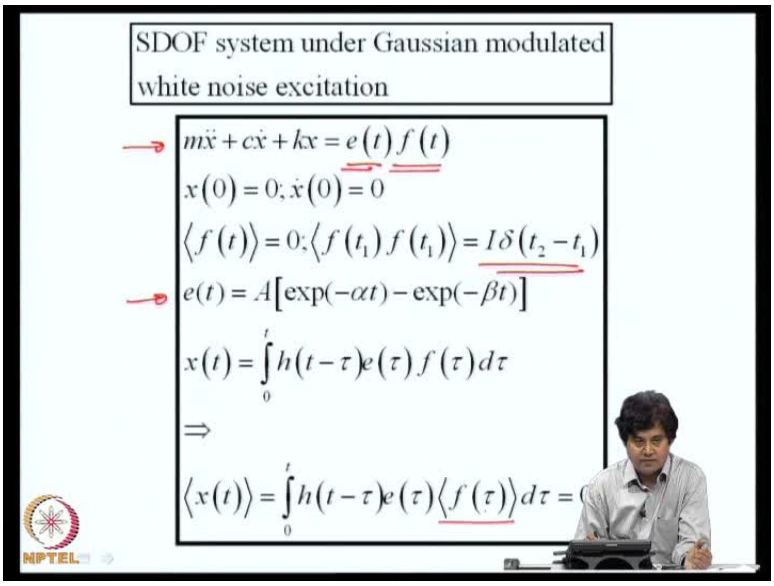

Now, what happens if the excitation is non-stationary; so, if you consider now, is a single freedom system and under a Gaussian modulated white noise excitation, by that mean, the excitation is obtained by multiplying a 0 mean Gaussian white noise process with modulating function or envelope function e of t; this e f t is given here, it is some of t exponentials and f of t is a stationary white noise. The right hand side here although it has stationary component, the process itself is non-stationary. Now, what is x of t, the system starts from rest, therefore Duhamel's integral we gives the complete solution, therefore, this is 0 to t h of t minus tau e of tau into f of tau t tau. What do you expected value of x of t, it takes expectation on both sides e of t is the deterministic, so I get the expected value of x of t is 0 to t h of t minus tau e of tau which is deterministic, therefore the outside, this expectation operator and we have expected value of f of tau e of tau equal to 0, because expected value of f of tau is 0.

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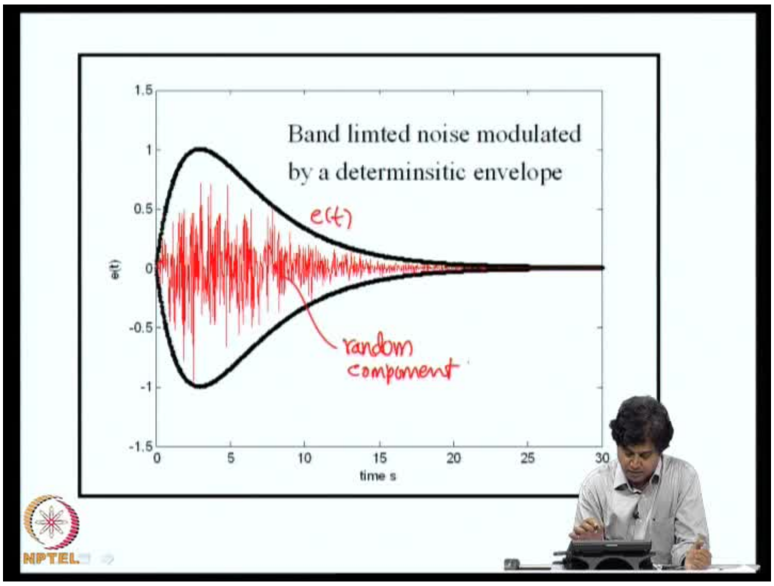

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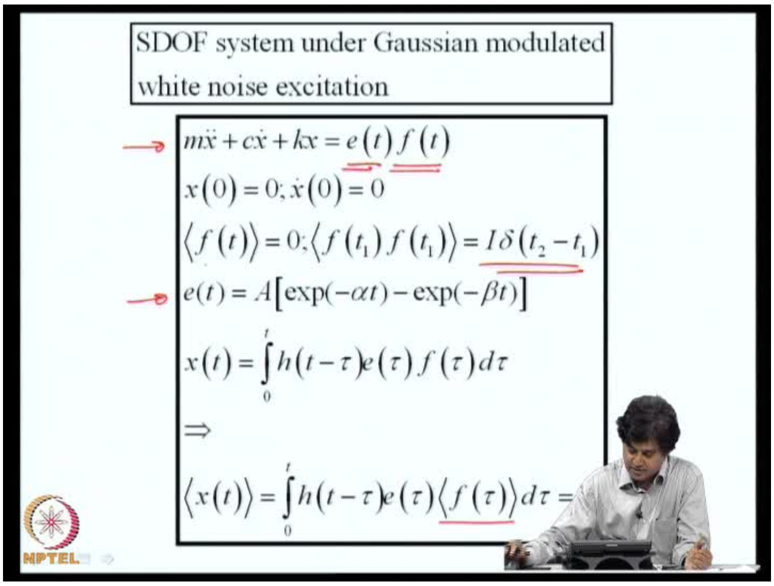

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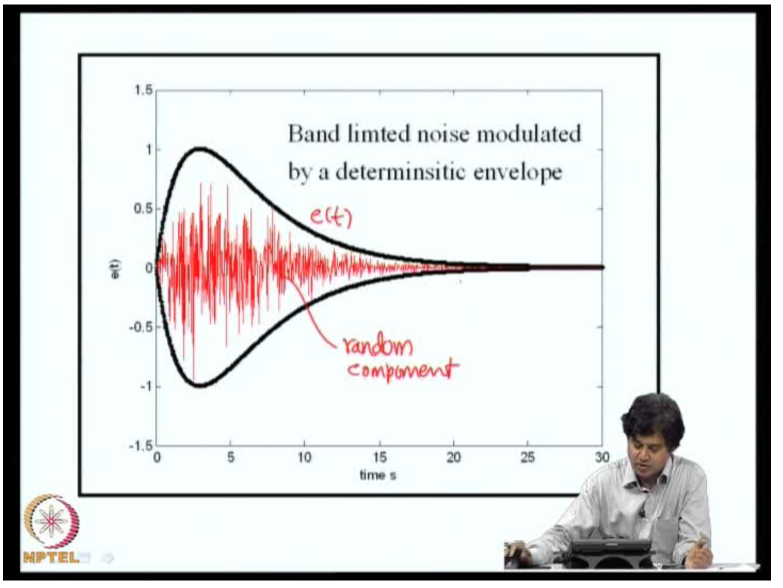

This is a kind of a notional representation of the excitation process; this practical line this is the e of t and this is the random excitation. This actually strictly speaking does not corresponding f of t being the white noise, because white noise as unwanted variance, we cannot really that is not physically releasable. So, what is shown in the red here, can be viewed as a band limited noise, where the band width is sufficiently large and therefore we can notionally represent that; so, in any case, in excitation looks like is red color.

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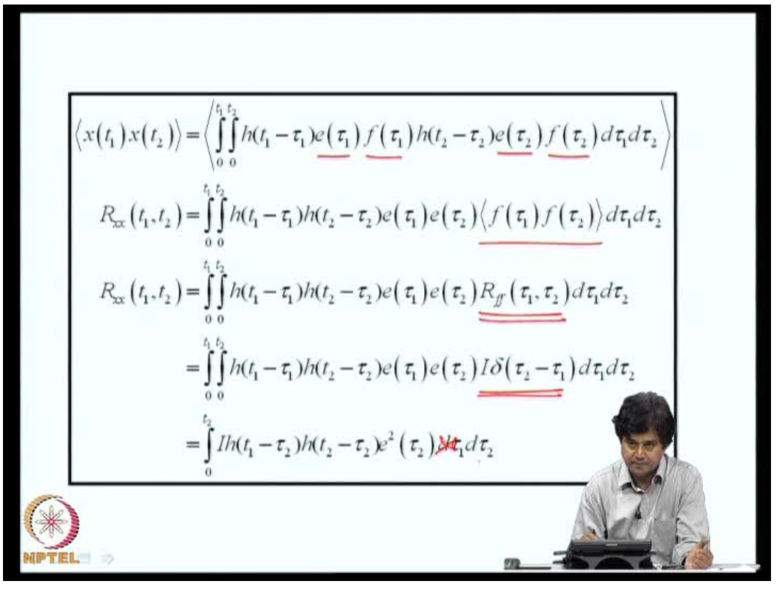

Now how does a covariance looks like, if we now take a expectation x of t 1 and x of t 2, there will be a expectation of double integral and I will have a e of tau 1 and e of tau 2, f of tau 1, f of tau 2, rearrangement of this will lead to expression for the output auto covariance in terms of input and auto covariance f of tau 1, tau 2, this is nothing but i into delta of tau 2 minus tau 1. Again, we can carry out the one of integrations by replacing tau 2 to tau 1 and we are left with an integration with respect only tau 2.

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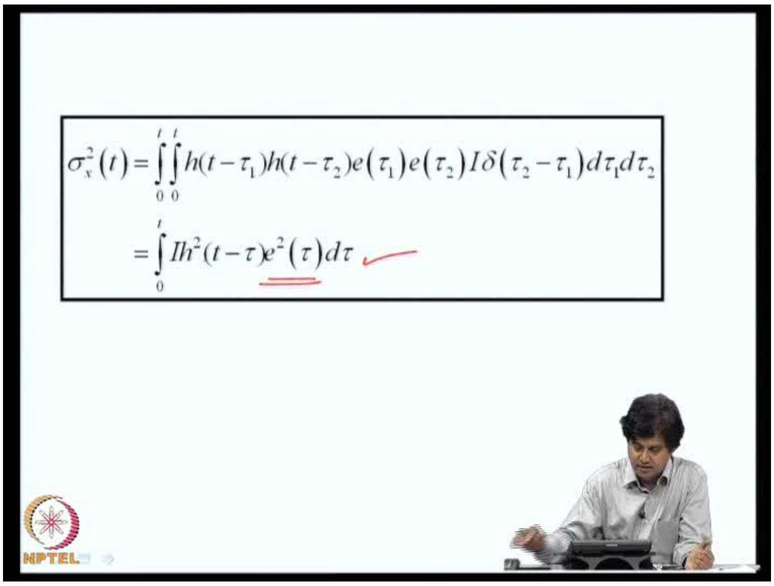

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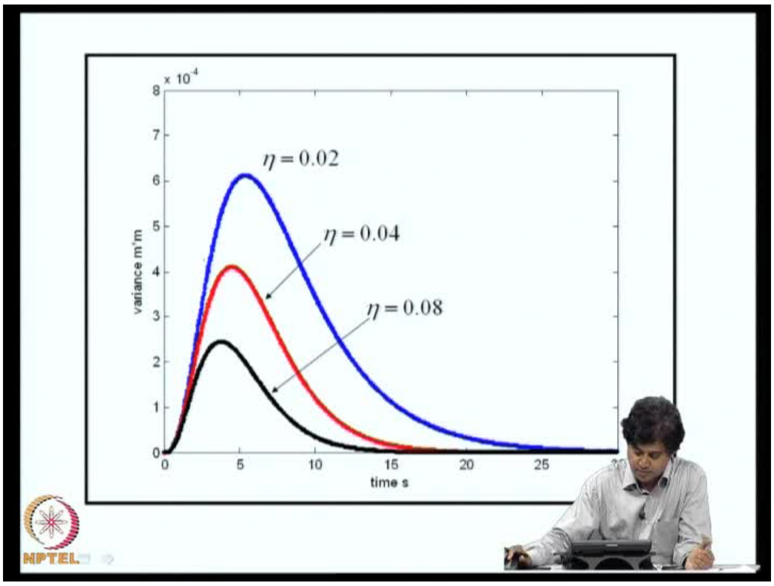

This again can be integrated, if the model for e of tau 1 we can easily evaluated this integral and if we put t 1 equal to t 2, we get the expression for the variance and this is the expression for variance. Now, we have an integrant, not only the h square of t minus tau but also multiple of e square of tau, h of t again t 1 in terms of sin in exponential function. We can plot, we can derive the expression for variance of the output and I will as shown here, the plots of variance of the response for the different values of eta. And we can easily make out, that the response is the non-stationary random process, this is to be the expected, because input itself is non-stationary; so, naturally the output is very likely to be the non-stationary that indeed is what is observed through the analysis.

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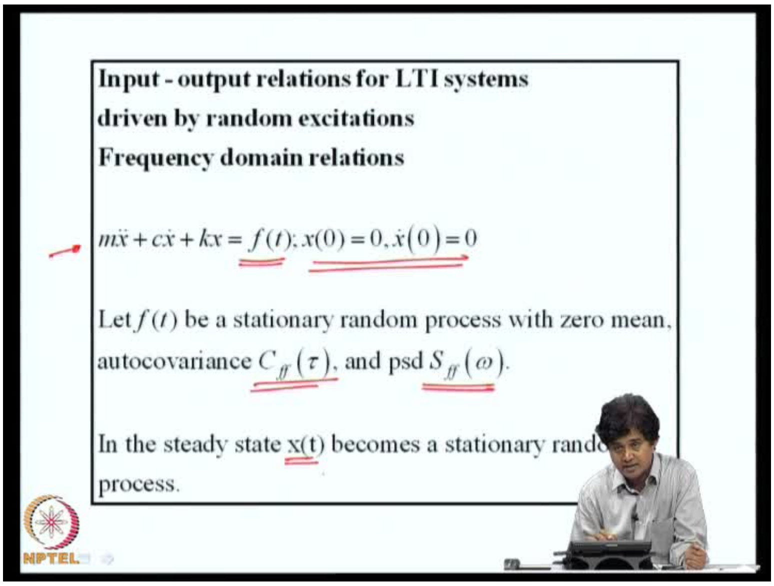

We now move onto the description of input output relations, in terms of frequency domain description of the system. So, we consider again the system driven by f of t, where f of t is the non-stationary random process. As we saw the motion of power spectral density function is valid only for stationary random process, so we are restricting input to be stationary and we assume the system starts from rest. So, f of t is the stationary random process is 0 mean and auto covariance c of f of t tau and power spectral density s f of omega and we restricted attention only to steady state response of x of t. As we already seen in this steady state, x of t is the stationary random process and therefore I can talk about is auto covariance, as well as it is power spectral density function. So, the question is now am going to discuss is, what happens to the power spectral density of the response? There are different ways of solving this problems, one is to right the develop, the model for auto covariance of x of t as function of t 1 and t 2 and the allowance the limits of t 1 and t 2 go to infinity with t 2 minus t 1 to be finite, you will get the auto covariance of the output, then you can consider the Fourier transform of the auto covariance.

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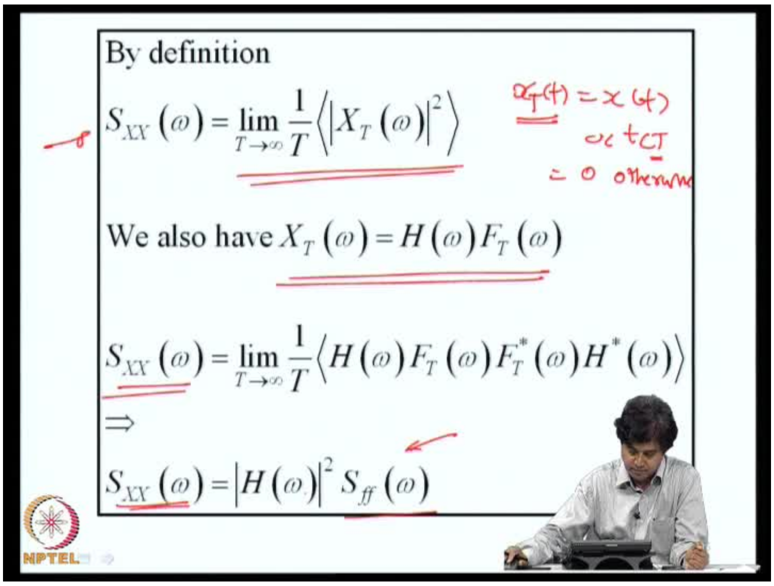

A simpler alternative would be to consider the definition of the power spectral density function, assuming the system is already in steady state, a transient have been dissipated and system is responding to the steady state, so the power spectral density of X of T is given by this, this is by definition. Now, X T of omega is the Fourier transform of sample of X of T, where capital T denotes the fact, that we are defining a associated function x T of T to be x of t for T, T and 0 otherwise. So, as we already done, we will analyze the Fourier transform of this function and then allow this limit t to infinity. So, this is the input of output relation, for the sample of Fourier transform of f of t and the output x of omega, using this, I can now construct the expression for power spectral density function, their response and this is what I get, h of omega is the frequency response function of the system. So, here we have h of omega into x star of omega, which becomes modules h of omega whole square and I have f t of omega into f t star of omega limit t 2 into infinity on by t h of omega is deterministic; therefore, it gets pulled out of the expectation, and what remains inside, nothing but the definition of power spectral density function of f of t, thus I have known the output power spectral density function S X X of omega, in times of the input power spectral density function S f f of omega and this is obtain by multiplying the input power spectral density function, by the square of the amplitude of the frequency response function of the system.

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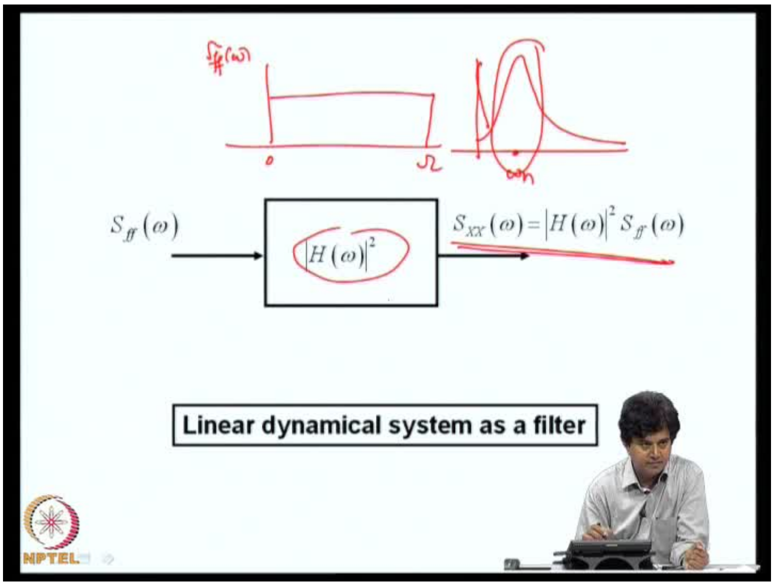

So, that is pictorially depicted here, this is the input and output relation for power density function of this system, this is the output power spectral density, this is input power spectral density and this is system of function, which is relevant for this input output relation. So, it is h of omega whole square which serves as the system function or the transfer function, this should mean that, if we carefully look at that h of omega whole square, it has the peak near the system natural frequency, and if S f f of omega, for instance, if it is a band limited function, the output power spectral density function will be product of these two; so, that would mean, this system will permit only those frequency components in the input, which are near the large values of system transfer function to be manufactured, in the output power spectral density function or in other words, this system behave as a, if it is a filter, to pass the frequency components in the input, selectively through its transfer function.

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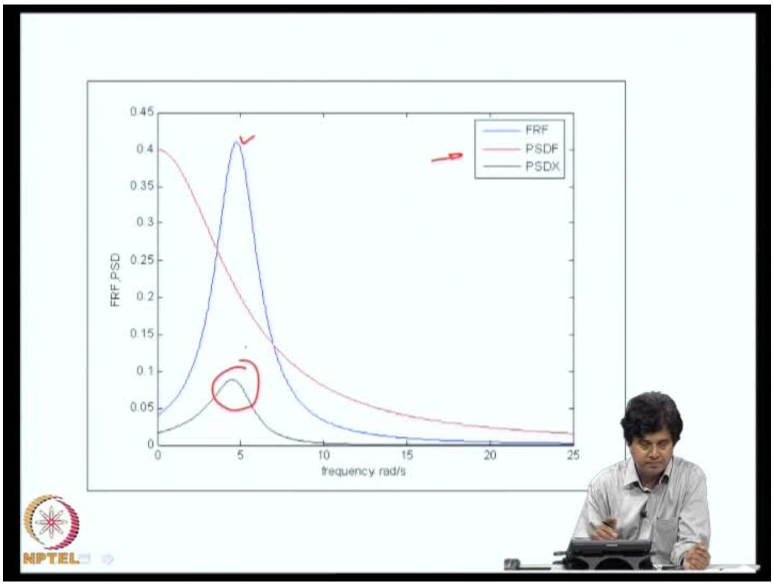

So, to see that we can consider now, a few input output relation, suppose red line is the power spectral density function of excitation, this is this, and blue line is the amplitude of square, of the amplitude of system frequency response function, the product of this will be this. So, in the output most of the power is concentrated in this region, although the input has frequencies, a several other frequencies, the output chooses to have higher power near the system natural frequency, so that sense the linear system is behaving like a filter.

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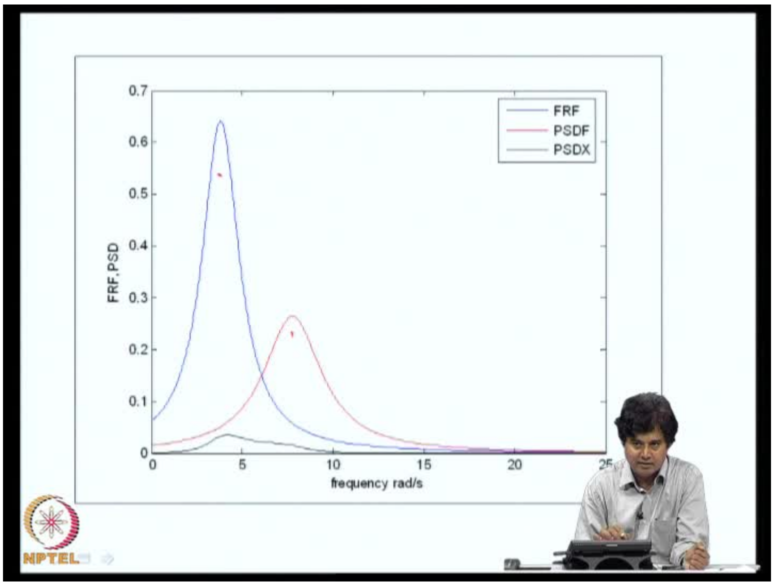

So, this is a different choice of relative values of dominant frequency in an excitation and dominant frequency, in the system transfer function, and output, now peak at a frequency which is neither the, a natural frequency of the system or the dominant frequency of the output. So, this is off resonance, the input and the system transfer function or off resonance.

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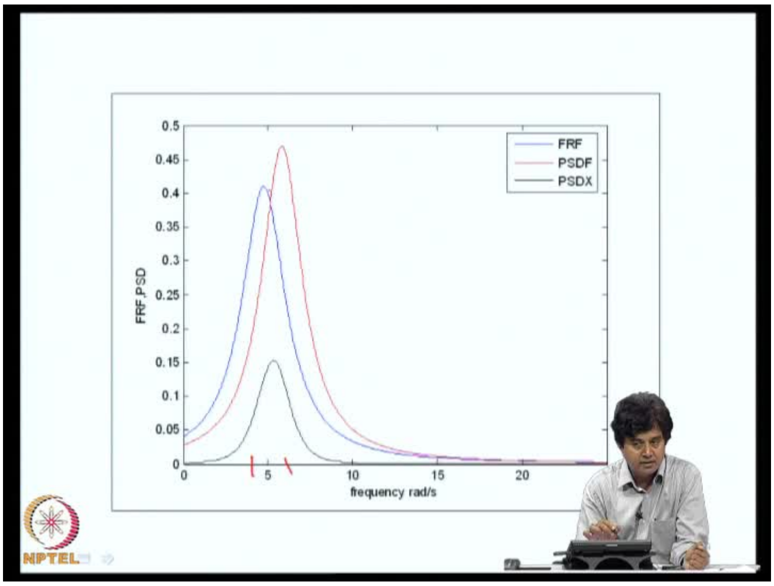

Now, if their peaks come too close to each other, the input as well as the system transfer function, again there substantial increase in the energy, and you can see that, is almost like a resonance condition in deterministic analysis. So, the output power, the average power in output process, tends to get magnify in those frequency regions, where there is a higher energy in higher power, in the input and the value of the transfer function is high.

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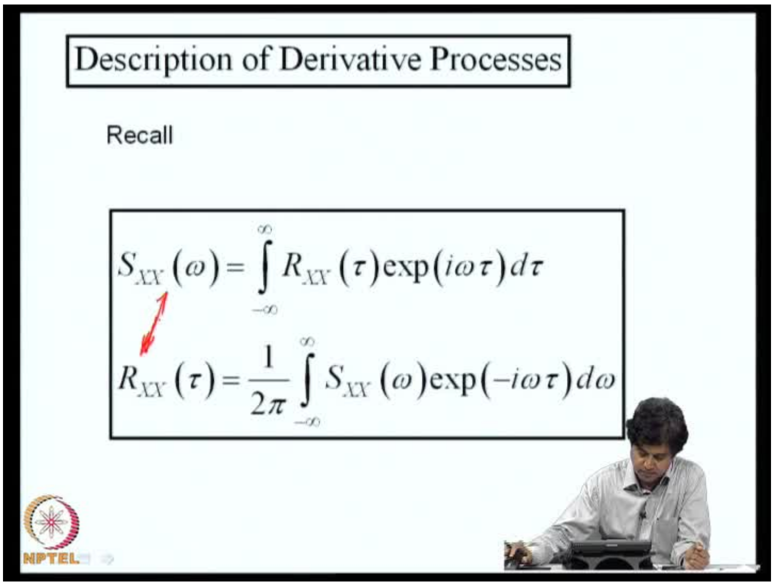

We have been talking response in terms of only displacement, include also as response in terms of velocity or reaction transfer to this support, so on and so forth; so, that would require, now, description of derivative processes, I already introduce the notion of mean square derivative for a random process. Now, let us see what is simplification on frequency domain description; you recall, now the power spectral density of x of t and auto covariance of x of t related through this Fourier transform, this pair of equations; these two functions for a Fourier transform pair.

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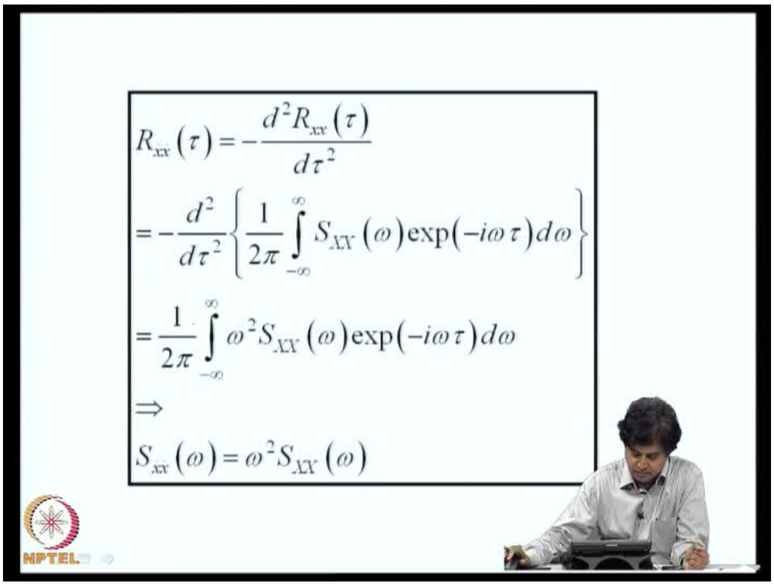

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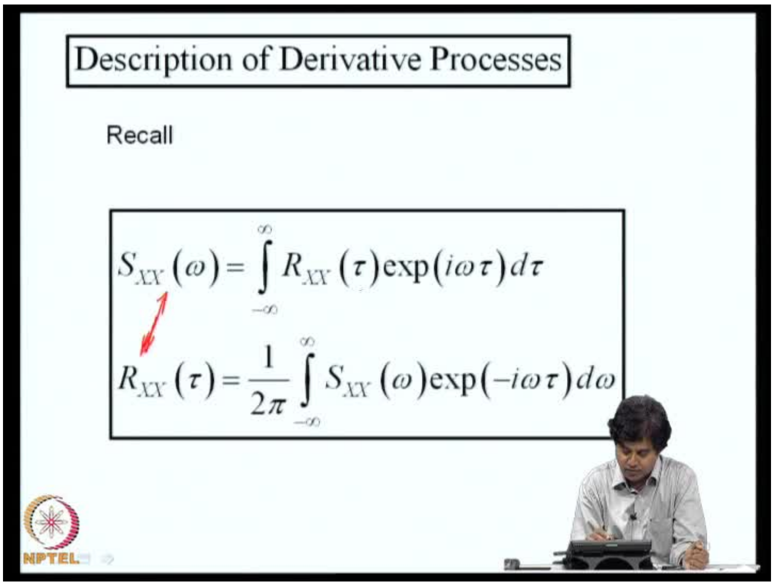

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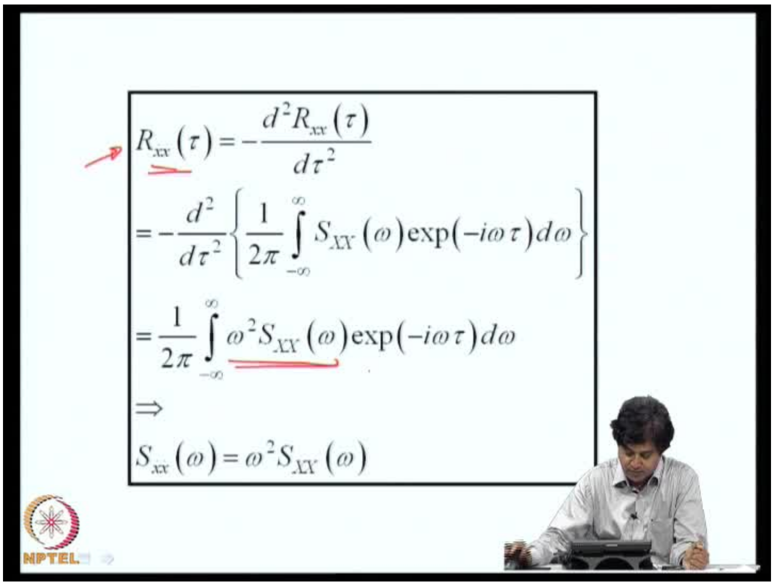

Now, we have already defined the auto covariance of the derivative process can be obtained a differentiating the auto covariance of parent process twice, in this manner. Now, for R x x of tau, I will use the representation in terms of power spectral density function, and if I now implement the differentiation, I get R x dot R x dot tau to be 1 by 2 pi minus infinity into plus infinity omega square S x x of omega into exponential i omega t d omega. Now, if you compare this worth, the definition of Fourier transform pair, you will see that the Fourier transform of the auto covariance of the derivative process is nothing but a power spectral density would be omega square S x x omega; this is the power spectral density function of the derivative process.

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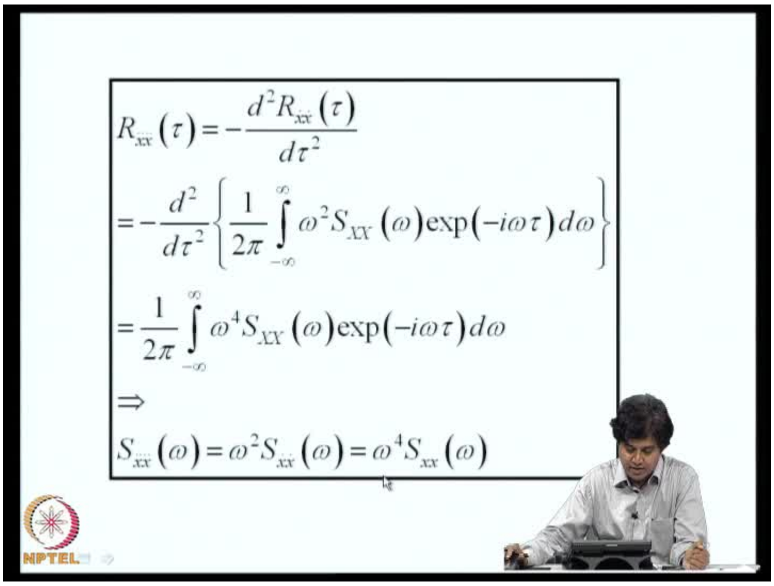

Similarly, you can consider the auto covariance of acceleration process, if x of t displacement x double dot will be acceleration; this will be derivative second derivative of the auto covariance of the velocity process. So, using a similar logic, you can show that, the power spectral density function of the acceleration process is given by omega to the power of 4 S x x omega.

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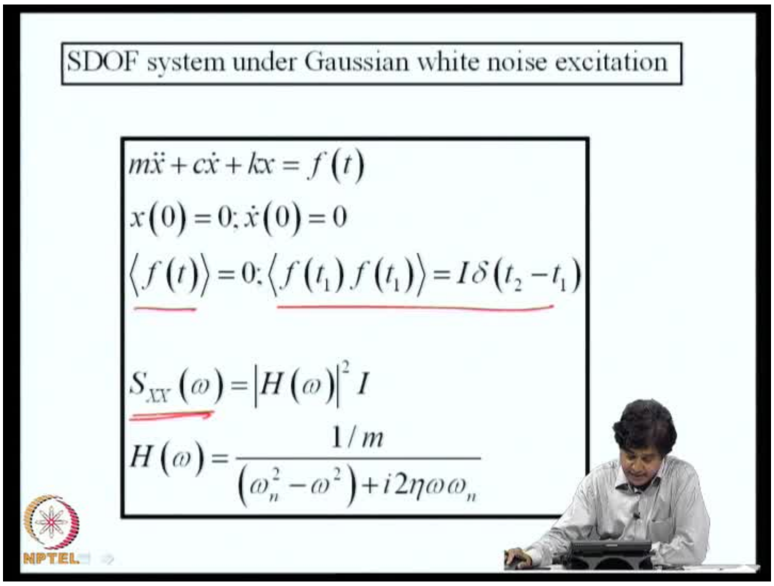

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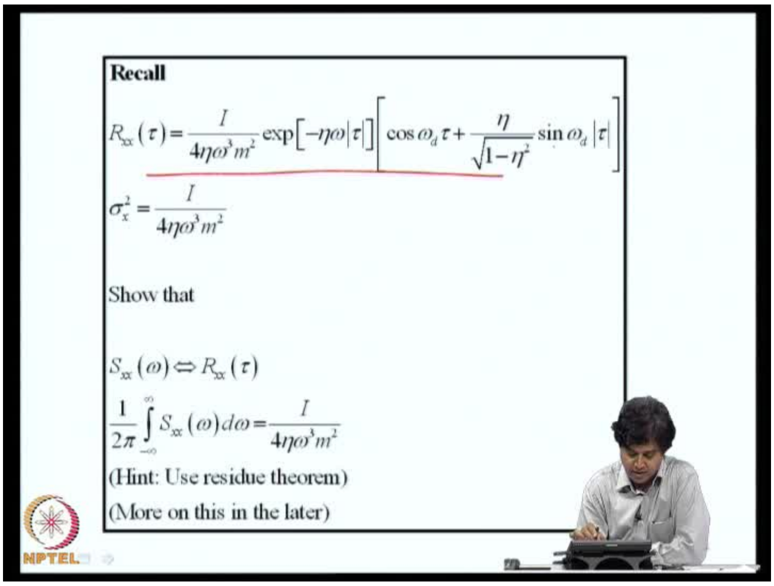

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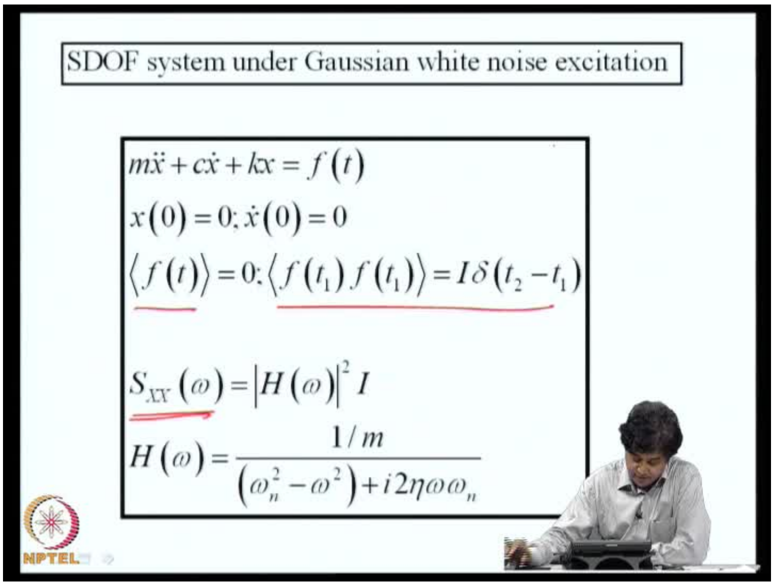

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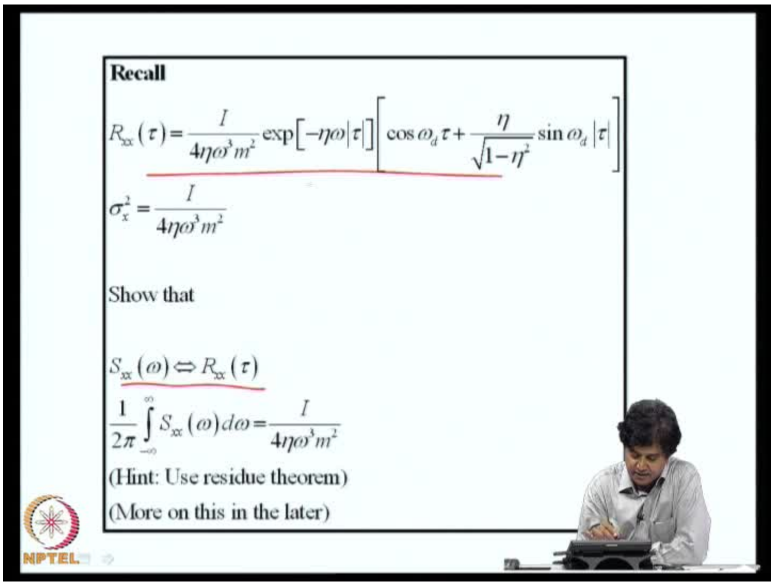

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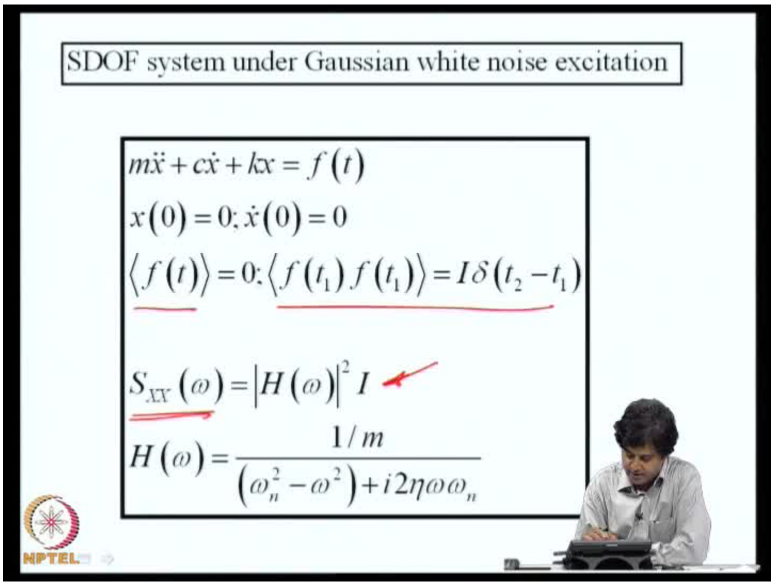

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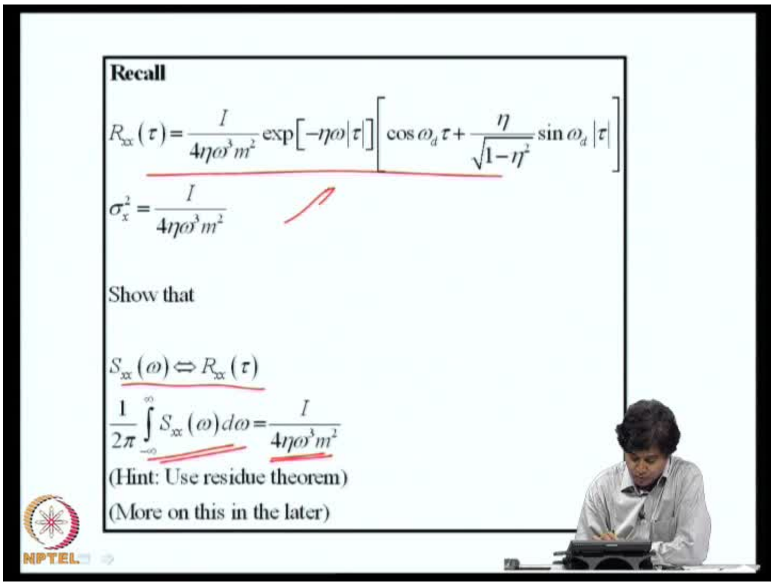

Now, let us return to the problem of single degree freedom system under Gaussian white noise excitation and consider the response analysis in frequency domain. So, we have the system equation, here m x double dot plus c dot plus k equal to f of t, this system starts from rest, the expected value of the expected value of f of t is 0, is auto covariance, is given by the directed delta function as shown. Now, the input and output relation will S x x of omega into h of omega whole square s f f of omega, where s f f of omega constant, so I get i into h of f omega whole square, therefore, and h of omega here is 1 by m omega n square minus omega square plus i 2 eta omega, omega. So, this is the power spectral density function of the output process, if you recall we have already derive the auto covariance response process, by using time domain representations and steady state and I already shown that, the auto covariance function is given by this function. Now, I have now derived the power spectral density function of the response in the steady state, I will leave it, in a exercise to show that, if you take the Fourier transform of this indeed get the auto power spectral density function, that we are just now derived or another words, show that the auto covariance given by this and the power spectral density function given by this, for the Fourier transform pair. Similarly, we know that R x x of 0 is the variance and that is given by i 4 eta omega cube m square, the area under power spectral density function is also related to the variance; so, the variance in steady state can also be obtain by a pure frequency domain analysis, and the exercise is to show that, the area under the power spectral density function is indeed the variance which is this, which has been obtained by a time domain analysis. Now, evaluation of this type of integrals requires the use of Cauchy's residue theorem and will returned to this, is the one of the next lectures, but to prove this, the hint is that we have to use that residue theorem.

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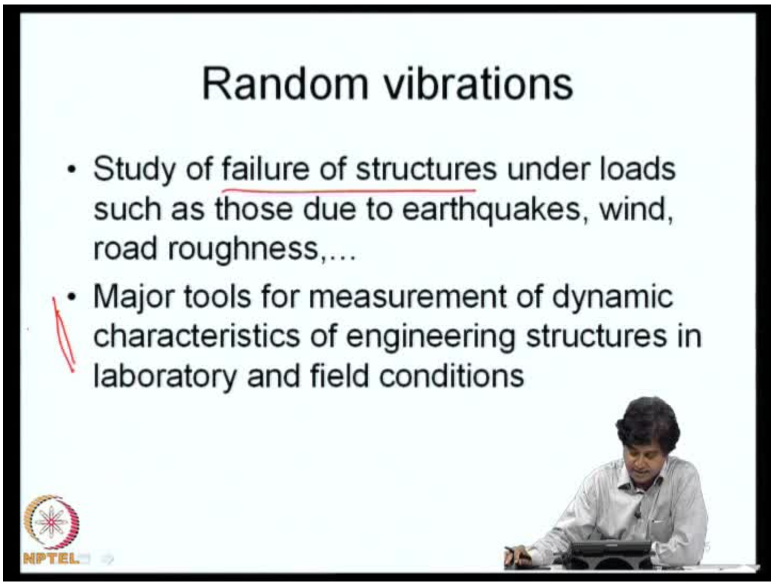

Now, the general remark at this stage is in order, the steady of random vibrations can be viewed from two perspectives; one is motivated by our interest is study of failures of structures under loads, such as earthquakes, wind, waves guide, way unevenness and so on and so forth, but there is yet another important application of random vibration principles and that occur in laboratory work. Actually, one of the basic issues in experimental steady of linear dynamical system is the measurement of frequency response functions. The whole idea of the whole process of measurement of dynamic characteristics of engineering systems, that would mean, frequency response functions and from that the natural frequency model, damping mode, shapes etcetera also these based on principles of random vibrations.

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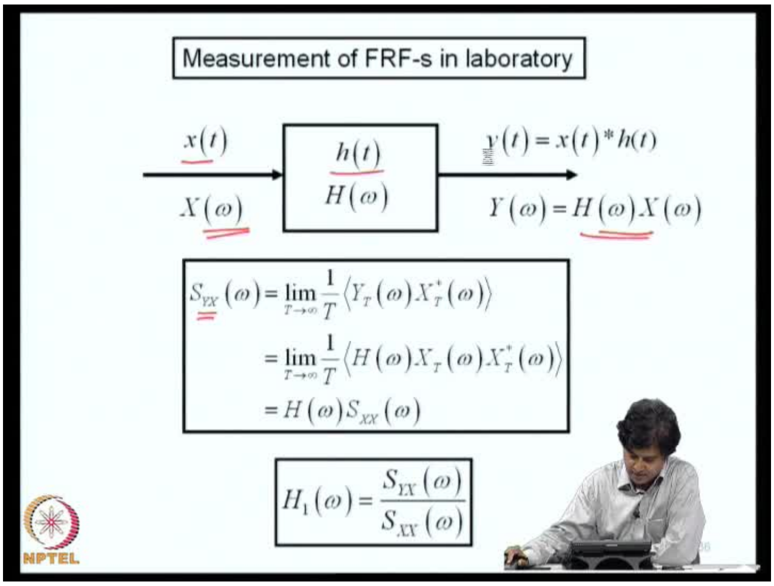

Now, there are certain results in this context that can be derived as this stage, based on whatever we have learnt so far and they can be listed like this. So, let start with input output relation in time, in frequency domain. So, if x of t is a input for system with impulse response h of t, y of t is convolution between x of t h of t in; similarly, if X of omega is the Fourier transform of the input, the Fourier transform of the output is obtained by multiplying this system, frequency response function with the input Fourier transform, that is this. And the input output relation for power spectral density function we already define, but now let us consider cross power spectral density function between y of t and x of t, this by definition is this, limit 1 by T Y T of omega into X T star of omega. Now, I know y of omega H of omega into X of omega, I will substitute that here and I get X Y X of omega is h of omega into S x x of omega; based on that, I get an estimate the formulae for the frequency response function, which is S y x of omega divided by S x x of omega. So, if you are doing an experiment, you have gather time histories of X of T, Y of T and use statistical procedure, to estimate S y x of omega and S x x of omega, these are fairly standard tools, we will discuss with later. And if you take ratio of that you get H 1 of omega, whenever we measure x of t of t, there will be measurement noises that will create certain problems.

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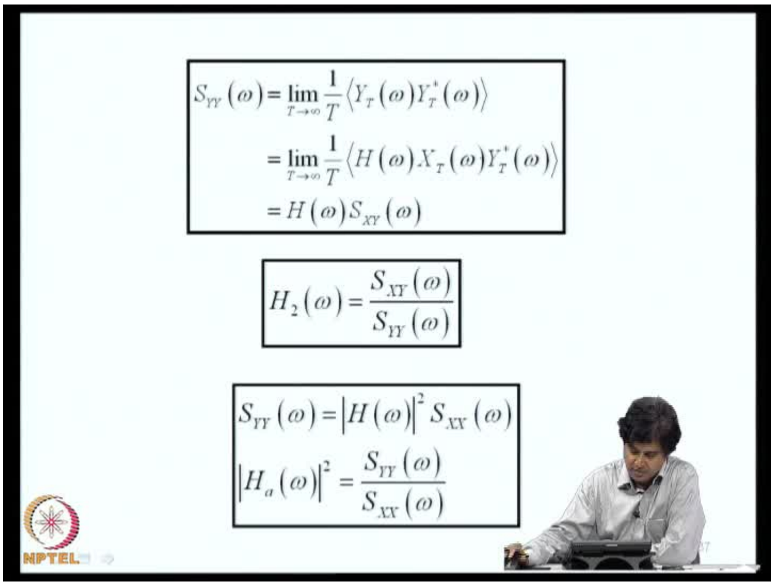

Before I come that, we can also consider another relation that is exist namely, now if you consider power spectral density function of the output, this is the definition, and if Y T of omega, if I write h of omega into X T of omega, and examine in this relation, I get S Y Y of omega to be H of omega into S X X of omega. So, I get another definition for the frequency response function, which is S X Y of omega divided by S Y Y of omega. Mathematically H 1 and H 2 are the same, but an experimental what we want do the same, because the measurement of noise and another issues; we also have S Y Y of omega as the square of the frequency response function multiply by S X X of omega. So, if you are interested only the amplitude of frequency response function, this is yet another formula, for that we divide the output power spectral density function divided by the input power spectral density, you get this function.

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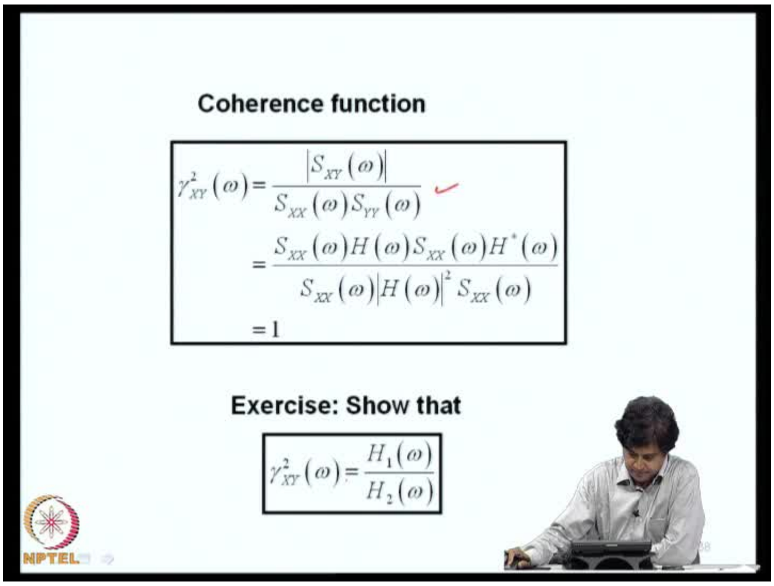

We can define now the coherence function between X and Y, if you do that this is actually the definitions square of the coherence, if you substitute all the formulae, just now the, derive, you can show that coherence function is 1 between X and Y, this is not surprising because system is linear; you can show that, this coherence function is a ratio of H 1 and H 2, this is small exercise. In a experimentally world, if you are to measure the coherence and plot it, you will see that, it will not be uniformly equal to 1, because of presence of measurement noise or because of structural non-linearity is so on and so forth; so, the plot of coherence function and is departure from unity is taken as a quality of measurement of frequency response function, if it is deviates too much from one and certain frequency, the same frequency, the transfer function is not measured acceptably.

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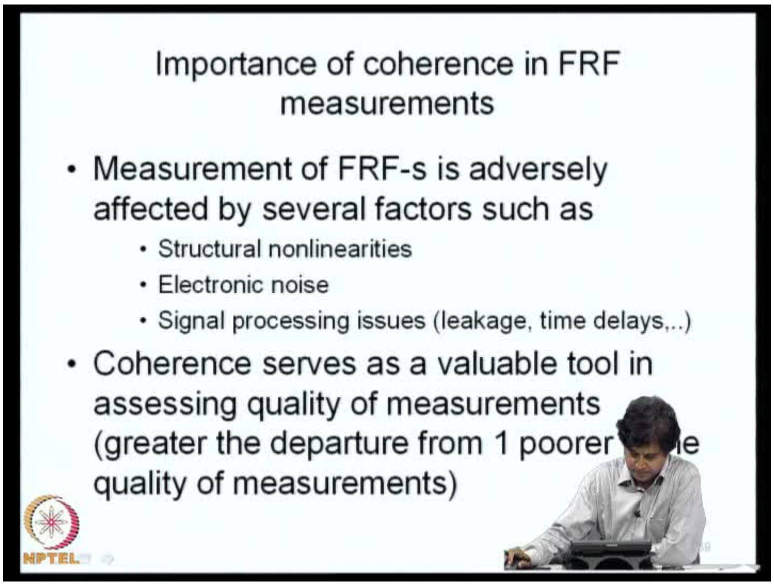

So, we can summarize the important of coherence of FRF measurements; if you consider this question, we should notice the measurement of FRFs is adversely affected by several factors, such as structural non-linearity, electronic noise and there are several signal processing issues, lateral in the course we may have to discuss this, so, by because of this, the coherence will never be equal to unity, I mean, not always be equal to unity; so, therefore, coherence serves as a valuable tool in assessing the quality of measurements, greater the departure from unity, one poorer the quality of measurements.

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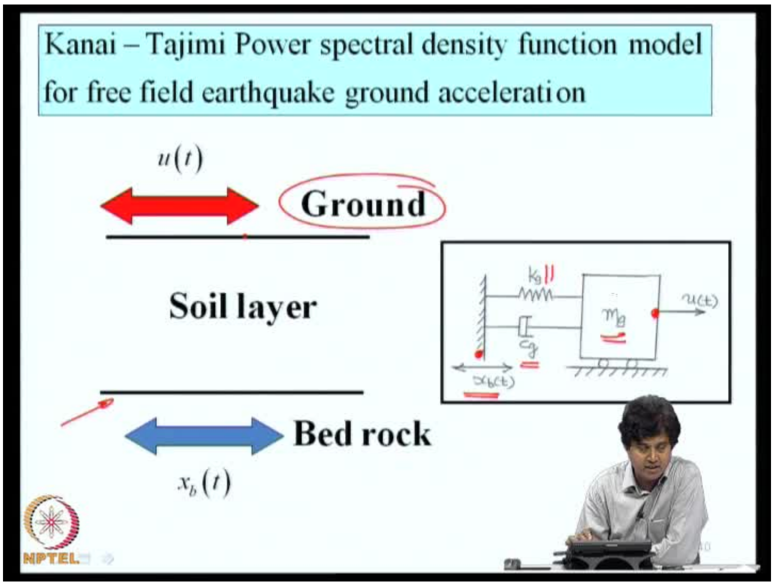

Now, an application of dynamics of single degree freedom of system under white noise excitations; this application arises in the context of the modeling earthquake ground motion, imagine that, this is the bed rock and overlying this, there is a soil layer and when an earthquake occurs, whatever displacement, that occurs at bed rock level propagates through the soil layer and appears as the ground shaking. We are basically interested in way the ground shakes, because the buildings are housed on this. Before the construction of any structure, if you study the ground motion, we will call it as free field ground motion, in the sense, there is no engineering structure yet constructed on that structure; so, most of the course of practices define this free field motion in some manner. Conceptually, to make a model for free field ground motion, one of the model is proposed by Kanai Tajimi, in this model, it is assume that the bed rock level the ground acceleration is the white noise and the soil layer is modeled as a single degree freedom system, where m g is the mass of the soil layer, k g is the stiffness and c g is the damping and this is the free field point and this is the bed rock point. The free field motion, thus that characterized in terms of the power spectral density of the base motion, multiplied by this transfer function, associates with this soil layer and how does take place.

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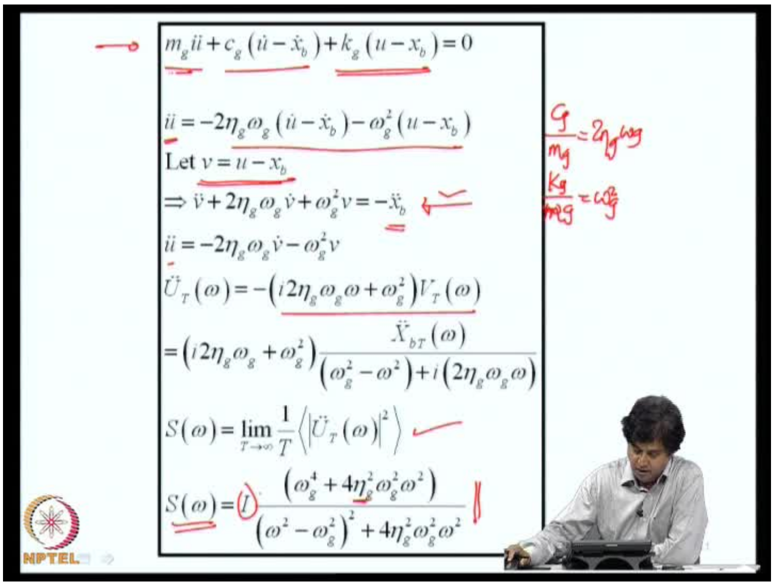

If you write the equations now, this is the equation for the total displacement of the ground, that is the total displacement of the mass, so this is m g e double dot, c g is the damping force which is the function of u dot minus x b dot, similarly k g is spring force which is function of e minus x b. So, if you now write from this, this expression for absolute acceleration in the ground level, we get this expression, where c g y m g is written as 2 eta g omega g and k g by omega g is written, k g by m g is written as omega g square; so, this is the expression for the total acceleration at the ground level. Now, you can as well as define a relative displacement u minus x b and the relative displacement is governed by this equation, where in the right hand side, I get the acceleration level at the bed rock, whereas here, on the right hand side, you will get damping into the velocity plus stiffness into the displacement at the bed rock level. Now, we can do a frequency domain analysis of this absolute acceleration, and if I do that, we get the Fourier transform of the absolute ground acceleration, can be written in this form and based on this, V T of omega is now obtained as through the input output relation of this equation, if you substitute that here and use the definition of power spectral density functions in terms of this expectation, I get the power spectral density of the ground acceleration to be in this form; this is the well known Kanai Tajimi power spectral density function. This derivation is based on how a single degree of freedom system response to support motions where acceleration support, but acceleration are modeled as white noise processes; omega g is the natural frequency of the ground, eta g is the damping in the soil layer, i is the intensity of shaking at the bed rock level.

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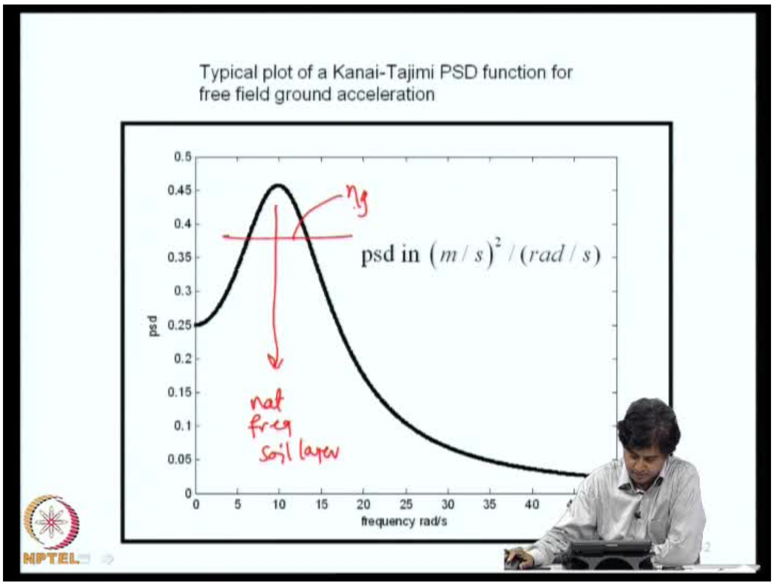

A typically, the power spectral density function appears in this form, it is new model and this peak occurs at the natural frequency of a soil layer and this bandwidth depends upon eta g, the damping in the soil layer.

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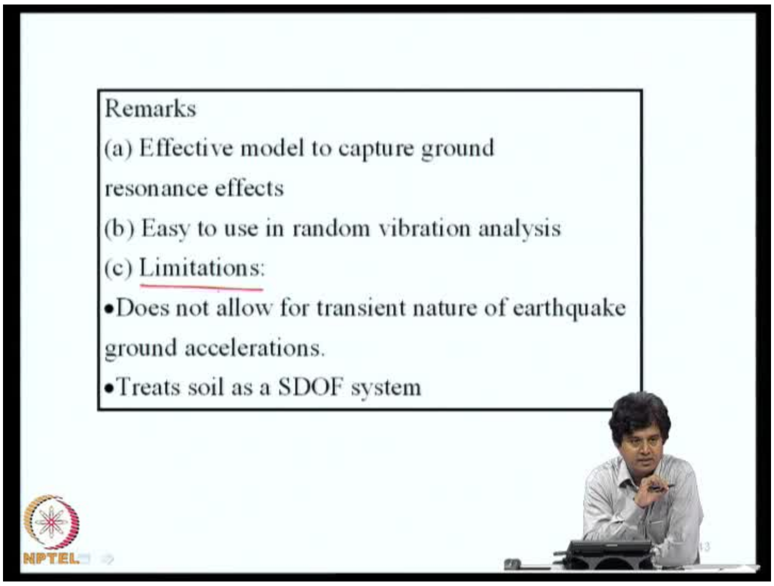

So, this power spectral density function, essentially captures the possibility of ground resonance, because this system is modeled as a single degree freedom system, there is one possibility to allow for one resonance and this is very easy to use random vibration analysis, because in frequency domain analysis, the power spectral density functions are obtain by simple multiplication of transfer function square into the input power spectral density function; so, it is very easy to use the problem here or that this model does not recognize, that the ground acceleration is actually the non-stationary random process and the soil layer itself is taken as single degree freedom system, whereas soil layer is a continuum, it can have probably more than one nature frequency, in the frequency range of interest; this can be remedy, this can also be remedy.

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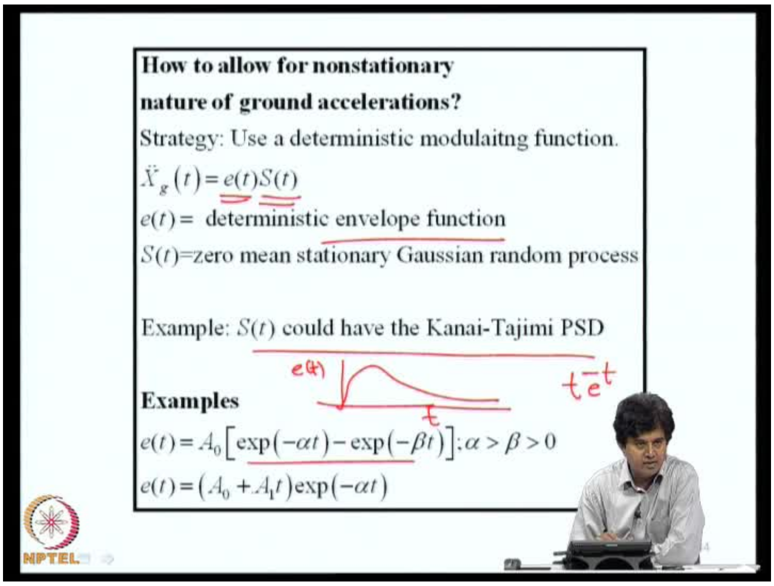

How to introduce non stationary into ground acceleration? We can multiply the stationary random process, whose power spectral density is Kanai Tajimi power spectral density by deterministic envelope function; these are deterministic envelope function, captures the non-stationary trend that is observed in earthquake ground motions. So, an example could be S of t could have a Kanai Tajimi power spectral density and this envelopes, for example, could be sums of this, we have seen, just a seen while before and a similar curve like this, in the sense, they essentially capture this type of behavior in time, the envelope functions either they can be expressed as differences between exponential or even if you take e raised to minus eta, also has qualitatively this type of behaviors. So, both this type of models has been proposed in the literature to capture transient behavior.

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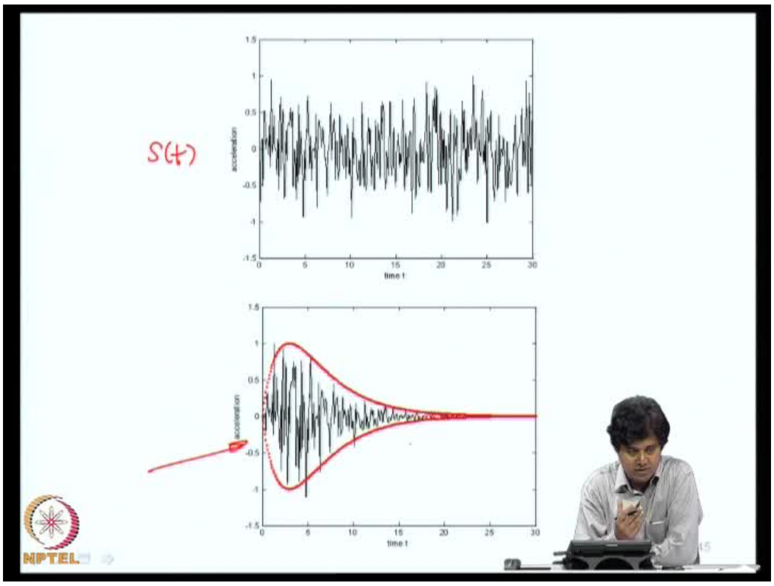

So, this is how conceptually looks like, this is sample of S of t and upon multiplication by E of t, I get the transient realization of a non-stationary random process; this red line that is shown here is the envelope. So, this models that we are develop, we are able to now proceed a bit further and we are able to develop simple models for earthquake ground motions.

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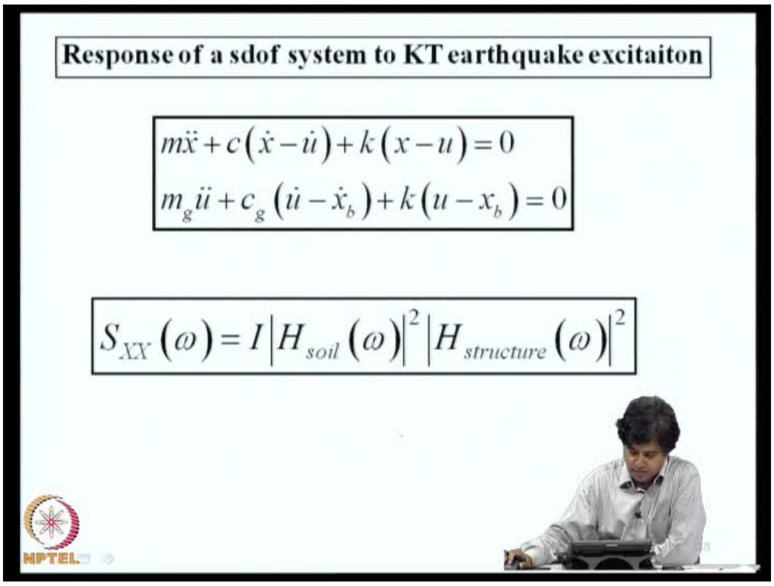

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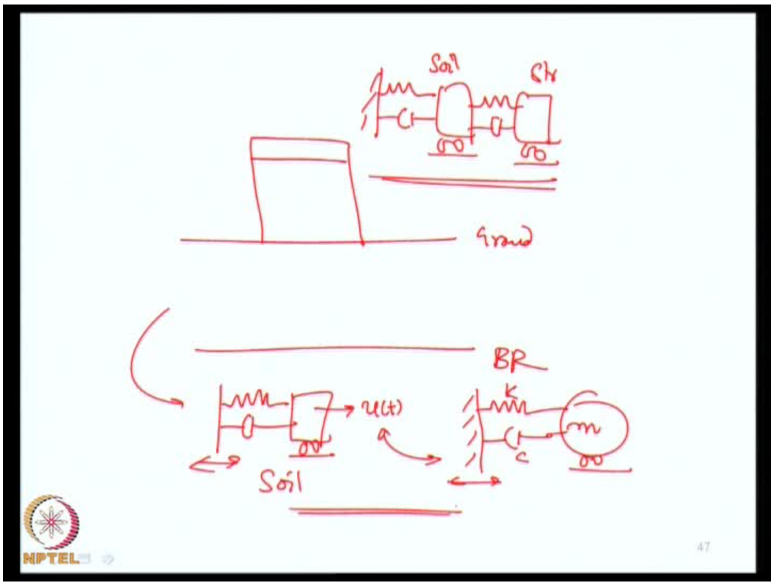

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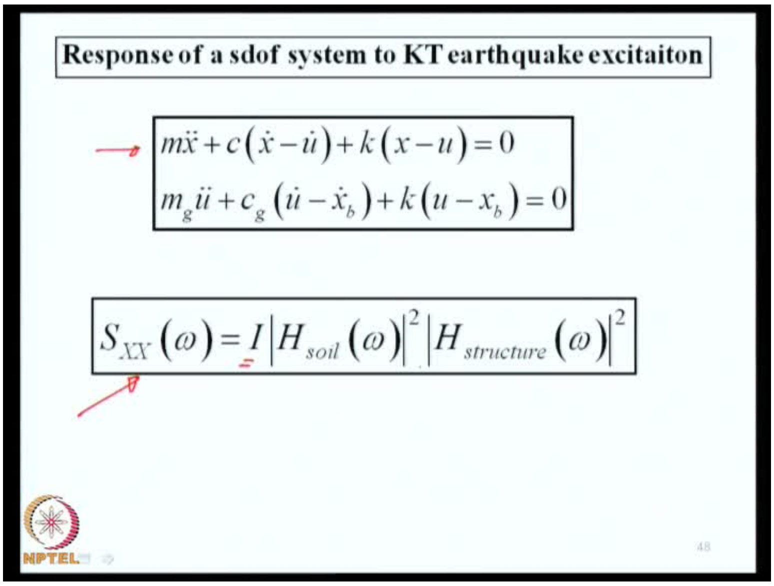

Now, how does a system itself response to Kanai Tajimi ground motions; so, this is my structure, that means, I have bed rock, this is ground and on this I have my structure, the way we model, it is, this itself is modeled as this is soil; the output of this is say u of t, a simple version of modeling would be, to this is my structure, so call it as m k c and this u of t appears as support motion for this structure, that means, there is a kind of a cascading assumption. We could as well as make a 2 degree freedom system. this is soil, this is structure, in this model we are allowing for possible interaction between structure and the soil, whereas here we are ignoring, that if we consider this simple situation, first it is quite straightforward to deduce the output power spectral density function, by simply multiplying the transfer function of the soil, with the transfer function of the structure and the power spectral density at this bed rock level. So, this is so-called cascading assumption which can easily be implemented, in this particular case. With this, we will conclude this lecture.