Stochastic Structural Dynamics

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Lecture No. # 10

Random Vibrations of SDOF System-2

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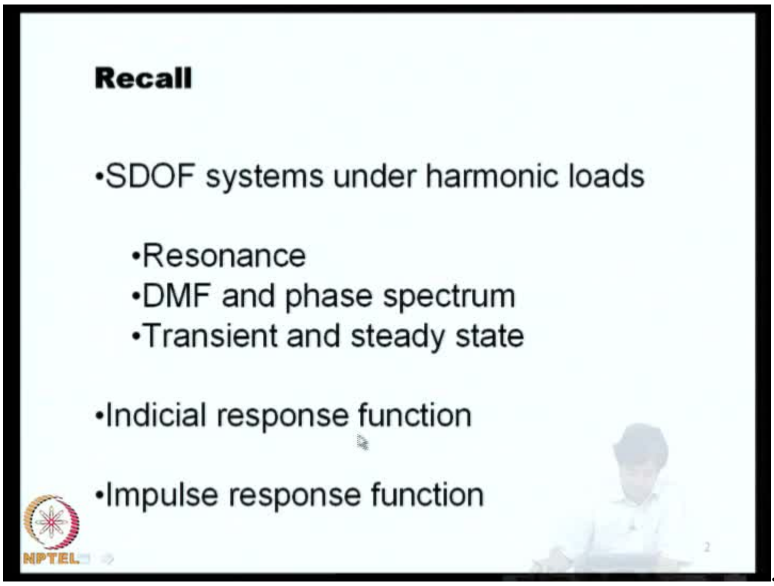

In the previous lecture, we completed reviewing the theory of probability and theory of random processes, to the extent that we will be need in this course. We also began reviewing some basics of modeling - linear single degree freedom systems. So, we studied the response of single degree freedom systems under harmonic loads, under step loads and under impulsive loads.

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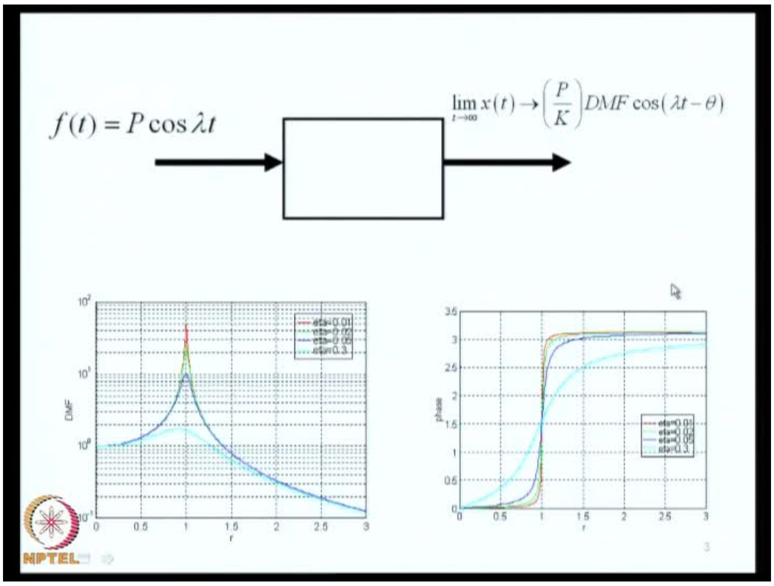

So, under harmonic load, that is, if we apply P cos lambda t as time becomes large the response becomes harmonic with frequency coinciding with the driving frequency. And the amplitude of the response varies as function of mass damping and stiffness of the system. And this DMF is a dynamic magnification factor, which represents the ratio of amplitude of dynamic response to static response and that it shows a characteristic behavior here. So, in the neighborhood of the driving frequency being close to the system natural frequency, there is a significant dynamic amplification and this condition known as resonance. And at resonance, the phase that is this theta undergoes a rapid change, that is one of the signatures of occurrence of resonance.

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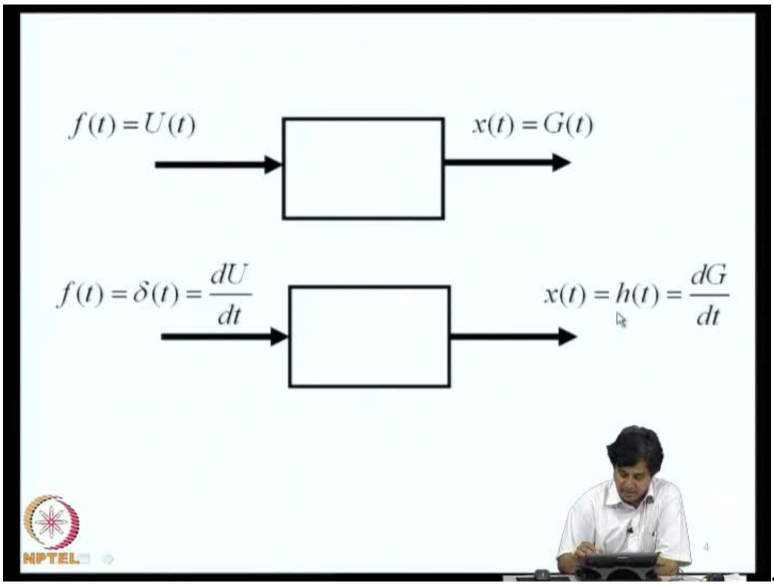

We called the response of the system to u side step function, as the initial response and we denoted it as capital G of t. And we have seen already, the derivative of U side step function can be interpreted as Dirac's delta function. And Dirac's delta a function model for impulsive load. So, if you apply an unit impulse at t equal to 0, we call that response as the impulse response function and we showed in the last class that this is actually the time derivative of the indicial response.

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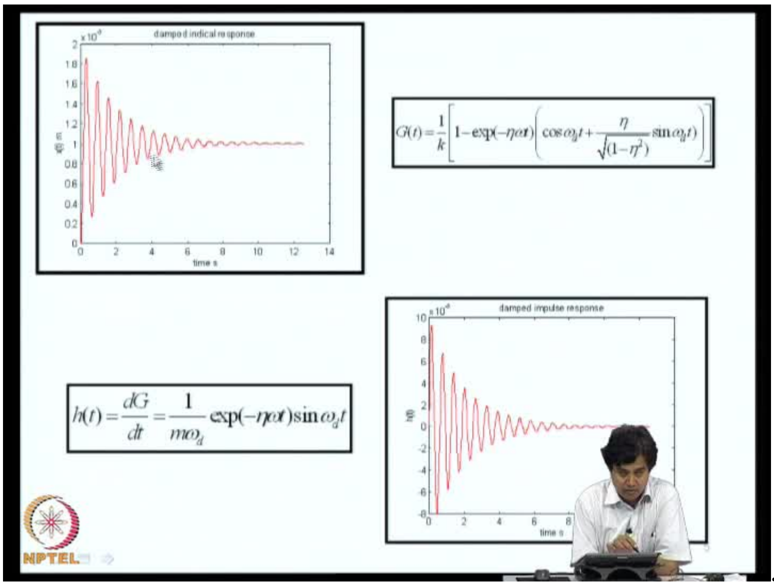

So, this is how the indicial response looks for a typical single degree freedom system. So, the displacement amplitude is one and this wave is in a steady state, it reaches this value is equal to the static displacement value, as you can see from here, as t becomes large this exponential decays to 0 and we get G of t as 1 by k, which is nothing but the static response - under this unit load. The time derivative of this is shown here the oscillation is about 0, whereas here oscillation about 1 and this is actually a damped sinusoidal function.

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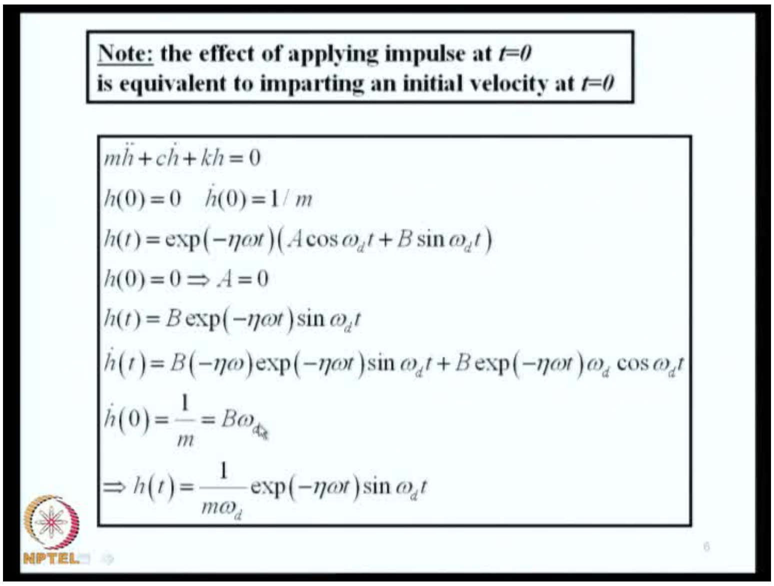

Now, we will continue discussing, some aspects of dynamical characterization of a single degree freedom systems, before we take up the problem of response of system to random excitation here, we are noting that the effect of applying impulse at t equal to 0 is equivalent to imparting an initial velocity at t equal to 0, that can be verified, if you consider the response of a single degree freedom systems under in free vibration with initial displacement 0 and initial velocity as 1 by m, if you write the complementary function in particular, there is no particular integral complementary function. There are two orbiter constants and they can be evaluated using these initial conditions, if you were to do that you will get A to be, because h of 0 is 0 and it turns out that B is actually 1 by m omega d, following this we get the solution to this equation as 1 by m omega d exponential minus theta omega t sin omega d t, which we have already obtained by an independent means as time derivative of indicial response. So, in constructing impulse response functions for dynamical systems with this approach is more easily implementable.

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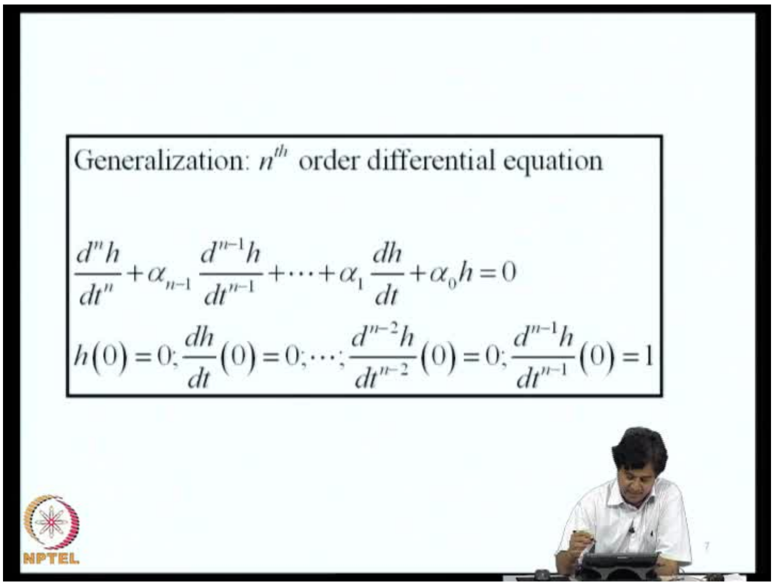

The definition of impulse response can be generalized to systems governed by nth order differential equation. So, we have considered second order differential equation. Suppose, we have an nth order differential equation as shown here with alpha n minus 1 alpha n minus 2 alpha 1 and alpha naught as coefficients, which are independent of time with the impulse response of this system is given by the free vibration response of the system under these set of initial conditions. Here, the field variable h its first n minus 2 derivatives are 0 at equal to 0 and the n minus 1th derivative is unity. So, if you solve all this differential equation under these initial conditions the resulting function will be the impulse response function for this nth order system.

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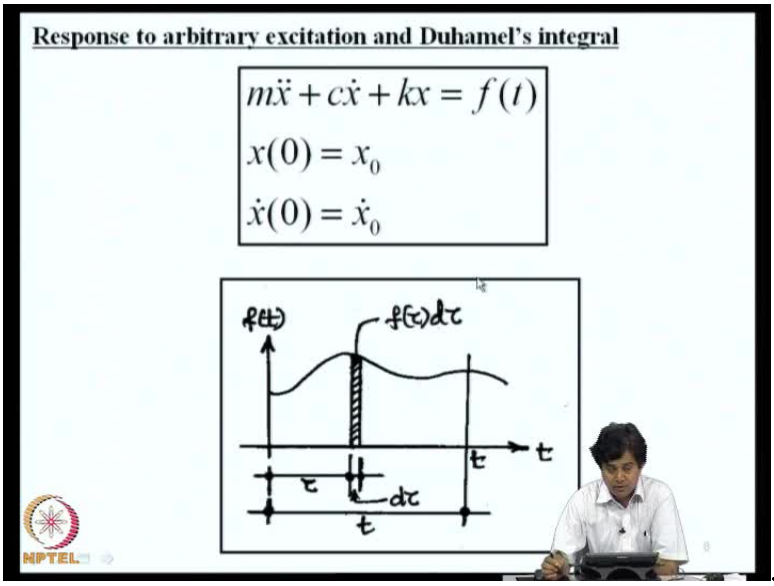

Now, what is the use of impulse response function? It can be used to model loads that act over short duration. The duration here is short vis-à-vis the time period of this system, but that is not its main use. The main use of impulse response function is in constructing solution of the system response of the system to arbitrary loads f of t, for example, f of t is an arbitrary load, this could be for example, a load induced when earthquake or a wave or a wind, where we do not have any functional representation in terms of sine's and cosines and exponentials so on and so forth. So, how do we functionally write the solutions? See one can easily write the complementary function that would not change, but when it comes to writing particular integral, the way we have been proceeding is that we construct the particular integrals based on knowledge of f of t, if it is sine omega t particular integrals is such an such and so on and so forth, but if it is an arbitrary function, how to proceed? Now - to appreciate that - we can consider a related problem from statics.

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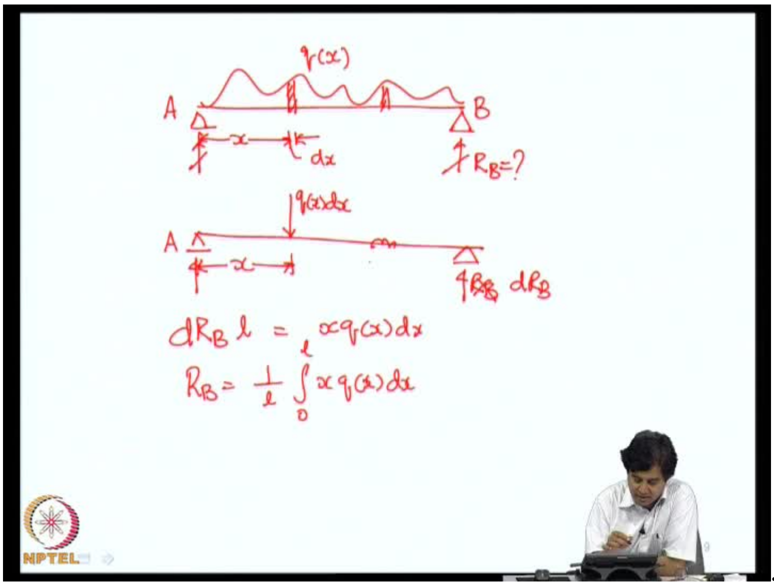

Suppose, we consider is simple supported beam static and some load q of x. So, let us call this end as A, this end as B, the question is, what are the reactions? Specifically, I want to know, say for an example, what is reaction at support B? This load is an arbitrary load, so, what the strategy we flow is we consider an elementary strip located at distance x and this is dx, we consider this load q of x into dx is area under the curve q of x is right of loading, load per unit length. So, q of x into dx will have the units of force and we first consider the response of the beam to this concentrated load, then we interpret this distributed load as a train of concentrated loads. So, if I want to find the reaction now, I will take moments about point A and this reaction is actually an incremental reaction, this is not the reaction R B. So, the moment of this reaction about this end suppose l is a span will be x into q of x dx, this is the contribution to reaction at B due to this strip of loading.

Now, if you consider another strip, there will be another contribution and the total contribution will some of all this. So, we get R B is equal to 1 by l integral 0 to l x dx. So, here, if you see, we have utilized the notion of a concentrated force to construct a solution for a distributed load. The concentrated load model in its own right has some merit, it can model loads that act to over short, you know areas in delay short in relation to the span, a short distributed load can be approximated as a concentrated load, but the main advantage of using the notation of concentrated loads is not so much to model such kind of patch small patches of loading but to construct solutions for distributed loading.

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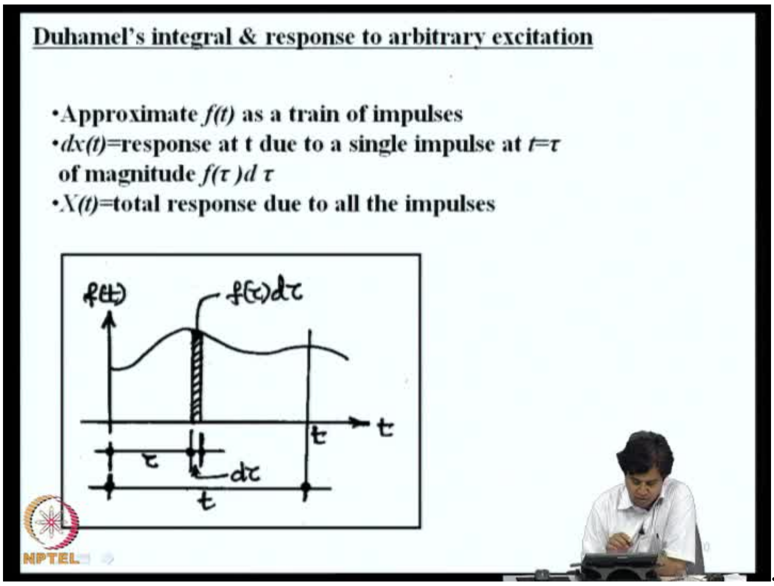

Now, in the same spirit, what we do is when it comes to an arbitrary load in time acting on a single degree freedom system, what we do is we divide the time axis into a series of impulses say, we can consider a time instance tau and a increment d tau and this area. Under, this curve is f of tau d tau this is an impulse, now we approximate f of t as a train of impulses, suppose I am interested in constructing response at time t. So, what I do is first I find out response a time t due to this single impulse. So, call that as dx of t, this is a response at time t response at time t due to this impulse at t equal to tau and its magnitude is what f of tau d tau magnitude in the sense here end of the curve is f of tau t tau. Now, what is capital X of this X of t it is response at time t due to several such impulses. So, first and foremost is we have to construct response due to this single impulse and then integrate that from 0 to t.

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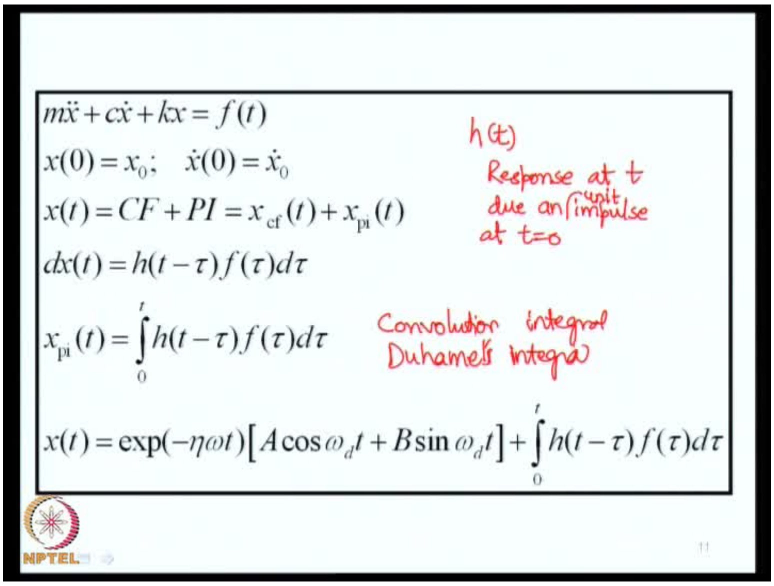

Now, bearing that in mind, we can now consider the response of single degree freedom system and arbitrary loads and certain specified initial conditions, we have a complementary function and particular integral. The complementary function continues to be for example, exponential eta omega t A cos omega t plus B sin omega t. The particular integral is what we are constructing dx of t is the response due to unit, due to impulse whose magnitude is f of tau d tau at applied at tau, see what we have seen, what is the interpretation for h of t h of t is the response at time t due to an impulse at t equal to 0. Actually, this impulse is a unit impulse.

Now, there are things that are different here, namely it is not an unit impulse instead, the magnitude is f of tau d tau. Secondly, the impulse is not applied at t equal to 0 but t equal to tau. So, you have to shift time to t minus to tau and multiply the response by f of tau d tau, if the impulse were to be applied at t equal to 0 and the impulse was an unit impulse, the answer would be straightaway h of t, but now we are applying a t equal to tau and magnitude is f of tau d tau. Now, this is response due to that single impulse. Now, the total response is summation of response due to the train of impulses and that becomes integral 0 to t h of t minus f of tau d tau. This integral is known as the convolution integral or the Duhamel's integral.

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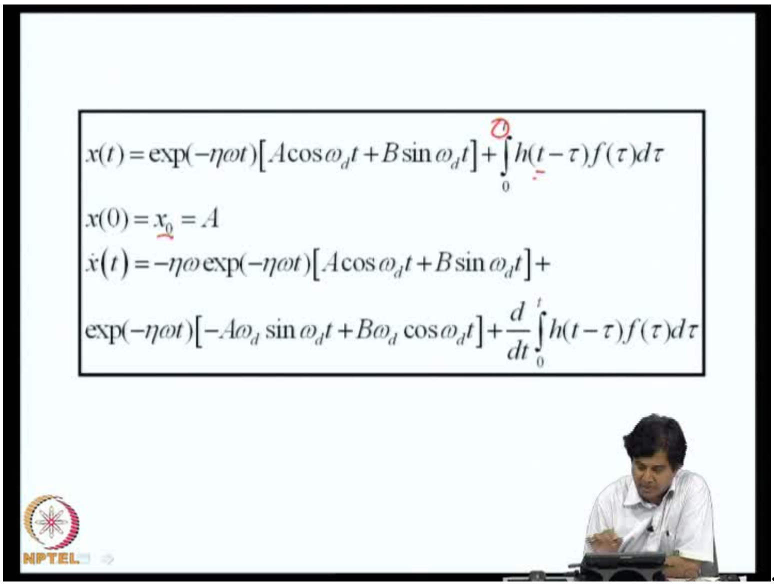

Now, how do we evaluate the constants A and B. So, at x equal to 0, I have x naught to be the initial condition. So, using that here I get A. Now, to find x dot of 0 have to differentiate this with respect to t there is a slight problem here, you can differentiate this with respect to t easily, first you differentiate this, keep these terms inside the bracket as it is, then exponentially term as it is and the differentiation of this, the first two terms are straightforward. But here, the difficulty is the time t appears not only in the integrant, but also as a limit. So, you should know how to differentiate an integral with respect to its limit and a parameter inside in the integrant.

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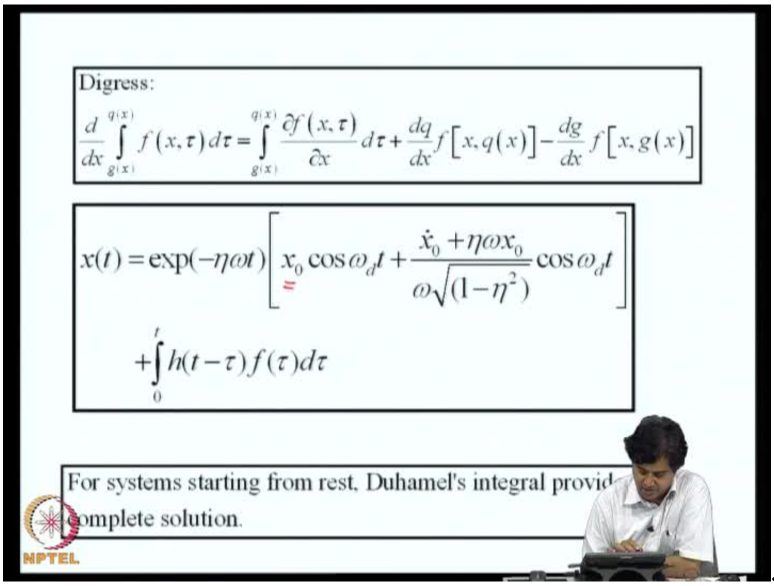

So, there this is the basic theorem, if you have an integral g of x to q of x f of x, tau d tau and you want differentiate this with respect to x. This is the rule for that. So, this is one of the results in integral and differential calculus that you should be aware of now. Based on that I can evaluate the initial A and B and if we do that we get this to be the solution. So, x naught the initial condition x naught is here, x naught not dot is here and x naught is here, and this is the response due to f of tau.

Now, if you carefully look at this solution, if this system starts from rest, that is, if x naught equal to 0 - that is, this is 0 and this 0 and if x naught dot equal to 0, the solution is given by the Duhamel's integral. That means for systems starting from rest Duhamel's integral provides a complete solution it also incorporates in its whole the part of the solution, which corresponds to the initial conditions x of 0 is 0 x dot of 0 is 0 in that sense it differs from the way we wrote particular integrals in the previous lecture.

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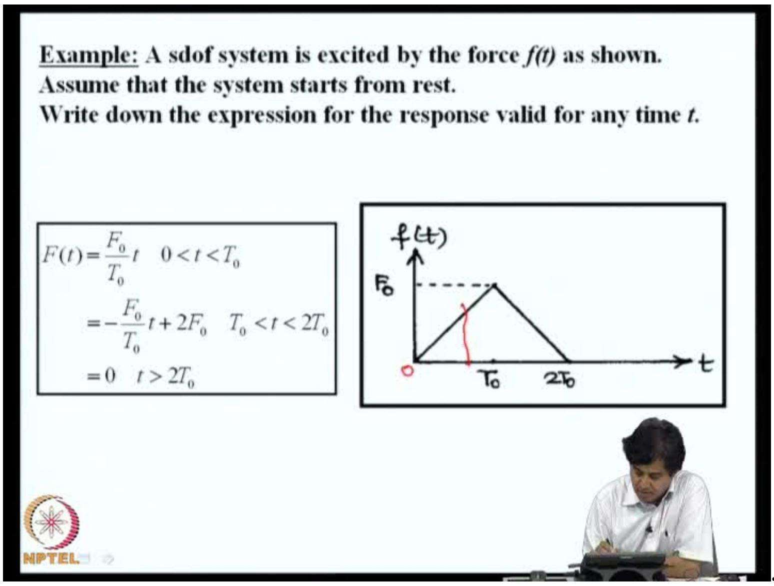

A small exercise, let us assume that a single degree freedom system is excited by a force f of t as shown here, it is a triangle, this is our region, this is the time T naught 2T naught. The question is I will assume that system starts from rest. Now, write the solution in terms of Duhamel's integral.

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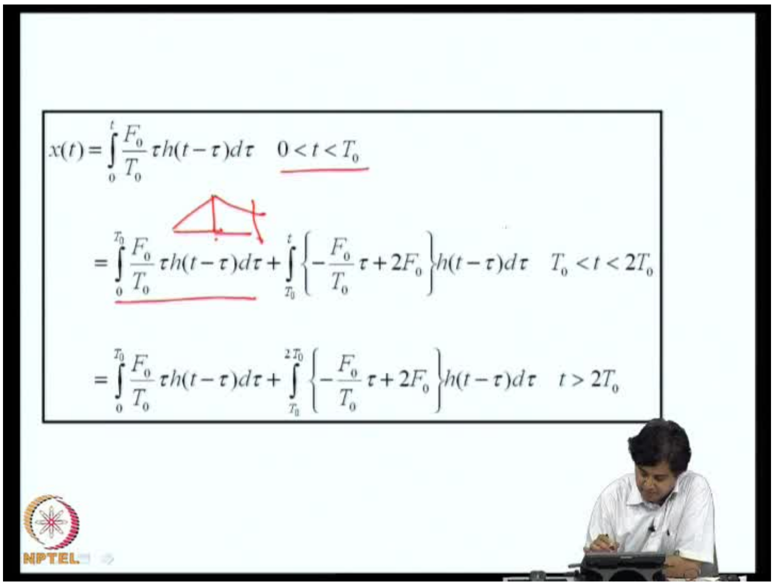

Now, if you if you are in time duration, this is if you are in 0 to T naught, you consider any time T you have to add the impulses that lie in this interval. So, there several impulses will be here, you have to add moment we come here, you have to so if you are in the region t from 0 to T naught this will be the Duhamel’s integral, because in that region the function is climbing as tau and this is this. Now, if you are in the region t from T naught to 2T naught the first expression represents the response due to the rising part that is up to this part. The second one is from T naught to 2T naught that is contained here if you cross 2T naught, then that is if you are somewhere here, then the function excitation is 0 from 2T naught onwards. So, the integration will be from 0 to T naught for this part and T naught to T naught from this part and then simply for the vibration d k. So, that is what you get here, so you could use the Duhamel’s integral in this manner to construct solutions for arbitrary loads.

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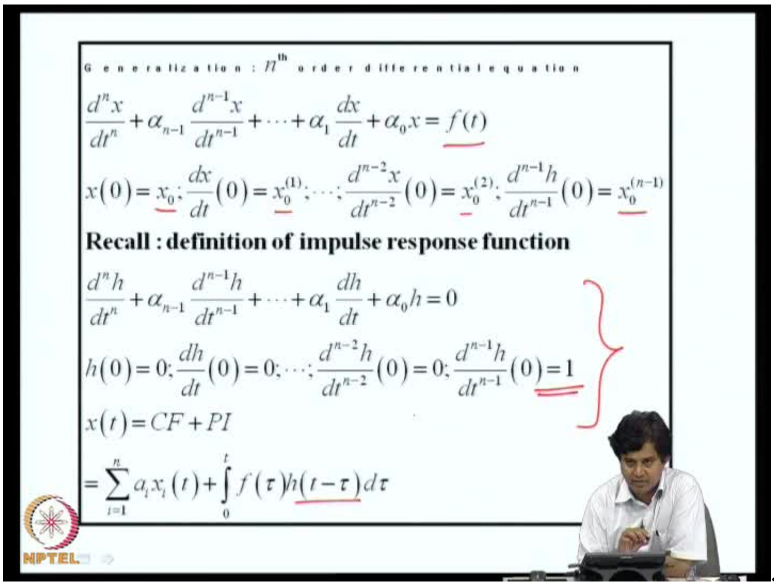

Now, how do we generalize the notion of Duhamel’s integral to nth order differential equations. Suppose, I have a nth order dynamical system with excitation as f of t and the initial condition are x naught x naught 1 x naught 2 so, x naught n minus 1 so on and so forth. Now, we you require just a while before we defined the impulse response function for these kinds of systems using these initial conditions. The n minus 1 derivative at t equal to 0 as one rest all were equal to 0. So, h of t would be available for us and we can continue to use the same logic that we did just now and write the solution as a complementary function plus particular integral.

Now, the complementary function will have now n arbitrary constants and n components in your complementary function plus 0 to t f of tau h of t minus tau d tau. This h of t is now solution of this problem. So, the theory of impulse response function and Duhamel's integral and construction of particular integrals for under arbitrary load is more generally valid it is not just for second order differential equations.

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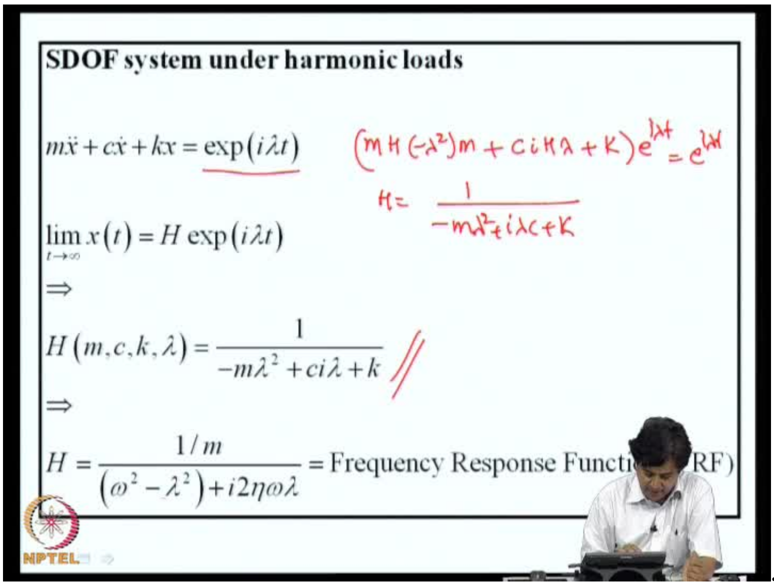

Now, will now revisit the problem of single degree freedom system under harmonic loads and based on this discussion, we will try to now introduce certain frequency domain descriptions of dynamical systems. This impulse response function can be viewed as a time domain description of a dynamical system. So, what is the frequency domain description of dynamical systems?

So, we know consider the single degree freedom system, but there we know write the harmonic load in terms of complex exponentials and we consider only the steady state response. So, as t tends to infinity, since the system is linear it has time invariant parameters and it is driven harmonically. In steady state, the response would also be harmonic at the driving frequency, but with a different amplitude. So, we assume the solution to be of this form and if we substitute now into this equation, so what we get is m H minus lambda square m plus c i H lambda plus k e raised to i lambda t equal to e raised to i lambda t.

So, consequently, this function H becomes we call H is the amplitude. So, H will be 1 divided by this is what I am write here minus m lambda square plus i lambda c plus k. So, this is the amplitude of response in steady state, we can take out this m and rewrite this function in the form, where omega is the natural frequency it has the damping ratio and this function is known as the frequency response function. So, it is the time domain description of a dynamical system it is a complex quantity its amplitude would be related to the dynamic magnification factor, which we discussed and its phase will be related to the phase angle that we discussed in the previous talk.

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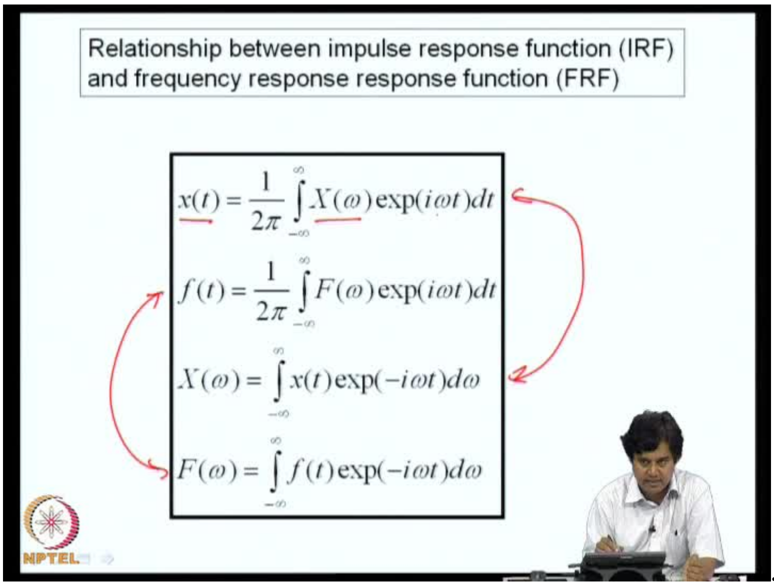

Now, the question we can ask is, what is the relationship between the impulse response function and the frequency response function, are they related? One is a time domain description other is a frequency domain description, we know that a time domain description of function and its frequency domain description is related through the Fourier transform pair, I mean they form a Fourier transform pair suppose x of t is the time signal x of omega is the Fourier transform. They are related by this pair of relations. Similarly, now this is the Fourier transform description of the response. This is the Fourier transform description of this say the excitation. Let us see what we get from this omega, please note is not the natural frequency it is the frequency parameter used in defining the Fourier transformer.

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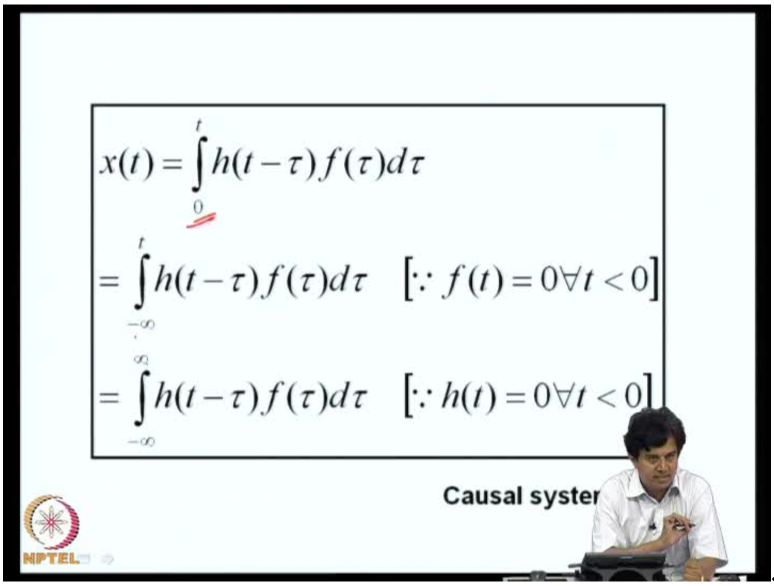

So, what we will do now is we will reconsider this Duhamel's integral. So, this is a response of the system under when the system starts from rest under the action of load f of t. Now, we would like to rewrite in a slightly different form we first thing is I want to write this lower limit as minus infinity, this is admissible because, we define the force to act on the system from t equal to 0 and when t is negative we take f of t to be 0, if this is acceptable, I can as well write 0 as minus infinity. Now, you could also write the upper limit as infinity, simply because h of t is a impulse is response of the system applied at t equal to 0. So, if this argument becomes negative t minus tau becomes negative that would happen when this tau crosses t, when tau is greater than t this argument will be negative that means from t to infinity. The argument of this function will be negative that means it is the response of the system to an impulse which is likely in to occur in future. So, that would be 0 such systems are known as causal systems they would not respond till a load is applied. So, h of t will be 0 for negative t. So, we can therefore write 0 to t has minus i infinity to plus infinity.

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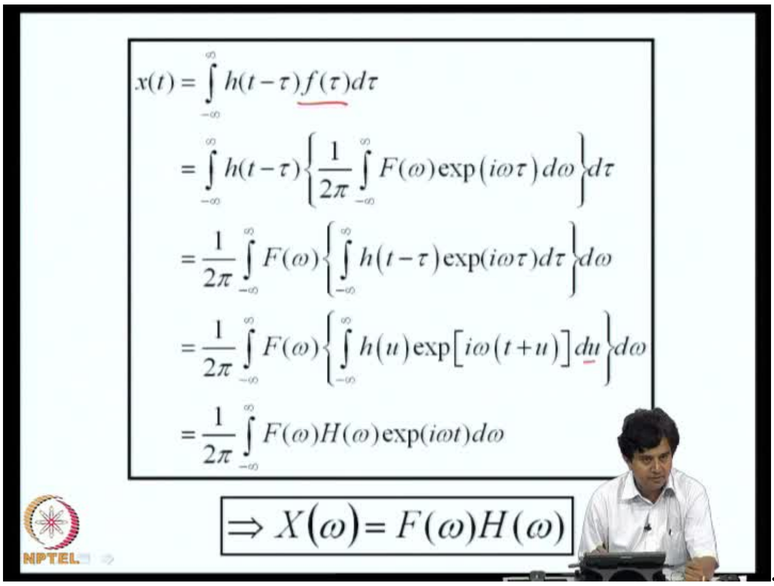

So, we will start with that we have minus infinity to plus infinity h of t minus tau f of tau d tau for f of tau, I will write its Fourier transform in terms of f of omega and we will rearrange this term, I will first integrate with respect to time tau and then with respect to frequency, I will change the order of integration. Now, look at the integral inside the braces. So, I will make a substitution t minus tau is u and there will be consequent changes here and this i omega u i omega t will come outside, because this integral is with respect to u and if you look at that it is nothing but the Fourier transform h of t what remains inside the braces. This is the Fourier transform of impulse response function h of omega.

Therefore, now, if you compare this expression with the expression for the Fourier transform definition x of omega this, you will identify that x of omega is nothing but, f of omega in to h of omega. So, in this viewgraph, this h of omega is the Fourier transform of h of t.

Now, this integral is also known as convolution integral. Now, as you see here we began with a convolution operation and we showed that instead of… if you are interested in x of omega. Instead of x of t that means, if you are interested Fourier transform of x of t not in x of t, but its Fourier transform then this convolution operation can be replaced by a multiplication operation. So, multiplication is a far easier exercise than evaluating these integral this integral.

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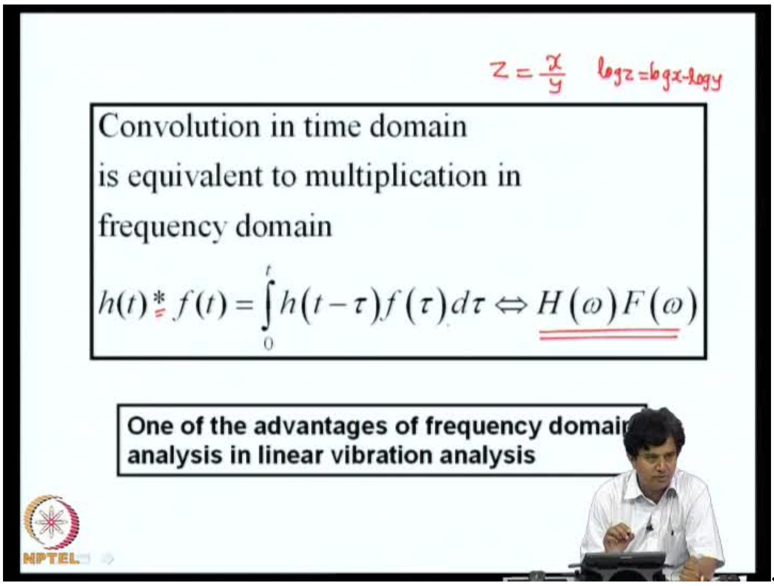

So, actually convolution in time domain is equivalent to multiplication in frequent zero. So, this notation star is use to denote this convolution operation, when I say h of t convolves with f of t, it means that the value of this quantity is this integral. Now, a convolution operation in time domain is equivalent to a multiplication in frequency. So, this is one of the major advantages of frequency domain analysis in linear vibration analysis. Analysis in frequency domain is far easier than analysis in time domain, this is like multiplying or dividing 2 real numbers. The best way to do it is to take logarithms, the difficult process of division, now becomes a process of subtraction. So, in the logarithmic domain you can easily find logarithm of Z and if you are equipped with a table of log logarithms and the so-called antilogarithms, you can find out Z by working only in the logarithmic domain in the same sense, we will not try to evaluate this integral in time, but we will go to the frequency domain and find the instead finding x of t I will find its Fourier transform, then this is equivalent to finding log Z and then I will do this so-called inverse Fourier transform and get x of t as desired.

So, this will be effective, if and only if, the movement from time to frequency and frequency to time is easy just like as you have a log table and antilog table it should be equipped with either a table of integrals or efficient algorithms to for moving from time frequency domain and frequency to time domain. The fact is that the very efficient algorithms known as fast Fourier transform algorithms, which enable you to move from time to frequency and frequency to time. So, here in lies the value of the frequency domain analysis, which we will be extensively using in a random vibration analysis. Again, let me, emphasize this is valid only for linear system because the construction of Duhamel's integral is basically dependent on the system being linear, because we are essentially using a principle of superposition and that is valid only if system is linear.

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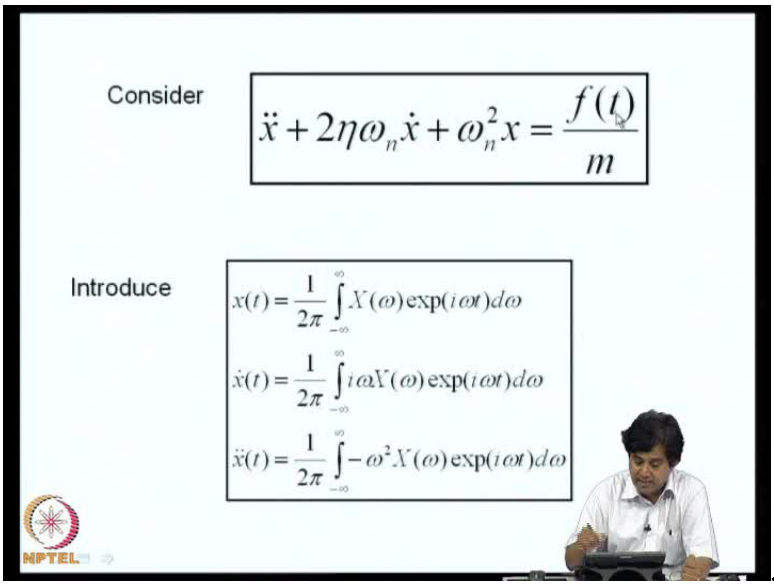

Now, till now, I have defined h of omega as the Fourier transform of h of t, but we also introduce a frequency response function, we were trying to define, what is the relationship between impulse response and frequency response function? Now, we will try to continue this discussion, suppose, if we consider the equilibrium equation and now for x of t i will write the Fourier transform and if I want x dot of t. I will differentiate this. So, this becomes i omega x double dot of t will be minus omega squared into this and i will substitute this into this and write f of t in terms of its Fourier transform.

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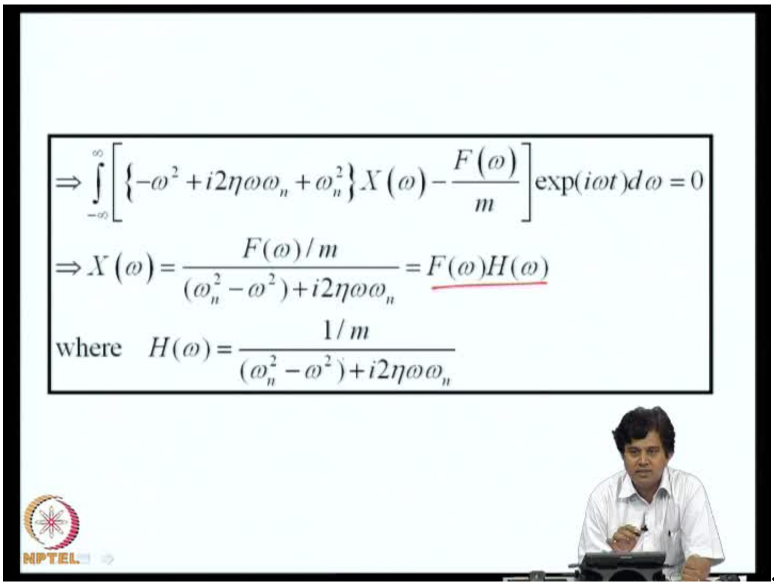

If I do that I get this expression and we get this x of omega, which is the amplitude of the response to be F of omega by m so on and so forth. And this we already know is f of omega into H of omega. So, if you now compare these 2 we see that H of omega is one by m omega n square minus omega square and so on and so forth. Now, this is, we have now two interpretations; one is that it is its frequency response function, other one is it is Fourier transform of H of t. So, that would mean the frequency response function and impulse response functions form a Fourier transform pair.

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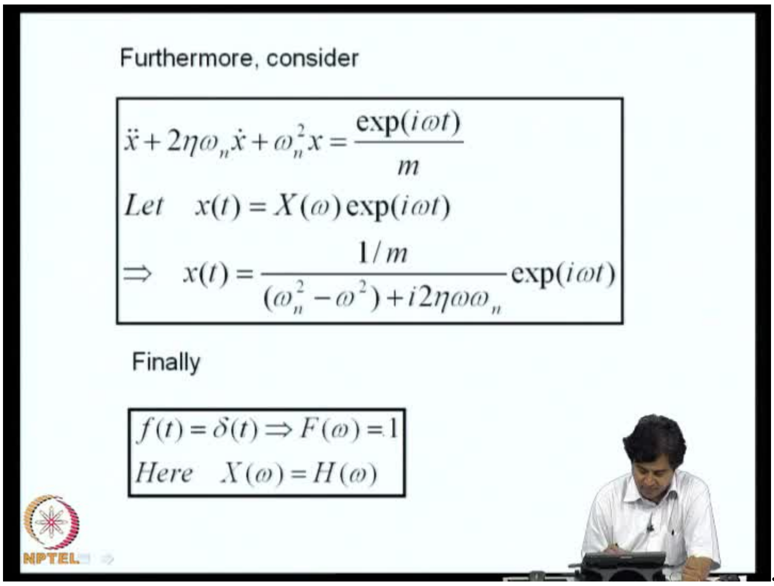

So, that is what I have written here, this is a frequency response function. And finally, we will notice that if f of t is a unit impulse, the response Fourier transform will be one and here x of omega will simply H of omega.

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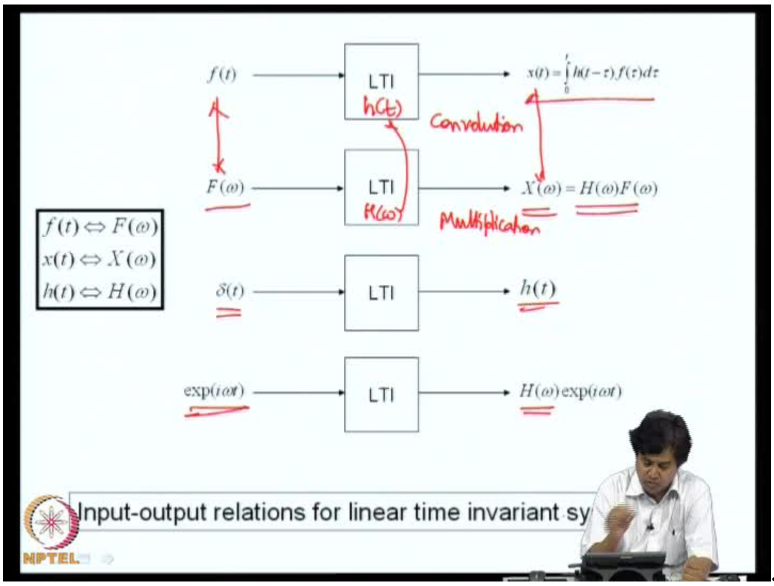

So, based on all these, the summary of this is that LTI is a linear timer invariant system that means system parameters are not functions of time. So, if a load f of t acts on this system and assume that system starts from rest, the solution is given by convolution integral. This is a time domain input-output relation, if you are not interested in time domain, but if you are specifying now, the input in terms of its Fourier transform and you are interested in Fourier transform of the output. The corresponding system parameter here is H of omega. So, this is if you know H of omega, you will get the Fourier transform of the response, here should know h of t, if you know h of t, I will convolve f of t with h of t and get my x of t. So, input output relation here is through convolution here input output relation is through multiplication, what are these h of t and H of omega h of t is nothing but response of the system to a unit impulse. This is h of t H of omega is nothing but amplitude of the response, when you apply a unit harmonic excitation this is H of omega.

So, this f of t and F of omega form Fourier transform pairs that is this and x of t and X of omega form Fourier transform pair and this h of t that is this and this also form for it Fourier transform pair. So, this is a nice you know a picture that emerges which in a nutshell constitutes the input output relations for linear time invariant systems in time and frequency domains, here am taking about description of the system. Therefore, the question of excitation being random or not does not arise. So, we have been defining that in terms of impulse excitation and harmonic excitation.

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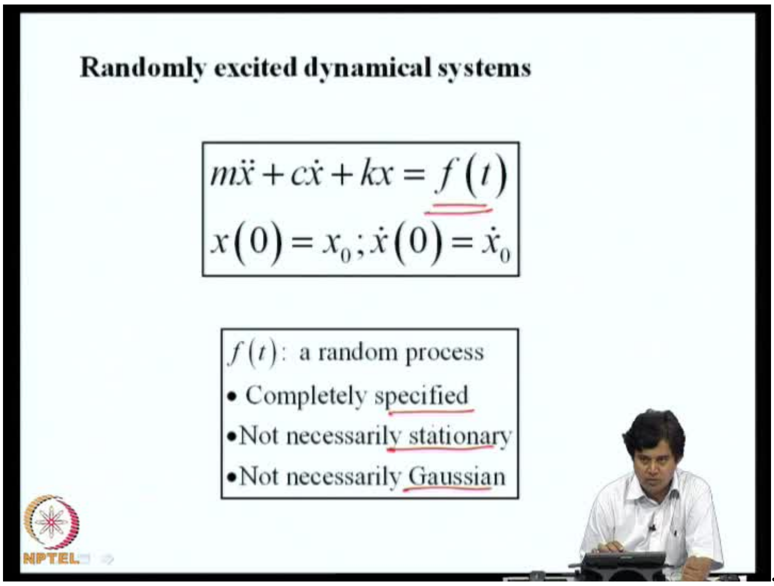

Now, we start discussing about response of the system, if excitation is the random process. So, the equilibrium equation from of the equilibrium equation, once remain the same. Here, this f of t is a random process to start with will assume that it is completely specified that means its nth order joint probability density function is more, it is not necessarily stationary, it is not necessarily Gaussian. So, induce course will be limiting our attention to stationary Gaussian random processes in which case the complete specification is to mean and covariance, but right now, while formulating the problem that is no such restrictions, what is the meaning of this equation, what is what does it mean.

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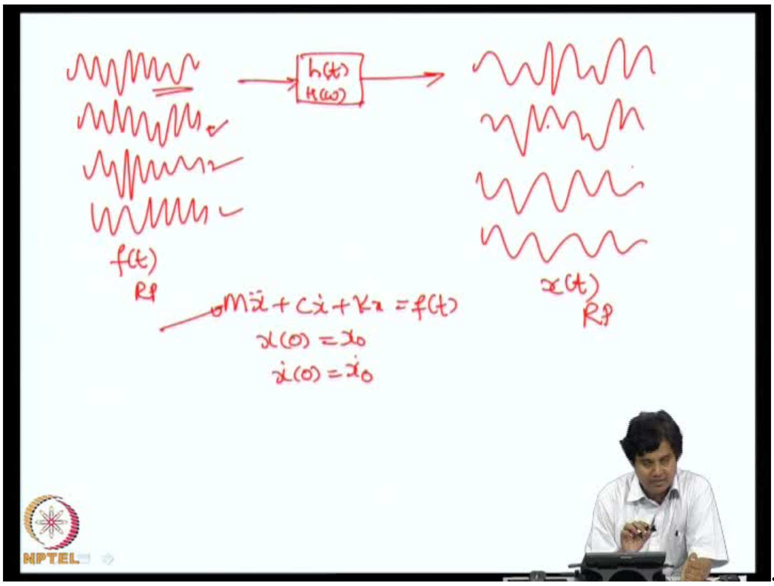

Here, we have a system, f of t is random process, may this is a collection of time is to use. So, you can assume that the system is excited by this time history f of t and it produces as the response x of t. This sample put produce one more response time history, this sample will produce another one. This will produce at another, so this system itself is character in terms of h of t or H of omega or it is equilibrium equitation in time domain. So, this will convolution with h of t and produce x of t. This will convolve with h of t and produce this function. So, if this is a random process that if f of t is a random process, x of t also a random process, this also random process. So, in a write this equitation mx double dot pulls cx dot plus kx equal to f of t with the associated initial conditions h unit. This equation itself represents an on some sample of equilibrium equation, because f of t is a sample consequently f of t also a sample. So, this representation of family of differential equations in burred force analysis, we can as well take a sample of f of t get a corresponding sample f of f x of t. So, if the problem is given to understand f of t, how to find x of t? It can be a versed as a large collection of deterministic analysis, but that is not what we mean by random vibration analysis.

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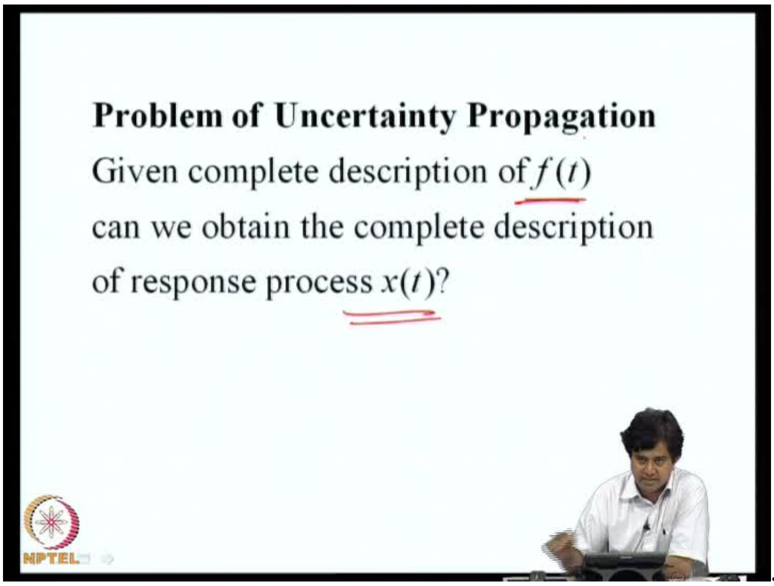

What we mean by random vibration analysis, is that we are modeling f of t is a random process. That means f of t has certain uncertainty associate with that and we are characterizing those uncertainty, in terms of theories of probability random variables, random processes. So, we are modeling f of t is a random process, implicit in that statement is that certainty in f of t and quantified; we want a similar description of x of t that means, how does uncertainty measures associated with f of t propagate through the system and produce uncertainty in the response. So, this problem is known as problem of uncertainty propagation, what it means?

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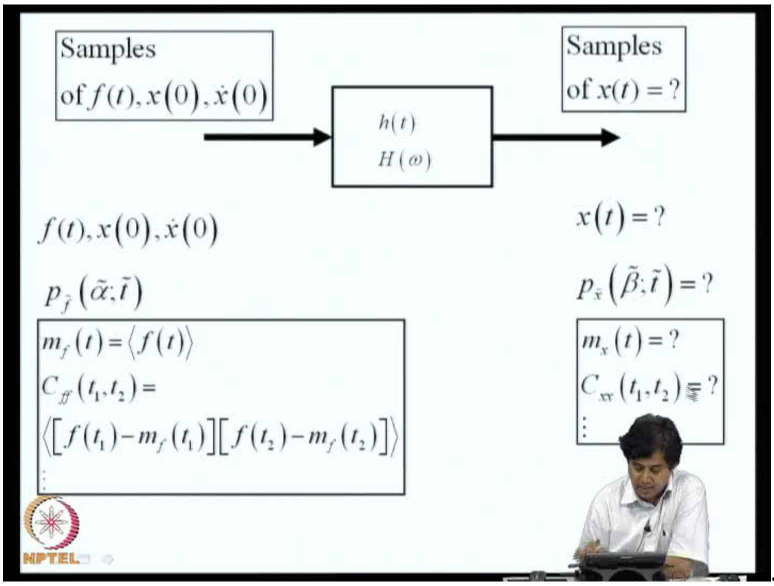

So, again I will consider the linear time invariants system, which are character impulse response function are complex frequency response. So, one way to look at is just as we saw, now we have samples of f of t and initial conditions and we need samples of x of t – right - but other way if looking at is that means given f of t x of 0 x dot of 0. What is x of t? Other question, that we can ask is given the nth order probability density function of f of t, which actually constituted the complete description of f of t is a random process. what is the complete description of x of t as a random process that is what is the nth order probability density function of x of t; set of much simpler question would be if m of t is mean of f of t that is expected value of f of t? What is the expected value of the response, if you know the expected value of the input, what is the expected response? Similarly, if you know what is the covariance of f of t that means you select 2 time is t 1 and t 2 you are 2 random variables and consider the this expectation if you are given this c of f of t 1, t 2 that is the covariance of the excitation. How do we get the covariance of t1, t2? Now this is the question that we need to know address.

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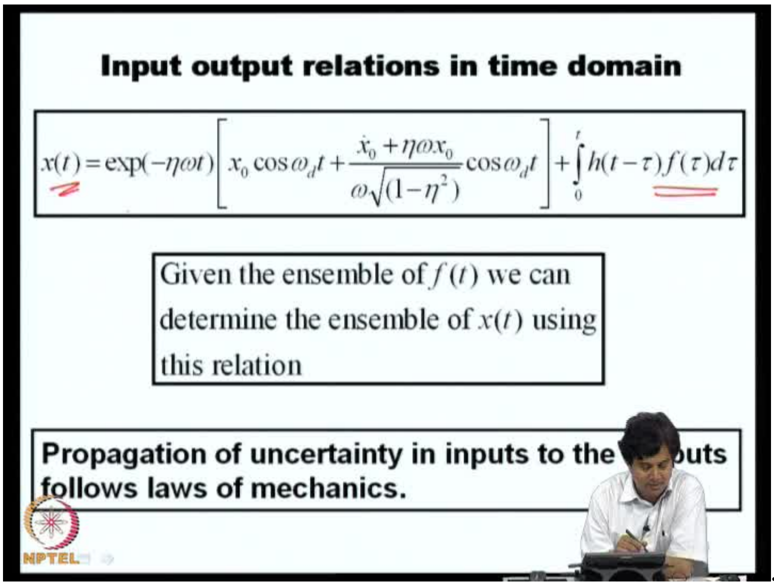

So, we starting point for this would be the input output relation in time domain will start it time domain, because its more general it include transient, it also alerts for excitation which cannot be represented in terms of Fourier transform so on and so far so. The input-output relation, we are just now discussed can be given in terms of this expression. The first term here is contribution due to non-zero initial conditions and second one is the Duhamel's integral, which represent as a complete solution systems start form rest. Now, look here f of t is here, our excitation is here. So, this is actually non-symbol of excitation and that at least to non-symbol of x of t. So, given non-symbol of f of t we can determine non symbol of x of t using this relation, but what is most important to notice here is that the uncertainty associated with f of t propagates through the system a produce x of t and the this propagation uncertainty in inputs to the outputs follows loss of mechanics, if f of t is Gaussian, how do we say what should be power to distribution x of t, to answer that question, we have to write equations of motion is inter elements principles of some variation argument and then only you can answer that question that means the subject. Now, combines the quantitative description of uncertainties in the inputs with the theory of vibration analysis to obtain the quantitative description of resulting uncertainties in x of t. So, this is the basic problem in so calls to as stochastic structural dynamics are random vibration analysis. How does uncertainties in inputs propagate obeying the loss of mechanics of the problem, you cannot arbitrarily impose a model on x of t. Let x of t be log normal or Gaussian set that kind of feel we do not have you have to start with modeling in the force and allow that to propagate through to the dynamic to the system and then arrive at model for x of t.

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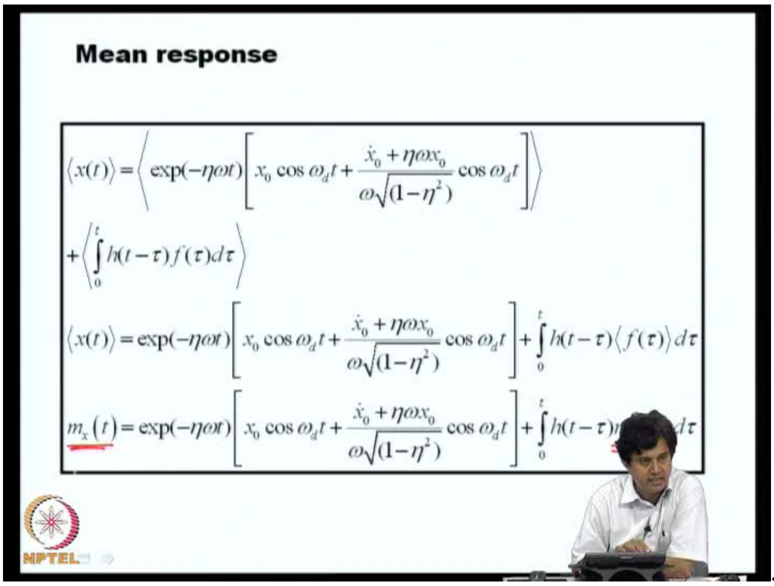

So, let us start now, suppose, I am interested in mean of x of t, so i will take a expectation of x of t, which is expectation of the first term and the expectation of the second term, if you assume initial conditions to be deterministic they could be as usual to random, but for our analysis, let as assume initial condition are deterministic this an expatiation of a constant. Therefore, that is remains as it is now, this expatiation includes this f of tau therefore, i can write this second term as h of t minus tau expected value of f of tau d tau expected value of f of tau d tau is a deterministic quantity which is nothing but mean of f of tau. So, the mean of the response is related to the mean of the load through this relation.

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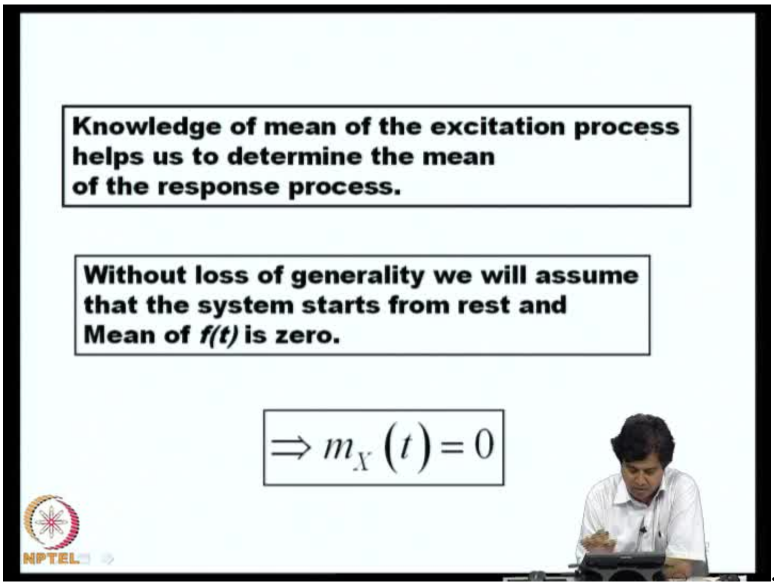

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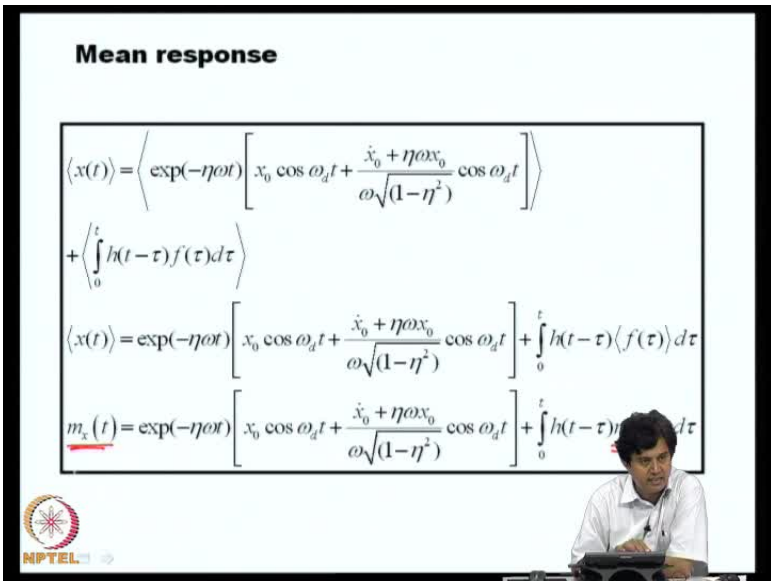

So, if you know the mean of the excitation, you can find mean of the response, that is, knowledge of mean of the excitation process helps us to determine the mean of the response process. Now, further discussion, what will do is will assume that mean of f of t is 0 and will also assume that system start from rest, that would mean x naught and x naught dot are 0 and m of tau is 0. Therefore, m x of t is 0, if they are not 0 for instance, if m f of tau is not 0 or x naught or x naught dot are not 0. This is the prescription for finding the mean. So, we are not really losing any generality in our approach, if you now set this to 0 if it is not 0, we can always add this component.

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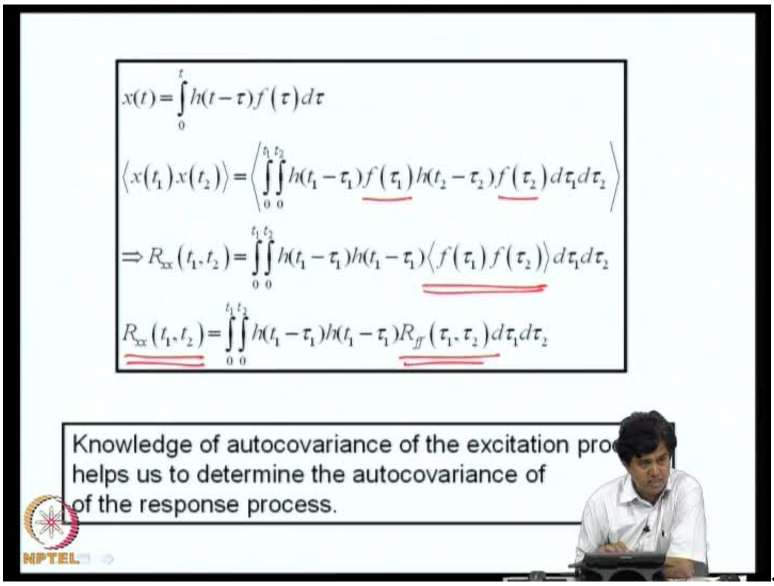

Now, with that mind, we will now proceed; the response, will now consist of only the Duhamel's integral, because system start from rest. Therefore, this is complete solution of the system and now I will consider the expected value of x of t 1 into x of t 2 that is nothing but the auto correlation function of x of t. The since means is 0, the auto correlation function is also auto covariance function. So, that is given by expected value of this product of this integral that is becomes double integral and that is shown here. The f of tau 1 is here f of tau 2 is here I can rearrange the terms and will assume that this integration and the integration associated with the expectation are interchangeable, if i do that the expectation operator can be taken inside the integral and I get this as the expectation. Now, this is the deterministic quantity, because it is nothing but the auto covariance of f of t, if I know that through this relation, I can get auto covariance of x of t. So, how does auto covariance of the input translate into auto covariance of the output it is through this relation, which is nothing but the Duhamel's integral which has roots in mechanic, so knowledge of auto covariance of the excitation process helps us to determine the auto covariance of the response process.

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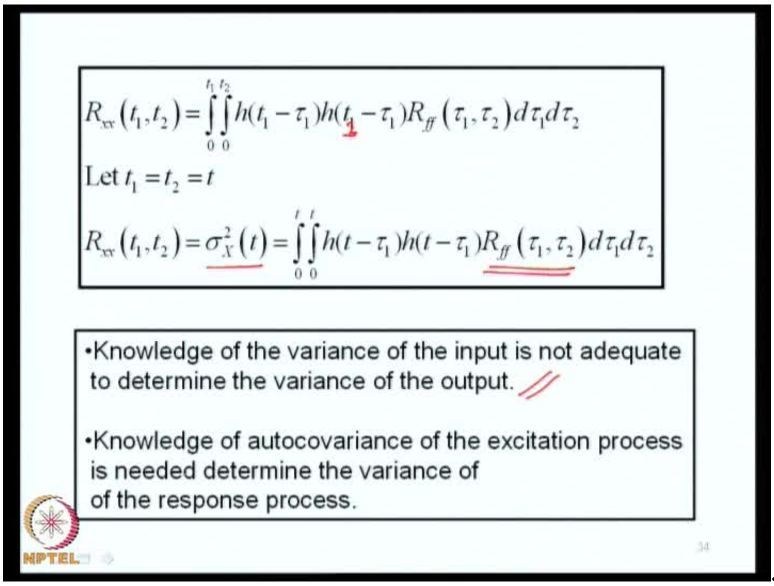

Now, if you now let this t 1 to be equal to t 2, the auto covariance function is nothing but, the variance. And variance of the response can be written in this form. So, this t 1 and i think this is t 2 the t 1 and t 2, become the same I call it as t but I still need to know the auto covariance of input that means, if you interested in variance of the response and if you happen to know only the variance of the input, you will not be able to determine the variance of the output that means given the variance of excitation, you cannot find variance of the response. So, to find variance of the response, you need the auto covariance of the excitation, but if you interested in auto covariance of the response auto covariance of the excitation is adequate. So, knowledge of the variance of the input is not adequate to determine the variance of the output.

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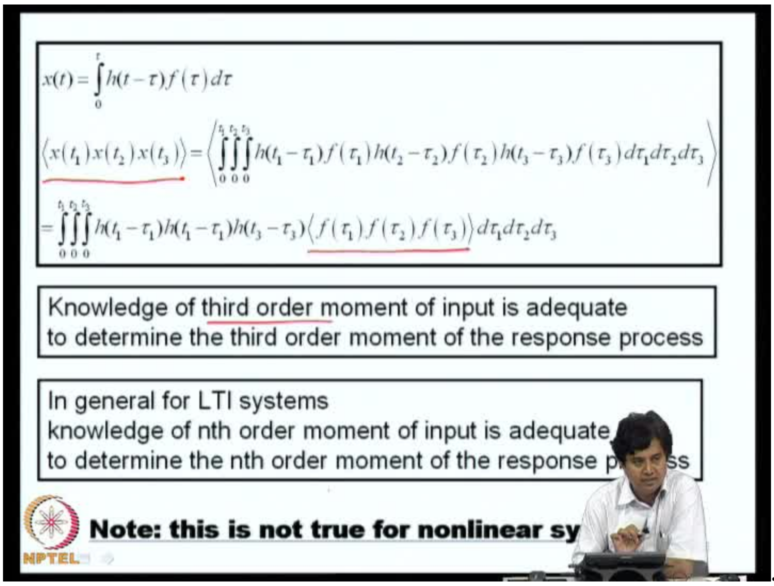

Now, we can continue this argument and we can also find higher order moments. Suppose, you want third moments expected value of x of t 1 into expected value of x of t 2 sorry expected value of x of t 1 into x of t 2 into x of t 3 it will be an expected value of triple integral. Now, if you know the third order moment of f of t, you can find the third order moment of x of t. This is higher order correlation function, we can say call them by that name, so knowledge of third order moment of input adequate to determine the third order moment of the response process. So, you can generalized this and say that for linear time in variance system knowledge of nth order moment of the input is adequate to determine the nth order moment of the response process, you must note that this is not generally true, this is true only for linear systems and if the system is non-linear, you will not be able to do this, if you want to find mean of the response, you will have to know that you will not able to find that. So, later in the course, I will elaborate me are knowledge of mean of the input is not adequate to find mean of the response for non-linear system.

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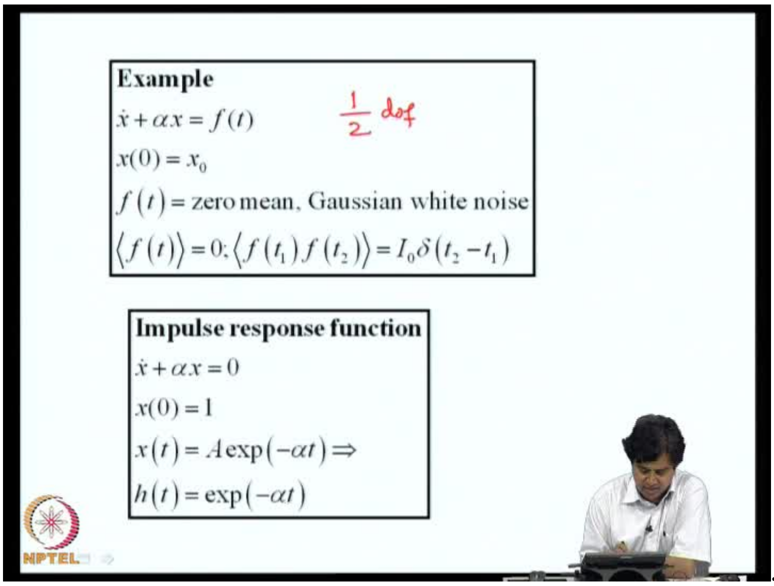

Now to explain the details of what we discussed till now. Let us consider a very simple example, we will consider a dynamical system, which is governed by a first order differential equation you can think of this as a single degree freedom system, where mass is extremely small inertial effects are negligible. So, you can think of this as a half degree of freedom system, if a second order differential equation describes a single free degree system. This can be taken as describing a half a degree freedom system. So, x dot plus alpha x is x of t x is scalar x of 0 is x naught.

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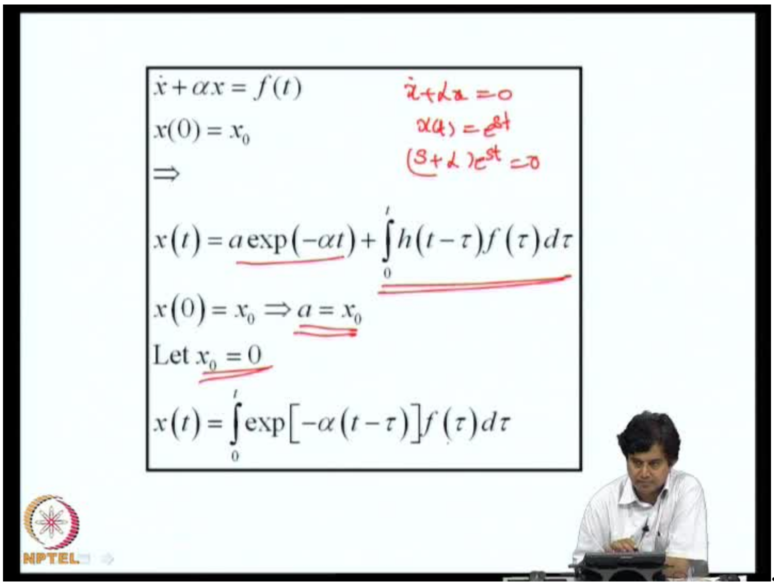

Suppose, f of t is 0 mean Gaussian white noise process by that I mean its mean is zero expected value of f of t 0 and if you find the covariance, it is a Dirac delta function. So, a question that now am going to ask is characterize x of t 2, first and foremost is we have to write the Duhamel’s integral that relates x of t to f of t for that I need an impulse function. So, I use the generalized definition impulse response function. So, this is x dot plus alpha x equal to 0 and this is n equal to 1 and n minus 1 nth derivative should be 1. So, x of 0 is 1; so, based on that I get h of t is exponential minus alpha t for this system.

Now, therefore, x of t would be the complementary function to construct complementary function I take x dot plus alpha x equal to 0 and take x of t as sum e raised to st and substitute here, I get s plus alpha into e raised to st equal to 0. Therefore, s equal to minus alpha and the complementary function is a exponential minus alpha t the particular integral is expressed. Now, in terms of the Duhamel’s integral - mind you, h of t e raised to minus alpha t it is not that 1 by m omega t sin omega t d t e raised to minus eta omega t. Now, have to find this constant of integration x of 0 is x naught. So, that would be mean a is x naught for purpose of simplification of the discussion, we will take that x naught is 0 if that happens x of t is 0 to t exponential minus alpha t minus tau f of tau d tau.

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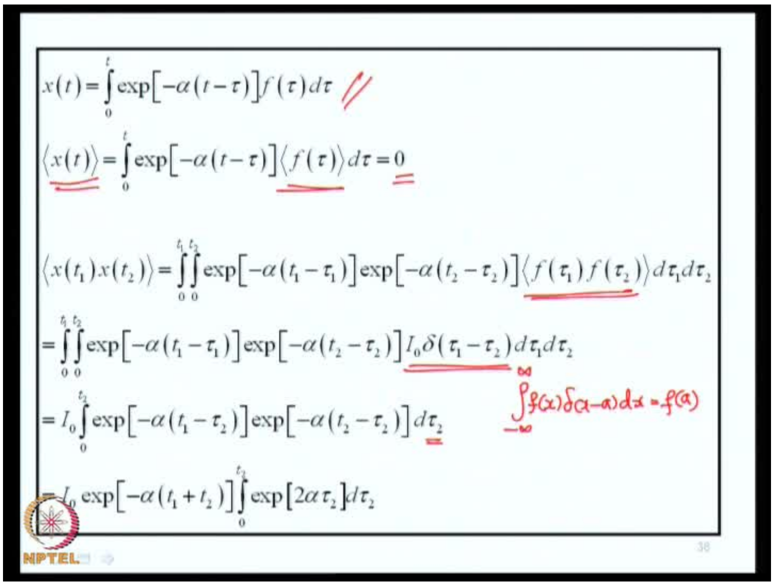

Now, so I have the input output relation in time domain given by this expression f of t as you know is a 0 mean Gaussian white noise random process. So, we will consider the expected value of x of t and this will be expected value of x of t into expectation of this integral and if we now take the expectation operator inside that will operate on f of tau f of tau d tau and this we know is 0 f tau is 1. Therefore, expected value of x of t is 0. Now, how about covariance? To do that I take expected value of x of t 1 into x of t 2, this is this double integral and the expectation operation is inside here and this, we know since f of t is a Gaussian white noise process, I can write this covariance in terms of Dirac's delta function.

Now, if you recall the definition of Dirac's delta function is delta of x minus a dx is f of a. So, integration when there is a Dirac delta function in the integrant is a very straightforward exercise. So, one of these double integrals can easily be evaluated. So, i will replace tau 1 by tau 2 and this integral becomes this. Now, this is the reasonably simple enough integrant. So, we can evaluate this, so I will rearrange the terms I will take out t 1 and t 2 terms outside, because this integration is with respect to tau 2 and I get this expression, this can easily be integrated.

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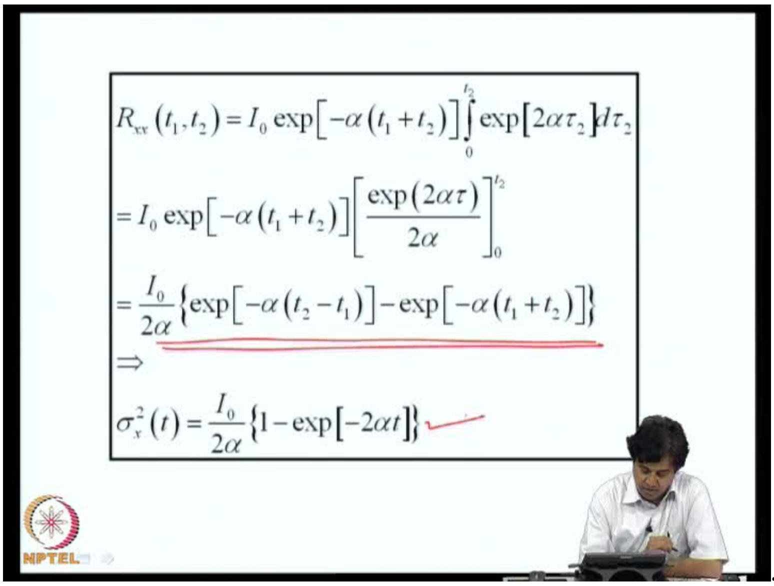

So, I get R xx of t 1, t 2 this is integral and if I do this integration, I get the covariance function to be given by this, if I now take t 1 equal to t 2, I get sigma x square of t is given by this expression. So, what we can say about x of t now, x of t is a non-stationary random process although the excitation was a stationary random process; the response is non-stationary.

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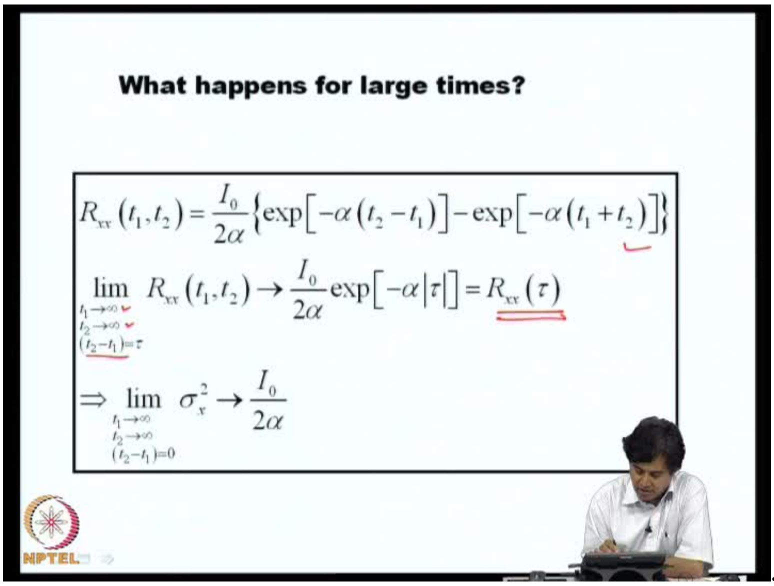

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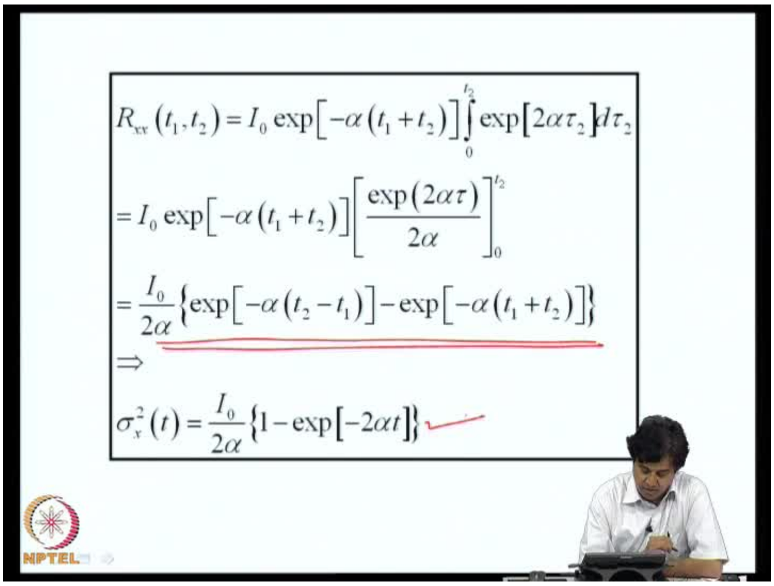

But what happens for large times? That is, we have this expression for covariance in this. Suppose, if I take t 1 becoming very large and t 2 becoming very large, but the time lag t 2 minus t 1 is tau, tau doesn't become large t 1 and t 2 can become large. If I do that you can see here the first term here will be exponential minus alpha tau and the second term as t 1 becomes large and t 2 becomes large goes to 0. So, under this limiting operation the covariance function, which is function of t 1 and t 2, now becomes a function of only tau. And what happens to the variance? You put t 1 equal to t 2. This tau becomes 0 and this quantity is 1. Therefore, this becomes 1, that you can also here in this expression as t becomes large, this exponential minus 2 alpha t goes to 0 and I will be left with I naught by **2** alpha.

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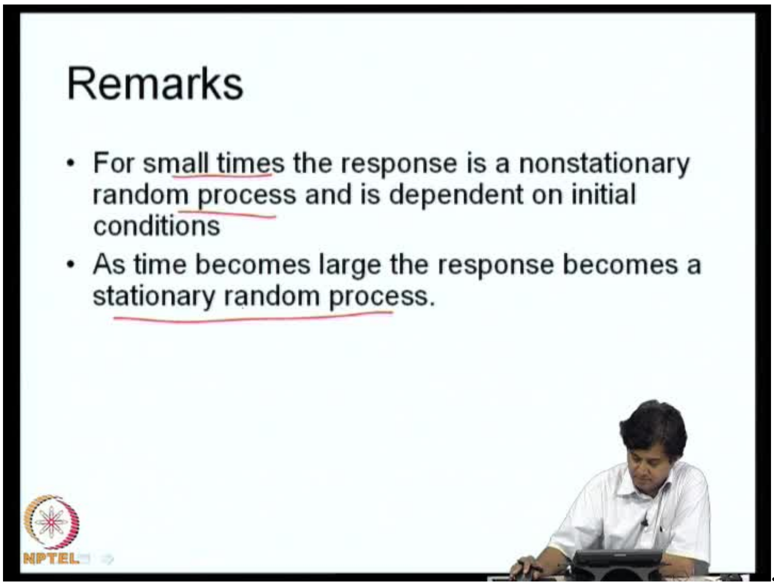

So, how does this look like - that means - we can make some remarks the now that: for small times the response is a non-stationary random process and is dependent on initial conditions. As time becomes large the response becomes a stationary random process. And is independent of initial conditions, so this reminds us of the steady state that we talked under harmonic excitations for linear systems, so when this happens that means as time becomes large, the mean is anyways 0 and the covariance becomes function of time difference; we say that the system has reached a stochastic steady state.

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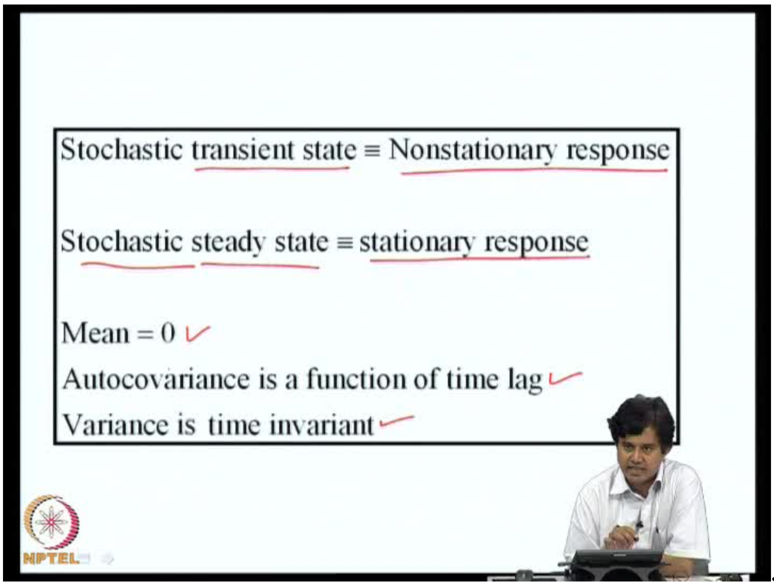

So, therefore, we talk about a transient state and a steady state. In transient state that is in stochastic transient state, the response is a non-stationary random process, in the steady state the response becomes a stationary random process that is mean is 0 and auto covariance function is a function of time lag and variance becomes time invariant. So, this is the definition of a wide sense stationarity and we agreed that this is our default definition of stationarity.

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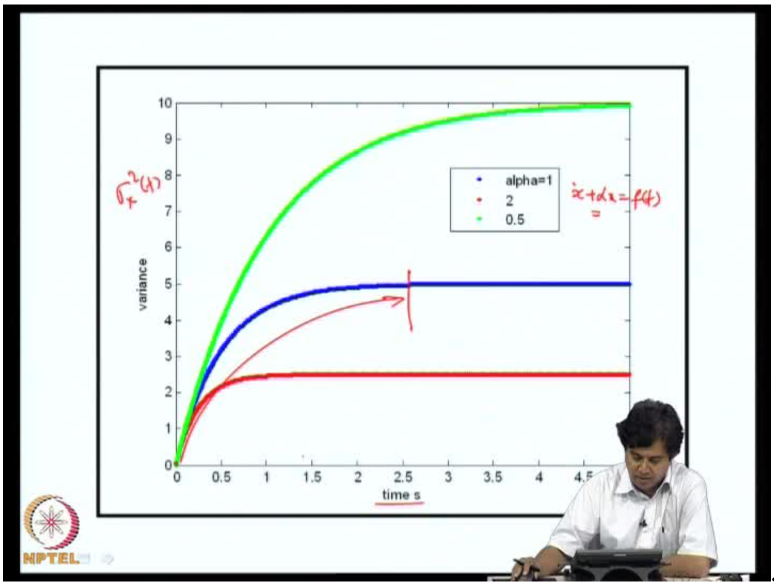

Here is a plot of variance that this is actually sigma x square of t as a function of time. The system starts from rest and there are different alpha here. Alpha, if you recall this system is x dot plus alpha x is equal to f of t and this alpha refers to this. So, if you see here, suppose, if you follow this red line, we see that for time say up to say 1.5 seconds the variance is growing and after 1 second it becomes a constant. Similarly, for a different value of alpha, this was for red was for alpha equal to 2, the blue say alpha equal to 1 indeed as time becomes large it reaches a different steady state and not only that it takes a longer time to reach the steady state. So, in this transient phase here the variance is still increasing.

So, if alpha becomes still smaller it reaches a higher steady state, all right, but it takes a longer time to reach that steady state. So, by depending on value of alpha there are different steady state possible not only that the time to reach steady state also changes.

So, that is fairly obvious, here, if you look at the expression, the time required for sigma x square of t to become constant depends on how fast this function decays to 0 and that is essentially governed by the parameter alpha.

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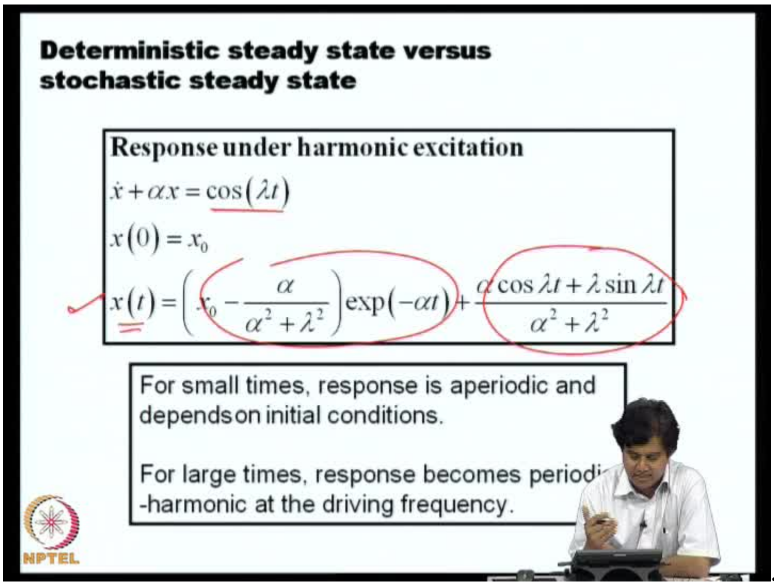

Now, we can quickly recall what happened, what would happen, if this system is driven harmonically. So, this is a reasonably straightforward exercise, you can write a complementary function and a particular integral and evaluate the arbitrary constant using initial conditions, if you do that response of this first order system can be shown by can be shown to given by this is deterministic. So, here again if you observe this expression, you will see that x of t is aperiodic, because that is the exponential minus alpha term, this part is still periodic, but this part is aperiodic that means for small times the response is aperiodic and depends on initial condition that is effect of x naught is still felt.

As time times becomes large this exponential alpha t starts decaying and this terms goes to 0 and we reach the harmonic steady state. So, there is a good analogy between harmonic steady sate and stochastic steady state transients and you know what is transient here is non-stationarity there, what is harmonic steady state is stationary.

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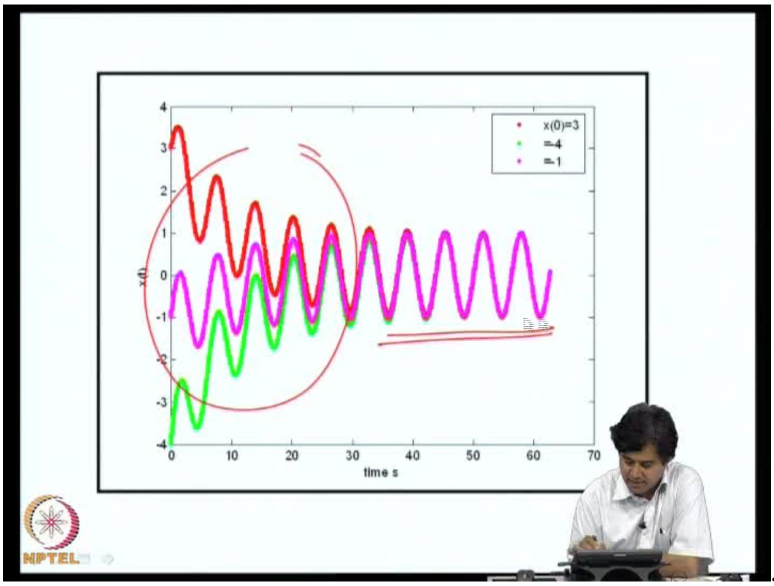

So, if you were to plot time histories of x of t, for same value of alpha but different initial conditions, we see here that the alpha is same for all these three trajectories, but different initial conditions are given. So, they take certain time to reach steady state. So, here we have reached steady state here all the 3 trajectories are almost sitting on each other but, they have different transients here, that means for small time the response depends on initial conditions and is aperiodic, but for large time it becomes independent of initial conditions and it reaches as steady state in the sense amplitude of this response and the phase difference with excitation becomes constant.

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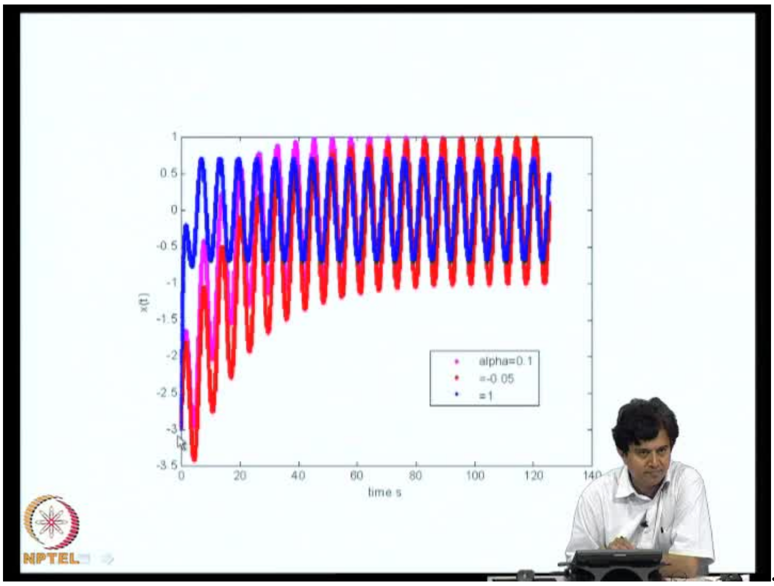

Now, there is another plot here, where I have different alphas but with same initial condition. How different systems take, you know different pass to reach different steady states. So, red graph is for alpha equal to point naught 5 it reaches a different steady state; blue reaches a different steady state and this magenta also reaches a different steady state. All of them start from same initial condition, so this is how we have a kind of analogy between deterministic steady state and stochastic steady state.

So, we will continue this in the next lecture.