

Structural Reliability
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Lecture –62
Joint Probability Distributions (Part - 13)

(Refer Slide Time: 00:27)

Functions of random variables

Structural Reliability
Lecture 7
Joint
probability
distributions

Example: Function(s) of several random variables

Sum of two IID uniform RVs (by convolution)

$X_1 \sim U(0,1), X_2 \sim U(0,1)$ and X_1, X_2 are independent of each other.
 Find the distribution of their sum, $Y = X_1 + X_2$

$$F_Y(y) = \iint_{0 \leq x_1, x_2} I(x_1 + x_2 \leq y) f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2$$

Differentiating,

$$f_Y(y) = \iint_{0 \leq x_1, x_2} \delta(x_1 + x_2 - y) f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2$$

$$= \int_{0 \leq x_1} \int_{0 \leq x_2} \delta(x_2 = y - x_1) f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2$$

$$= \int_{0 \leq x_1} f_{X_1}(x_1) f_{X_2}(y - x_1) dx_1$$

Since the PDFs of X_1 and X_2 are non-zero only in the interval $[0,1]$:

$$f_Y(y) = \int_{x_1=0}^1 (1) I(0 < y - x_1 < 1) dx_1$$

For $0 < y < 1$, we need to restrict $0 < x_1 < y$, yielding,


$$f_Y(y) = \int_0^y (1)(1) dx_1 = y, \quad 0 < y < 1$$

For $1 < y < 2$, x_1 does not need any restriction, yielding,

$$f_Y(y) = \int_0^1 (1) I(y - 1 < x_1 < y) dx_1 = \int_{y-1}^1 (1)(1) dx_1 = 2 - y, \quad 1 < y < 2$$

Thus $f_Y(y) = \begin{cases} y, & 0 < y < 1 \\ 2 - y, & 1 < y < 2 \end{cases}$

which is the triangular distribution



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In the second set of problems we look at the sum of two IID uniform random variables. We will solve this problem in two methods the first one is by convolution which is what you see on this slide and in the next one we are going to solve by transformational variables. So, X_1 and X_2 are two uniform random variables independent and identically distributed between 0 and 1. So, we want to find the distribution of their sum.

So, let us start with the basic definition of the CDF of Y which is the sum of X_1 and X_2 and we just write it as the double integration over X_1 and X_2 of the region in which the sum is less than little y for which we have used the indicator function as you see. So, now we can differentiate this to get the PDF of Y . Let us do that and when we differentiate the indicator function we get the delta function.

And if we write it out in terms of X_2 then we can get rid of the integration with respect to X_1

and end up with a one dimensional integration involving X_1 only y is a parameter that is not dependent on X_1 uh. So, now we make use of the fact that X_1 and X_2 are uniform and f_{X_1} and f_{X_2} are defined only in the interval 0 and 1. So, once we use that fact we can simplify the density of y in terms of just one indicator function.

So, we are now integrating over all values of x_1 but making sure that $y - x_1$ is positive on one end and less than one on the other ah. So, now it is inconvenient to split this integration for 2 sets of values of y . So, one in which y is between 0 and 1 and the other in which the y is between 1 and 2. So, in the first range the density of y simplifies to y itself that is between 0 and 1. And in the second range between 1 and 2 the density integrates to $2 - y$.

So, that is between 1 and 2. So, putting these two things together it is clear that the sum of two independent standard uniforms is the triangular random variable. So, its density function is a triangle symmetric about the point one and in fact this is kind of a precursor to what we will see later when we add several such uniforms. So, if we added a few more uniforms it is going to look more and more like the normal density function and just by adding two of them you kind of start getting the shape.

Now we will solve the same problem by transformation of variables which we looked at a couple slides ago.

(Refer Slide Time: 04:42)

Functions of random variables

Example: Function(s) of several random variables

Sum of two IID uniform RVs by transformation of variables

$$X_1 \sim U(0,1), X_2 \sim U(0,1)$$

and X_1, X_2 are independent of each other.

$$\begin{aligned} \text{Let } Y_1 &= X_1 + X_2, & \text{Then, the inverse transform is:} \\ \text{and } Y_2 &= X_2, & X_1 &= Y_1 - Y_2, \\ & & X_2 &= Y_2 \end{aligned}$$

and the Jacobian of the transformation is:

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

Hence the joint density of Y_1, Y_2 :

$$f_{Y_1, Y_2}(y_1, y_2) = \frac{f_{X_1, X_2}(x_1, x_2)}{|J(x_1, x_2)|} = \frac{f_{X_1}(x_1) f_{X_2}(x_2)}{1}$$

using the independence between X_1, X_2

Expressing the x's in terms of the y's:

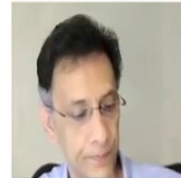
$$\begin{aligned} f_{Y_1, Y_2}(y_1, y_2) &= f_{X_1}(y_1 - y_2) f_{X_2}(y_2) \\ &= 1 \cdot I(0 \leq y_1 - y_2 \leq 1) \cdot 1 \cdot I(0 \leq y_2 \leq 1) \end{aligned}$$

The marginal density of Y_1 :

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{\max(0, y_1-1)}^{\min(y_1, 1)} f_{Y_1, Y_2}(y_1, y_2) dy_2 \\ &= \int_{\max(0, y_1-1)}^{\min(y_1, 1)} (1) dy_2 \end{aligned}$$

$$= \begin{cases} \int_0^{y_1} (1) dy_2, & 0 \leq y_1 \leq 1 \\ \int_{y_1-1}^1 (1) dy_2, & 1 < y_1 \leq 2 \end{cases}$$

$$= \begin{cases} y_1, & 0 \leq y_1 \leq 1 \\ 2 - y_1, & 1 < y_1 \leq 2 \end{cases} \text{ as before}$$



So, it is the same problem but now we have X_1 and X_2 on one side and just y on the other side. So, we need to introduce one sort of dummy variable as we discussed. So, to make the problem 2 by 2. So, let Y_1 be $X_1 + X_2$ and then we define a new variable Y_2 which is identical to X_2 . And now we can we need to invert that that relation also. So, X_1 in terms of Y_1 and Y_2 and X_2 in terms of Y_1 and Y_2 in this case only X_2 is equal to Y_2 .

So, this lets us write the Jacobean and it is a constant equal to 1 and that lets us describe the joint density of Y_1 and Y_2 in terms of the joint density of X_1 and X_2 and we conveniently make use of the fact that X_1 and X_2 are independent. So, the joint density is the product of the marginal densities. Once we do that we should express the X_1 and the X_2 in terms of Y_1 and Y_2 . So, that gives us the joint density of Y_1 and Y_2 in terms of the product of two indicator functions one involving $Y_1 - Y_2$ and the other involving just Y_2 .

So, the marginal density of Y_1 which is what we need we can get it simply by integrating out Y_2 from the joint density function. So, now that is that is what we do and now you know we just need to make sure that the indicator functions are properly accounted for. So, the limits of the integration can be found to be as I have written on the on the screen. So, the lower limit is maximum of 0 and $y_1 - 1$ and the upper limit is minimum of y_1 and 1.

And if you complete the integration then you get again two ranges of y the first one is 0 to 1 and the second is 1 to 2 and you get the same triangular density function as before.