Advanced Foundation Engineering
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Lecture No -34
Beams on Elastic Foundation- VIII

So, last class I have discussed that how you will get the equation for beam resting on two-parameter model then, I discussed also the differences between the beam resting on two-parameter model and beam resting on spring or one-parameter model and under different conditions that means under plane-strain condition or beam with finite width, you have to modify the b^* value and E^* value.

And I have given the equation for both the conditions that means beam resting on spring and beam resting on two-parameter model. Now, today I will discuss how I can solve this equation by using finite difference method and because I have already given the idea that how you can solve the infinite beam and semi-infinite beam in case of beam resting on spring and similarly you can do it for the finite length beam also.

And, for two-parameter models also you can solve and you will get the closed form solution. But as you will go for one-parameter model to two-parameter model or beam resting on spring to beam resting on two-parameter model, your solution technique will become more complex. And for example, here most of the soil parameters are considered linear or the stress strain behavior is considered the linear.

But, if we want to incorporate the nonlinearity then also getting closer solution is very difficult. So, in such case we can take the help of numerical methods and I will discuss the finite difference method only. So, that means here, you can see that getting solution for closed form solution for infinite beam is easier compared to the semi infinite and compared to the finite length beam also.

So, that means infinity is most, easiest solution procedure to get the closed form solution and as you go for the finite length beam or I will go for the beam on two-parameter model or I will incorporate the non-linearity then the solution will become more complex. So, in such case getting closed form solution is difficult. So, we can go for the numerical. So, now first I will discuss how I will get the finite difference coefficients.

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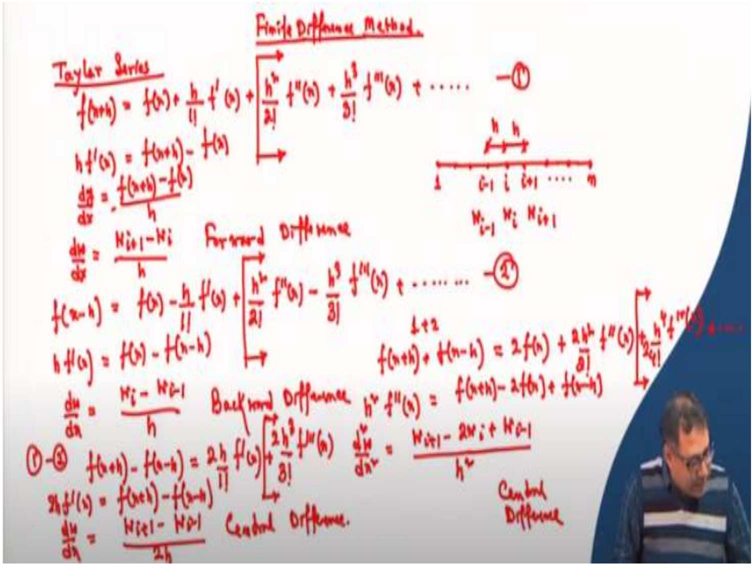

So, the finite difference method you can write by Taylor series, that is $f(x + h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \cdots$. So, now if I neglect this h^2 onward terms, I can write that $hf'(x) = f(x + h) - f(x)$. So, now suppose you have a beam at this point and that is say, i^{th} point this is $(i + 1)^{\text{th}}$ point this is $(i - 1)^{\text{th}}$ point.

And, at i^{th} point your deflection is W_i and $(i + 1)^{\text{th}}$ point your deflection is W_{i+1} and here deflection is W_{i-1} and the distance between these nodes or the points is h. So, that means we have divided this beam with number of nodes and each node as it is named. So, that is the i^{th} node, $(i + 1)^{\text{th}}$ and so on the last node will be the n^{th} node and the first node will be the first node and deflection of each node because when we apply the load, there will be deflection of each node, you can write y also and here deflection is written as W.

So, W_i, W_{i+1} and W_{i-1} to W_1 and W_n. So, this is the deflection of each node. So, now I can write this equation in terms of this deflection, this is $f'(x)$ that means I can write $\frac{dy}{dx} = \frac{f(x+h)-f(x)}{h}$ or I can write that $\frac{dw}{dx} = \frac{W_{i+1} - W_i}{h}$. So, this is called forward difference scheme method. So, that means here we are going in forward direction.

So, this way, I can write $\frac{dw}{dx}$ in finite difference form and that is forward difference scheme. Now, similar to Taylor series, if I write this expression $f(x-h) = f(x) - \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \cdots$. Now again, if I remove or neglect this h^2 term, then I can write that $hf'(x) = f(x) - f(x-h)$ or I can write that $\frac{dw}{dx} = \frac{W_i - W_{i+1}}{h}$. So, this is called backward difference method. So, here we are going in the backward direction, i to $i-1$.

Now if we have this is equation number 1, this is equation number 2 and if I add these two equations 1 and 2, then I will get that $f(x + h) + f(x - h) =$ we are adding that is $2f(x)$ then this will cancel out $+\frac{2h^2}{2!}f''(x)$ then + then $\frac{h^3}{3!}f'''(x)$ will also cancel. So, there will be $\frac{2h^4}{4!}f''''(x)$ and so on.

So, now I can neglect this term; 4 term then I will get that this 2, 2 will cancel out. So, this will be $h^2 f''(x)$ that is equal to $f(x + h) - 2f(x) + f(x - h)$. So, I can write this is $h^2 f''(x)$. So, this $\frac{d^2w}{dx^2} = \frac{W_{i+1} - 2W_i + W_{i-1}}{h^2}$. So, this is called central difference method because here it is central, we are taking $2W_i$, W_{i+1} as well as W_{i-1}, this is called central difference.

Now, if I subtract 2 from 1, then I will get $f(x + h) - f(x - h) =$ subtracting so that the first term will cancel and so this will be $\frac{2h}{1!}f'(x) + \frac{2h^3}{3!}f'''(x)$. If I neglect this h^3 term and higher then I can write $2hf'(x) = f(x + h) - f(x - h)$ or I can write $\frac{dw}{dx} = \frac{W_{i+1} - W_{i-1}}{2h}$, h is the difference between these two nodes.

So, now this is also called central difference scheme. So, you can see that in central difference scheme we are neglecting h^3 and higher but in forward and backward difference scheme we are neglecting a square and higher term as h has very small value, so if I am using the h^3 and higher so, basically our error is less compared to when we are neglecting a square and higher term. So, in that sense the central difference scheme is less erroneous compared to the forward difference and backward difference scheme.

Here obviously there are some errors because we are neglecting some terms. Because our aim is to get the solution which is very close to the closed form solution, obviously closed form solution will give me the actual solution but sometimes as I mentioned it is very difficult to get. So, that is why we are taking the help of numerical tools and our aim is to get that closed form solution as close as possible because you can see we are neglecting some terms.

So, definitely there are some errors, but if I use the central difference scheme our error will be less compared to the backward and forward difference scheme and one more thing I want to mention that do not mix these two schemes when you using them in your problem either you always try to use the central difference scheme in all the cases in a particular problem. So, now let me apply this problem or this finite difference scheme.

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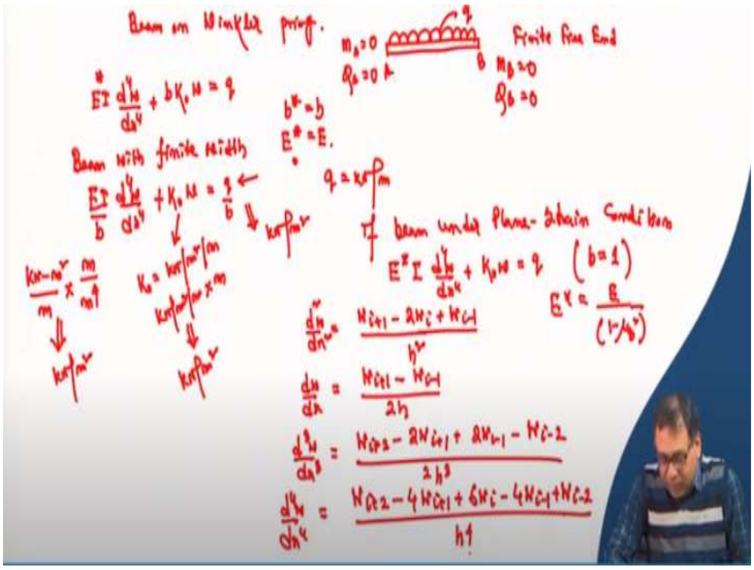

So, beam on Winker Spring and it is a finite beam and it is a free end beam, it is a finite free end beam, both the ends are free and so, I can write this is A point and this is B point, so bending moment at A will be zero and here shear force at A will be zero again bending moment at B will be zero and shear force at B will be zero because it is a free end beam.

So, now my basic equation is $EI\frac{d^4W}{dx^4}$ because now it is $E^* +$, because it is a Winkler model so whether it is on two cases, it will be always b so, bk_0W; so that is equal to q. So, now we are taking that infinite length, so that means in such case b with finite width, so b with finite width my $b^* = b$ and $E^* = E$. So, I can write this is beam with finite width.

So, I can write this is $EI\frac{d^4W}{dx^4}$, this is your b and this one is $+k_0$. So, this is $+k_0W$ that is equal to $\frac{q}{b}$. So, that means in such case your q is in kN/m^2. So, q is in kN/m^2 divided by b. So, now it will be in kN/m and here also you can see again q is in kN/m.

So, finally it will be kN/m^2 unit q is in kN/m, this is the q, which is in kN/m. So, finally it will be kN/m^2 because we are dividing by b and this k_0 is kN/m^2/m. So, if I multiply with the W, so this will be kN/m^2/m × m. So, this will be in kN/m^2.

And you come to this part this EI is in kN/m^2 then b is in m × d^4W means m^4 divided by dx^4 which is in m^4. So, this will be kN/m^2. So, you can write this expression in any form but when you write, write them properly, so that the dimensions in the left side and right side will match. So, in this way you have to express. So, then you can write k directly in that case, this will be kb or k_0b.

So, that means here you just express these in different forms, but you have to express these properly and another thing you remember that the k_0 unit is kN/m^2/m and k is $b \times k_0$. So, k is in kN/m^2 but here k we are not using that equally we are using k_0 and we are taking b divided by 2 other parts. So, now this is your equation.

So, finally I can write the equation in this way that if the beam with finite length or finite width then this is the expression and then I can finally write that if beam is under plane-strain condition, because this is being finite width under plane-strain condition, this will be $E^*I\frac{d^4W}{dx^4} + k_0W = q$, because under that condition $b = 1$ because it is under plane-strain condition and $E^* = \frac{E}{1-\mu_b^2}$.

So, I will solve this equation; basically beam under plane-strain condition. You can solve the beam with finite width also but in that case, there will be b value but here I have taken b as unity because it is under plane-strain condition. So, my equation is this one and then I will express in similar way the $\frac{dW}{dx}$ for finite difference scheme as $\frac{dw}{dx} = \frac{W_{i+1}-W_{i-1}}{2h}$ and $\frac{d^2w}{dx^2} = \frac{W_{i+1}-2W_i+W_{i-1}}{h^2}$.

So, this is equal to E^* and $\frac{dW}{dx}$ is the expression. So, I can write $\frac{W_{i+1}-W_{i-1}}{2h}$ similar way I can write $\frac{d^3w}{dx^3} = \frac{W_{i+2}-2W_{i+1}+2W_{i-1}-W_{i-2}}{2h^3}$ and $\frac{d^4w}{dx^4}$ I can write in central difference scheme as $\frac{W_{i+2}-4W_{i+1}+6W_i-4W_{i-1}+W_{i-2}}{h^4}$. So, these are the expressions of $\frac{dW}{dx}$ to $\frac{d^4w}{dx^4}$ in central difference scheme. So, now my equation is this one.

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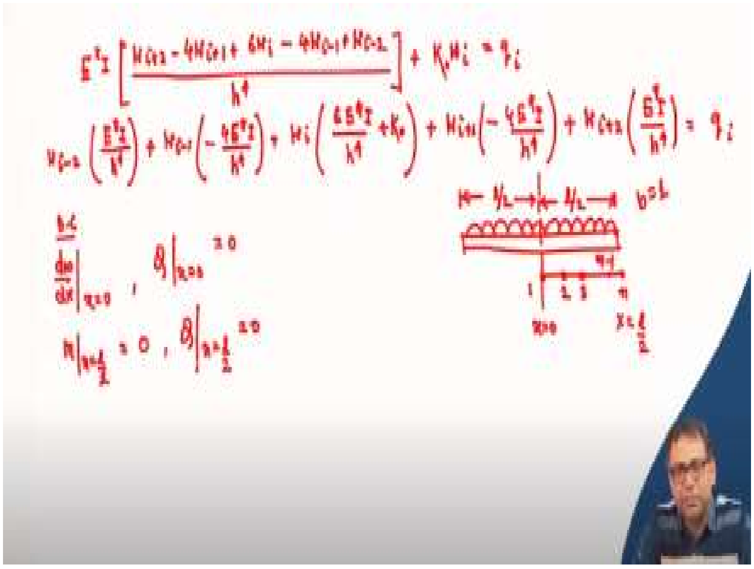

So, that means here I have to write E^*I and this equation is $\frac{W_{i+2}-4W_{i+1}+6W_i-4W_{i-1}+W_{i-2}}{h^4}$, then + this is k or k_0W_i that will be equal to q_i because this is my equation. This equation I am expressing in finite difference form and now I take all the coefficients in a bracket. So, coefficient of W_{i+2}, or if I take this side minus coefficient of W_{i-2} that coefficient will be $\frac{E^*I}{h^4}$ then the $+W_{i-1}$ this coefficient will be – 4.

$\frac{E^*I}{h^4}$, then W_i coefficient is from here it is $\frac{6E^*I}{h^4}$ from here, it will be k_0 then this side form, W_{i+1}, this will be again $\frac{-4E^{*}I}{h^{4}}$ then $+W_{i+2}$, this will be $\frac{E^{*}I}{h^{4}}$, that will be equal to q_i. So, now we have divided suppose this is my beam, which is a finite beam subjected to UDL and this is the center and it is a symmetric problem so I can write this is point is $x = 0$.

So, if I take these points only and if I divide these points into number of node, that is first node at $x = 0$, second node, third node and this is my n^{th} node at $x = \frac{l}{2}$ I can write that this is my beam length; so, this is $x = \frac{l}{2}$, so this is $\frac{l}{2}$, this is also $\frac{l}{2}$. So, it is under plane-strain condition, so $b = 1$ I have discussed. So, these are my different nodes.

So, at every node I have to apply these equations. So, now the problem, I know that W_{i-2}, W_{i-1}. Suppose, if I apply it in third node then W_{i-1} is the second node W_{i-2} is the first node, so that is no problem so up to third node and onwards there is no issue but if I want to apply them in the first node then I have to get one i - 2 node that mean - 1 node which is not present.

So, that means for first node and second node we cannot apply this but from three nodes onward we can apply. Similar thing will happen for n^{th} node and $(n-1)^{\text{th}}$ node. So, from $(n-1)^{\text{th}}$ node, so at these four nodes we will have problems that mean first node and the second node and the n^{th} node and $(n-1)^{\text{th}}$ node these four nodes we cannot directly apply this equation, because we need more nodes to apply this equation on these four nodes.

The other node from 3rd to $(n-2)^{\text{th}}$ node, I can apply this equation no issue. So, then; what to do? So, here we have to apply the boundary conditions so now it is a finite beam, free end beam. So, we have four boundary conditions so what are those four boundary conditions plus here now we have taken the half portion of the beam and the loading is symmetric.

So, as the loading is symmetric, boundary conditions are: slope at $x = 0$ is 0 and it is UDL and in the center shear force at $x = 0$ is 0. Now at the end, what is at the end? At the end, your bending moment at $x = \frac{l}{2}$ will be 0 and shear force at $x = \frac{l}{2}$ will be 0 because it is a free end similarly hinged beam or the fixed end beam you have to apply the boundary conditions, I have discussed during semi-infinite beam, so accordingly those boundary conditions you have to apply but it is a free end beam so at $x = \frac{l}{2}$ bending moment is 0 and the shear force is also 0.

So, the next class I will discuss that how I can apply these boundary conditions to solve these particular problems. So, this boundary condition is valid for this particular problem where beam is finite beam and it is a free end beam, both ends are free and it is a symmetric condition. So, that is why you have taken only half portion of the beam. So, next class I will give you the idea how we can apply these boundary conditions and how we can apply these equations on these four nodes, that is node 1, node 2, n^{th} node and $(n-1)^{\text{th}}$ node. Thank you.