

Structural Reliability
Prof. Baidurya Bhattacharya
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture –65
Joint Probability Distributions (Part - 16)

(Refer Slide Time: 00:27)

Joint distribution examples

Example: correlation and non-linear transformation

Let Y_1 and Y_2 be jointly distributed normal random variables with zero mean, unit variance and correlation coefficient ρ between them. Find the correlation coefficient ρ' between their squares, Y_1^2 and Y_2^2 .

Let X_1 and X_2 be two independent standard normal variables. Then $Y_1 = X_1$, and $Y_2 = \rho X_1 + \sqrt{1-\rho^2} X_2$ are bivariate normal with zero means, unit variances and correlation coefficient ρ , as specified in the problem statement.

The correlation coefficient, ρ' , between Y_1^2 and Y_2^2 is:

$$\rho' = \frac{E[(Y_1^2 - E(Y_1^2))(Y_2^2 - E(Y_2^2))]}{\sqrt{\text{var}(Y_1^2)}\sqrt{\text{var}(Y_2^2)}}$$

MGF of standard normal: $G_X(s) = \exp(s^2/2)$, which yields: $E(Y_1^4) = \frac{d^4}{ds^4} G_X(s) \Big|_{s=0} = 3$
 $\Rightarrow E(Y_1^2) = [e^{s^2/2} + e^{-s^2/2}]_{s=0} = 1$
and $E(Y_2^4) = [e^{s^2/2} + 6e^{s^2/2} + 3e^{s^2/2}]_{s=0} = 3$

Hence the variance of Y_1^2 and Y_2^2 are:
 $\text{var}(Y_1^2) = E(Y_1^4) - (E(Y_1^2))^2 = 3 - 1^2 = 2$

Thus ρ' simplifies to:

$$\rho' = \frac{E[(Y_1^2 - 1)(Y_2^2 - 1)]}{\sqrt{2}\sqrt{2}} = \frac{E(Y_1^2 Y_2^2) - 1}{2}$$

$$E(Y_1^2 Y_2^2) = E\left[X_1^2 \left(\rho^2 X_1^2 + (1-\rho^2) X_2^2 + 2\rho\sqrt{1-\rho^2} X_1 X_2\right)\right]$$

$$= \rho^2 E[X_1^4] + (1-\rho^2) E[X_1^2 X_2^2] + 2\rho\sqrt{1-\rho^2} E[X_1^3 X_2]$$

Since X_1 and X_2 are independent standard normals, the numerical values of the expectations are:


$$E[X_1^4] = 3, E[X_1^2 X_2^2] = E[X_1^2] E[X_2^2] = 1 \times 1 = 1$$

$$E[X_1^3 X_2] = E[X_1^3] E[X_2] = 0 \times 0 = 0$$

Yielding the result, $E(Y_1^2 Y_2^2) = 3\rho^2 + (1-\rho^2) = 2\rho^2 + 1$

Hence, $\rho' = \frac{2\rho^2}{2} = \rho^2$

Structural Reliability
Lecture 7
Joint
probability
distributions



Let us solve some examples involving joint normal random variables. Let us take a minute to read the problem Y 1 and Y 2 are joint standard normal's they have unit variance they have zero mean and rho as their correlation coefficient. If we square them we get two new random variables as we have seen earlier in this lecture and what would be the correlation coefficient because of this non-linear transformation of Y1 and Y2.

So, we will call them call that rho prime. So, to start with let us define Y1 and Y2 in terms of two independent standard normal's. So, that the statistics of Y1 and Y2 are satisfied. So, let X1 and X 2 be these two individual independent standard normal's and we define Y1 as equal to X1 and Y2 in terms of a linear combination of X1 and X2 as you see on the screen and let us just take a minute to make sure that we get what we wanted.

So, obviously the statistics of Y1 is mean 0 variance one that is clear the statistics of Y2 is also

mean 0 because the mean of Y_2 would be ρ times 0 plus $\sqrt{1 - \rho^2}$ times 0 and the variance of Y_2 would again be the sum of the variances of X_1 and X_2 multiplied by ρ and $1 - \rho^2$ respectively and that also checks out to be equal to 1.

And what would be the correlation coefficient between Y_1 and Y_2 we need to find the covariance. So, that would be $E(Y_1 Y_2)$ that turns out to be just ρ from that we subtract the product of the means which is zero and we divide that by the individual standard deviations each of which is 1. So, it does check out. So, let us now define ρ' in terms of these new random variables Y_1 and Y_2 .

So, ρ' as you see requires the mean of Y_1^2 the mean of Y_2^2 the variance of Y_1^2 and variance of Y_2^2 but it also requires the expectation of $Y_1^2 Y_2^2$. So, we will need to derive them one by one. Let us first get the expectations of Y_1^2 and also Y_1 to the power of 4 and likewise for Y_2 and Y_2^2 and Y_2 to the power of 4 for that we are going to use the moment generating function of the standard normal random variable which we did a few lectures back and which is $e^{s^2/2}$ and the k th derivative of that evaluated at 0 gives me the k th moment or the k th raw moment of Y_1 and Y_2 .

So, if we go through the steps the expectation of Y_1^2 is 1 which is also expectation of Y_2^2 and expectation of Y_1^4 turns out to be 3 which is also the same for expectation of Y_2^4 . This would let us write the variance of Y_1^2 and Y_2^2 each of them which turns out to be 2. So, we have of all the quantities that we needed in ρ' we have derived most of them except the one that remains which is expectation of $Y_1^2 Y_2^2$.

And that we will now take up and to do that one of the straightforward steps would be to write Y_1 and Y_2 in terms of X_1 and X_2 uh. So, let us do that. So, $Y_1 = \rho X_1 + \sqrt{1 - \rho^2} X_2$ and we

multiply them when we express them in terms of X 1 and X 2 and this is what we get. So, the expectation of Y 1 squared and times Y 2 squared is given in terms of expectations of X 1 to the power of 4 and X 1 squared X 2 squared and so, on that you see on the screen.

We know most of these actually we know all of these because X 1 and X 2 are independent. So, the expectation of X 1 cubed being an odd moment is 0 because X is the standard normal the others we have already found out through the moment generating function. So, we can simplify this expression and it turns out that expectation of Y 1 squared times Y 2 squared is a simple function of rho. So, it's twice rho squared plus 1.

So, now we plug this value back in the top expression for rho prime and the answer turns out to be rho squared. So, the non-linear transformation of Y 1 and Y 2 reduces the correlation coefficient between them because as we know the correlation coefficient measures the degree of linear dependence uh. So, rho prime becomes rho squared which in the range of 0 to one or -1 to plus one is always less than the absolute value of rho.

(Refer Slide Time: 07:58)

Joint distribution examples

Example: correlation and non-linear transformation

X and Y are jointly Normal $X \sim N(\mu_x, \sigma_x)$, $Y \sim N(\mu_y, \sigma_y)$ with correlation coefficient ρ_{XY} .

What would be the correlation coefficient between A and B where $A = e^X$ and $B = e^Y$

Structural Reliability
Lecture 7
Joint
probability
distributions

A and B are lognormal random variables with the first two moments given by:

$$\mu_A = e^{\mu_x + \frac{1}{2}\sigma_x^2} \quad \sigma_A^2 = \mu_A^2(e^{\sigma_x^2} - 1)$$

$$\mu_B = e^{\mu_y + \frac{1}{2}\sigma_y^2} \quad \sigma_B^2 = \mu_B^2(e^{\sigma_y^2} - 1)$$

The correlation coefficient between them is:

$$\rho_{AB} = \frac{E[AB] - E[A]E[B]}{\sigma_A \sigma_B}$$

Define $C = AB = e^{X+Y} = e^Z$ where $Z = X + Y$.

Z is a normal random variable, $Z \sim N(\mu_z, \sigma_z)$ with moments:

$$\mu_z = \mu_x + \mu_y \quad \sigma_z^2 = \sigma_x^2 + \sigma_y^2 + 2\rho_{XY}\sigma_x\sigma_y$$

Hence, C is lognormal random variable with mean:

$$\mu_C = e^{\mu_z + \frac{1}{2}\sigma_z^2} = \exp\left(\mu_x + \mu_y + \frac{1}{2}(\sigma_x^2 + \sigma_y^2 + 2\rho_{XY}\sigma_x\sigma_y)\right)$$


Then the correlation coefficient between A and B becomes:

$$\rho_{AB} = \frac{\mu_C - \mu_A \mu_B}{\sigma_A \sigma_B} = \frac{\mu_A \mu_B [\exp(\rho_{XY}\sigma_x\sigma_y) - 1]}{\mu_A \mu_B (e^{\sigma_x^2} - 1)^{0.5} (e^{\sigma_y^2} - 1)^{0.5}}$$

$$= \frac{e^{\rho_{XY}\sigma_x\sigma_y} - 1}{\sqrt{(e^{\sigma_x^2} - 1)(e^{\sigma_y^2} - 1)}} = \rho_{XY}$$

If X, Y are joint standard normals,

$$\rho_{AB} = \frac{e^{\rho^2} - 1}{e - 1}$$



© Daidarya Bhattacharya | IIT Kharagpur | www.facweb.iitkgp.ac.in/~baidaryaj
197

The next example we take up also involves bivariate normal's and correlation coefficient we actually have seen this in the previous lecture. So, we would not spend a lot of time on this but if you want to rederive these especially in terms of x and y being standard normal then that would

be a good exercise and let us go through the steps. We what we did was we found the the mean of a and b and the variances of A and B in terms of those of X and Y.

And then we started deriving the row between A and B and for that we had to define a new log normal random variable C and we went through the steps and ended up with an expression which was definitely different from ρxy but it would be interesting to see what the answer would be for the standard normal case when x and y are joined standard normal's and you would get the answer as exponential of $\rho - 1$ divided by $e - 1$. So, that would be the result of this particular nonlinear transformation.