NPTEL ONLINE CERTIFICATION COURSES

EARTHQUAKE SEISMOLOGY

Dr. Mohit Agrawal

Department of Applied Geophysics , IIT(ISM) Dhanbad

Module 01 : Basic Seismological Theory, Waves on a String , Stress and Strain and seismic waves

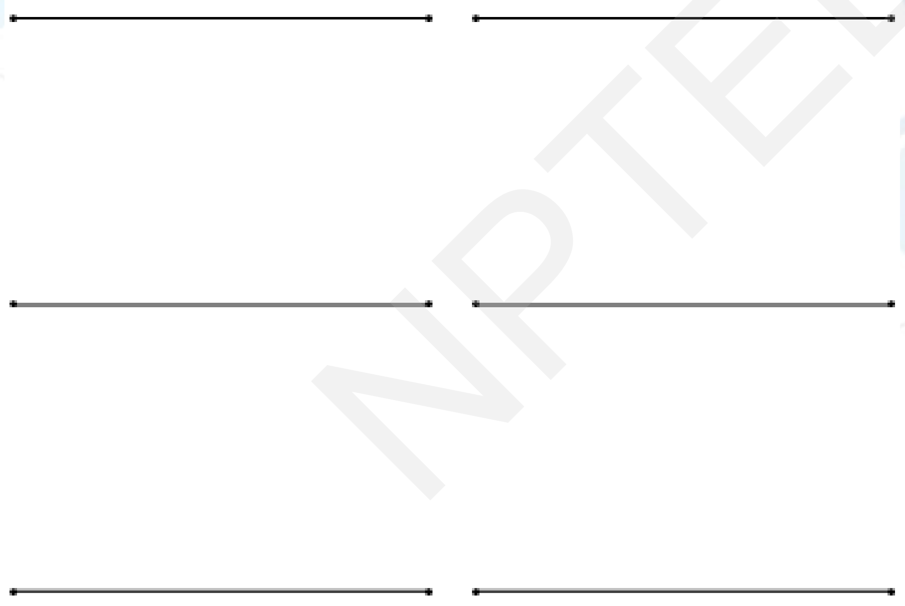
Lecture 02: Waves on a String, Wave Equation, Energy and Normal Modes

CONCEPTS COVERED

- **Waves on a String**
- **Wave Equation**
- **Energy and Normal Modes**

Waves on a String

Goal: To derive the wave equation on a string to help guide our thinking for the 3-D wave equation.



Derivation of the 1-D wave Equation

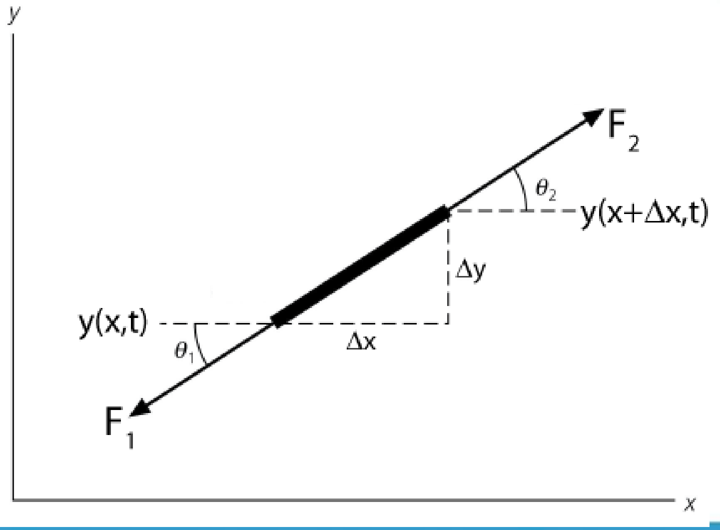
Assumptions:

1) The string is perfectly uniform, so properties, such as mass per unit length (ρ) don't change in x.

2) The string only moves up and down, in the y-direction, and the y-amplitude is small

3) Gravity is ignored (but it's not too hard to add).

Figure 2.2-1: Tensions on a string segment.



Start with Newton's Law $\vec{F} = m\vec{a}$

Decompose it into the x and y components

$$F_x = ma_x$$

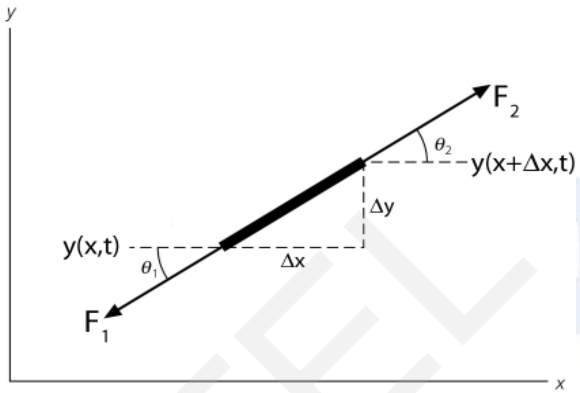
$$F_y = ma_y$$

Now consider a small element of the string at time, *t*

1) If the string is not moving in the x-direction, then the forces in the x-direction must balance.

$$F_1\cos\theta_1 = F_2\cos\theta_2 = F_x$$

Figure 2.2-1: Tensions on a string segment.



2) In the y-direction

$$F_y = ma_y$$

3) Here, $m=\rho dx$ and $a_y = \dfrac{\partial^2 y}{\partial t^2}$

4) **F_y is the sum of the forces in the y-direction**

$$F_y = F_2\sin\theta_2 - F_1\sin\theta_1$$

5) Steps 2 and 4 can be combined to get

$$F_2\sin\theta_2 - F_1\sin\theta_1 = \rho\Delta x\frac{\partial^2 y}{\partial t^2}$$

6) Divide this by F_x, but use a different term for each part of the equation.

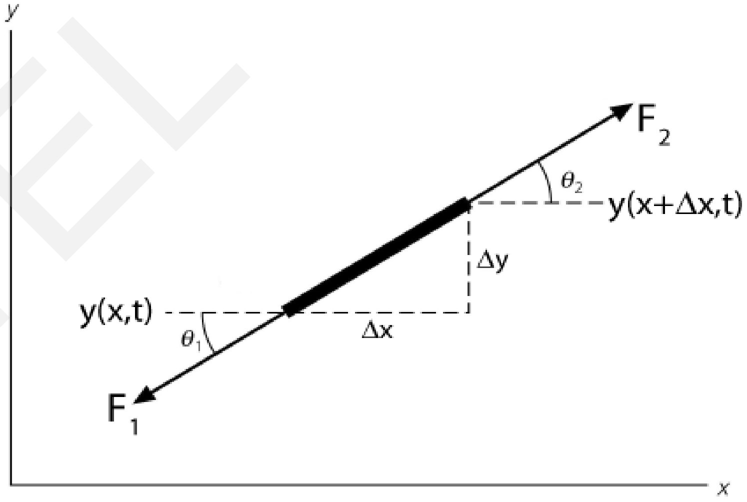
$$F_2 \sin\theta_2 - F_1 \sin\theta_1 = \rho \Delta x \frac{\partial^2 y}{\partial t^2}$$

(Recall that F_x is: $F_1 \cos\theta_1 = F_2 \cos\theta_2 = F_x$)

This gives us:

$$\underbrace{\frac{F_2 \sin\theta_2}{F_2 \cos\theta_2}}_{\tan\theta_2} - \underbrace{\frac{F_1 \sin\theta_1}{F_1 \cos\theta_1}}_{\tan\theta_1} = \frac{\rho \Delta x}{F_x} \frac{\partial^2 y}{\partial t^2}$$

Figure 2.2-1: Tensions on a string segment.



7) Recall that tan is just the ratio of the opposite to the adjacent sides for angle θ. In other words,

$$\tan\theta_1 = \left.\frac{\partial y}{\partial x}\right|_x$$

$$\tan\theta_2 = \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}$$

8) Substitute the derivative forms of tan into our working equation

This

$$\underbrace{\frac{F_2 \sin\theta_2}{F_2 \cos\theta_2}}_{\tan\theta_2} - \underbrace{\frac{F_1 \sin\theta_1}{F_1 \cos\theta_1}}_{\tan\theta_1} = \frac{\rho \Delta x}{F_x} \frac{\partial^2 y}{\partial t^2}$$

then becomes this

$$\left[\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x} \right] = \frac{\rho \Delta x}{F_x} \frac{\partial^2 y}{\partial t^2}$$

9) Divide through by Δx

$$\frac{\left[\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x} \right]}{\Delta x} = \frac{\rho}{F_x} \frac{\partial^2 y}{\partial t^2}$$

As x -> 0, this is just the 2nd derivative.

$$\frac{\left[\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right]}{\Delta x} = \frac{\rho}{F_x}\frac{\partial^2 y}{\partial t^2}$$

Hence,

$$\lim_{\Delta x \to 0} \frac{\left[\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right]}{\Delta x} = \frac{\partial^2 y}{\partial x^2}$$

which gives us the 1-D wave equation

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{F_x}\frac{\partial^2 y}{\partial t^2}}$$

Waves on a String

How motion of waves on a string mimics the wave propagation in the Earth?

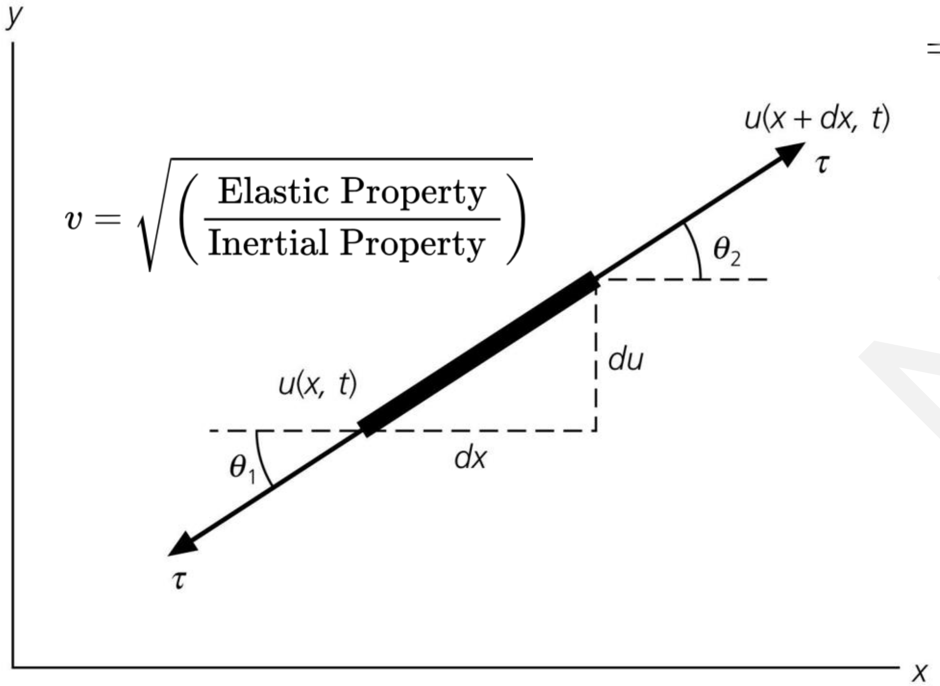
Here we are assuming state of the stress in earth, which is quite similar to string with tension ‘τ’.

$$f(x,t) = \tau \sin\theta_2 - \tau \sin\theta_1 = \rho dx \frac{\partial^2 u(x,t)}{\partial t^2} \implies \tau\left(\frac{\partial u(x+dx,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right) = \rho dx \frac{\partial^2 u(x,t)}{\partial t^2}$$

$$\implies \frac{\partial^2 u(x,t)}{\partial x^2} = \left(\frac{\rho}{\tau}\right)\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

Using Taylor’s expansion

$$v = \sqrt{\left(\frac{\text{Elastic Property}}{\text{Inertial Property}}\right)}$$



This equation gives the relationship between the time and space derivatives of the displacement $u(x,t)$ along the string. This coupling between the two partial derivatives gives rise to waves propagating along the string with a velocity v.Thus the stress in string or earth has clear impact wave propagation.

Fig. 1.4

Solving the 1-D Wave Equation

The solution is relatively simple: $y(x \pm vt)$

where $v = \sqrt{\frac{F_x}{\rho}}$ **or in the notation of the book** $v = \sqrt{\frac{\tau}{\rho}}$

F_x **or** τ **represent the tension in the string.**

The equation can be rewritten in terms of v **to get:**

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Let's test our solution to the wave equation, and see if y(x+vt) solves the equation.

So, we'll plug in this solution and see if both sides of the equation are equal

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Let's do a substitution: *u = x + vt*

Now, apply the chain rule: $$\frac{\partial y(x+vt)}{\partial x} = \frac{\partial}{\partial u}(y)\frac{\partial u}{\partial x}$$

Since $\frac{\partial u}{\partial x} = 1$,

$$\frac{\partial y(x+vt)}{\partial x} = \frac{\partial y}{\partial u}$$

and

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u}\right)\frac{\partial u}{\partial x} = \frac{\partial^2 y}{\partial u^2}$$

This gives us a value for the LHS of the equation.

Likewise, we can do the same for the time-derivatives

$$\frac{\partial y(x+vt)}{\partial t} = \frac{\partial}{\partial u}(y)\frac{\partial u}{\partial t}$$

In this case, since $u = x + vt$, $\frac{\partial u}{\partial t} = v$

Hence,

$$\frac{\partial y(x+vt)}{\partial t} = v\frac{\partial y}{\partial u}$$

Taking the second derivative with respect to time, we get:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial u}\left(v\frac{\partial y}{\partial u}\right)\frac{\partial u}{\partial t} = v^2\frac{\partial^2 y}{\partial u^2}$$

Now, we have: $\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2}$ and $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial u^2}$

Let's plug this into the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

One gets:

$$\frac{\partial^2 y}{\partial u^2} = \frac{1}{v^2} v^2 \frac{\partial^2 y}{\partial u^2}$$

Both sides are equal, which proves that y(x+vt) is a valid solution to the wave equation. You could do the same things with y(x-vt).

Principle of Superposition

It can be shown that the sum of two solutions to the wave equation is also a solution to the wave equation.

Since a function in the form $y(x \pm vt)$ can readily be decomposed in terms of sine or cosine waves, this means that solutions to the wave equation can be considered in terms of superposition of these waves.

So, we will be spending a lot of time looking at waves with one frequency, but with the understanding that we can create an arbitrary function out of the sum of harmonic waves of different frequencies.

Harmonic Wave Solution and Wave Parameters

A useful way to characterize the solutions to the wave equation is in terms of *harmonic* waves.

$$y(x,t) = u(x,t) = Ae^{i(\omega t \pm kx)} = A\cos(\omega t \pm kx) + iA\sin(\omega t \pm kx)$$

Here, y(x,t) or u(x,t) is the vertical displacement

The wave velocity is: $v = \omega / k$

A harmonic wave thus can be characterized by three parameters, such as:

- **Amplitude (A)**
- **Angular frequency (ω)**
- **Wavenumber (k)**

Some properties of a harmonic wave

Consider the real part of the harmonic wave equation, for a wave propagating in one direction: .

$$u(x,t) = A\cos(\omega t - kx)$$

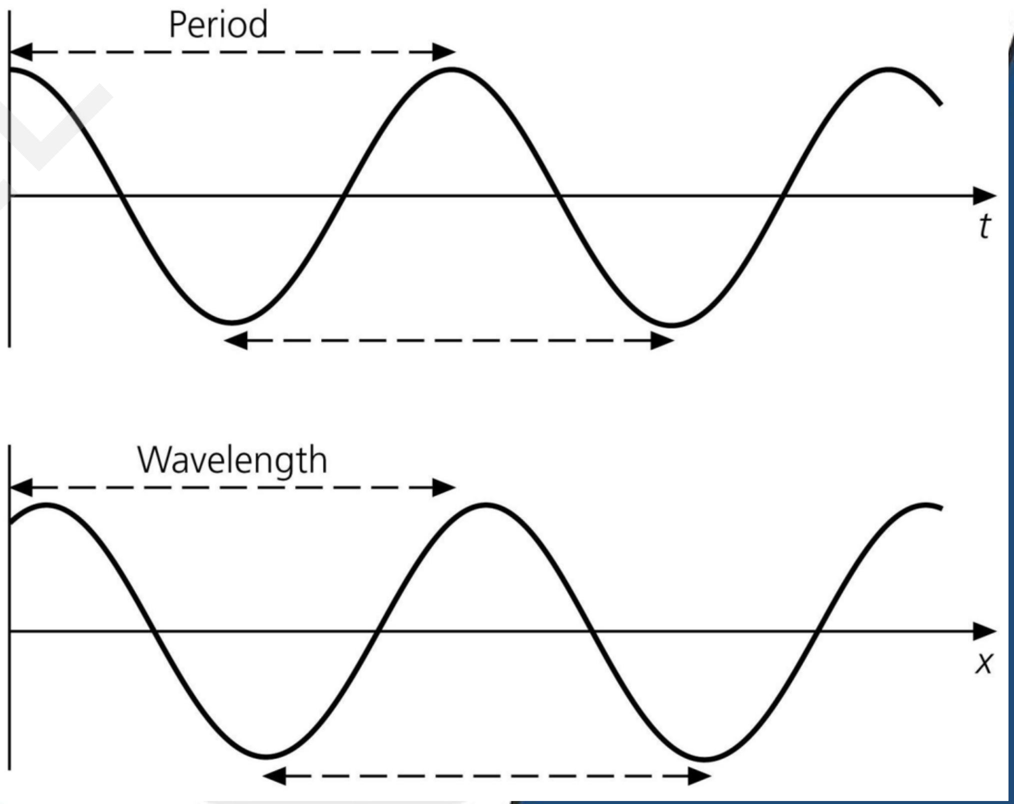
u is constant when the phase, $(\omega t - kx)$, is constant.

At a given location, x_0, $u(x_0,t) = A\cos(\omega t - kx_0)$ will have the same value every time ωt changes by 2π.

The wave period is characterized by T = $2\pi/\omega$ or $1/f$.

Alternatively, the *frequency* gives the number of oscillations in a given time $f = 1/T = \omega/2\pi$.

Figure 2.2-4: Harmonic wave, $u = A\cos(\omega t - kx)$.



Useful Summary of Wave Parameters

Quantity	Units	Equations
Velocity	distance/time	$v = \omega / k = f\lambda = \lambda T$
Period	time	$T = 2\pi/\omega = 1/f = \lambda/v$
Angular Frequency	time^{-1}	$\omega = 2\pi/T = 2\pi f = kv$
Frequency	time^{-1}	$f = \omega/2\pi = 1/T = v/\lambda$
Wavelength	distance	$\lambda = 2\pi/k = v/f = vT$
Wavenumber	distance^{-1}	$k = 2\pi/\lambda = \omega/v = 2\pi f/v$

Wavenumber may be an unusual concept. It is the number of wavelengths per unit distance, in radians.

For instance, if the wavelength is 16km for a 8 km/s P-wave, the frequency, f = .5Hz, and the wavenumber is pi/8 = ~.4 km^{-1}

Reflection and Transmission

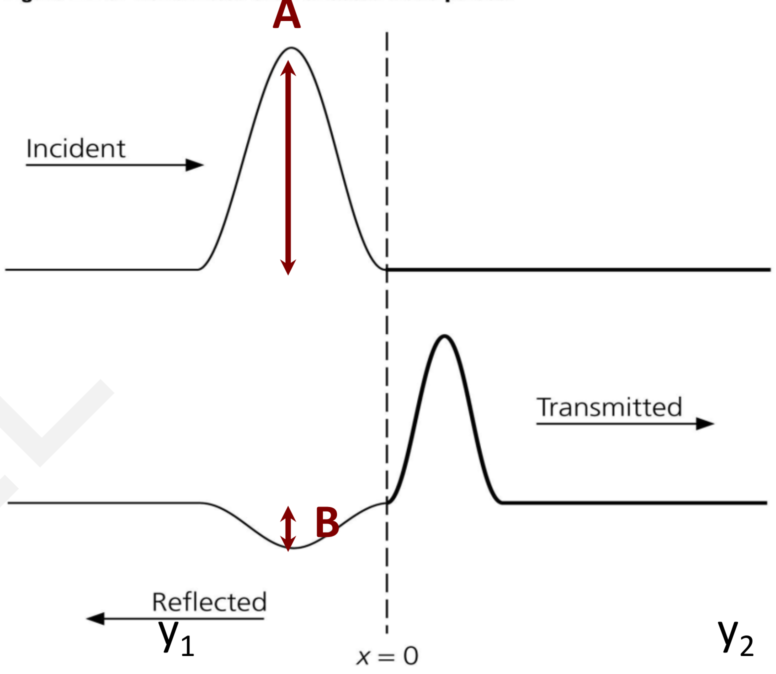
Consider two strings with different properties—the left string has density ρ_1 and velocity v_1 , and the right string has density ρ_2 and velocity v_2.

For x<0 (the left hand side of the string), the displacement can be represented as the sum of the

- Incident wave moving in the + x direction, with amplitude A
- The reflected wave, moving in the –x direction, with amplitude B.

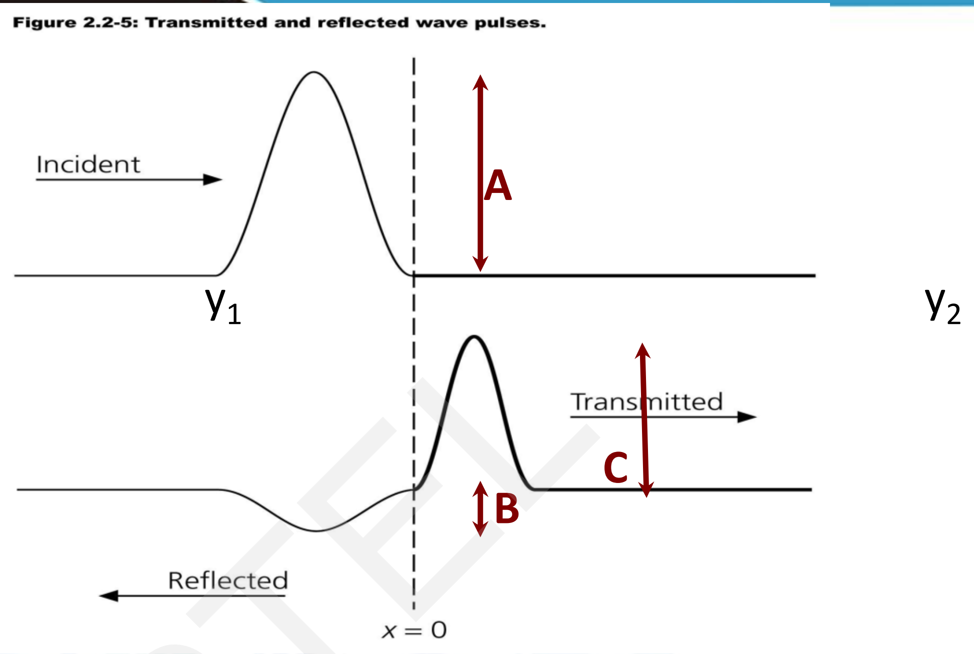
$$y_1(x,t) = Ae^{i(\omega_{1a}t - k_{1a}x)} + Be^{i(\omega_{1b}t + k_{1b}x)}$$

Figure 2.2-5: Transmitted and reflected wave pulses.



For x>0 (the RHS of the string), the displacement can be represented as the transmitted wave, with amplitude C.

$$y_2(x,t) = Ce^{i(\omega_2 t - k_2 x)}$$



To get a feel for the movement of these waves, recall that constant phase implies ***Φ*** = constant, where ***Φ*** represents the phase.

If we assume, for convenience, that ***Φ=0***, then for the negative ***k*** case: ***ωt-kx***=0. This implies ***ωt = kx***

Consequently, as ***t*** gets larger, ***x*** must get larger for the phase to stay constant. Hence, ***ωt - kx*** implies a wave moving in the ***+x*** direction, and ***ωt+kx*** would be a wave moving in the –x direction.

At *x = 0*, we can establish two conditions:

1) The string is continuous: $y_1(x=0,t)=y_2(x=0,t)$

2) The string has continuous $\frac{\partial y}{\partial x}$. To understand this, consider if the string had a discontinuous slope at x = 0. This means the 2nd derivative is infinite:

$$\frac{\partial^2 y}{\partial x^2}=\infty$$

which means

$$\infty=\frac{\rho}{F_x}\frac{\partial^2 y}{\partial t^2}$$

And if tension (F_x) is constant, $\frac{\rho}{F_x}$ will also be constant, which implies

$$\infty=\frac{\partial^2 y}{\partial t^2}$$

But this is the vertical acceleration. Since $F_y = m\frac{\partial^2 y}{\partial t^2}$

a discontinuous slope would imply an infinite force in the y-direction

Indeed, any tendency to kink and have a discontinuous slope would be met with increased acceleration to reduce the slope. Hence, we have our 2nd boundary condition:

$$\left.\frac{\partial y_1}{\partial x}\right|_{x=0} = \left.\frac{\partial y_2}{\partial x}\right|_{x=0}$$

With these constraints, and some thinking, we can start to solve the problem. Constraint 1 implies:

$$Ae^{i(\omega_{1a}t)} + Be^{i(\omega_{1b}t)} = Ce^{i(\omega_2 t)}$$

For this to work for all ***t***, $\omega_{1a} = \omega_{1b} = \omega_2$

Also, at *t* = 0, $A + B = C$

Now, let's apply constraint #2. Note that since ω is constant, then the wavenumber *k* controls the velocity of each string. Hence, $k_{1a} = k_{1b} \neq k_2$**.**

$$-ik_1 Ae^{i(\omega t)} + ik_1 Be^{i(\omega t)} = -ik_2 Ce^{i(\omega t)}$$

Remove common terms to get

$$k_1(A - B) = k_2 C$$

We can put this in terms of the velocity, since $k = \omega / v$

$$\frac{\omega}{v_1}(A - B) = \frac{\omega}{v_2} C$$

Now recall that $v = \sqrt{\tau / \rho}$

We can cancel out the angular frequencies, and then multiply the LHS by:

$$\frac{v_1^{\;2}}{v_1^{\;2}} = \frac{\rho_1 v_1^{\;2}}{\tau}$$

and the RHS by:

$$\frac{v_2^{\;2}}{v_2^{\;2}} = \frac{\rho_2 v_2^{\;2}}{\tau}$$

This gives $$\rho_1 v_1 (A - B) = \rho_2 v_2 C$$

Since $C = A + B$, then $$\rho_1 v_1 (A - B) = \rho_2 v_2 (A + B)$$

Reflection and Transmission coefficients

A little bit of algebra gets us to the reflection coefficient:

$$R_{12} = \frac{B}{A} = \frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2}$$

Similarly, we can get the transmission coefficient:

$$T_{12} = \frac{C}{A} = \frac{2\rho_1 v_1}{\rho_1 v_1 + \rho_2 v_2}$$

Impedance

- The quantity *ρv* is called acoustic impedance, and is often denoted as Z, although sometimes I use *I*.

- $I_1 = \rho_1 v_1$ can equal $I_2 = \rho_2 v_2$ even though $v_1 \neq v_2$.

- If $I_1 > I_2$ then $T_{12} > 1$.   As we shall see below, amplitudes are not necessarily conserved, but energy, and energy flux is.

Wavelength and Interfaces

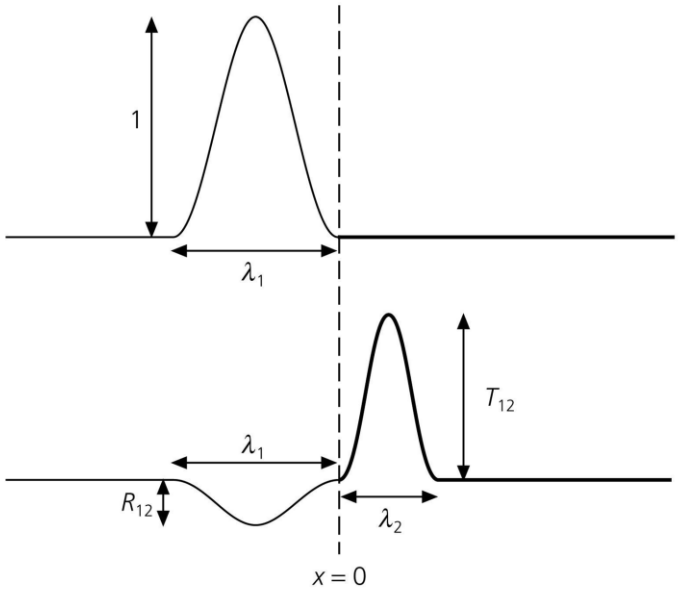
Because the angular frequencies of the two strings are the same

$$\omega = v_1 k_1 = v_2 k_2 = \frac{v_1 2\pi}{\lambda_1} = \frac{v_2 2\pi}{\lambda_2}$$

which means

$$\frac{\lambda_2}{\lambda_1} = \frac{v_2}{v_1}$$

Figure 2.2-7: Reflected and transmitted amplitudes.



Energy in a Harmonic Wave

The total energy in a system is the sum of the potential energy (PE) and the kinetic energy (KE).

Let's consider the KE first. From physics, we know that:

$$KE = \frac{1}{2} m v^2$$

This isn't too hard to put into our string formalism: $v = \frac{\partial y}{\partial t}$ $m = \rho dx$

Thus,

$$KE = \frac{\rho}{2} \left(\frac{\partial y}{\partial t} \right)^2 dx$$

Kinetic Energy

Over a wavelength, the total KE is (the λ in the denominator is because we are averaging over a wavelength)

$$KE = \frac{\rho}{2\lambda}\int_0^{\lambda}\left(\frac{\partial y}{\partial t}\right)^2 dx$$

If $y(x,t) = A\cos(\omega t - kx)$, **then** $\frac{\partial y}{\partial t} = -A\omega\sin(\omega t - kx)$

and

$$KE = \frac{\rho A^2\omega^2}{2\lambda}\int_0^{\lambda}\sin^2(\omega t - kx)dx$$

Kinetic Energy

To solve this, we'll first use the identity $\sin^2(x) = \frac{1}{2}(1 - \cos 2x)$

So,

$$KE = \frac{\rho A^2 \omega^2}{2\lambda} \int_0^{\lambda} \left(\frac{1}{2} - \frac{1}{2}\cos[2(\omega t - kx)] \right) dx$$

Let's integrate the first part of this equation:

$$\frac{\rho A^2 \omega^2}{2\lambda} \int_0^{\lambda} \left(\frac{1}{2} \right) dx = \frac{\rho A^2 \omega^2}{2\lambda} \left(\frac{x}{2} \right) \Bigg|_{x=0}^{x=\lambda} = \frac{\rho A^2 \omega^2}{2\lambda} \frac{\lambda}{2}$$

To integrate the second part of the equation $\frac{\rho A^2 \omega^2}{2\lambda} \int_0^{\lambda} \left(\frac{1}{2}\cos[2(\omega t - kx)] \right) dx$

we'll use a u-substitution,
where

$$u = 2(\omega t - kx)$$

$$du = -2k dx$$

$$dx = -\frac{1}{2k} du$$

Kinetic Energy

This gives us

$$KE = \frac{\rho A^2\omega^2}{2\lambda}\int_{x=0}^{x=\lambda} -\frac{1}{4k}\cos u\, du$$

$$= \frac{\rho A^2\omega^2}{2\lambda}\left[-\frac{1}{4k}\sin(u)\Big|_{x=0}^{x=\lambda}\right]$$

$$= \frac{\rho A^2\omega^2}{2\lambda}\left[-\frac{1}{4k}\big(\sin(2\omega t - 2k\lambda) - \sin(2\omega t - 0)\big)\right]$$

Since $k\lambda = \frac{2\pi}{\lambda}\lambda = 2\pi$ the quantity $\sin(2\omega t - 2k\lambda) - \sin(2\omega t - 0) = \sin(2\omega t - 4\pi) - \sin(\omega t - 0)$

These two terms are 4π or exactly 2 wavelengths apart, so they will equal each other, and thus

$$\sin(2\omega t - 4\pi) - \sin(2\omega t - 0) = 0$$

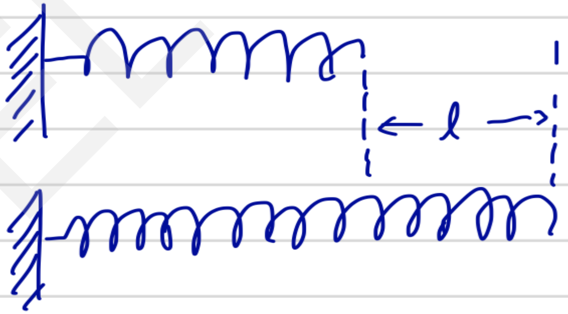
This means that

$$KE = \frac{\rho A^2\omega^2}{4}$$

Potential Energy

Let's examine the potential energy in a spring thanks is stretched a distance *l*.

$$\Delta PE = Work = \int F(l)dl$$



where *F(l)* is the force needed to stretch the spring,

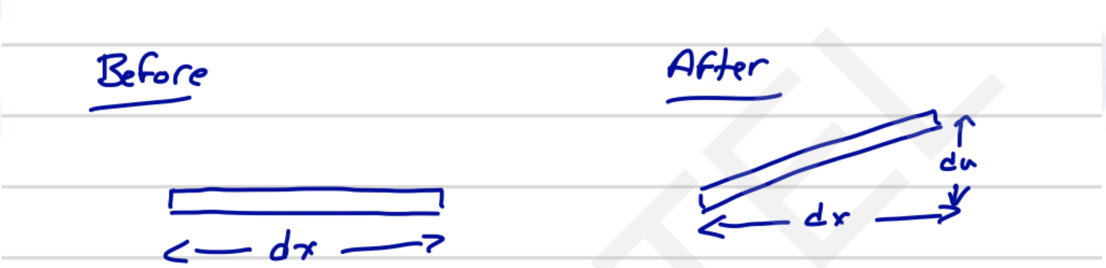
We know this from Hooke's Law

$$F(l) = kl$$

So the potential energy in a spring is: $$PE = \int kl\,dl = \frac{1}{2}kl^2$$

Potential Energy

With this in mind, let's consider the stretching of a string.



The change in length, *dl*, is

$$\Delta l = \sqrt{dx^2 + du^2} - dx$$
$$= dx\left(\sqrt{1 + \frac{du^2}{dx^2}} - 1\right)$$

Potential Energy

Since the Maclaurin Series $(1+a^2)^{1/2} \approx 1+\frac{1}{2}a^2$, then for small $\left(\frac{\partial u}{\partial x}\right)$, we can approximate

$$\sqrt{1+\frac{du^2}{dx^2}} \text{ as } 1+\frac{1}{2}\frac{du^2}{dx^2}$$

$$(1+a^2)^{1/2} \approx 1+a^2$$

which gives us

$$\Delta l = \left(1+\frac{1}{2}\frac{du^2}{dx^2}-1\right)dx$$

$$= \frac{dx}{2}\left(\frac{du}{dx}\right)^2$$

The force required to stretch the string is the tension, τ, which makes the potential energy

$$PE = \frac{\tau}{2}\int\left(\frac{du}{dx}\right)^2 dx$$

Potential Energy per Wavelength

Averaged over a wavelength the PE is

$$PE = \frac{\tau}{2\lambda}\int_0^{\lambda}\left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{\tau A^2 k^2}{2\lambda}\int_0^{\lambda}\sin^2(\omega t - kx)dx$$

Solving this like we did for the KE, we get

$$PE = \frac{1}{4}A^2\omega^2\rho$$

Total energy transported, averaged over a wavelength, is

$$E = KE + PE = \frac{1}{2}A^2\omega^2\rho$$

Energy Flux

The rate of energy transport, or energy flux, is the energy times the velocity.

$$\frac{dE}{dt} = \dot{E} = \tfrac{1}{2} A^2 \omega^2 \rho v$$

Let's show that energy, and not amplitude, is conserved at an interface.

Consider (for simplicity), the case where the equations for the incident, reflected, and transmitted waves are as follows:

$$u_I(x,t) = \cos(\omega t - k_1 x)$$
$$u_R(x,t) = R_{12} \cos(\omega t + k_1 x)$$
$$u_T(x,t) = T_{12} \cos(\omega t - k_2 x)$$

Energy Flux

Since the amplitude of the R and T waves are R_{12} and T_{12} respectively, the energy fluxes are:

$$\dot{E}_I = \frac{\omega^2 \rho_1 v_1}{2}$$

$$\dot{E}_R = \frac{R_{12}^2 \omega^2 \rho_1 v_1}{2}$$

$$\dot{E}_T = \frac{T_{12}^2 \omega^2 \rho_2 v_2}{2}$$

Let's sum the reflected and transmitted energy fluxes.

$$\dot{E}_R + \dot{E}_T = \frac{\omega^2}{2}\left(R_{12}^2 \rho_1 v_1 + T_{12}^2 \rho_2 v_2\right)$$

Energy Flux

To keep the math simple, define the impedance as I=vρ, so $I_1=v_1\rho_1$ and $I_2=v_2\rho_2$. Then,

$$R_{12} = \frac{I_1 - I_2}{I_1 + I_2} \text{ and } T_{12} = \frac{2I_1}{I_1 + I_2}$$

So,

$$\dot{E}_R + \dot{E}_T = \frac{\omega^2}{2}\left[\left(\frac{I_1 - I_2}{I_1 + I_2}\right)^2 v_1\rho_1 + \left(\frac{2I_1}{I_1 + I_2}\right)^2 v_2\rho_2\right]$$

$$= \frac{\omega^2}{2}\left[\left(\frac{I_1 - I_2}{I_1 + I_2}\right)^2 I_1 + \left(\frac{2I_1}{I_1 + I_2}\right)^2 I_2\right]$$

$$= \frac{\omega^2}{2}\left[\frac{I_1\left({I_1}^2 - 2I_1I_2 + {I_2}^2\right) + 4I_2{I_1}^2}{(I_1 + I_2)^2}\right]$$

$$= \frac{\omega^2}{2}\left[\frac{{I_1}^3 + 2{I_1}^2I_2 + I_1{I_2}^2}{(I_1 + I_2)^2}\right]$$

Energy Flux

$$= \frac{\omega^2}{2}\left[\frac{I_1\left(I_1{}^2 + 2I_1I_2 + I_2{}^2\right)}{(I_1 + I_2)^2}\right]$$

$$= \frac{\omega^2}{2}\left[\frac{I_1(I_1 + I_2)^2}{(I_1 + I_2)^2}\right]$$

$$= \frac{\omega^2}{2} I_1 = \dot{E}_I$$

So, $\dot{E}_R + \dot{E}_T = \frac{\omega^2}{2} I_1 = \frac{\omega^2}{2}\rho_1 v_1 = \dot{E}_I$

which means energy flux is conserved, even though T can be > 1.

Perhaps more importantly, energy flux, and total energy, goes up as the square of the frequency. So more energy exists in higher frequency waves.

Normal Modes of a String

We've been looking at travelling waves as a solution to the 1-D wave equation.

A completely valid alternative is to seek solutions to the wave equation with a cos(ωt) dependence, such that:

$$y(x,t) = Y(X,\omega)\cos(\omega t)$$

For a constant property string, one solution is where the Y(x,ω) term is

$$Y(x,\omega) = \sin(\omega x/v)$$

If the string is fixed at *x*=0, and *x*=L, then these boundary conditions imply the only frequencies that work are

$$\omega_n = n\pi v / L$$

Normal Modes on a String

Since the string can only vibrate at these discrete frequencies, these frequencies are called *eigenfrequencies*.

These eigenfrequencies correspond to the spatial terms of the solution

$$Y_n(x,\omega_n) = \sin(\omega_n x/v)$$

The complete solution is

$$Y(x,t) = \sum_{n=0}^{\infty} A_n Y_n(x,\omega_n)\cos(\omega_n t)$$

where each term (*n*) is called a normal mode.

Normal Modes on a String

The normal modes are orthogonal, which means

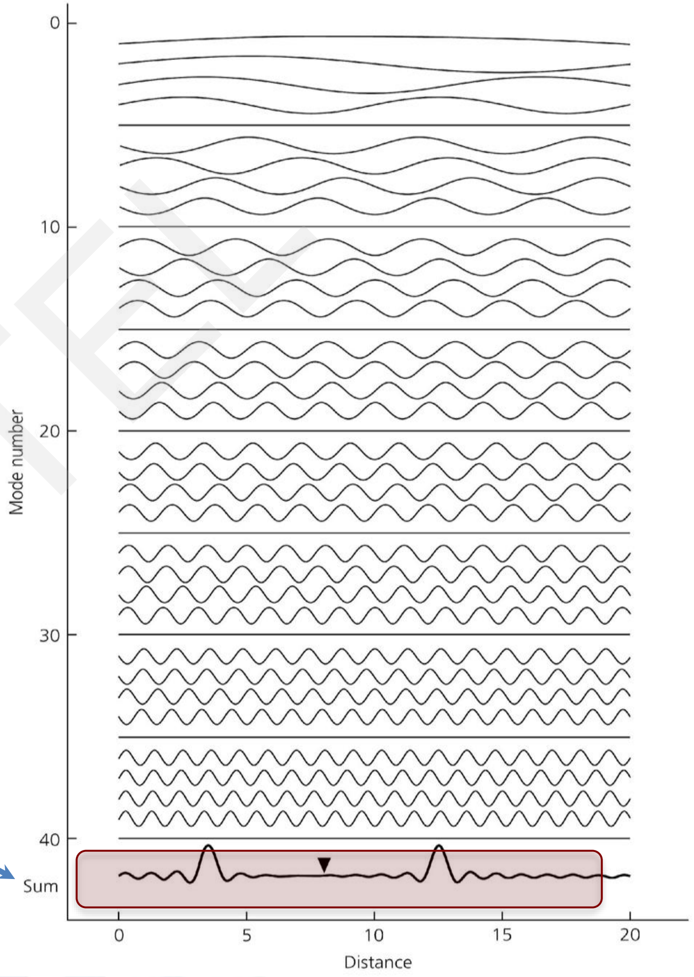
$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{2}\delta_{mn}$$

Normal Modes on a String

It may seem counterintuitive, but the normal modes are a completely valid and equivalent way to model a wave on a string, or in the earth.

Here is an example wave that is described approximately by 40 normal modes.

Figure 2.2-8: Waves on a string as a summation of modes.



Principle of Reciprocity

The principle of reciprocity: the equations for displacement of a string, and seismic waves in the Earth, are such that under the appropriate conditions, the same displacement occurs if the source and receiver are interchanged.

This is often used in exploration seismology.

How motion of waves on a string mimics the wave propagation in the Earth?

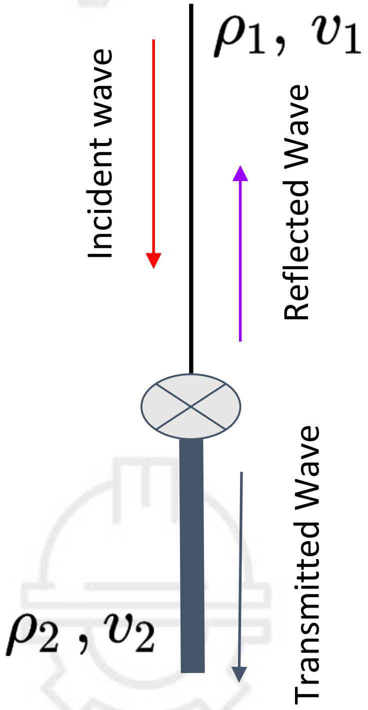


Fig. 1.5

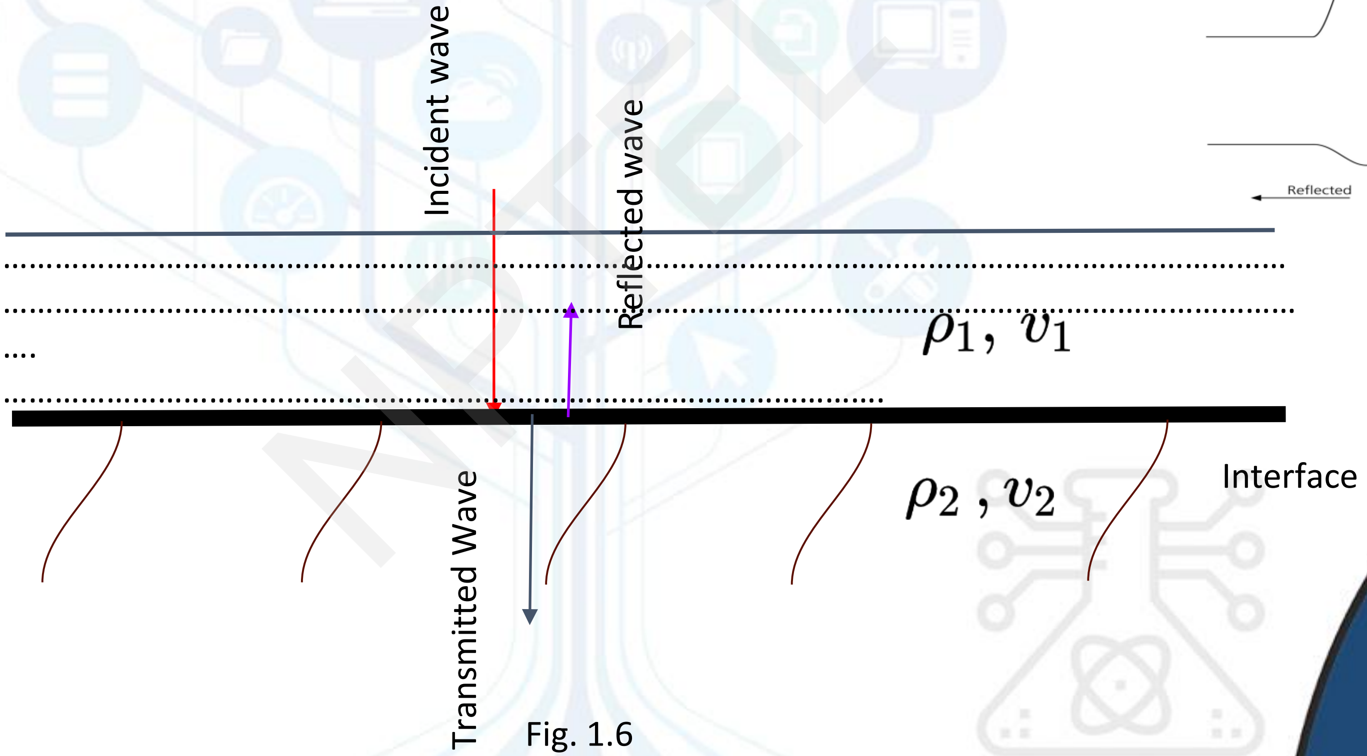
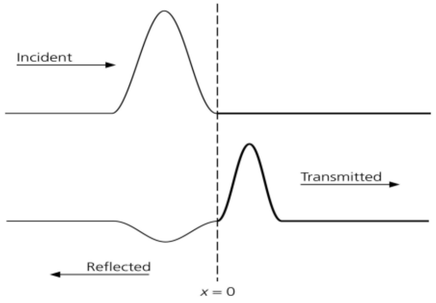


Fig. 1.6



What have we done?

- We used force balance to derive the 1-D wave eq.
- We did an overview of parameters that describe harmonic waves. The wave number, k, may be new to you.
- Using C^1 continuity, we derived reflection and transmission coefficients.
- We looked at KE and PE averaged over a wavelength.
- An alternative method of solving a differential equation is through normal modes.

How does this fit in?

- We will use similar methods to develop and solve the wave equation in a continuum.
- The harmonic wave parameters (modified) will be useful for 3-D wave propagation.
- Reflection and Transmission coefficients can also be calculate for a layered earth.
- For a continuum, kinetic and potential energy will have similar functions.
- The earth has normal modes.   These can be used to infer structure or create synthetic seismograms.

REFERENCES

- Stein, Seth, and Michael Wysession. An introduction to seismology, earthquakes, and earth structure. John Wiley & Sons, 2009.
- Lowrie, William, and Andreas Fichtner. Fundamentals of geophysics. Cambridge university press, 2020.
- Kearey, Philip, Michael Brooks, and Ian Hill. An introduction to geophysical exploration. Vol. 4. John Wiley & Sons, 2002.
- https://geologyscience.com/geology-branches/structural-geology/stress-and-strain/
- https://www.wikipedia.org/

THANK YOU !