NPTEL ONLINE CERTIFICATION COURSES

# EARTHQUAKE SEISMOLOGY


Dr. Mohit Agrawal

Department of Applied Geophysics , IIT(ISM) Dhanbad


Module 02 : Snell's law, Plane wave reflection and transmission

Lecture 04: Boundary Conditions, Types Of Interfaces, Ray Theory, Reflection/Transmission, Snell's law

# CONCEPTS COVERED

- Recap
- Boundary Conditions
- Types Of Interfaces
- Ray Theory
- Reflection/Transmission
- Snell’s law
- Summary

# Recap from previous lecture

- The total energy and flux are proportional to the square of the amplitude and the frequency.

  Hence, waves of the same amplitude, the higher frequency transports more energy.
- Displacement of the P-wave can be obtained by solving $\nabla\Phi(x,z,t)$.
- Displacement of the SV-and SH-wave can be obtained by solving $\nabla\times\psi(x,z,t)$ and $\nabla\times\nabla\times\chi(x,z,t)$, respectively.
- P-and SV waves are coupled as the displacement correspondings to these waves lies in the same plane.
- Displacement corresponding to SH-wave lies in the plane orthogonal to the plane of displacement of SV-and P-wave.

# Boundary Conditions

**We will discuss about four types of interfaces in seismology, which are as follows.**

## Types of Boundaries

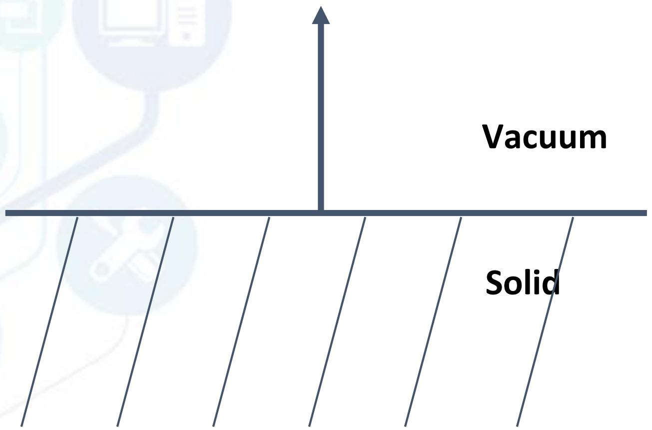
- **Solid-Solid**
- **Solid-Liquid**
- **Solid-Free surface (Air)**
- **Liquid-Liquid**

# The Free Surface Condition

- This is a good approximation for the Earth's surface
- There will not be any traction on the boundary due to vacuum so the tractions resolved on the boundary are zero.
- The normal vector , the vector is $\hat{n} = [0 \quad 0 \quad 1]$

  Then by cauchy's stress theorem $\sigma\hat{n} = [0 \quad 0 \quad 0]$

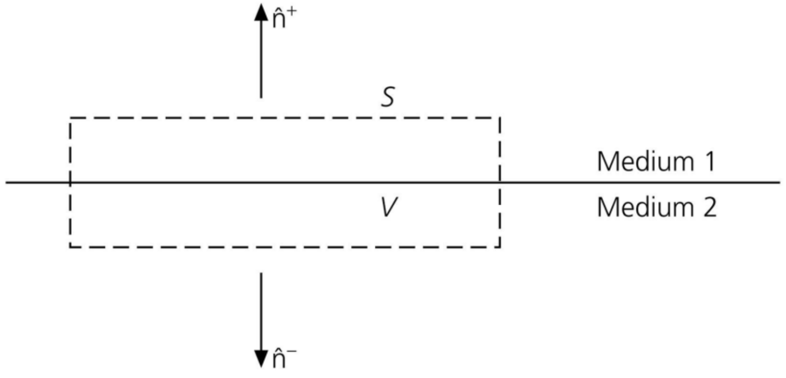
This produce zero traction at the free surface




$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \sigma_{13} = 0 \\ \sigma_{23} = 0 \\ \sigma_{33} = 0 \end{cases}$$

Let's talk about other boundary conditions, for this purpose, we will use the gaussian pillbox .

We will integrate the homogeneous equation of motion over the volume of pillbox.




$$\sigma_{ij,j} = \rho \ddot{u}_i$$
$$or \ \sigma_{ij,j} - \rho \ddot{u}_i = 0$$

Then integrate it for box's volume

$$\int_v (\sigma_{ij,j} - \rho \ddot{u}_i) dV = 0$$

Now we can use Gauss divergence theorem to convert one of the terms to a surface integral.

According to this theorem, the outward flux through a closed surface is equal to the integral volume within the surface of the divergence over the area.

$$\int_s \hat{n} . \overrightarrow{u} \, ds = \int_v \nabla . \overrightarrow{u} \, dV \qquad (1)$$

Vector Field

Normal unit vector to the surface

**In Einstein summation,**

$$\hat{n}.\overrightarrow{u} = n_i u_i = u_i n_i$$

$$\nabla.\overrightarrow{u} = \frac{\partial u_i}{\partial x_i} = u_{i,i}$$

**So, equation (1) can be written as**

$$\int_s u_i n_i dS = \int_v u_{i,i} dV$$

**By analogy, we can write,**

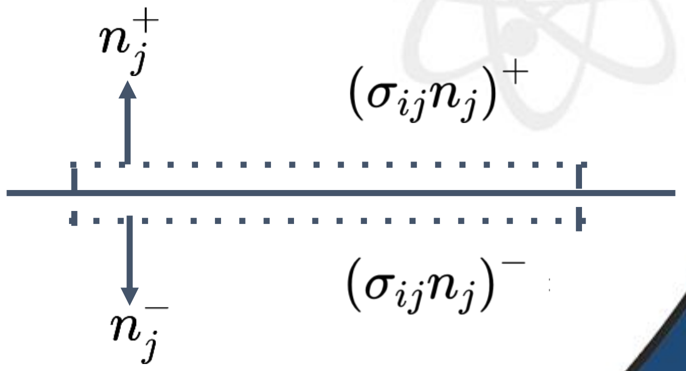
$$\int_s \sigma_{ij} n_j dS = \int_v \sigma_{ij,j} dV$$

**so,**

$$\int_s \sigma_{ij} n_j dS - \int_v \rho \ddot{u}_i dV = 0$$

If we squeeze the pillbox then, thickness $\rightarrow$ 0 and volume $\rightarrow$ 0

so, $$\int_v \rho \ddot{u}_i dV \rightarrow 0 \quad and \quad \int_s \sigma_{ij} n_j dS = 0$$




Because thickness of the pillbox is zero, the contributions from the top (+) and bottom (-) surfaces have to cancel each other out.

For this to happen, at each point, $$(\sigma_{ij} n_j)^+ + (\sigma_{ij} n_j)^- = 0$$



$n_j^-$ $and$ $n_j^+$ are opposite direction, we can say that $\sigma_{ij} n_j$ is the same at both sides of the interface . Therefore $T_i = \sigma_{ij} n_j$ , tractions must be continuous at an interface.

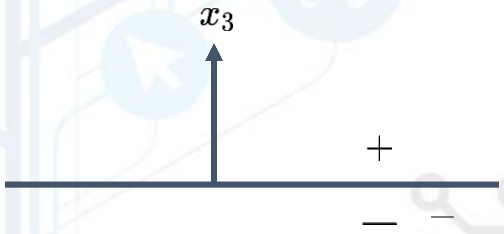
# Solid-Solid Interface

**It is also known as welded interface, because two solids are combined at the boundary.**

**Things should be connected /welded at this interface, so it implies continuity of displacement, so**

$$u_i^+ = u_i^-$$

$$T_i^+ = T_i^-$$



# Solid-Liquid Interface

$\mu = 0$ in the liquid, so shear tractions are zero in the liquid. By continuity they must be zero in the solid boundary too.

$$T_1 = 0$$
$$T_2 = 0$$
$$T_3^+ = T_3^- \quad , \quad u_3^+ = u_3^-$$

| $Interface$ | $Boundary\,Conditions$ |
|---|---|
| $Solid - Solid$ | $T_i^+ = T_i^-$<br>$u_i^+ = u_i^-$ |
| $Solid - Liquid$ | $T_3^+ = T_3^-$<br>$T_2 = T_1 = 0$<br>$u_3^+ = u_3^-$ |
| $Free\,Surface$ | $T_i = 0$ |

# Ray Theory and Reflection/Transmission

- Ray theory is an approach in which a point on the wavefront is tracked rather than the complete wavefield.
- Ray theory is extensively used due to its simplicity, rapidness and applicability to a wide range of problems
- It is valid both for shallow and deep earth, from industry seismics in a sedimentary basin to the P-waves in the outer core.
- Ray theory is an approximation and it does not work well where heterogeneity or the order of wavelength are less, exists. For examples diffractions, steep velocity gradients etc.

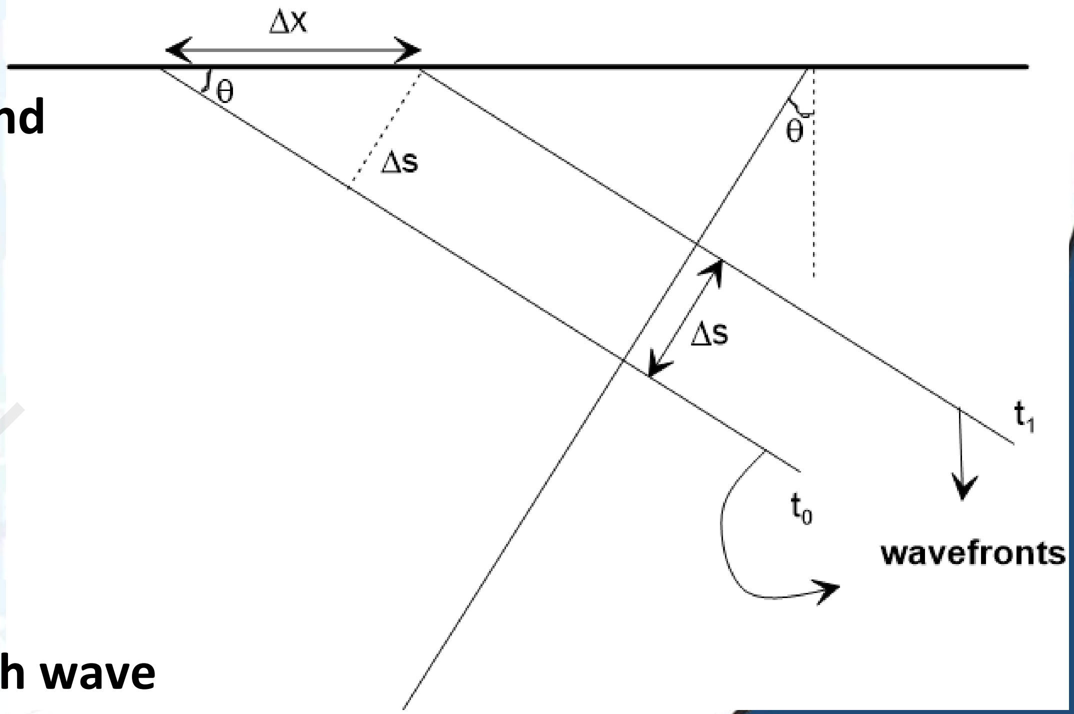
# Ray Theory and Reflection/Transmission

**Note:**

1. **If an SH-wave is vibrating parallel to the interface, then it will produce reflected and transmitted SH-wave.**

1. **P -and SV- waves are coupled to each other, So a P-wave can produce SV-wave and vice-versa**

# Ray Theory and Reflection/Transmission




**Let us suppose there is a propagating wavefront at times $t_0$ and $t_1$. During this period it travels distance Δs.**

$$\sin\theta = \frac{\Delta s}{\Delta x} \qquad \text{and} \qquad \Delta s = v\Delta t$$

$$\Longrightarrow \frac{\Delta t}{\Delta x} = \frac{\sin\theta}{v} = p\,(\text{ray parameter})$$



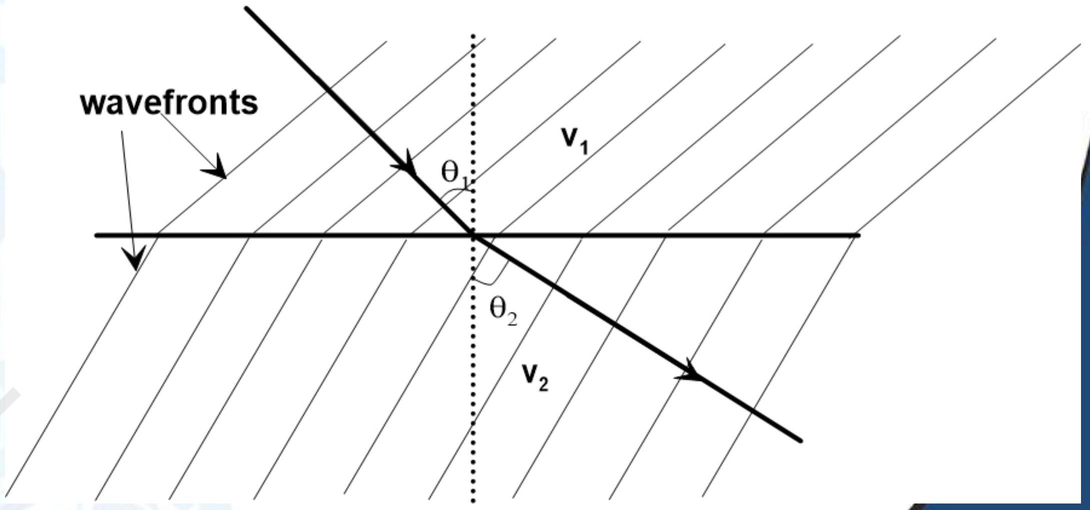
**Consider apparent horizontal velocity is $c_x$, is the rate at which wave travels along the X-direction such as surface of the Earth**

**Note that:**

$$\frac{1}{p} = \frac{\Delta x}{\Delta t} = c_x$$

$$\text{slowness} = u = \frac{1}{v} = s$$

$$\text{then} \quad \frac{\sin\theta}{v} = u\,\sin\theta$$




**For wavefronts to be continuous across the interface, their horizontal apparent velocity along the interface has to be same**

$$p_1 = \frac{\sin\,\theta_1}{v_1} = u_1 \sin\theta_1 = p_2 = \frac{\sin\,\theta_2}{v_2} = u_2 \sin\theta_2$$

**OR**

$$\boxed{\frac{\sin\,\theta_1}{v_1} = \frac{\sin\,\theta_2}{v_2}}$$

**This is known as Snell's law**

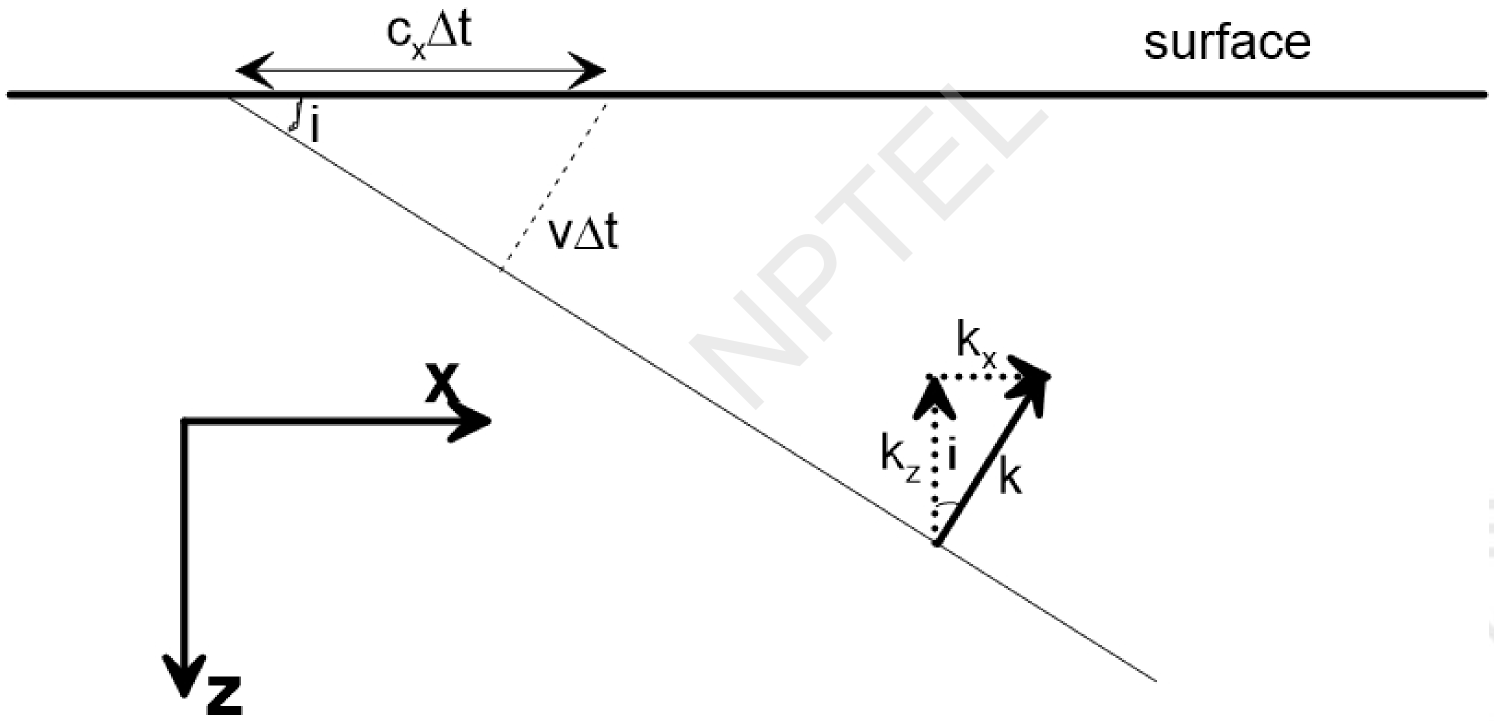
**Note that ray parameter of the ray remains constant along its path**

**Wave number vector defines the direction of the ray**

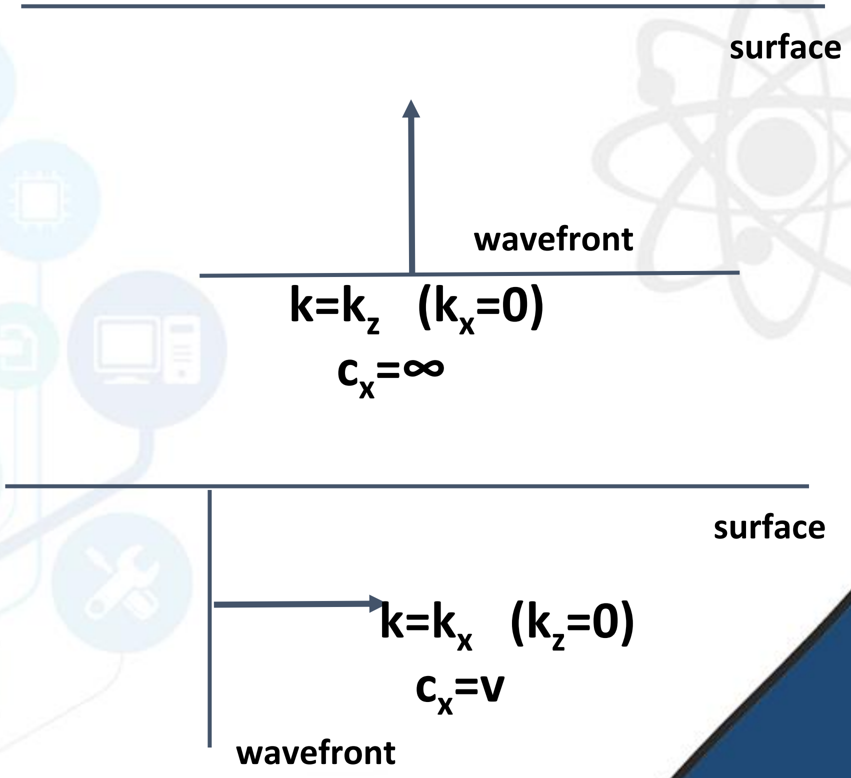
**It has two components: one along horizontal and another along vertical direction**

$$k_z = k \cos i$$

$$k_x = k \sin i$$



If $c_x$ and $c_z$ are apparent velocities along horizontal and vertical directions



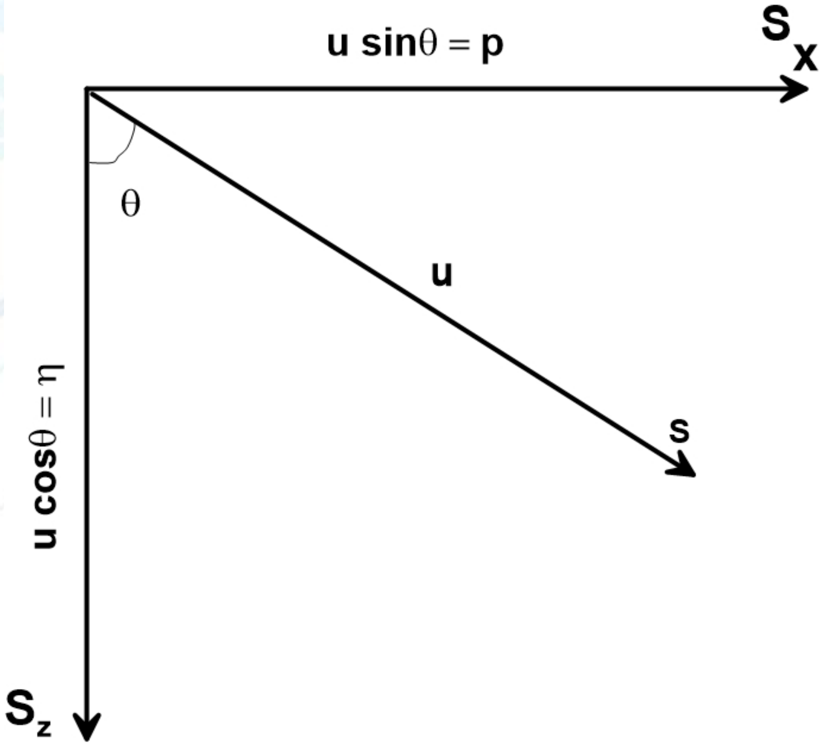

**Note:**

- The apparent horizontal velocity is always greater than or equal to the medium velocity, $\alpha$ for P-wave and $\beta$ for S-wave
- A horizontally propagating wave, with $i = 90^0$, has an apparent velocity equal to the medium velocity.
- A vertically incident plane waves arrives everywhere on the surface at the same-time, so it has infinite apparent velocity

**Slowness vector can be decomposed into vertical and horizontal components**

$$\vec{u} = (p, \eta) = \left(\frac{\sin\theta}{v}, \frac{\cos\theta}{v}\right) = \left(\frac{1}{c_x}, \frac{1}{c_z}\right) = \left(\frac{k_x}{\omega}, \frac{k_z}{\omega}\right)$$

**Horizontal slowness (p) remains constant for a ray while vertical slowness (η) varies with velocity of the layers.**




Also, $$\frac{1}{v^2} = \eta^2 + p^2$$

or

$$\eta = \sqrt{\frac{1}{v^2} - p^2}$$

$$u^2 = \eta^2 + p^2$$

or

$$u = \left|\sqrt{\eta^2 + p^2}\right|$$

# Summary

- For free surface, $T_i = \sigma_{ij} n_j$ tractions must be continuous at an interface.
- For the solid-solid interface $u_i^+ = u_i^-$, $T_i^+ = T_i^-$ and for the solid-liquid Interface $T_3^+ = T_3^-$ , $u_3^+ = u_3^-$
- If an SH-wave is vibrating parallel to the interface, then it will produce reflected and transmitted SH-wave.
- P -and SV- waves are coupled to each other, So a P-wave can produce SV-wave and vice-versa.
- The apparent horizontal velocity is always greater than or equal to the medium velocity, α for P-wave and β for S-wave and a vertically incident plane waves has infinite apparent velocity.
- Horizontal slowness (p) remains constant for a ray while vertical slowness (η) varies with velocity varies with velocity of the layer.
- Snell's law $\boxed{\dfrac{\sin\,\theta_1}{v_1} = \dfrac{\sin\,\theta_2}{v_2}}$ and $\vec{u} = (p, \eta) = \left(\dfrac{\sin\theta}{v}, \dfrac{\cos\theta}{v}\right) = \left(\dfrac{1}{c_x}, \dfrac{1}{c_z}\right) = \left(\dfrac{k_x}{\omega}, \dfrac{k_z}{\omega}\right)$

# REFERENCES


- Stein, Seth, and Michael Wysession. An introduction to seismology, earthquakes, and earth structure. John Wiley & Sons, 2009.
- Lowrie, William, and Andreas Fichtner. Fundamentals of geophysics. Cambridge university press, 2020.
- Kearey, Philip, Michael Brooks, and Ian Hill. An introduction to geophysical exploration. Vol. 4. John Wiley & Sons, 2002.
- https://geologyscience.com/geology-branches/structural-geology/stress-and-strain/
- [https://www.wikipedia.org/](https://www.wikipedia.org/)
- Seismology course, Professor Derek Schutt, Colorado State Univ., USA.



# THANK YOU !