Structural Reliability
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Lecture –70
Monte Carlo Simulations (Part - 02)

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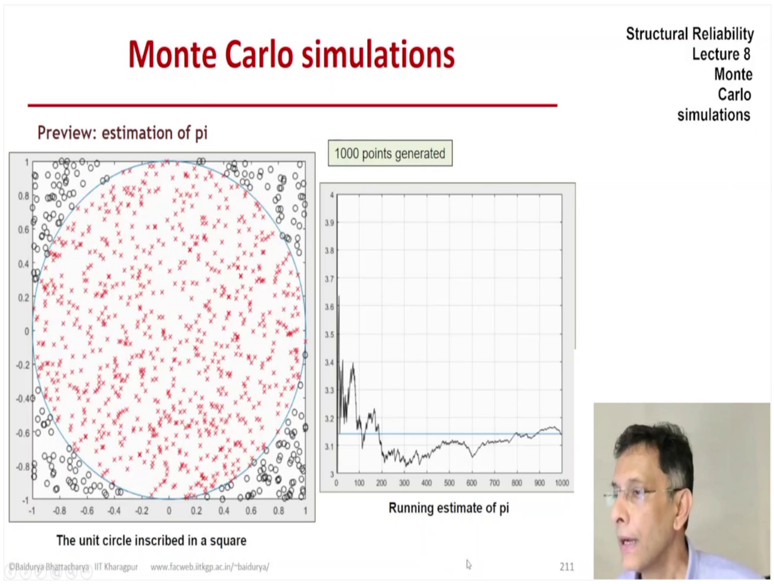

Before getting into the details of random number generation the algorithms and what to do with the simulated random samples let us walk through an example and identify some of the features that will be relevant in our problems. Let us say we have a square of side 2 and the unit circle is inscribed in the square. Let us suppose that we have the ability to generate points randomly in this square.

And the red points are those that lie within the unit circle we can count them and the ratio of the red points to the total number of points it is intuitive that if the points are completely random then that ratio will be the ratio of the area of the circle to that of the square which we know would be pi over 4. So, if we are able to count that ratio and multiply that by 4 we should have an estimate of pi and interestingly this estimate is completely independent of the infinite series based approximations to pi that comes from calculus.

So, let us see what we get and as the sample size increases we have generated 1000 samples here. Let us see if we can get a sense of the estimate. So, let us estimate pi as four times that ratio of number of points and as we see here as the number of samples increases the estimated value reaches the known approximation for pi which in this case is somewhere between 3.14 and 3.15.

So, now let us let us formalize what we did and see if we can get an estimate of the accuracy of what we are doing.

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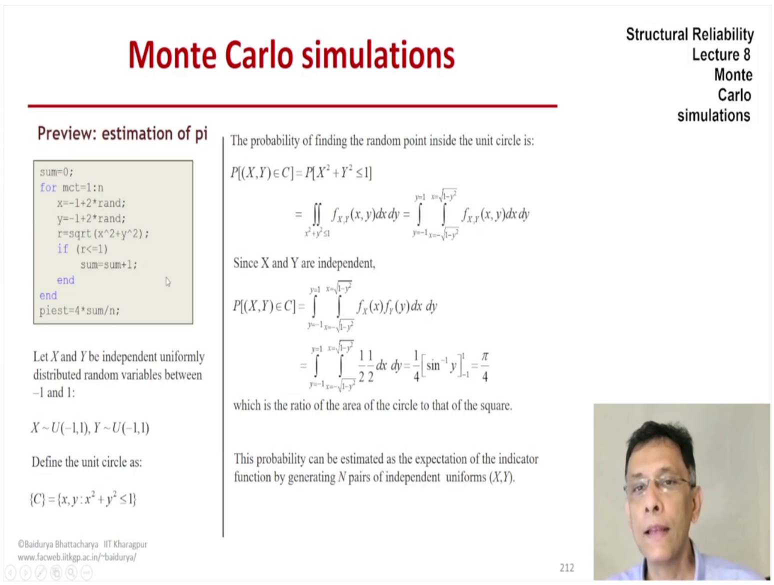

So we start with Y and Y the independent uniformly distributed random variables between -1 and +1 which is what we are generating points on the square of side two and we are interested in the probability of finding these points within the unit circle. So, this

presents itself as a problem of a joint of a double integration which is what you see on the screen y going from -1 to +1 and Y between the corresponding limits depending on Y. And the double integration because X and Y are independent the joint density function is the product of the marginals and the marginals in turn they are each equal to half between the limits -1 and +1. So, if you go through the steps the estimated the answer is indeed pi over 4 which was also the ratio that we thought of in the previous slide.

So, if we are able to estimate that ratio and multiply that with 4 we should be able to get an estimate of pi. And let us now take a look of at the code the Matlab code in this case that generated those points and estimated the value of pi. So, here is the code and it is very simple there is a loop where the index mct goes from one through n and for each value for each trial we generate one X coordinate between -1 and +1 uniformly between -1 and +1.

The command rand is the Matlab command for the standard uniform deviate. So, we generate one X we generate one Y independently and then compute the distance from the origin. And if the distance is less than or equal to 1 which means the point is within the unit circle we increment a counter which is sum by 1 that counter was initialized at 0 outside of the loop and once we are done with the n simulations the estimated value of pi is 4 times the ratio sum over n.

So, this is the process that we followed. Now let us see if we can look at the accuracy of the estimate as a function of the number of samples the sample size which is n in this case.

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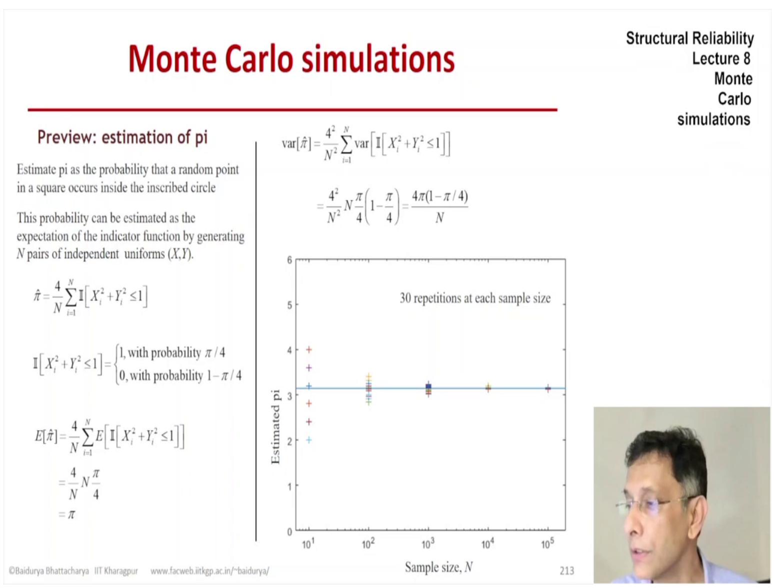

So, this is the problem statement we are estimating pi by counting points within the unit circle and uh. So, this is the estimated value of pi, pi hat the indicator function the truth function it's equal to 1 if the point is within the unit circle and 0 otherwise. So, it is a binary random variable and the probability of the truth function taking on the value 1 is as we found out in the previous slide it is pi over 4 and the complement is the indicator function is 0 and the probability is 1 minus pi over 4.

So, pi hat which is a random variable we let us find out the mean of that random variable the expectation of pi hat we because all the indicator functions they are identically distributed with the same probability of success of pi by four we can find out the expectation of pi hat as the sum of all those individual expectations they are all equal. So, the answer comes to pi. So, the estimate that we have set up is unbiased that is good to know.

Now let us see how that is affected by the sample size because that is the kind of thing we will do a lot in this course that we are going to find out some probability or some other function of a set of random variables and we would like to know how accurate that estimate is. So, the variance of the estimate pi hat would be the sum of the variances of the individual indicator functions and that's because we make use of the second very important property of the samples that they are mutually independent.

So, we can just add the individual variances and multiply the sum with the coefficient 4 by N whole squared. So, once we put in the values of p and q the variance of the estimate is found to be a constant divided by the sample size. So, as the sample size increases the uncertainty in the estimate does indeed go down and in the limit it equals the estimate equals the true value of pi and this as we remember from our previous lecture that this is indeed the strong law of large numbers.

Now let us see what would happen pictorially to the estimate in terms of the sample size the movie that we saw that had just one sample size of 1000 just one realization one set of samples. Here we have sample sizes going from 10 all the way to 100000 and at each sample size we are repeating it 30 times. So, when there are only 10 samples you see there are six possible values that have come up in these 30 reputations which basically means that of all the 11 possible values there could be only six were encountered.

When the sample size increases to 100 you can see the scatter reduce significantly and kind of assume values around the true value of pi which is 3.1415 and. So, on and as the sample size increases the scatter indeed comes down you can see for 1000 for 10 000 and when it is about 100000 the all the 30 samples all the 30 values of the estimated pi are quite close to the true value which is also the mean in this case.