Stochastic Structural Dynamics

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Module No. # 01

Lecture No. # 01

Introduction to Probability and Random Variables

This is the first lecture on the course on stochastic structural dynamics.

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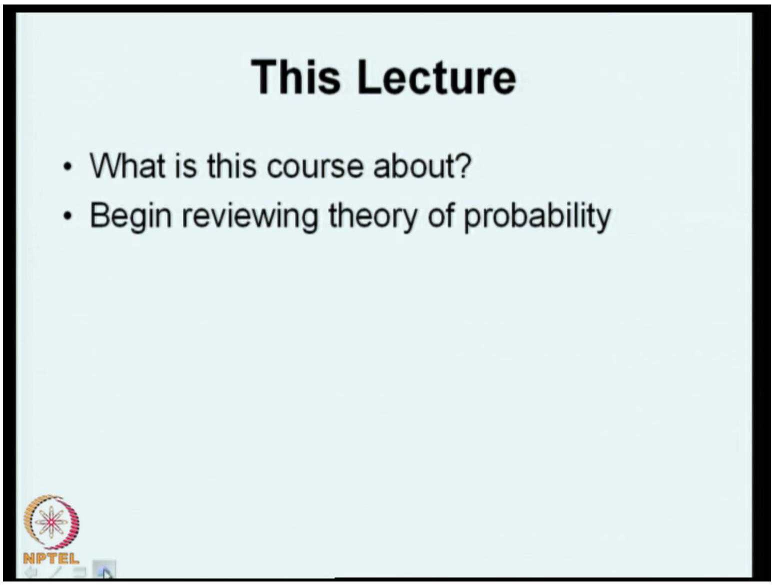

In this lecture, what I will do is, I will tell what is this course is about and will begin reviewing theory of probability, which is the basic mathematical tool required for this course.

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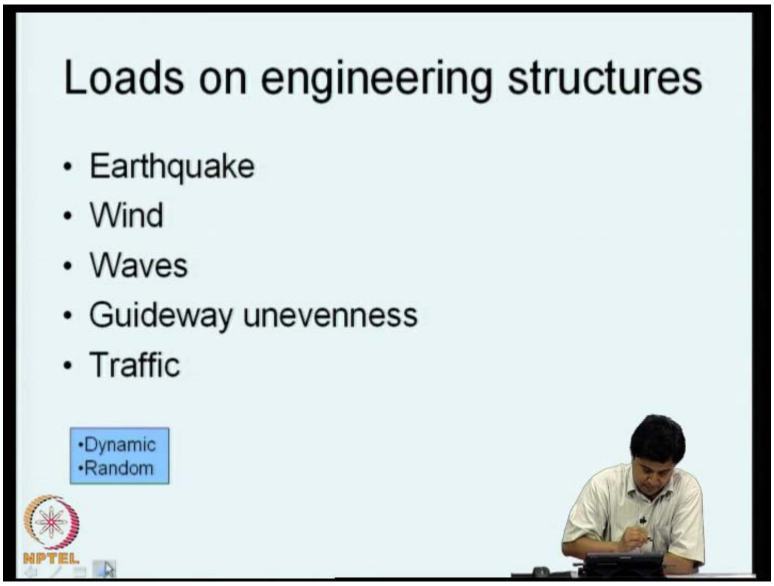

So, let may begin by outlining the scope of this course. Engineering structures are subjected to various kinds of loads such as earthquake, wind, waves, guide way unevenness for vehicles, and traffic loads on bridges. One common feature of all these loads is that they essentially are dynamic in nature. And, another thing that you would notice that they are random.

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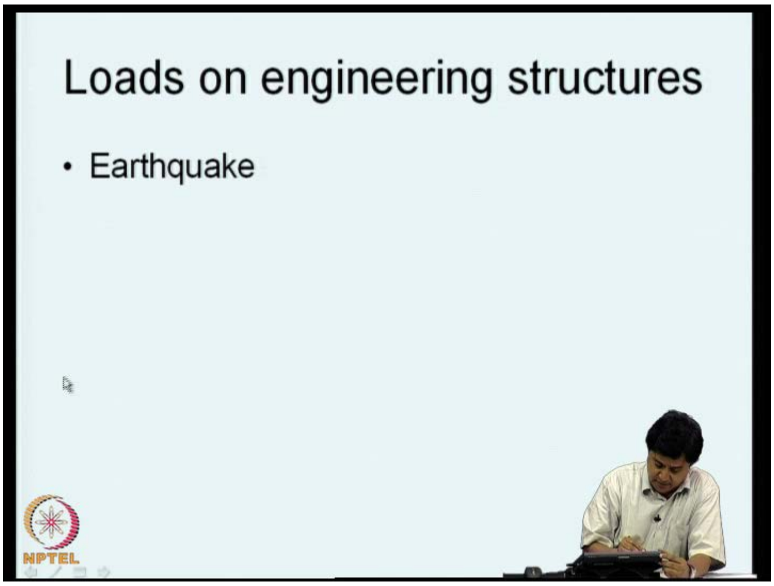

So, I need to explain what this annotation means. So, let us begin with the case of earthquake loads on engineering structures.

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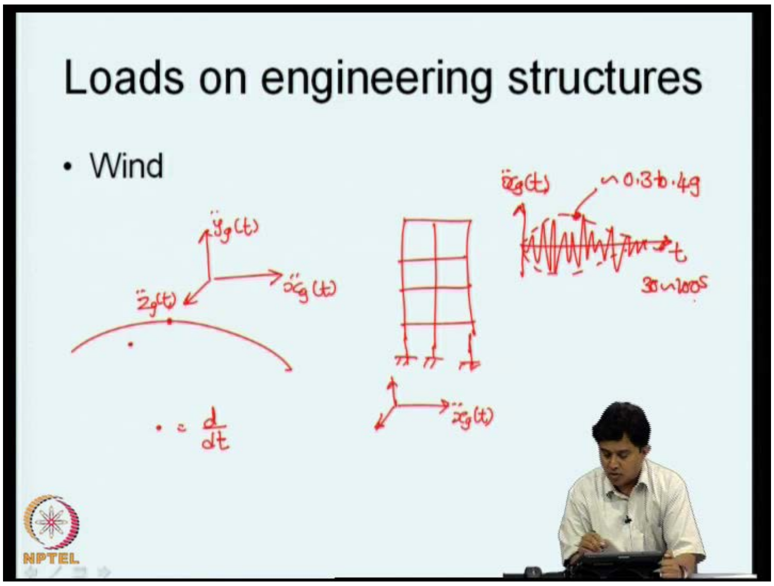

See earthquake is a natural phenomenon. Imagine this is surface of the earth. Somewhere in the crest of the earth, there will be energy release and this energy gets accumulated due to various tectonic processes that take place in the entry of the earth. And, this energy release propagates as waves. And, any point on the structure during the event of an earthquake, undergoes time dependent displacements. And, associated with that displacement, there will be acceleration. So, at any point on the surface of this structure, there is acceleration. And, as you know, acceleration is a vector, which can be resolved into three components. So, let us call this as x g double dot of t, y g double dot of t and z g double dot of t. The g here refers to the fact that I am talking about ground acceleration. t is a time variable; and, a dot has we all use in theory of vibrations indicates differentiation with respect to time.

In the event of an earthquake, all structures, which are in touch with the ground, for example, a multi storable building frame will be subjected to these accelerations. Due to these accelerations, the structure will now displace and these displacements are dependent on time. Therefore, the structure not only displaces, it also accelerates. And, the inertia of the structure will oppose this acceleration. And, this complex interplay between opposition to displacement through the stiffness and opposition to acceleration due to inertial properties results in the phenomena of vibrations. So, if you now look at one of these components; typically, if we plot the time history of (Refer Slide Time: 03:58) this ground acceleration, it has an appearance of kind of something like this. So, this is acceleration on y-axis. And, these peak values are typically of the order of 0.3 to 0.4 of acceleration due to gravity. And, the time duration over which these oscillations take place is about 30 to about 100 seconds. These are typical; they can be more or less.

Now, you have noticed that the time history here is random. I am sure, intuitively, you would appreciate what that random. When I say it is random, you would appreciate what it means. So, in due course, we will make that meaning of that phrase more explicit. So, on account of these inputs at the support level, the structure would also oscillate in a somewhat similar manner and we say that the structure is undergoing random vibration. Predominantly, during an earthquake, it is found that the ground acceleration in the horizontal direction, namely, these two will be significantly higher than this vertical component. That would mean that in the event of an earthquake, this structure would be subject to horizontal accelerations. And therefore, the structure needs to resist horizontal inertial forces. And, that is one of the challenges in doing earthquake-resistant design.

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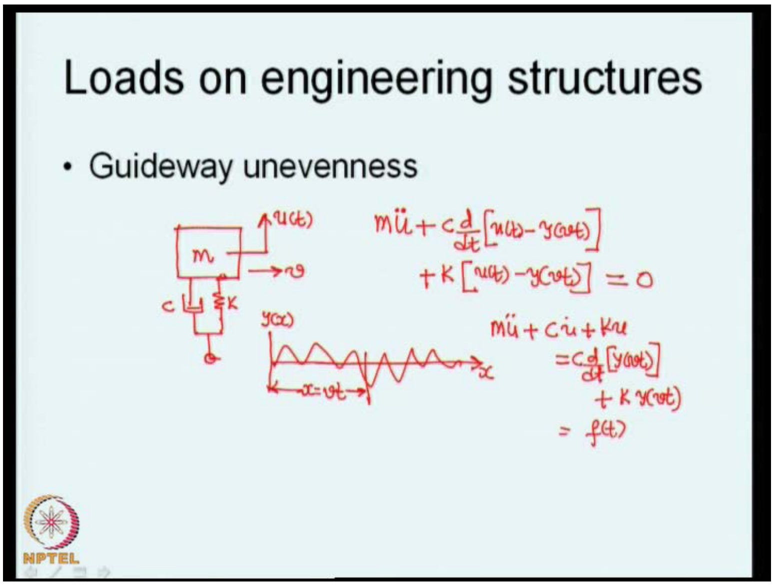

Let us look at vibrations due to guide way unevenness. So, here what we do is, let us consider a simple model for a vehicle. This is one of the simplest model for vibrating systems. This is the mass element; this is the damper element; this is stiffness element; and, this is the wheel of the vehicle; this is the mass of the vehicle. And, we assume that the vehicle oscillates in the vertical direction. And, this vehicle – let us consider what happens as it traverses on a road whose surface is undulating.

Let us assume that this vehicle is moving with the velocity v. Now, if we write the equation of motion for this vehicle oscillation, you can easily imagine that this vehicle will oscillate due to passage over this rough road. So, if you write the equilibrium equation, you can see that it will be of the form – this is a inertial force plus the force in the damper; force in the damper would be proportional to the velocity – actually the relative velocity between these two points (Refer Slide Time: 07:10). This point is moving as u of t; whereas, this point is moving on this rough road. So, at any time t, if it is moving with constant velocity, it will be at a distance v t. So, this term will be u of t minus y of v t; d by d t of this is the velocity and this is the damping coefficient. Similarly, the forced in the spring will be u of t minus y of v t equal to 0. This is an equilibrium equation.

Now, we can rearrange this as m u double dot plus c u dot plus k u is equal to c d by d t of y of v t plus k into y of v t. So, this (Refer Slide Time: 08:15) if we take as a forcing function f of t, here the nature of f of t depends on the geometry of these undulations and also with the velocity with which this vehicle is moving; not only that, the stiffness of the vehicle, the damping of the suspension, so on and so forth. So, all of these could be uncertain. Consequently, the force that acts on this structure, that is, this vehicle would itself be answered. So, we say that this vehicle is undergoing random vibration due to the guide way unevenness.

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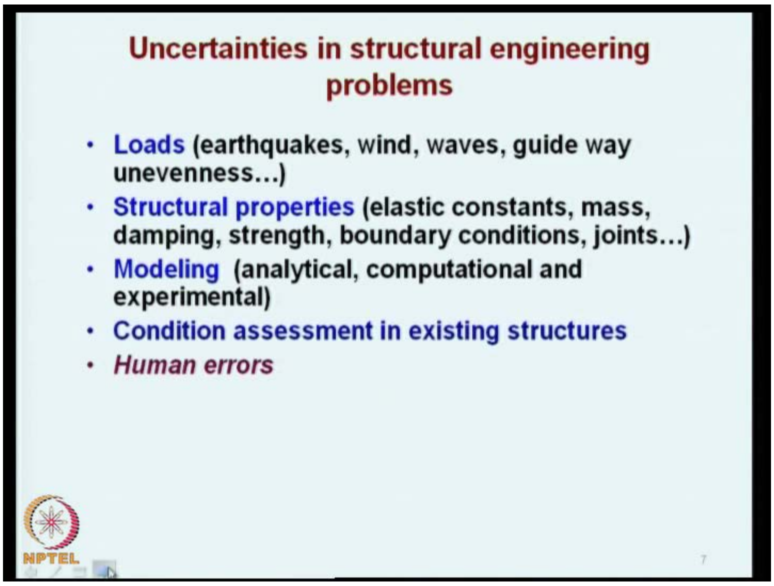

Now, we talked about earthquakes and guide way unevenness. So, we could also talk about wind.

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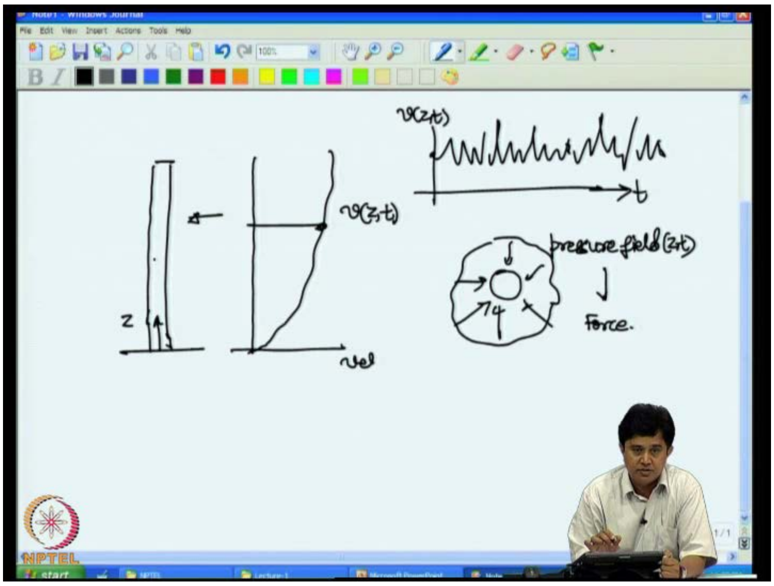

In case of wind, what happens – for purpose of discussion, let us assume that we are talking about a tall chimney and wind is blowing in this direction. Suppose if I plot the wind velocity along this direction and suppose this is coordinate z, the wind velocity profile typically is something like this. So, this is v of z at a time t. It is a snapshot taken at time instant t and this is how it varies in space. This is known as the atmospheric boundary layer; where, the velocity of wind flow is zero on the surface and it reaches steady value as we go higher up in the atmosphere. So, this structure will be immersed in this flow velocity. If you now take a point on the structure at an elevation z and plot the velocity of wind as the function of time now, z is fixed. So, this will be… There will be some mean component about which there will be oscillation. So, this goes on and on and on. So, this is velocity.

Now, if I look at this chimney in plan, (Refer Slide Time: 10:37) imagine that this is a cylinder. So, there is a flow past this cylinder and because of which there will be some pressure field that will be set up, which acts on the structure. This is the pressure field. This pressure field is function of both z and t. And, if you integrate this pressure field over this surface, you will get a force. It can be shown that this force is proportional to square of this velocity. So, consequently, any erratic behavior of this velocity will manifest as a erratic load on the structure. And, we say that this structure is undergoing random vibration under wind-induced loads.

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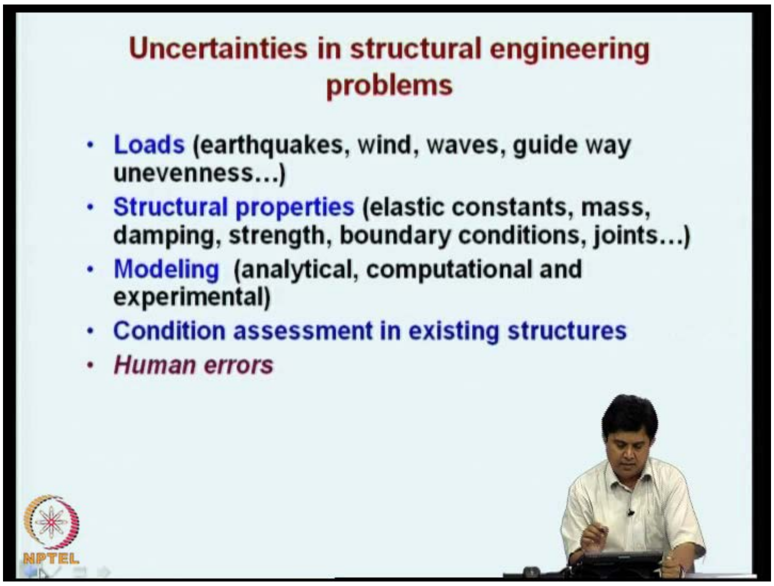

I talked about uncertainties in loads – earthquake, wind, guide way unevenness. You could also imagine what would happen to say a tension-like platform in a ocean or ship. You could also imagine other context such as wind flow across an aircraft wing. So, they all have a common feature, namely, the loads acting on the structure will be random in nature. They are uncertain. This is a first order description of these actions. We will quantify them in more precise terms as we go along in the course. It is not only the loads, even the structural properties like elastic constants, inertial properties, damping, strength, boundary conditions, joints, etcetera are also uncertain. For example, if you take a joint in a truss member, say, a lattice girdle bridge truss, we do not know whether it is a perfectly pinned end or it is a perfectly fixed end. For mathematical idealization, one could assume that they are pinned, but in reality, there could be a partial fixity. So, there is uncertainty about the flexibility of the joints. Similarly, boundary condition, typically in modeling, we assume they are free or fixed or hinged. But, in reality, they can be quite complex. So, they also introduce uncertainties in our modeling.

Of course, the mathematical modeling itself is subject to uncertainties. To model a beam, one engineer may use Euler Bernoulli beam theory; another person may choose (( )) beam theory. Similarly, in modeling, energy dissipation characteristics – one person may assume proportional viscous damping; another person may model that using friction, colon model and so on and so forth. So, in any modeling exercise, the analyst will have to take certain decisions. And, these decisions or not canonical in the nature; in nature in the sense that these decisions can vary from person to person. So, again there is a question of uncertainty in modeling. There is another area of modern structural engineering, namely, condition assessment of existing structures. Here again there are uncertainty of a slightly different kind. The uncertainties here are on measurement of properties of the structure as it exists – measurement of the response of the structure to applied actions; the level of degradation that has taken place in the structure, because structure has existed for several years before we are actually assessing its conditions, so on and so forth. There is yet another source of uncertainties in structural engineering, namely, human errors. By very definition, human errors need not have any definite patterns; they can be erratic. So, they again are uncertain in nature.

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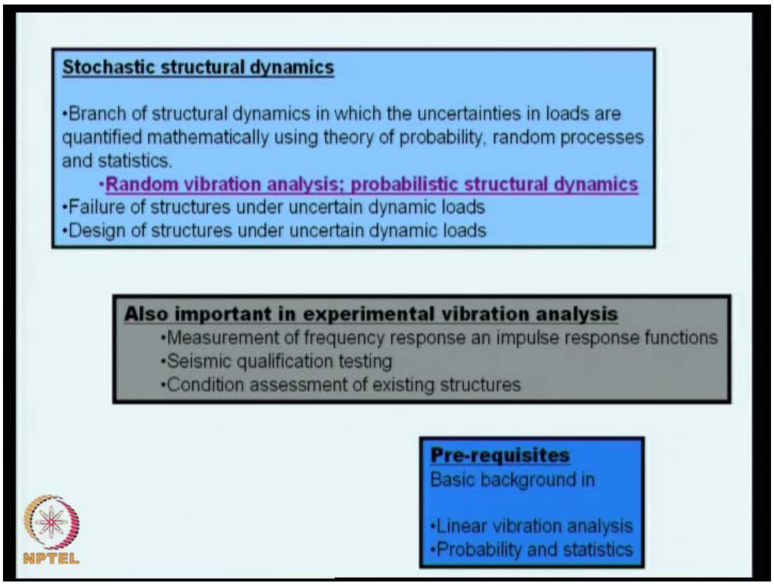

Now, the subject of stochastic structural dynamics can be described as here. We can view it as a branch of structural dynamics in which the uncertainties in loads are quantified mathematically using theory of probability random processes and statistics. So, the basic issue here is we recognize that there are uncertainties in our problem, but then, we have to quantify. How do we quantify uncertainties? So, the mathematical theory that would help us in achieving this – these theories are theory of probability, theory of random processes and theory of statistics. The subject of stochastic structural dynamics is also known by few other names, namely, random vibration analysis, probabilistic structural dynamics. So, we have titled our course as stochastic structural dynamics. The important objectives of this course are – we are interested in characterizing failure of structures under dynamic loads. How do structures fail if there is un…? How do characterize failure of structures if the loads acting on the structure are dynamic in nature and they are random? This leads to a question of design of structures under uncertain dynamic loads. How do you design? As engineers, we are interested in failure and objective of design is to when we say as condition, which leads to failure and prevent them in some sense.

The subject of stochastic structural dynamics also forms an important ingredient in experimental vibration analysis. For example, if you are interested in measuring frequency response or impulse response functions of linear systems, the measurement techniques are founded on principles of random vibration analysis. Similarly if you are doing qualification testing of equipment or structures for earthquake loads, there is quite a bit of random vibration principles involved in specifying the input and interpreting the results. As already said in condition assessment of existing structures, we would need to model uncertainties, so that we could take rational decisions. So, this is the overview of the course. So, at the end of the course, we should be able to answer typically the questions on the failure of structures under uncertain dynamic loads; and, what will be the consequence of such modeling on design of the structure. There are certain pre requisites for this course. I would except that you would have done a course on linear vibration analysis and also you are familiar with some aspects of probability and statistics. I would be quickly reviewing some of these topics as much as is needed for the course, but it would be helpful if you prepare yourself independently of this review.

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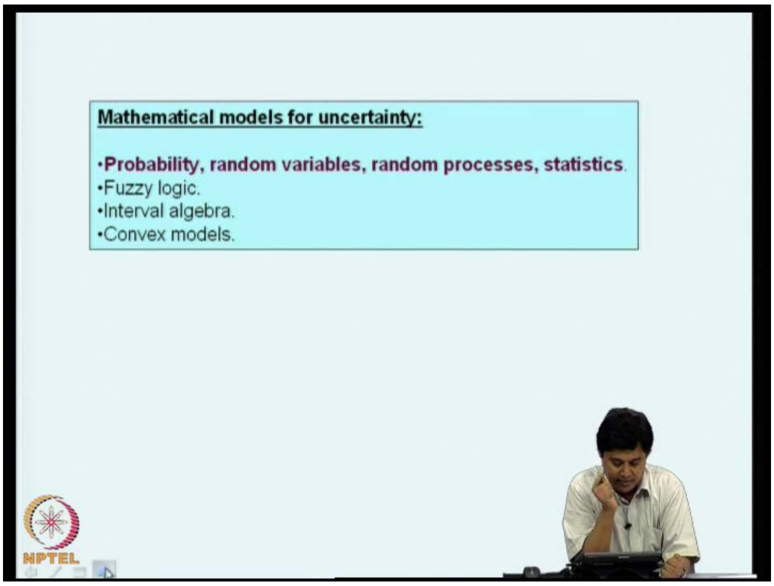

So, with this preamble, let us start now with questions on mathematical models for uncertainty. There are various tools for this. One tool is drawn from the basket of probability, random variables, random processes and statistics. There are other tools, for example, fuzzy logic, interval algebra, convex models, etcetera. As far as this course is concerned, we will be using probability, random variables, random processes and statistics.

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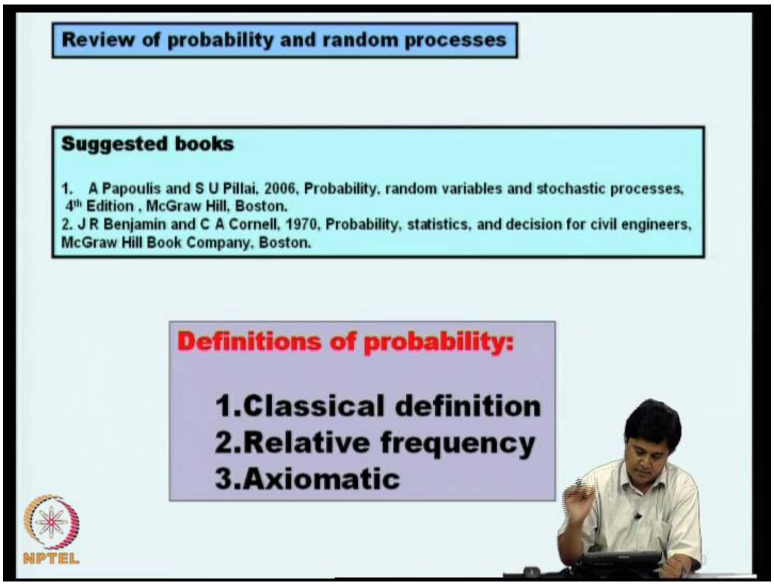

We will begin the course with a review of probability and random processes. There are two books, which I am suggesting: the one is by Papoulis and Pillai; other one is by Benjamin and Cornell for this review of probability and random processes. So, let us begin now by considering how to define probability. In the existing literature, there are three definitions: one is so-called classical definition, the other one is what is known as relative frequency definition, and third one is known as axiomatic.

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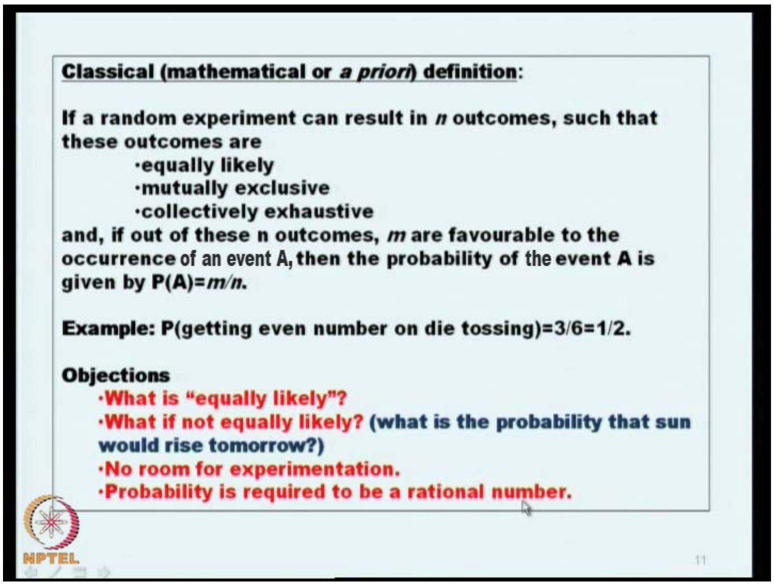

So, let us start with the classical definition of probability. According to this, the definition of probability is as follows. If a random experiment can result in n outcomes, such that these outcomes are equally likely, mutually exclusive and collectively exhaustive; and, if out of these n outcomes, m are favorable to the occurrence of an event A, then the probability of event A is given by m by n. So, this is the definition. So, you can quickly understand what these terms mean by considering an example.

Suppose you throw a die; a die has a six faces and each faces marked with numbers 1, 2, 3, 4, 5, 6. When I say outcomes are equally likely, what it means is the die is fair; you would toss a die; you could expect to get 1, 2, 3, 4 or 5 with equal chance. Mutually exclusive means if one turns up, no other number would turn up. So, the occurrence of this event will exclude the occurrence of all other events. Collectively exhaustive means these n outcomes are all that would happen when you toss a die. You will get 1, 2, 3, 4, 5, 6; that is all to it.

Now, let us consider the event of getting an even number on tossing a die. What is n here? n is 6; we can get 1, 2, 3, 4, 5 or 6; n is 6. What is m? Of these six outcomes, how many of them are favorable to observing an even number? 2, 4 or 6. So, m is 3. So, the probability of getting an even number is half. Now, are we satisfied with this definition? We could raise some objections. What is meant by equally likely? We are trying to define probability and we are already using a notion of probability in saying that outcomes are equally likely. So, it is a circular definition. What if not equally likely? If the die is not fair, how do we define what is the probability of even number or any other event? This definition does not allow for that. For example, what is the probability that sun would rise tomorrow? So, if you argue, there are two outcomes: it will rise; it will not rise. But, thing is they are not equally likely. So, it would be something like n divided by n plus 1; where, n is the number of days or which we have observed that sun has risen; that is likely to be answer. It is not certainly half.

In this definition, there is another problem; there is no room for experimentation. The fact that we have seen sun rising for so many years is not allowed for in this. Then, probability is required to be a rational number, because you are taking ratios of two integers. Those of you, who are mathematically inclined, will take objection to this. There are so many irrational numbers between 0 and 1. They cannot be probability according to this definition. These are the objections.

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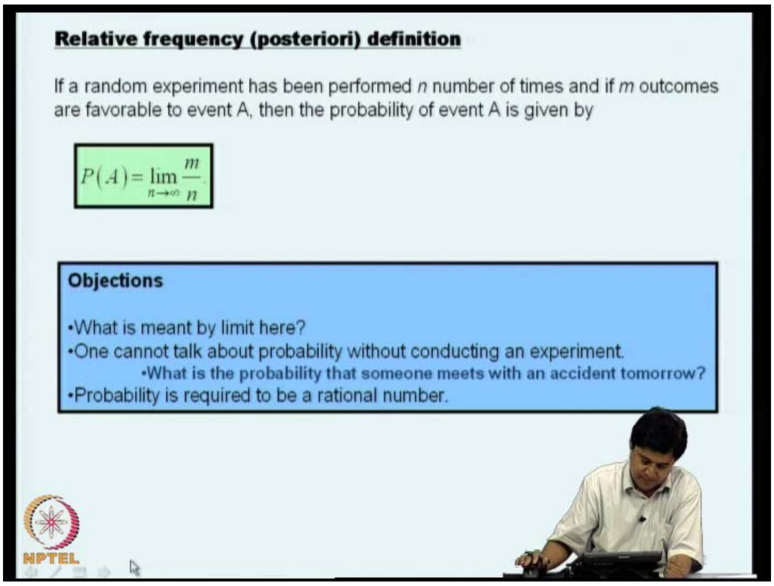

Now, we will go to the next definition, that is, relative frequency or so-called posteriori definition. Here the definition is as follows. If a random experiment has been performed n number of times and if m outcomes are favorable to event A, then the probability of event A is given by limit of this number of trials going to infinity of m by n. So, you toss a die for a very large number of times and see how many times you get head. So, the number of times you have got head divided by number of trials is probability of getting head. So, this definition is purely based on experimentation.

Now, there are again few objections that one can take to this definition. It is not clear what is meant by this limit here. It is not the limit that we talk about in the calculus. It only says that n is sufficiently large. You cannot verify whether this limit in a classic sense will not be able to verify that. Other problem here is that we cannot talk about probability without conducting an experiment. So, the questions, for example, what is the probability that someone meets with an accident tomorrow? Cannot be answered within the frame work of this definition; again, here probability is required to be a rational number. Now, the two definitions that I discussed – some of fall short of what we would like to do in engineering. In engineering, if you want to quantify uncertainty, we better have a good definition for probability, which are free from this kind of objections.

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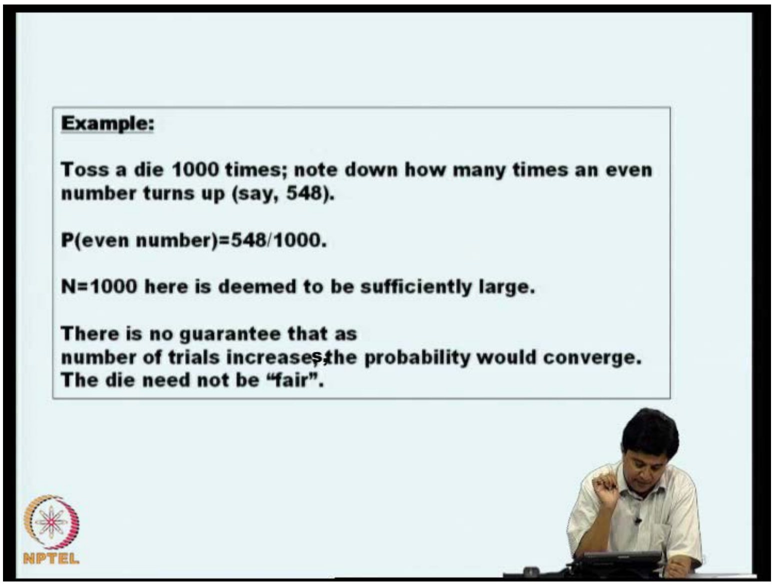

So, to clarify what I said just now, we will run through an example. Toss a die thousand times. Note down how many times an even number turns up. Say we get 548. So, probability even number is 548 divided by 1000. Now, here 1000 is deemed to be sufficiently large. Therefore, this is an acceptable estimate for probability of even number. There is no actually guarantee that as a number of trials increase, the probability would converge. Of course, this allows for the fact that die is not fair. I may be tossing thousand times a die in a particular manner. And, if I start tossing a different way, there could be some systematic changes. So, this sense of convergence is not very clear.

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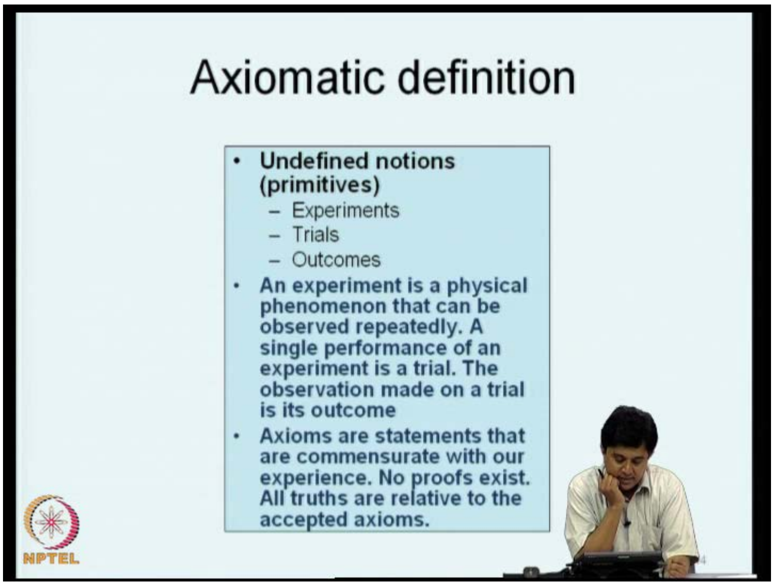

So, we now move to a more acceptable definition of probability. That is known as axiomatic definition. Here what we do is we begin with certain undefined notions. For example, we say that terms like experiments, trials, outcomes – these are the terms for which there are no definition. But, we have certain description, for example, experiment. An experiment is a physical phenomenon that can be observed repeatedly. A single performance of an experiment is a trial. The observation made on a trial is its outcome. These are descriptions; these are not definitions in a mathematical sense.

The word axiom itself… The axioms are statements that are commensurate with our experience of the physical world. And there were no proofs for the axioms. So, all truths that we derived based on these axioms are relative to the accepted axioms. So, if in course of using this theory, if you come across situations where these axioms are violated, you have to start a fresh with a new theory. So, these are valid as long as our physical experience is commensurate with these axioms. So, there are two things: one is what are axioms and other one is what are these so-called primitive notions, which we do not actually define systematically.

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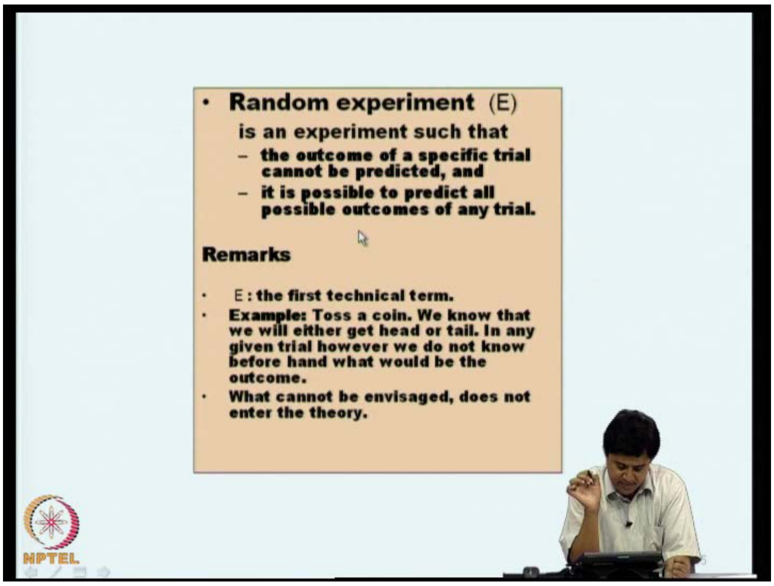

The first technical term in theory of probability is known as random experiment. So, random experiment is an experiment such that the outcome of a specific trial cannot be predicted; and, it is possible to predict all possible outcomes of any trial. This is the definition of a random experiment. Again, we will take up example of tossing a coin. We know that we will either get a head or a tail. In any given trial, however we do not know beforehand what would be the outcome. If I toss a coin, I know I am going to get head or tail. But, in a given trial, I would not know whether I will get head or tail. Therefore, this is a random experiment.

One more important aspect of probability theory can be clarified at the outset. When we say that it is possible to predict all possible outcomes of a trial, the scope of the theory would be limited to what we envisaged as possible outcomes. Something that cannot be envisaged does not exist in the theory. Just to give an example, which is slightly exaggerated; when you toss a coin, there are other eventualities like coin may… As it falls on the ground, it may stand vertically; or, you may toss a coin with such a ferocious force that it can escape the gravity of earth and never return to earth. Then, you will not even be able to observe the outcome. So, these are rare events. So, this is one of the issues that we talk about when we discussed about rare events like attacks on buildings and so on and so forth, which may… or it is very difficult to envisage what will happen at the stage of design. If you do not envisage, it does not exist in the theory.

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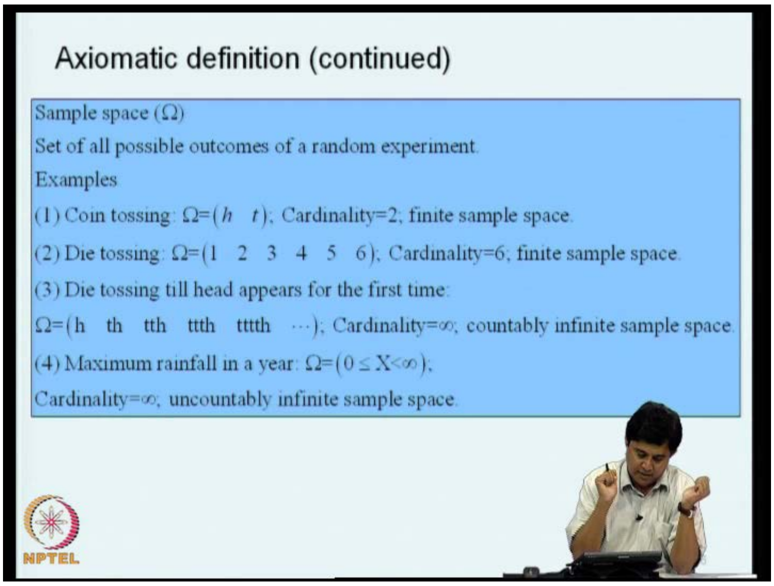

So, we have now talked about the random experiment. We come to the next technical term, namely, sample space. Sample space is set of all possible outcomes of a random experiment. Since we say that outcomes of random experiment in any given trial are all known, we assemble them in a set and give a name known as sample space. Again, few examples; you toss a coin, there are two outcomes – head or tail; and, we say that the sample space consists of head or tail. The number of elements here is the cardinality of this set; here it is 2. And, we say that sample space in this case is a finite sample space.

Similarly, in a die tossing experiment, there are six outcomes; the cardinality is 6 and this again is a finite sample space. Now, let us do one more experiment; we will toss a coin till head appears for the first time. So, if you get head, you will start. So, one possible outcome is on the first trial itself, you will get head. Next one is th, tth, ttth, tttth and so on and so forth. The coin will never pass our test; it will be tttttt forever. So, here this is accountably infinite set. On the other hand, if we consider another random experiment, where we note down the maximum rainfall in a year in a given location, this can be any real number. Therefore, this is uncountably infinite sample space, because sample space is a real line.

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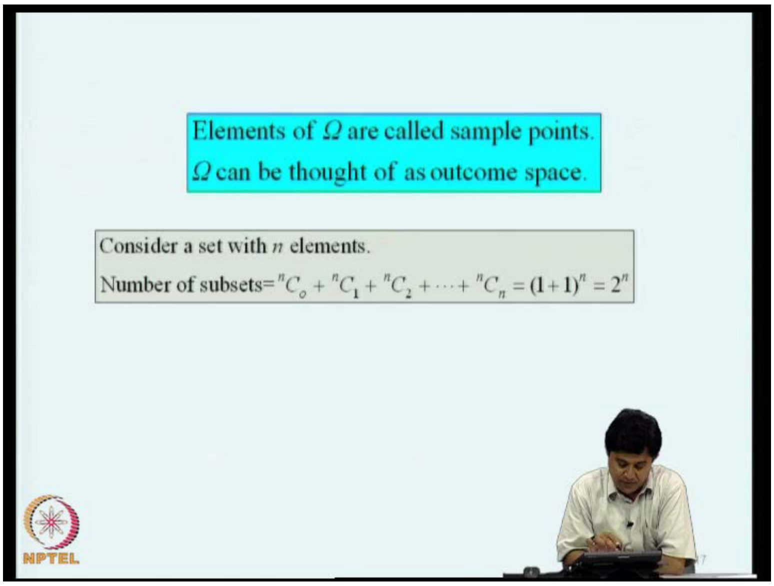

These elements of sample space are called as sample points. The sample space can be thought of as outcome space also. This is a small exercise here. If you have a set with n elements, you can show that the number of subsets from that we can form from these elements is actually 2 to the power of n. So, you may try this. We will need this result shortly.

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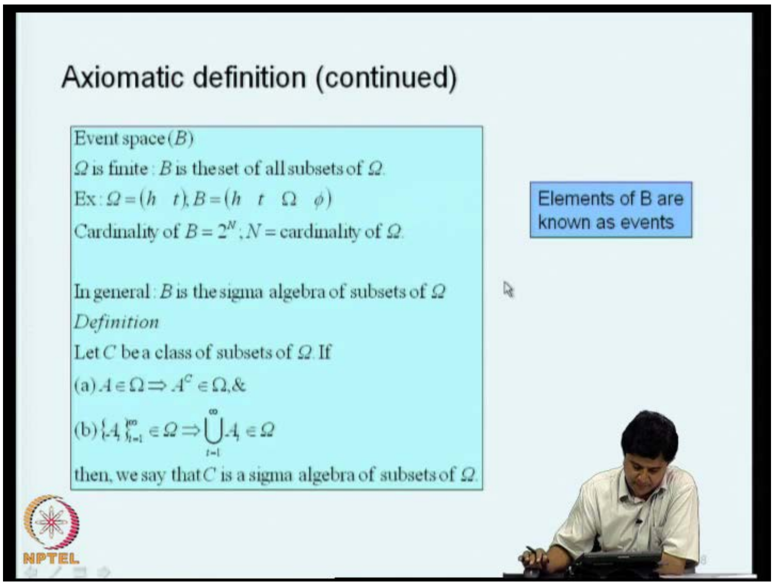

We have now talked about random experiment; we have talked about sample space. So, the next technical term is what is known as event space. Let us begin by considering sample space to be finite. When sample space is finite, we say that we define the event space as the set of all subsets of sample space. So, for coin tossing experiments, the sample space is h t and the event space is h t; the sample space itself and a null set. Null set is like zero in a number system; we include that. So, cardinality of b is 2 power of n; where, n is cardinality of omega. This is 4 in this case.

If omega is not finite, what we do is we consider the event space to be a sigma algebra of subsets of omega. What is the meaning of sigma algebra? Let us C be a class of subsets of omega. If A belongs to omega implies that A complement also belongs to omega. And, if A 1, A 2, A 3, A infinity belong to omega, their union also belongs to sample space. Then, we say that C is a sigma algebra of subsets of omega. I will just clarify why we need this in a valve. The elements of the event space are known as events.

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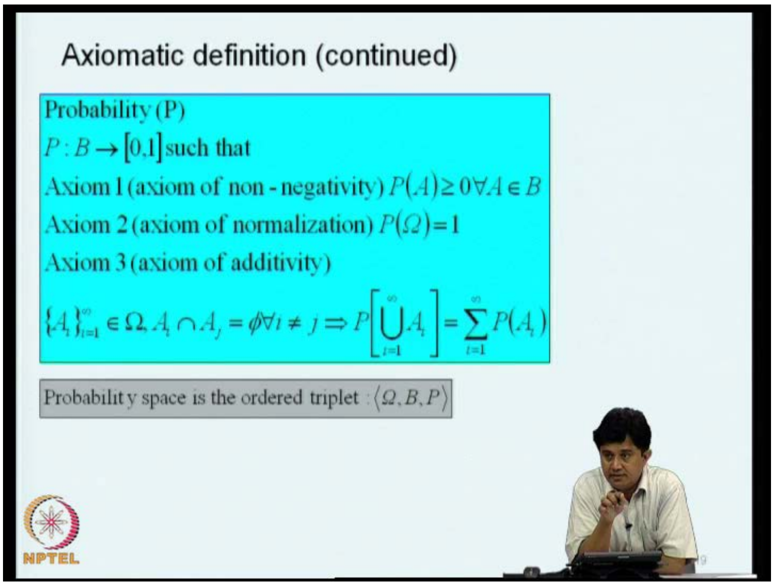

We have now introduced a random experiment, sample space and event space. Now, we introduce the so-called probability measure P. For every element in B, we assign a number between the 0 and 1, such that the three axioms that will be obeyed. The first axiom is known as axiom of non-negativity, which says that probability of A cannot be negative. For any element in B, you can assign a number, which has to be greater than or equal to 0. According to second axiom – axiom of normalization, probability of sample space itself should be 1. There is third axiom known as axiom of additivity. If we consider sets A 1, A 2, A 3, A infinity from the sample space, so that they are mutually exclusive – mutually exclusive means their intersection is a null set, then the probability of union of these sets is equal to the sum of probability of individual events. This tells us how we can add probabilities. This triplet of sample space, event space and this probability measure is called probability space.

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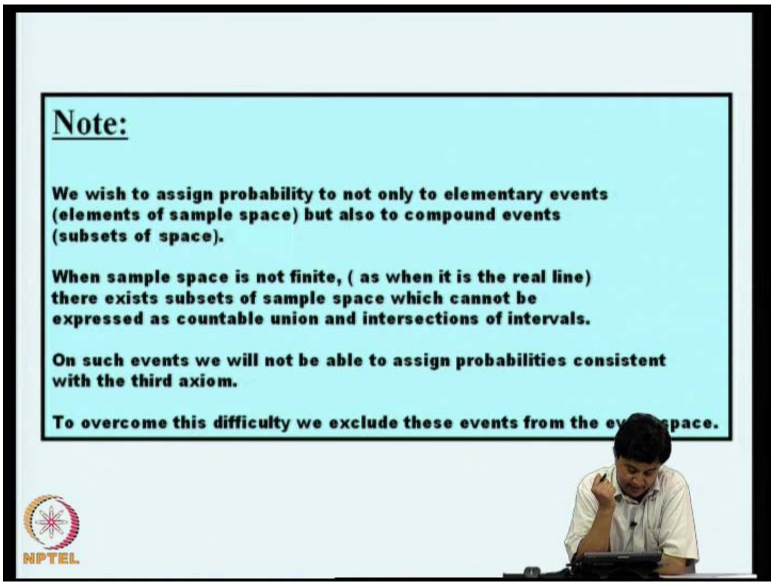

I talked about sigma algebra; a brief clarification on why it is needed can be made now. See if you look at this third axiom, (Refer Slide Time: 35:54) what we are doing here is for any sequence of subsets of sample space, we would like to assign probability on union of these sets. This union of these sets – typically, we would like also to be an event; it has to be a member of the event space.

When sample space is not finite; for example, as when it is on the real line, there exists subsets of sample space, which cannot be expressed as countable union and intersection of intervals. Now, actually on such events, we will not be able to assign probabilities consistent with the third axiom. So, we would like to exclude them from our consideration. So, it is a mathematical (( )) that we have to take into account at the outset. And therefore, we insist that the event space should be a sigma algebra of the subsets of sample space. If sample space is finite, all subsets of sample space will be in the event space; otherwise, we have to exclude some sets, because that is the logic.

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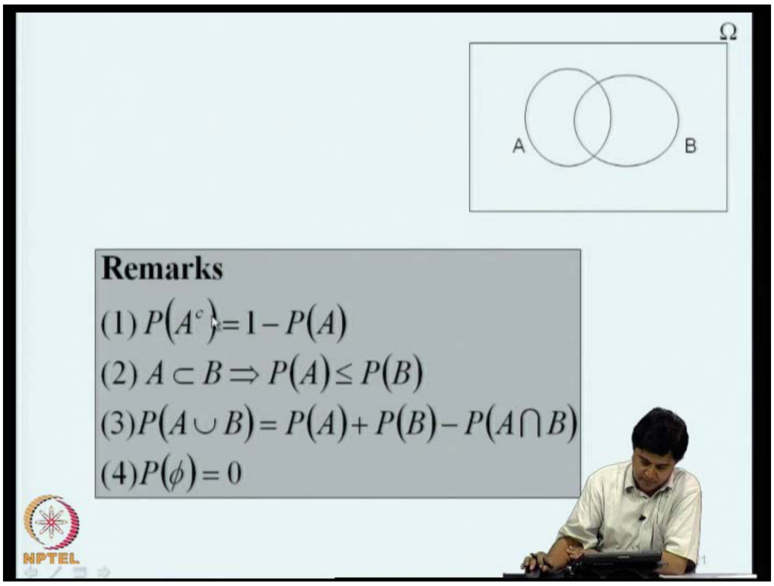

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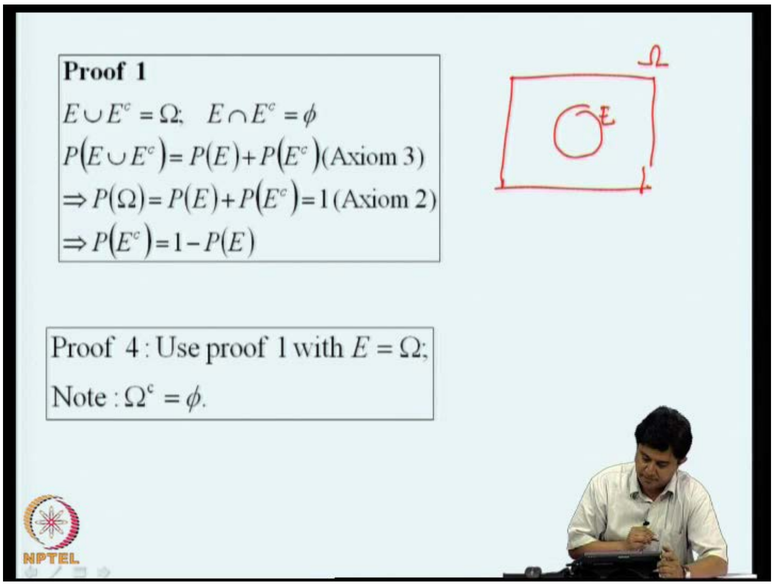

We have now three axioms, based on which the entire edifies of theory of probability will now emerge. We will start with some corollaries. Probability of A complement is 1 minus probability of A. We will see this and then return to the other one. How do you show that? For example, suppose this is the sample space and this is a subset E. What I need to show is probability of E complement, is 1 minus probability of E. So, how do I do that? You can write samples space as union of E and E complement. Now, E and E complement are mutually exclusive. Therefore, their intersection is a null set. Therefore, if you write now probability of E union E complement, it will be probability of E plus probability of E complement according to the third axiom of probability. This one is sample space (Refer Slide Time: 38:25). Therefore, according to the second axiom, probability of omega is 1. Therefore, I get probability of E plus probability of E complement is 1. Therefore, probability of E complement is 1 minus probability of E. So, I am using the axioms to show. I use that axiom in two places here as well as here (Refer Slide Time: 38:46).

Let us now come to the fourth axiom – probability of null set is zero. So, that can be shown by using this proof, where we take E as sample space itself. From that, we can show that probability of omega complement is a null set is 1 minus probability of E, which is probability of omega. Therefore, 1 minus 1 is zero. So, that is how.

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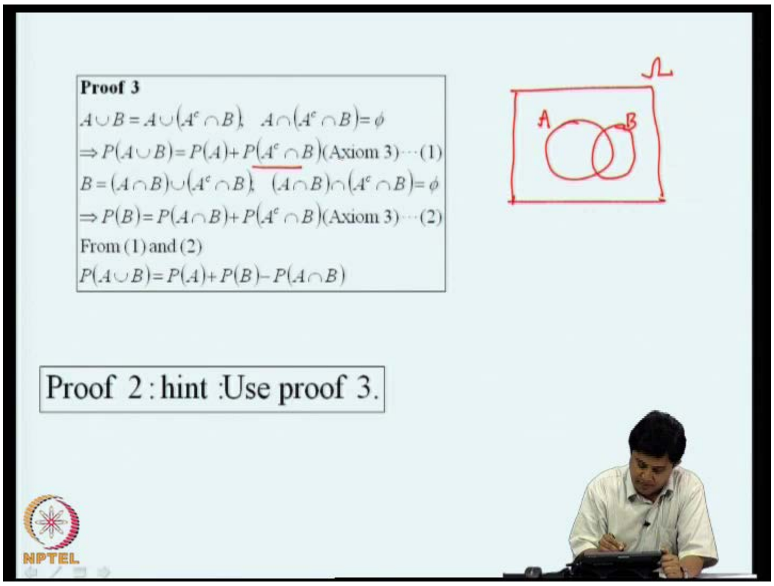

Now, there is one more corollary, that is, probability of A union B. A is probability of A plus probability of B minus probability of A intersection B. The axiom that we had was with respect to events A and B, which were mutually exclusive. Now, if they are not mutually exclusive, what to do is the question. So, here again, this is sample space; this is A; this is B. So, we are interested in probability of A union B. Now, A union B itself can be written as A union of this part in B, which is not in A, So, that is, A union A complement intersection B. And, these two sets – A and A complement intersection B are mutually exclusive, because their intersection is a null set. Therefore, probability of

A union B is probability of A plus probability of A complement intersection B. This is according to axiom 3.

Now, B itself can be written as this portion, (Refer Slide Time: 40:42) that is, A intersection B union A complement intersection B; that means the part of A, which is not in B. Again, these two sets are mutually exclusive. So, if you write probability of B now, it will probability of A intersection B plus probability of A complement intersection B. This is again according to axiom 3. Now, if you combine these equations: 1 and 2, we get requisites results; that probability of A union B is probability of A plus probability of B minus this. If A intersection B is a null set, we recover the third axiom. Now, the second corollary; I will leave it as an exercise. You can use this proof and arrive at proof for the second corollary.

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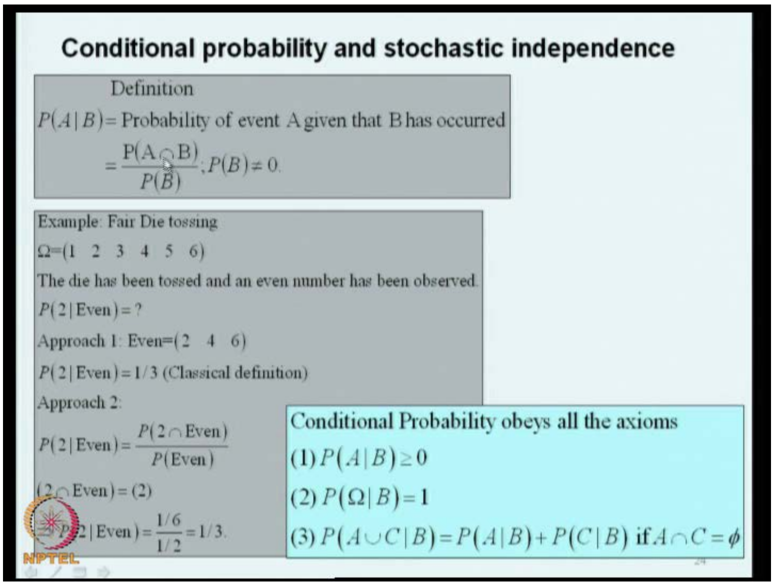

The next concept that we need to consider is the notion of conditional probability and stochastic independence. The definition of conditional probability is here. Probability of A conditioned on B is… This is how we should read – this vertical line is to be read as conditioned on. So, this is probability of event A, given that B has occurred. By definition, probability of A intersection B divided by probability of B with the condition of that probability of B itself not zero. A simple example – you take a fair die; it has six faces; and, this is a sample space. Now, a die has been tossed and an even number has been observed. Given that we have observed a even number, now, I ask the question –

what is probability of getting 2? So, one approach is what are even…? The event even consists of 2, 4 and 6. So, probability of 2 conditioned on even is 1 by 3, because there are three possible outcomes here. One is favorable to you; it is 1 by 3; and, die is fair. So, this definition yields the probability as 1 by 3.

In the second definition, we can use the definition of conditional probability. This is given by probability of 2 intersection even divided by probability of even. What is probability event 2 intersection even? It is 2. So, this is this (Refer Slide Time: 43:17). Then therefore, the probability of 2 conditioned on even is probability of getting 2, which is 1 by 6 divided by probability of getting even, which is 1 by 2; and, this is 1 by 3. So, this agrees with this. Now, one thing that we should notice is conditional probability actually obeys all the axioms of probability; that is, non-negativity, normalization and additivity; that is, probability of A conditioned on B is always a number, which is greater than or equal to 0. Probability of sample space conditioned on B is 1. And, if you take two sets – A and C, probability of A union C conditioned on B is probability of A conditioned on B plus probability of C conditioned on B if A intersection C is a null set.

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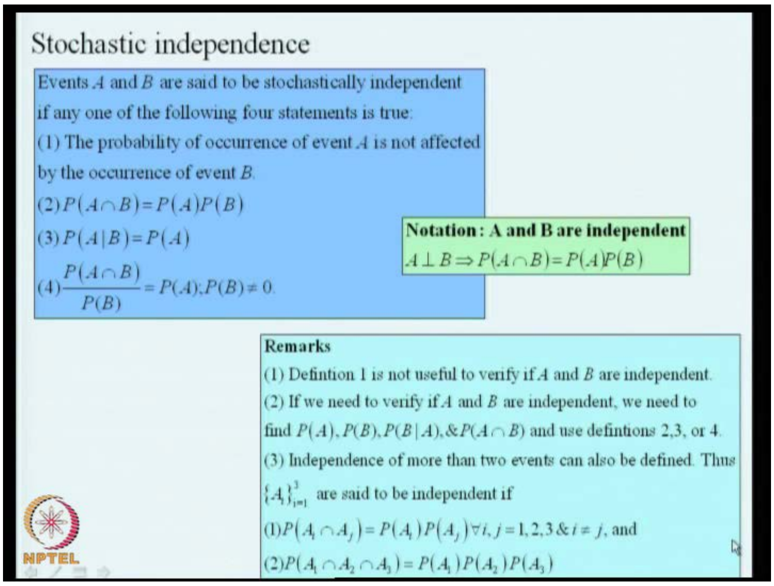

Now, using the notion of conditional probability, we can introduce the notion of stochastic independence. The notion of stochastic independence tells us how to, when we can multiply probabilities. So, let us consider again events A and B. We say that the events A and B are said to be stochastically independent if any one of the following conditions are satisfied. The first statement is the probability of occurrence of event A is not affected by the occurrence of event B. Then, we say A and B are independent. The second statement is probability of A intersection B is probability of A into probability of B. The third definition is actually a mathematical statement of the first statement – probability of A conditioned on the fact that B has occurred, is not affected. It remains as the probability of A itself; then, we say A and B are independent. Other definition is if probability of A intersection B divided by probability of B is nothing but probability of A; then, we say that A and B are independent. All these four definitions are equivalent; and, depending on the context, you can use whichever definition, which is convenient.

We write that the notation that we use is – this notation (Refer Slide Time: 45:52) is used to denote that A and B are independent. Now, let us reflect on these definitions for a while. The first definition is somewhat verbal and it is not actually going to help us in quantitatively verifying if A and B are independent. If we need to verify if A and B are independent, we need to find probability of A, probability of B, probability of B given A, and probability of A intersection B, and use the definition 2, 3 or 4. This is something that we can quantitatively verify. This is independence of two events – this definition. We can extend this definition to independence of more than two events.

Suppose we consider three events: A 1, A 2, A 3; we say that the events A 1, A 2, A 3 are independent if probability of A i intersection A j is probability of A i into probability of A j for i equal to i, j equal to 1,2,3; that means any combination of pair of A 1 and A 2 you take, this identity (Refer Slide Time: 47:07) should be valid. Not only that; probability of A 1 intersection A 2 intersection A 3 must be equal to probability of A 1 into probability of A 2 into probability of A 3. What this means is, it is not enough if A 1, A 2, A 3 are pair wise independent. They need to satisfied one more condition that when we take all three together, they should satisfied this additional requirement that probability of A 1 intersection A 2 intersection A 3 must be equal to this product (Refer Slide Time: 47:40).

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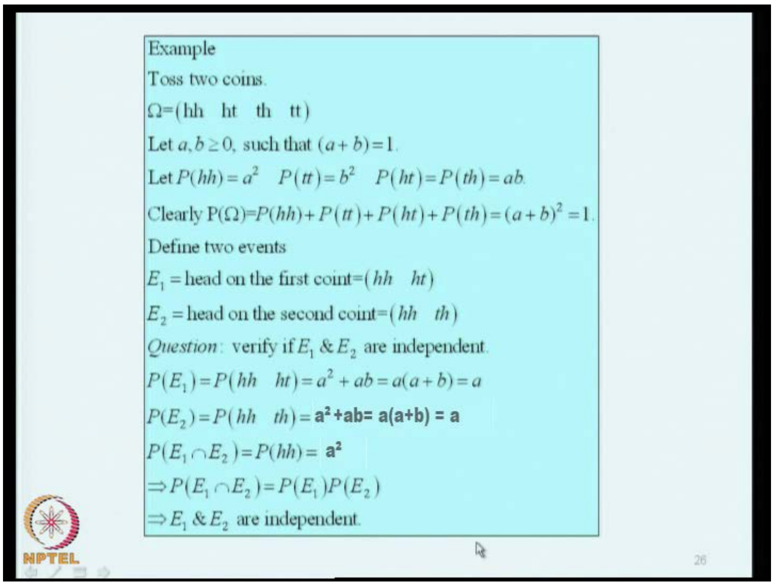

We will consider a simple example. We will toss two coins and we know sample space is hh, ht, th and tt; h stands for head; t stands for tail. Now, let us consider two numbers, such that their sum is equal to 1. And, I will assign the probabilities as follows. Probability of hh is a square; probability of tt is b square; and, probability of ht and probability of th are ab. Now, what is sample space omega? It is actually union of these three/four events and they are all mutually exclusive. Therefore, I can add the probabilities; this is actually a plus b whole square, which 1, because I have put a plus b equal to 1.

Now, let us define two events: E 1 – the event head on the first coin, that is, hh and ht; E 2 is head on the second coin, that is, hh and th. Now, the question is are E 1 and E 2 stochastically independent? How do you verify? We will find out the probability. Probability of E 1 is probability of hh plus ht, which is a square plus ab, which is a into a plus b, which is a, because a plus b is 1. Probability of E 2 is probability of hh and th, which is b square plus ab, which is again b. What is E 1 intersection E 2? It is hh. So, what is probability of hh? Which is given to be ab. So, now, let us look at probability of E 1 intersection E 2, which is ab; is it equal to probability of E 1 into probability of E 2? Yes, this is a and this is b. Therefore, varies the conclusion that E 1 and E 2 are independent.

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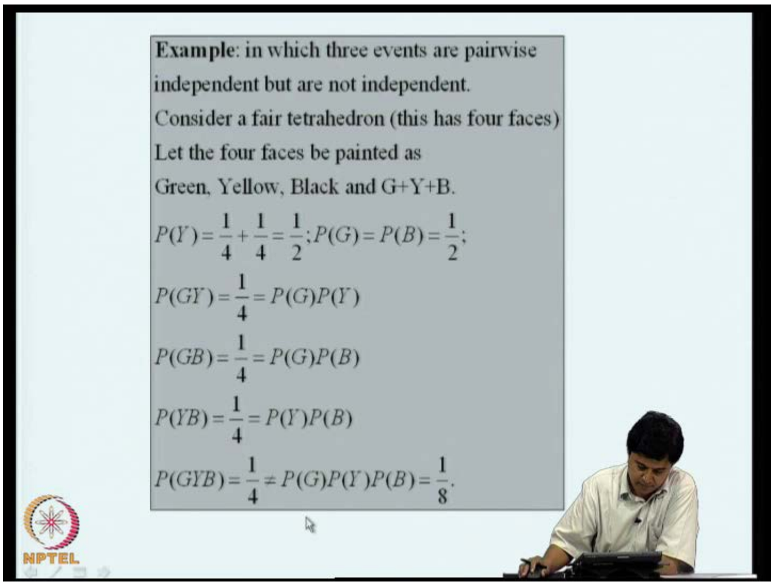

We can also construct an example, where there are three events, which are pair wise independent, but are not independent. So, this is a thought experiment. You can consider a fair tetrahedron, which has four faces; and, the four faces be painted as green, yellow, black and all three colors together – G Y B. Now, what is probability of Y? It is 1 by 4 plus 1 by 4, because there is a Y in the fourth face also; this is 1 by 4. Similarly, probability of green, blue is half. What is G Y? Which is 1 by 4, which is probability of G into probability of Y. What is G B? It is 1 by 4, etcetera. But, what is probability of G Y B? Which is 1 by 4, but that is not equal to 1 by 8. So, this is an example that may help to you understand what is meant by independence of three events.

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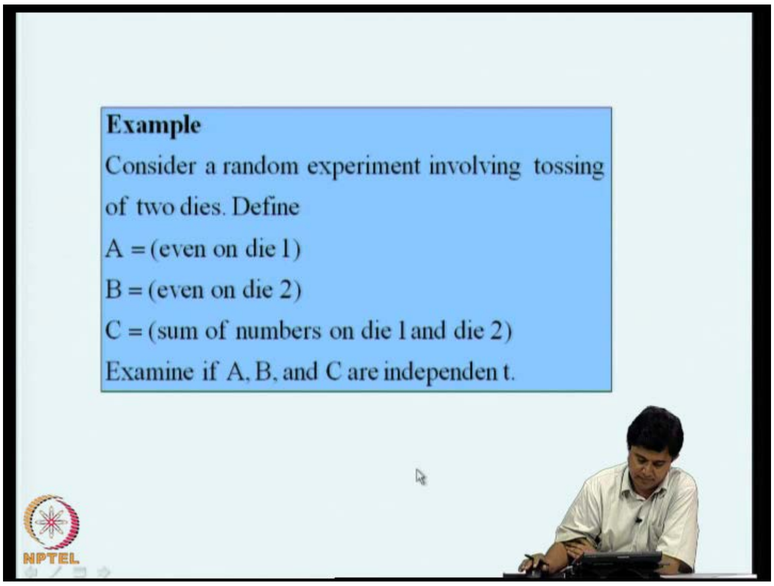

Now, this is an example that you can try. So, let us consider random experiment involving tossing of two dies. So, we define A as even on die 1; B as even on die 2; C as sum of numbers on die 1 and die 2. Now, the question is you need to examine if A, B and C are independent.

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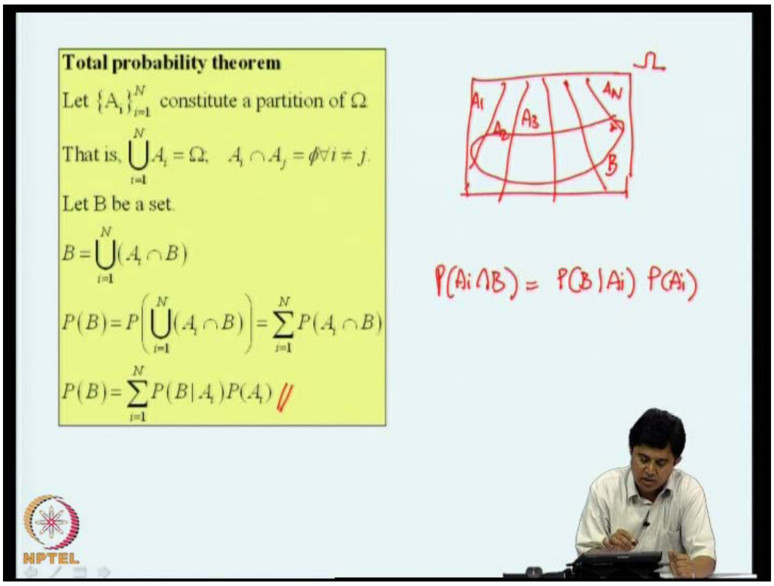

Now, we come to a useful theorem known as total probability theorem. To explain what this means, we will again consider a sample space; we will consider subsets A i – i running from 1 to N and we demand that these subsets constitute a partition of sample space. What it means is this A 1, A 2, A 3, A 4, A 5, etcetera are all mutually exclusive; that means their intersection is a null set. There are no common elements between any of these subsets. And, if you take union of all these A Ns, they become the sample space. Now, let us consider a set B; now, the B itself can be thought of as union of the common elements between B and A N, the common element between B and N minus 1, so on and so forth; common elements between B and A 1, that is, union of A i intersection B. And, these are mutually exclusive. Therefore, probability of B will be probability of union of A i intersection B by axiom of third axiom of probability. This is sum of the probabilities of these A i intersections. Now, for this probability of A i intersection B, I write this as probability of B conditioned on A i into probability of A i. So, if you substitute here, this statement is known as total probability theorem.

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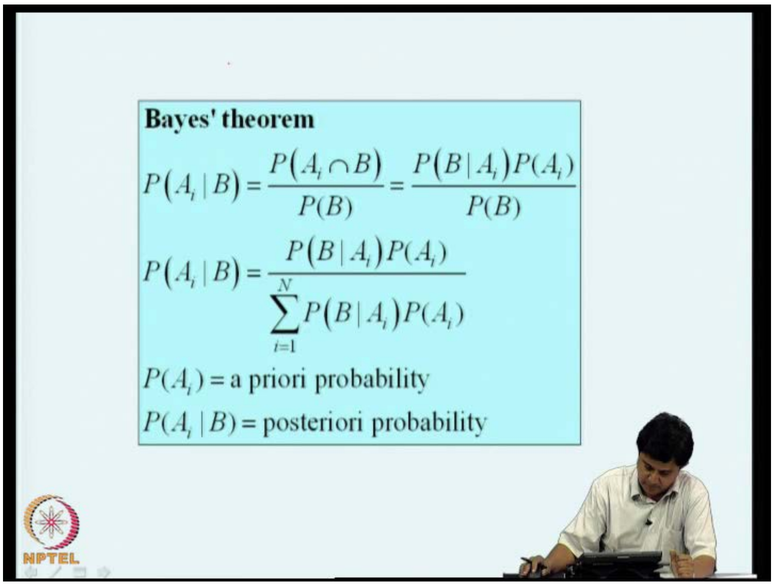

This is another theorem, which will be needing, is known as Bayes' theorem. This is an application of… This can be shown to be true by using total probability theorem. The statement of this theorem is as follows. Probability of A i conditioned on B is given by probability of B conditioned on A i into probability of A i divided by probability of B. How do we show this? Probability of A i conditioned on B is given to be this; probability of A i intersection B divided by probability of B. And, this itself can be written in this form.

Now, for probability of B i, we will use total probability theorem and write it in this summation form (Refer Slide Time: 54:18). Now, this statement is known as Bayes' theorem. But, what it tells you is that imagine you are making an observation of event B with a view to characterize probability of A i. Before you have made any observations on event B, you would have assigned to event A i, this – probability of A i. This is known as a priori probability. Now, you do an experiment; you observe B. Now, given that B has occurred, I have gained some information about the phenomena that I am studying. What can I now say about probability of A i? So, this is what Bayes' theorem helps you to… This is the question that Bayes' theorem answers. So, we will stop here for the day.

In the next lecture, I will introduce a notion of random variables.