Structural Reliability

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Lecture –59
Joint Probability Distributions (Part - 10)

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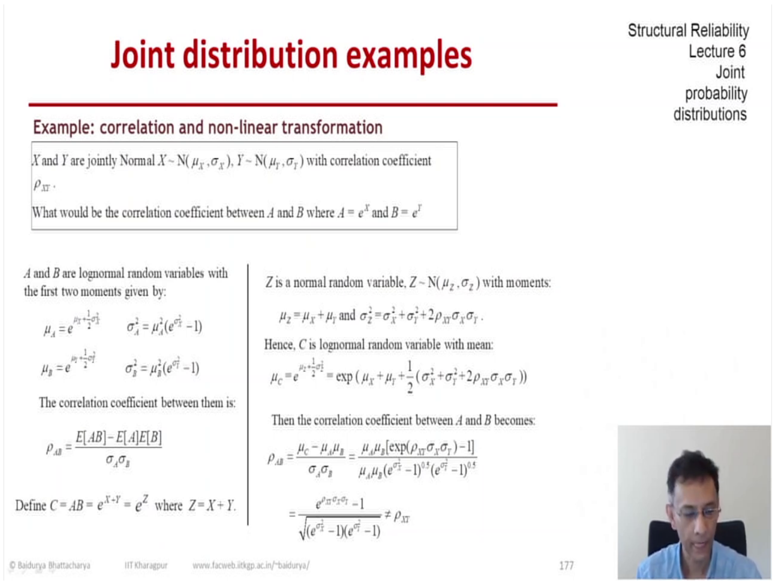

In this last example let us explore the effect of nonlinear transformation on the correlation coefficient. So, we have X and Y are joint normal random variables now obviously we have yet not covered them but this is just a preview. So, we are going to talk about joint normal's in the next lecture but this particular problem is this is the right time to just bring it up to find the effect of non-linear transformation on the correlation coefficient rho between X and Y.

$$\mu_A = e^{\mu_X + \frac{1}{2}\sigma_X^2} \qquad \sigma_A^2 = \mu_A^2(e^{\sigma_X^2} - 1)$$

$$\mu_B = e^{\mu_Y + \frac{1}{2}\sigma_Y^2} \qquad \sigma_B^2 = \mu_B^2(e^{\sigma_Y^2} - 1)$$

So, we perform a nonlinear transformation on each of them and obtain a as the exponential of x and b the exponential of y. So, A and B are two new random variables and we recognize them very well because we have looked at them before. So, A is not normal as is B. So, we are also familiar with the process of how to find the mean of A and B and the variance of a and b given those quantities for x and y. So, we get those and what we need to do is to find the correlation coefficient between A and B. $\rho_{AB} = \dfrac{E[AB] - E[A]E[B]}{\sigma_A \sigma_B}$

So, in this expression we know the mean of A which is there on the screen and the mean of B and the standard deviation of A and standard deviation of B the only thing we do not know is the expectation of the product. So, E AB is not known. So, let us see if we can find that and again to find that we need to use knowledge which we have not discussed yet but again this is just a preview. So, we define C as the product A, B and we can express this as we can just see that this is exponential of X + Y.

$$\text{Define } C = AB = e^{X+Y} = e^{Z} \text{ where } Z = X + Y.$$

Now the Z which is X + Y is also normal and this is what we have not discussed formally that the normal family is closed under linear combinations. So, if X and Y are jointly normal. So, is their sum z and we just need to be able to find the moments of Z and again this is something we are going to see in the next lecture. But the mean of Z is what you see on the screen the sum of the means and the variance of Z is the sum of the variance plus an additional term which which comes from the covalence term.

$$\mu_Z = \mu_X + \mu_Y \text{ and } \sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y.$$

So, that is twice rho x y sigma x sigma y if the rho was 0 if x and y were independent then we would just have the sum of the variance as the variance of the sum as the sum of the variance. Anyway so, because we assert that Z is normal exponential of C which is C is log normal. So, we can find the mean of C that is what we actually need expectation of AB is expression of C and if we knew that then we could compute rho of AB.

$$\mu_C = e^{\mu_Z + \frac{1}{2}\sigma_Z^2} = \exp\left(\mu_X + \mu_Y + \frac{1}{2}(\sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y)\right)$$

So, going through the math we can find the mean of C in terms of the mean the variances and correlation coefficients of X and Y and that is what you see on the screen and then we can plug everything back together. And Rho AB is what you see on the on this screen it is a nonlinear relationship involving rho XY. So, if you do a little bit of simplification then we find that rho AB is actually not equal to rho XY caused by the nonlinear transformation between X and A and Y and B.

$$\rho_{AB} = \frac{\mu_C - \mu_A \mu_B}{\sigma_A \sigma_B} = \frac{\mu_A \mu_B [\exp(\rho_{XY} \sigma_X \sigma_Y) - 1]}{\mu_A \mu_B (e^{\sigma_X^2} - 1)^{0.5} (e^{\sigma_Y^2} - 1)^{0.5}} = \frac{e^{\rho_{XY} \sigma_X \sigma_Y} - 1}{\sqrt{(e^{\sigma_X^2} - 1)(e^{\sigma_Y^2} - 1)}} \neq \rho_{XY}$$

Now for most cases if V the coefficient variation of A and B are not too large then this deviation between rho AB and rho XY is not too large also but will discuss this at some other time.