Structural Reliability
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Lecture –60
Joint Probability Distributions (Part - 11)

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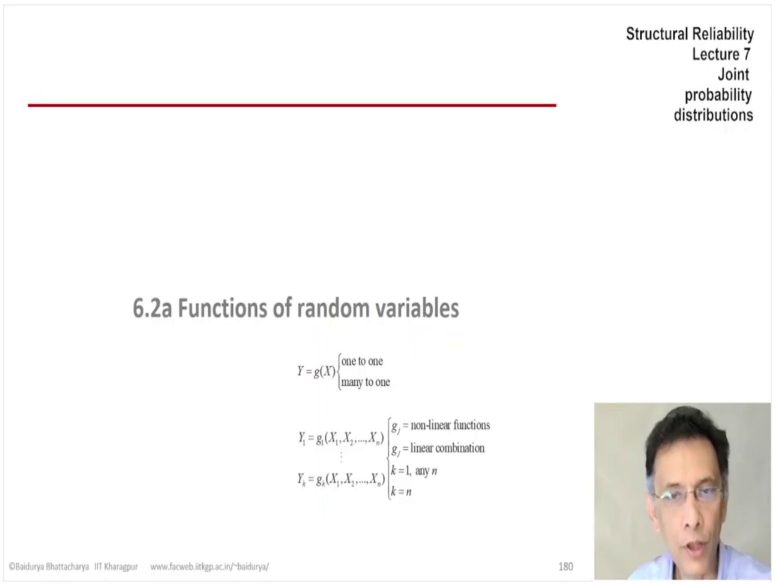

We continue with our discussion on joint distributions we will start with functions of random variables and we will look at mainly 2 types which occur frequently is y is a function of x, x is a random variable that function could be either one to one or many to one. And we will also look at several functions of several random variables. So here we have y 1 up to y k which are functions of x 1 through x n and we will pay particular emphasis on the g is being non-linear and linear and the special case that k is 1 and any value of n and the second cases k and n are equal.

So let us start with the first case that y is a function of a single random variable and we will first look at that map being 1 to 1.

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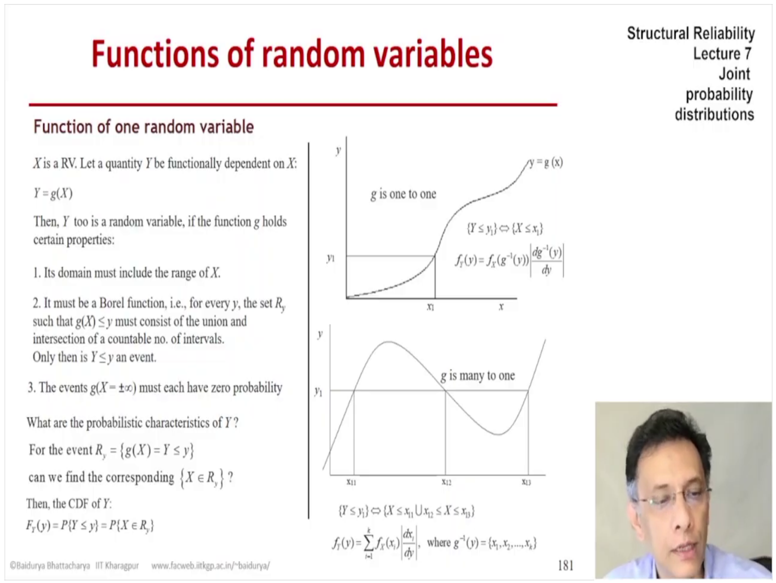

So if x is a random variable then if I define a function on it so y 2 is a random variable provided that function has certain well-behaved properties basically if we have a set of outcomes of y in mind can we map it back to a set of outcomes in x and do we have the probability measure for those outcomes in x. So if that is so then we are especially interested in events of the type that the random variable y is less than or equal to little y.

And if we can find the probability measure of that in the x space then we clearly have the CDF of y and once we have the CDF of y we know the entire probabilistic description of the new random variable y the function of x. As I said we will look at 2 cases the first one is that map g is 1 to 1. so here you see an example of that and so y being less than or equal to y 1 is simply that I take the inverse of that and that corresponds to x 1 and x being less than x less than or equal to x 1 is the same as y being less than or equal to y 1 with the same probability measure.

So we can show that the densities map the as you see on the screen basically it says that the elemental probability fy dy is equal to the elemental probability fx dx provided y maps onto x. If g is many to 1 then we have to be a little careful and make sure that we capture the entire region in the x space. So here if I am interested in y is less than or equal to y 1 as before there is not a single interval in x which gives me that but as you can see there is a 1 which is x is less than or equal to x 11.

And there is another disjoint interval where x is between x 12 and x13. And here the density function of y would take into account all such inverse maps so fy dy would be equal to fx dx summed at all the respective points. And now let us solve a few problems which would make these concepts very clear.

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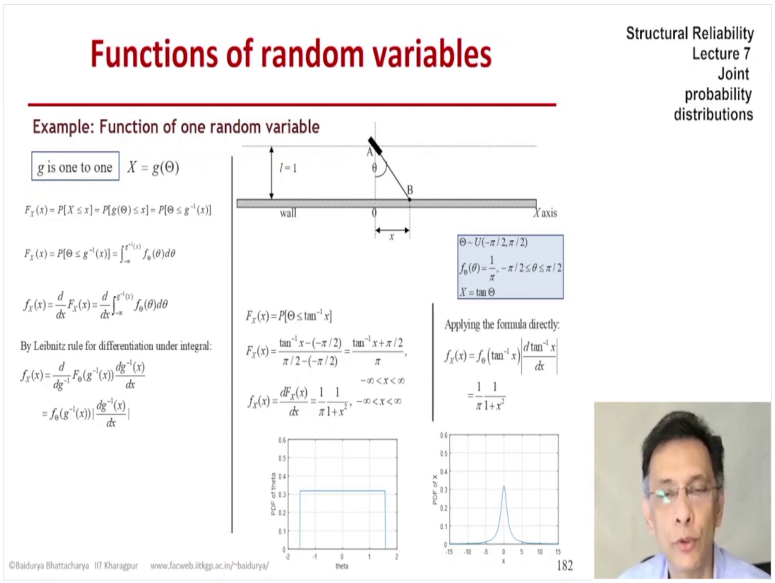

So we have seen this problem before theta is a random variable and x is a function of that in particular x is tan theta. So let us set the problem up that if we know the distribution of theta can we find the distribution of x. So here is the function relationship starting from the CDF of x and if we work through the steps I can describe the CDF of x in terms of an integration of the density function of theta.

And then I can take the derivative of that which if I apply Leibniz rule for differentiation under the integral sign I arrive at the formula that I showed in the previous slide. So let us go back to this example theta is a uniform random variable between minus pi by 2 and plus pi by 2 and as we know very well that this gives rise to the Cauchy distribution. So x is a function of theta tangent of theta and we can go through the same derivation as in the left panel and making use of the uniform CDF of theta.

We can arrive at the density function of x which is the well-known Cauchy density function we could also apply the formula directly and we see the same answer. Now just to show what the map does here on the left I have the density function of u the uniform density function of theta and because of that non-linear map x equals tan theta I get a density function of x that looks quite different and as you know that this particular density function is famous for not having any finite moments.

So the mean does not exist the variance does not exist and so on. But let us tweak this problem a little and see what we would get if we change the density function of theta somewhat if we move some density from the edges the minus pi by 2 and plus pi by 2 and concentrated more towards the middle part what is it that we would get.

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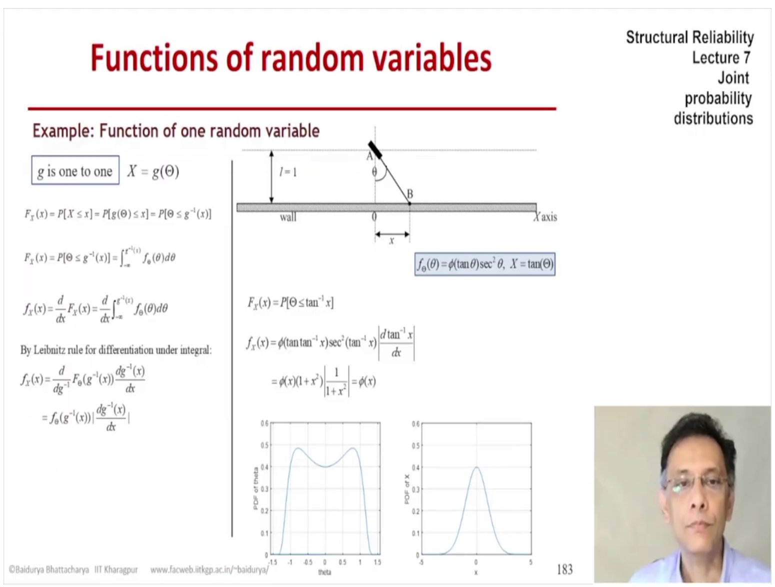

So we let us define the density function a little differently and as you see it is phi of tan theta phi being the normal density function the standard normal density function and x is tan theta as before. So, if this is the functional relationship between theta and x then we can go through the steps and we arrive at a different density function for x which in the previous case was the Cauchy density function.

So if we go through the steps here we find that the density function of x is actually the standard normal and that is very interesting and if you see the change so here the density function of the angle is no longer uniform it is different especially as you can see I have moved some mass from the edges towards the middle portion and that causes the density function of x to become Gaussian instead of Cauchy.

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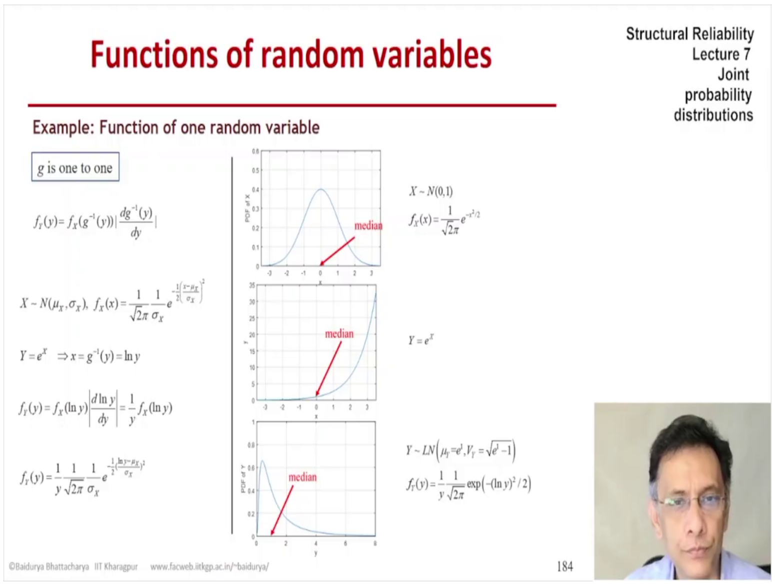

Let us look at one more example this is something also very familiar to us let us see what happens when we exponentiate the normal random variable. So x is normal with mean mu x and standard deviation sigma x and we define a new random variable y which is exponential of x. So it is a one-to-one map so x is g inverse of y so x is log of y and now I can apply the the formula that we derived so the density function of Y is simply 1 over y times the density function of x evaluated at log of y.

And we all know that if we expand we arrive at the well-known log normal PDF. Let us see what it does pictorially. So we first start with the standard normal let us say we are talking about x being the standard normal. So it is symmetric about 0 x equal to 0 is the mean and the median of the distribution. So if we exponentiate that because it is a one-to-one map the median also maps nicely. So here you see y is the exponential of x and the median of y is now e to the power 0 so the median of y is 1.

And this is the density function of Y and you can see how the median maps you can also see that the mean log normal is a positively skewed distribution. So the mean is e to the power 1 and the median is e to the power 0 and I have marked them in red so this is the result of the exponential the exponentiation of the standard normal random variable.

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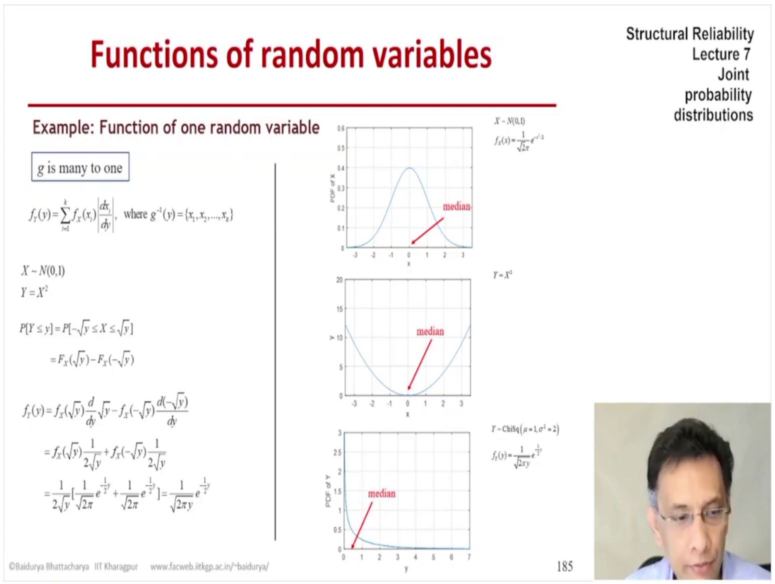

Now let us look at a function of many to one so when g is many to one. This is where we start with so the density function of y is carefully evaluated at all the inverse maps from y to x and the corresponding density function of those points in x we have to find out. So, let us go through an example X again is standard normal but unlike exponentiating it we square it. So now this map is no longer 1 to 1. So, if i define the CDF of y I have a range of x which is between minus square root of y and plus square root of y.

And I can easily find out the CDF of y in terms of the difference of the CDF of x at those 2 values and differentiating that and going through the algebra I arrive at the nice well-known chi-squared density function which is 1 over square root of 2 pi y times exponential minus y by 2. Again as we did before for the log normal case let us see what happens in the chi-square case we start with the standard normal.

And here because the map is not one to one the median does not map nicely so the median of y is something like 0.45 which you can work out and this would be the chi-square density function.