Structural Reliability
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Lecture –41
Common Probability Distributions (Part - 12)

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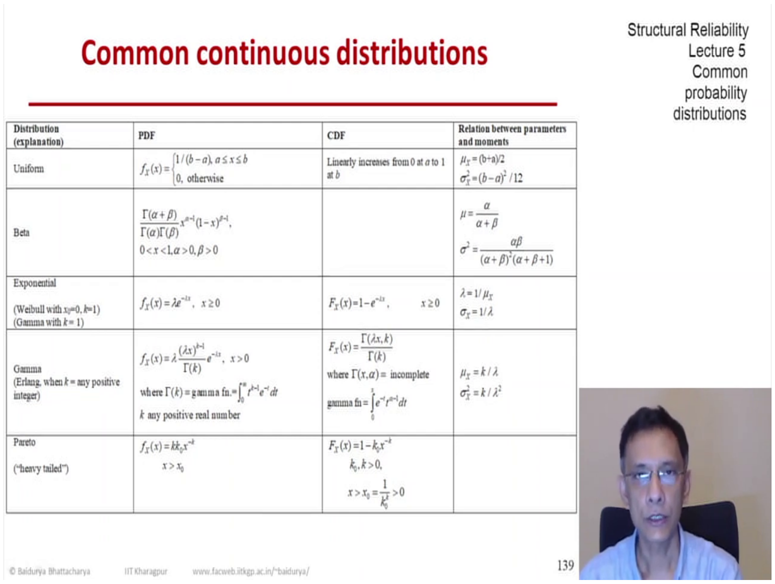

Distribution (explanation)	PDF	CDF	Relation between parameters and moments
Uniform	$f_X(x)=\begin{cases}1/(b-a), a\le x\le b\\ 0,\ \text{otherwise}\end{cases}$	Linearly increases from 0 at a to 1 at b	$\mu_X=(b+a)/2$ $\sigma_X^2=(b-a)^2/12$
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, $0<x<1, \alpha>0, \beta>0$		$\mu=\frac{\alpha}{\alpha+\beta}$ $\sigma^2=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Exponential (Weibull with x_0=0, k=1) (Gamma with k = 1)	$f_X(x)=\lambda e^{-\lambda x}$, $x\ge 0$	$F_X(x)=1-e^{-\lambda x}$, $x\ge 0$	$\lambda=1/\mu_X$ $\sigma_X=1/\lambda$
Gamma (Erlang, when k = any positive integer)	$f_X(x)=\lambda\frac{(\lambda x)^{k-1}}{\Gamma(k)}e^{-\lambda x}$, $x>0$ where $\Gamma(k)$ = gamma fn. $=\int_0^\infty t^{k-1}e^{-t}dt$ k any positive real number	$F_X(x)=\frac{\Gamma(\lambda x,k)}{\Gamma(k)}$ where $\Gamma(x,\alpha)$ = incomplete gamma fn $=\int_0^x e^{-t}t^{\alpha-1}dt$	$\mu_X=k/\lambda$ $\sigma_X^2=k/\lambda^2$
Pareto ("heavy tailed")	$f_X(x)=kk_0x^{-k}$ $x>x_0$	$F_X(x)=1-k_0x^{-k}$ $k_0, k>0$, $x>x_0=\frac{1}{k_0^k}>0$	

In this lecture we are going to look at some of the commonly occurring continuous random variables in structural reliability problems. I have this list which is also in your course material of such continuous random variables of these will pick up the uniform the exponential and the gamma for further discussion and in the normal family we are going to look at the normal and the normal distribution today.

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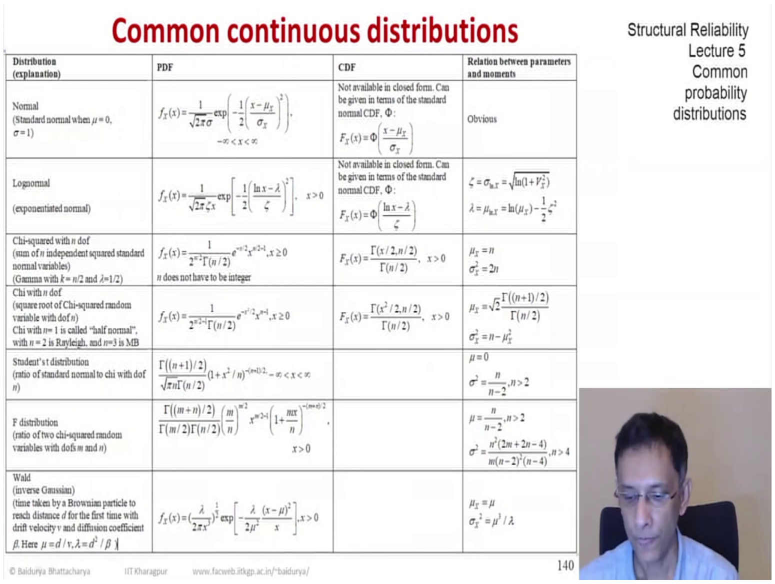

Distribution (explanation)	PDF	CDF	Relation between parameters and moments
Normal (Standard normal when $\mu=0$, $\sigma=1$)	$f_X(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right)$, $-\infty<x<\infty$	Not available in closed form. Can be given in terms of the standard normal CDF, Φ: $F_X(x)=\Phi\left(\frac{x-\mu_X}{\sigma_X}\right)$	Obvious
Lognormal (exponentiated normal)	$f_X(x)=\frac{1}{\sqrt{2\pi}\zeta x}\exp\left[-\frac{1}{2}\left(\frac{\ln x-\lambda}{\zeta}\right)^2\right]$, $x>0$	Not available in closed form. Can be given in terms of the standard normal CDF, Φ: $F_X(x)=\Phi\left(\frac{\ln x-\lambda}{\zeta}\right)$	$\zeta=\sigma_{\ln X}=\sqrt{\ln(1+V_X^2)}$ $\lambda=\mu_{\ln X}=\ln(\mu_X)-\frac{1}{2}\zeta^2$
Chi-squared with n dof (sum of n independent squared standard normal variables) (Gamma with $k=n/2$ and $\lambda=1/2$)	$f_X(x)=\frac{1}{2^{n/2}\Gamma(n/2)}e^{-x/2}x^{n/2-1}, x\geq 0$ n does not have to be integer	$F_X(x)=\frac{\Gamma(x/2,n/2)}{\Gamma(n/2)}$, $x>0$	$\mu_X=n$ $\sigma_X^2=2n$
Chi with n dof (square root of Chi-squared random variable with dof n) Chi with $n=1$ is called "half normal", with $n=2$ is Rayleigh, and $n=3$ is MB	$f_X(x)=\frac{1}{2^{n/2-1}\Gamma(n/2)}e^{-x^2/2}x^{n-1}, x\geq 0$	$F_X(x)=\frac{\Gamma(x^2/2,n/2)}{\Gamma(n/2)}$, $x>0$	$\mu_X=\sqrt{2}\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}$ $\sigma_X^2=n-\mu_X^2$
Student's t distribution (ratio of standard normal to chi with dof n)	$\frac{\Gamma((n+1)/2)}{\sqrt{\pi n}\Gamma(n/2)}(1+x^2/n)^{-(n+1)/2}, -\infty<x<\infty$		$\mu=0$ $\sigma^2=\frac{n}{n-2}, n>2$
F distribution (ratio of two chi-squared random variables with dofs m and n)	$\frac{\Gamma((m+n)/2)}{\Gamma(m/2)\Gamma(n/2)}\left(\frac{m}{n}\right)^{m/2}x^{m/2-1}\left(1+\frac{mx}{n}\right)^{-(m+n)/2}$, $x>0$		$\mu=\frac{n}{n-2}, n>2$ $\sigma^2=\frac{n^2(2m+2n-4)}{m(n-2)^2(n-4)}, n>4$
Wald (inverse Gaussian) (time taken by a Brownian particle to reach distance d for the first time with drift velocity v and diffusion coefficient β. Here $\mu=d/v, \lambda=d^2/\beta$)	$f_X(x)=(\frac{\lambda}{2\pi x^3})^{\frac{1}{2}}\exp\left[-\frac{\lambda}{2\mu^2}\frac{(x-\mu)^2}{x}\right], x>0$		$\mu_X=\mu$ $\sigma_X^2=\mu^3/\lambda$

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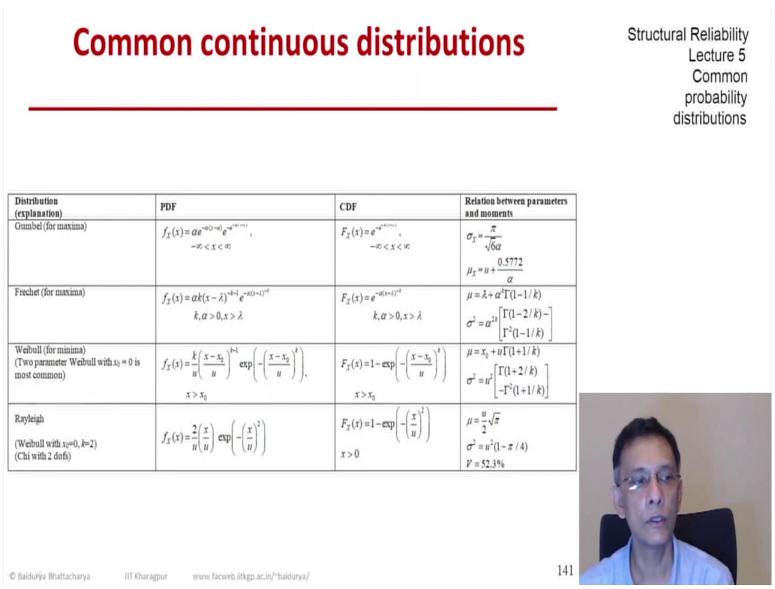

Distribution (explanation)	PDF	CDF	Relation between parameters and moments
Gumbel (for maxima)	$f_X(x)=\alpha e^{-\alpha(x-u)}e^{-e^{-\alpha(x-u)}}$, $-\infty<x<\infty$	$F_X(x)=e^{-e^{-\alpha(x-u)}}$, $-\infty<x<\infty$	$\sigma_X=\frac{\pi}{\sqrt{6}\alpha}$ $\mu_X=u+\frac{0.5772}{\alpha}$
Frechet (for maxima)	$f_X(x)=\alpha k(x-\lambda)^{-k-1}e^{-\alpha(x-\lambda)^{-k}}$ $k,\alpha>0, x>\lambda$	$F_X(x)=e^{-\alpha(x-\lambda)^{-k}}$ $k,\alpha>0, x>\lambda$	$\mu=\lambda+\alpha^k\Gamma(1-1/k)$ $\sigma^2=\alpha^{2k}\begin{bmatrix}\Gamma(1-2/k)- \\ \Gamma^2(1-1/k)\end{bmatrix}$
Weibull (for minima) (Two parameter Weibull with $x_0=0$ is most common)	$f_X(x)=\frac{k}{u}\left(\frac{x-x_0}{u}\right)^{k-1}\exp\left(-\left(\frac{x-x_0}{u}\right)^k\right)$, $x>x_0$	$F_X(x)=1-\exp\left(-\left(\frac{x-x_0}{u}\right)^k\right)$ $x>x_0$	$\mu=x_0+u\Gamma(1+1/k)$ $\sigma^2=u^2\begin{bmatrix}\Gamma(1+2/k) \\ -\Gamma^2(1+1/k)\end{bmatrix}$
Rayleigh (Weibull with $x_0=0, k=2$) (Chi with 2 dofs)	$f_X(x)=\frac{2}{u}\left(\frac{x}{u}\right)\exp\left(-\left(\frac{x}{u}\right)^2\right)$	$F_X(x)=1-\exp\left(-\left(\frac{x}{u}\right)^2\right)$ $x>0$	$\mu=\frac{u}{2}\sqrt{\pi}$ $\sigma^2=u^2(1-\pi/4)$ $V=52.3\%$

And finally in the extreme value family we are going to discuss how the Gumbel, the Frechet and the viable distributions arise.

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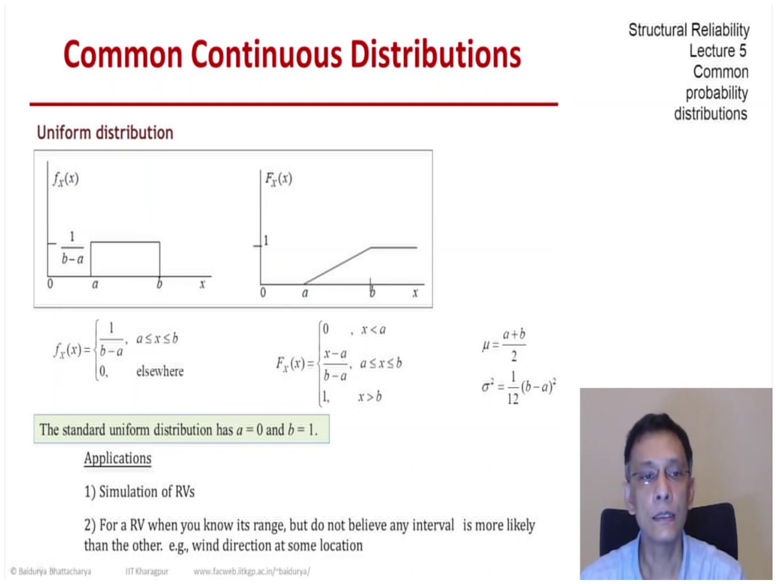

So, let us start with the uniform as we did in the discrete case like in the discrete case no outcome in the range is more likely than the others. So, the density function is a rectangle between a and b and the distribution function the CDF is a straight line between a and b going up from 0 to 1. And the mean is the average of a and b and the variance is the difference square divided by 12. If a 0 and b is 1 we have what is known as the standard uniform distribution which is very important in generation of random numbers which you will see next week.

And as I said the application is in random number generation and whenever we have no reason to prefer any outcome over another in a given range of values the uniform is the natural choice.

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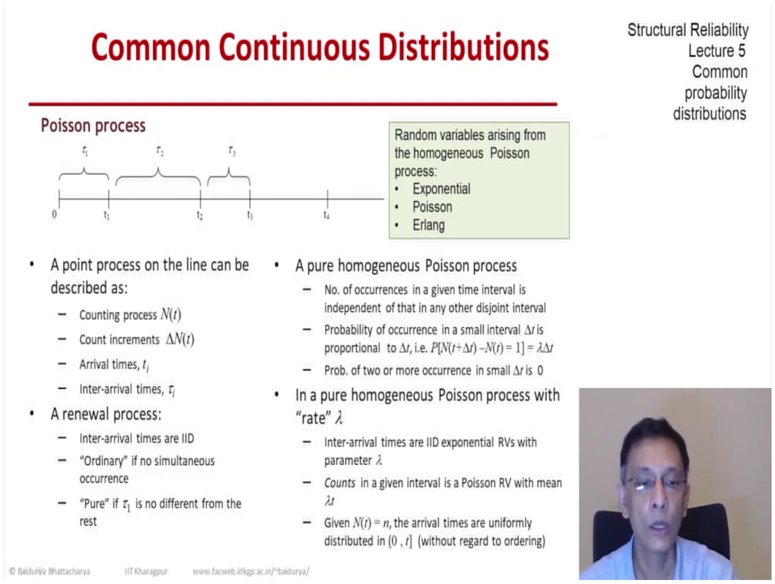

Let us discuss next the Poisson process. So, let us look at the timeline as you can see on the screen and let's start with defining a point process. So, a point process is basically occurring of random points and it can be described in any one of four equivalent ways the counting process the number of points up to the time t. The increments of the counts over an interval, The arrival times of the points the occurrence times or equivalently as I said the inter-arrival times.

So those are the tau's the arrival times are the tau's and the inter-arrival times of the tau's. Now a point process is a renewal process if the tau's are IID random variables further it is ordinary if there is no clustering of points and it is pure if the first occurrence time of the first inter interarrival time is no different than the others and. Now a pure homogeneous Poisson process is a special kind of renewal process which has the probability of one occurrence in a small interval *dt* equal to *a* constant times the length of that interval.

So, it is proportional to the interval length and there may not be two or more occurrences there can be either zero on one occurrence only in a small interval of size delta t. That gives us a pure homogeneous Poisson process and we get some very powerful results out of it that the inter arrival time the tau's are IID exponential random variables with that same parameter lambda. So, its mean is 1 by lambda.

The counts in a given interval of size t, is a Poisson random variable which we have already seen in the discrete case and that has a mean of lambda times t. So, lambda is the rate and t is the length of the interval and sometimes it is also very useful to use the fact that if we know the number of occurrences in the interval 0 to t then the occurrence times the t's are uniformly distributed in that interval.

So, we have three very important random variables occurring from a Poisson process the exponential which is basically the time to the first occurrence its discrete counterpart is the geometric the Poisson which we have already seen it is the number of occurrences in a fixed time interval. So, it is discrete counterpart will be binomial and the Erlang which is the time to occurrence number k and its discrete counterpart would be the negative binomial or the Pascal distribution.