

**Structural Reliability**  
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**Lecture –10**  
**Review of Probability Theory (Part -02)**

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**Basic set theory (review)**

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**Set algebra**

Let  $\Omega$  be any set. A non-empty collection  $\mathcal{A}$  of subsets of  $\Omega$  is an algebra of sets (i.e., a *field*) if: whenever  $A_1, A_2$  are in  $\mathcal{A}$ , so are  $\Omega \setminus A_1$  (i.e., complement of  $A_1$ ) and  $A_1 \cup A_2$  (and therefore  $A_1 A_2$  also). Generalizing, if  $A_1, A_2, \dots, A_n$  (n finite) are in  $\mathcal{A}$ , so are  $A_1 \cup A_2 \dots \cup A_n$  and  $A_1 A_2 \dots A_n$ .

Example: Let  $\Omega = \{a, b, c\}$ . Then we could define a field  $\mathcal{A}$  as:

$$\mathcal{A} = \{ \emptyset, \Omega, \{a\}, \{b, c\} \}$$

Alternately, we could define a different field  $\mathcal{A}_1$  as:

$$\mathcal{A}_1 = \{ \emptyset, \Omega, \{a, b\}, \{c\} \}$$

If we want to add  $\{b\}$  to  $\mathcal{A}_1$  we need to make sure we add all required unions and complements in order to complete the new field:

$$\mathcal{A}_2 = \{ \emptyset, \Omega, \{a, b\}, \{c\}, \{b\}, \{b, c\}, \{a\}, \{a, c\} \}$$

which also happens to be the power set of  $\Omega$ .

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When we want to assign probabilities we need to decide which events are relevant and which we would like to ignore say we want to cast a die a six-face die but we may not be interested in all the 6 outcomes all we might care about are the even faces. So, that is fine then all you need to do in addition to defining the set of even outcomes are the set of odd outcomes. Take another example a coin toss if you are interested in head as an outcome then the tail needs to be automatically defined.

Now if that is all then you have decided that the coin landing on its edge is not an outcome worth considering. So let us say we have the sample space consisting of three elements A, B and C then we could define a field script  $\mathcal{A}$  which is a collection of subsets of  $\Omega$  as the null set which must always be there and its complement. So, the sample space should also be there and then A and its complement BC. So, these four would constitute a valid algebra of sets from this sample space.

We could also define a different field script  $A_1$  and there we could have the null set the sample space and  $A$ ,  $B$  and  $C$ . So, these four would also be a valid collection of sets. Now we might want to add the member  $B$  the set  $B$  to the collection of sets script  $A_1$ . Now to do that; we must make sure that we add all the required unions and complements in order to complete the new field. So, the moment we add  $B$  we need to add the union of  $B$  and  $C$ .

So,  $BC$  has to be there and the moment we have  $BC$  its complement  $A$  should be there and the moment  $A$  is there  $C$  was already there. So, the union of  $A$  and  $C$  should also be there. So, in the end what we get for this just by wanting to add the single element  $B$  to the collection  $A_1$  we have ended up with the power set of the sample space. In the next slide let us work through a small example defining the smallest algebra that contains two sets of our interest.

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## Basic set theory (review)

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Set algebra (contd.)

*Example:*

Given  $S = \{1, 2, 3, 4\}$ , describe  $F$ , the smallest field containing  $\{1\}$  and  $\{2, 3\}$

Answer:

$F = \{ \emptyset, \{1, 2, 3, 4\}, \{1\}, \{2, 3\}, \{1, 2, 3\}, \{4\}, \{1, 4\}, \{2, 3, 4\} \}$

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So, given the sample space of 1, 2, 3 and 4 we want to define a field that contain 1 and 2, 3. So, then we have to start with the null set and the universal set and then 1 and 2, 3 are there but because they are there the union must be there then the complement of that union should be there. So, that is how we get 4 and the moment we have 4 then its union with 1 must be there and its union with 2, 3 should be there. So, that would be the smallest field that contained the 2 sets of our interest.

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## Basic set theory (review)

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Set algebra (contd.)

**$\sigma$  algebra (or  $\sigma$  field):** The algebra described above is a  $\sigma$  algebra of sets if it holds for a countably infinite collection  $A_1, A_2, \dots$ . That is, whenever, the sequence  $A_1, A_2, \dots$ , belongs to  $\mathcal{A}$ , so does  $\bigcup_{j=1}^{\infty} A_j$ . In other words, a  $\sigma$  algebra  $\mathcal{A}$  of subsets of a given set  $\Omega$  contains the empty set  $\phi$  and is closed with respect to complementation and countable unions.

A **finite sequence** (of size  $n$ ) is a function whose domain is the first  $n$  natural numbers. An **infinite sequence** is a function whose domain is the set  $\mathbb{N}$  of natural numbers. A set  $A$  is called **countable** if it is the range of some sequence (finite or infinite). The set  $A$  is **finite and countable** if it is the range of some finite sequence. The set  $A$  is **countably infinite** if it is the range of some infinite sequence.

When the events  $A_1, A_2, \dots$  are **countably infinite**, we can take the sigma field constituting the probability space to consist of *all* subsets of  $\Omega$ . When  $\Omega$  is **uncountably infinite** (e.g., the real line  $\mathbb{R}$ ), we do not want the sigma field to be the collection of *all* subsets of  $\Omega$  to constitute a probability space. In this case, we only consider events of the type  $X \leq x_i$  and the sigma field to consist of all finite intervals in  $\mathbb{R}$ .

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Now when you need to generalize this concept of set algebra to a countably infinite collection that is as we will see to include random variables either continuous type or discrete random variables whose range is say all positive integers. Then we have sigma algebra and define them in the most efficient manner.

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## Basic set theory (review)

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Measures:

**Measurable set:** A couple  $(X, \mathcal{A})$  is a measurable space where  $X$  is any set and  $\mathcal{A}$  is a  $\sigma$  algebra of subsets of  $X$ . A subset  $A$  of  $X$  is measurable with respect to  $\mathcal{A}$  if  $A \in \mathcal{A}$ .

**Measure:** A measure  $m$  on a measurable space  $(X, \mathcal{A})$  is a non-negative set function defined for all sets of the  $\sigma$  algebra  $\mathcal{A}$ , if it has the properties:

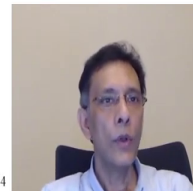
(i)  $m(\phi) = 0$ .

(ii) If  $A_1, A_2, \dots$  is a sequence of disjoint sets of  $\mathcal{A}$ , then  $m\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i)$ .

**Measure space:** A measure space  $(X, \mathcal{A}, m)$  means a measurable space  $(X, \mathcal{A})$  together with a measure  $m$  defined on  $\mathcal{A}$ .

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Then we need to define an appropriate measure on the sigma algebra which in our case will be the probability measure.

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## Basic set theory (review)

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Measures (contd.):

**Measurable function:** Let  $(\Omega, \mathcal{A})$  and  $(\Omega', \mathcal{A}')$  be two measurable spaces. Then the function (or map)  $f: \Omega \rightarrow \Omega'$  is called measurable if the inverse satisfies

$$f^{-1}(A') \in \mathcal{A}$$

In the special case that  $\Omega$  is the sample space and the range of  $f$  is the extended real line, i.e.,  $\Omega' = \mathbb{R}$  and  $\mathcal{A}' = \mathcal{B}(\mathbb{R})$  the sigma algebra of intervals on the real line, then  $f$  must satisfy any one of the following conditions in order to be a measurable function with respect to  $\mathcal{A}$ :

$$\{x: f(x) < \alpha\} \in \mathcal{A} \text{ for each } \alpha$$

$$\{x: f(x) \leq \alpha\} \in \mathcal{A} \text{ for each } \alpha$$

$$\{x: f(x) > \alpha\} \in \mathcal{A} \text{ for each } \alpha$$

$$\{x: f(x) \geq \alpha\} \in \mathcal{A} \text{ for each } \alpha$$

Then, as we will see in the next lecture,  $f$  is called a random variable.

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And when the outcomes of the random experiments can be defined in numerical terms that is the sample space can be mapped onto the real line and in particular onto boreal field of intervals on the real line we will call that map the function a random variable.