

Structural Reliability
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Lecture –73
Monte Carlo Simulations (Part - 05)

Once a random number generator has been selected the user may want to test the generator for various properties because the accuracy of Monte Carlo simulations depends on the supply of a large number of IID random samples. So, um here we discuss few very basic and general tests of randomness they are by no means complete.

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Monte Carlo simulations

Structural Reliability
Lecture 8
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simulations

Tests for randomness

Monobit frequency test:

Frequency test of zeros or ones in a sequence of size n .
 The number of zeros should be close to $n/2$
 ϵ_i is the i^{th} bit with $P[\epsilon_i = 0] = P[\epsilon_i = 1] = 1/2$
 Define $X_i = 2\epsilon_i - 1 \rightarrow E[X_i] = 0, \text{var}[X_i] = 1$
 Consider the sum, $S = X_1 + X_2 + \dots + X_n$

If the sequence $\{X_i\}$ is IID,
 $E(S) = 0 \times n = 0$
 $\text{var}(S) = n \text{var}(X_i) = n \times 1 = n$

Normalize $S_0 = \frac{S}{\sqrt{n}} \rightarrow N(0,1)$

$P\left[\left|\frac{S}{\sqrt{n}}\right| < z_{1-\alpha/2}\right] = 1 - \alpha$

Set up test of hypothesis,
 $H_0: \mu = 0, H_1: \mu \neq 0$ with level of significance α

Block frequency test:

Let there be n blocks of size m each. Then number of 1s in each block, X_i should be Binomial with mean $m/2$ and variance $m/4$.

$$\frac{X_i - m/2}{\sqrt{m/4}} \rightarrow N(0,1)$$


$$\left(\frac{X_i - m/2}{\sqrt{m/4}}\right)^2 \rightarrow \chi^2(1)$$

and $\sum_{i=1}^n \left(\frac{X_i - m/2}{\sqrt{m/4}}\right)^2 \rightarrow \chi^2(n)$

Set up test of hypothesis:

$$H_0: 4m \sum_{i=1}^n \left(\frac{X_i - 1}{m - 2}\right)^2 \rightarrow \chi^2(n)$$

Further reading:
 A Statistical Test Suite for Random and Pseudorandom Number Generators for Cryptographic Applications by Bassham et al., NIST, SP 800-22, 2016.



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So, let us start with the idea that there is a sequence of zeros and ones these might be the outcome of some physical process like tossing a fair coin many times or one of the pseudo-random number generators working on a computer. So, if we have such a sequence we expect that the number of zeros or number of one's should be close to n by 2. Now let us formalize this. So, let ϵ_i be the i th bit uh.

So, the probability of zero and 1 is each equal to half and then let us define a new random variable X which is twice ϵ_i minus 1. So, the mean of X you can work out is 0 and the variance of X is 1. Now let us consider the sum the sum of these n X 's, n being the length of the

sequence that we have and this sum S is also a random variable and its mean is zero and if the sequence if all the X 's are independent then the variance of S would be the sum of all the variances.

So, the variance of S is n . So, we can. Now set up a test of hypothesis. So, we normalize the sum by square root of n which reaches which approaches the standard normal distribution by central limit theorem and then we could set up the test that the estimated value of S would be between $z_{1-\alpha/2} \sqrt{n} - z_{1-\alpha/2} \sqrt{n}$ that would be our test of hypothesis with the level of significance α .

We could extend this idea to blocks of size m . So, if we have several such blocks each of size m . So, in each block the number of zeros or the number of ones if the epsilons are equally likely to be 0 and 1 would be the mean would be $m/2$ and the variance would be $m/4$. So, if I define X to be the number of such zeros in each block then again by central limit theorem X normalized by mean and the standard deviation should reach the standard normal.

And here let us square each of these. So, I get the chi-square the chi-squared distribution with degree of freedom 1 and if I then sum all of them I get a chi-square random variable with n degrees of freedom and just like I did for the monobit frequency test I could also set up a hypothesis test for the block frequency. And there is a very nice report from NIST which discusses many such tests if you like you might want to refer to it.

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Monte Carlo simulations

Tests for randomness (contd.)

Moments test:

If $\{U_i\}$ is an IID sequence of standard uniform deviates, the estimated mean and variance should be close to $1/2$ and $1/12$ respectively.

$$\hat{\mu} = \frac{1}{n} \sum U_i \quad E(\hat{\mu}) = \frac{1}{n} \sum E(U_i) = \frac{1}{n} n \mu_1 = \frac{1}{2}$$

$$\text{var}(\hat{\mu}) = \frac{1}{n^2} \sum \text{var}(U_i) = \frac{1}{n^2} n \mu_2 = \frac{1}{12n}$$

$$\hat{\mu} \rightarrow N\left(\frac{1}{2}, \frac{1}{12n}\right)$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum \left(U_i - \frac{1}{2}\right)^2 \quad E(\hat{\sigma}^2) = \frac{1}{n} \sum E\left(U_i - \frac{1}{2}\right)^2 = \frac{1}{n} n \mu_2 = \frac{1}{12}$$

$$\text{var}(\hat{\sigma}^2) = \frac{1}{n^2} \sum \text{var}\left(U_i - \frac{1}{2}\right)^2$$

$$= \frac{1}{n^2} (n \mu_4 - n \mu_2^2) = \frac{1}{180n}$$

$$\hat{\sigma}^2 \rightarrow N\left(\frac{1}{12}, \frac{1}{180n}\right)$$

Turning points test:

Given the sequence, $\{U_i\}$, create the sequence of turning points $\{X_i\}$:

$$X_i = \begin{cases} 1 & \text{if } U_i < U_{i-1} > U_{i+1} \text{ or } U_i > U_{i-1} < U_{i+1}, i=1, \dots, n-2 \\ 0 & \text{otherwise} \end{cases}$$

If $\{U_i\}$ is a sequence of IID uniform deviates, $\{X_i\}$ is 2-dependent, and

$$P\{i \text{ is a turning point}\} = P\{X_i = 1\} = \frac{2}{3}$$

$$\text{Thus } E(X_i) = \frac{2}{3}, \text{var}(X_i) = \frac{2}{9}$$

Now define the sum, $Y = \sum_{i=1}^{n-2} X_i$

$$E(Y) = \sum_{i=1}^{n-2} E(X_i) = (n-2) \frac{2}{3} \quad \text{var}(Y) = \frac{16n-29}{90}$$

$$\frac{Y - 2(n-2)/3}{\sqrt{(16n-29)/90}} \rightarrow N(0,1)$$

Further reading:
Advanced Theory of Statistics by Kendall and Stuart.



We now take these ideas to the sequence of uniform deviates. So, they are no longer bits of zeros and ones that we are talking about we are talking about uniform deviates between 0 and 1. So, these are an IID sequence U_1 up to U_n . So, one of the most intuitive tests would be that if this is indeed uniform then the mean of the sequence should be close to half and the variance should be 1 over 12 as we know.

So, we can estimate the mean and we can estimate the variance and each of these actually is a random variable. So, the mean of the estimated mean is half the variance of the estimated mean is 1 over $12n$ as you see. So, we could set up the test of whether μ had the estimated mean is normal with mean half and variance 1 by $12n$. Likewise the estimated variance is also a random variable its mean is 1 over 12 . Its variance can be computed to be 1 over $180n$ and we could likewise set up another test of hypothesis for the estimated variance as a normal random variable with mean of 1 over 12 and a variance of 1 over $180n$.

Now obviously you probably sense a problem with this test because even if we have a true random a good random sequence which passes this tests of moments the mean and the variance. If I just rank ordered them that sequence would obviously not be random anymore by any stretch of imagination but they would still equally well satisfy the moment's test. So, obviously we need to see whether the sequence itself has the appearance all the required appearance of randomness.

So, one very um interesting test is the turning point test. So, given the sequence U_n we can see if there is a turning point or not. So, turning point means if the sequence first rises and then falls at that point or falls and then rises. So, the point i is a turning point X_i equals one if U_{i+1} is greater than both the number before it and after it or it is less than the number before it or after it.

So, we have $n - 2$ such points turning points from a sequence of size n and if this original sequence U is an IID sequence of uniform deviates then this turning point sequence is actually no longer an independent sequence it is 2 dependent and we can work out that that X_i is a turning point it is the probability is $2/3$. It basically comes from the property that if you if there are 6 possible ways of having a sequence of 3 and then the 4 of them are turning points and 2 of them are not.

So, that is how that $2/3$ comes they are all equally likely and then. So, that the mean of X is $2/3$ and the variance is $2/3$ times one-third. So, that is $2/9$ and if we define the sum of all these turning points all these $n - 2$ turning points then the mean of the sum the mean of Y is $n - 2$ times $2/3$ the probability that X is one turning point the variance is little less than what would happen if they were all independent but as you know as I said the the X 's are 2 dependent.

So the variance of Y the sum is $16n - 29$ over 90 .. So, it is it is less than $n - 2$ times $2/9$ which would be if the sequences were completely independent. So, again invoking the central limit theorem the normalized value should approach the standard normal distribution and one could set up a test of hypothesis. A very nice a derivation in all of this is given in Kendall and Stewart's book.

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Monte Carlo simulations

Tests for randomness (contd.)

- Many more tests
 - Runs test
 - Difference sign test
 - Portmanteau test
 - Serial test
 - Up and down test
 - Etc.
- “Strange and unpredictable is not necessarily random.”
- In theory, the number of tests that need to be performed to test for uniformity is uncountably infinite.
 - Thus, it is relatively easy to spot bad generators, but theoretically impossible to identify a perfect one.
 - In practice, the user should at least test the generator on a similar problem with known answer, before trusting its results.

Further reading:
A Statistical Test Suite for Random and Pseudorandom Number Generators for Cryptographic Applications by Bassham et al., NIST, SP 800-22, 2010.
Time Series, by Brockwell and Davis.
Simulation and the Monte Carlo Method, by RY Rubinstein
Chaos and randomness by F. James in Chaos Solitons and Fractals, vol 6, pp 221-226, Pergamon, 1995.



So, there are obviously infinitely possible tests that one could do for randomness. Here we you have just a few mentioned on your screen the thing to remember is just because something is unpredictable it doesn't mean that they are necessarily random. So, one needs to look deeper and in theory it is much easier to test a bad generator a sequence that is not random than to satisfy oneself completely that a sequence is indeed random it is an IID sequence.

So, what is important is that for the application at hand one should test the generator and its output for accuracy. And preferably test it against some known solutions there are many excellent resources on this subject and I have listed a few of them and this they should give you a lot more information on tests of randomness.