

**Strength of Materials**  
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**Indian Institute of Technology, Kharagpur**  
**Lecture No. # 19**  
**Torsion - II**


Welcome to the 2nd lesson of 4th module Torsion Part II.

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In the last lesson which was the first part of torsion we have seen some aspects of the torsional moment which were applied on a bar and we considered the cross section of which is a circular one and is a solid circular cross section. We have seen the effect of the torsional moment in such cross sectional bar. Today, we are going to study the effect of the twisting moment on such a bar where the cross section is not a solid one but a hollow one. We will also look into what will be the consequence of the twisting moment on a bar, the cross section of which is a tubular one instead of a solid shaft.

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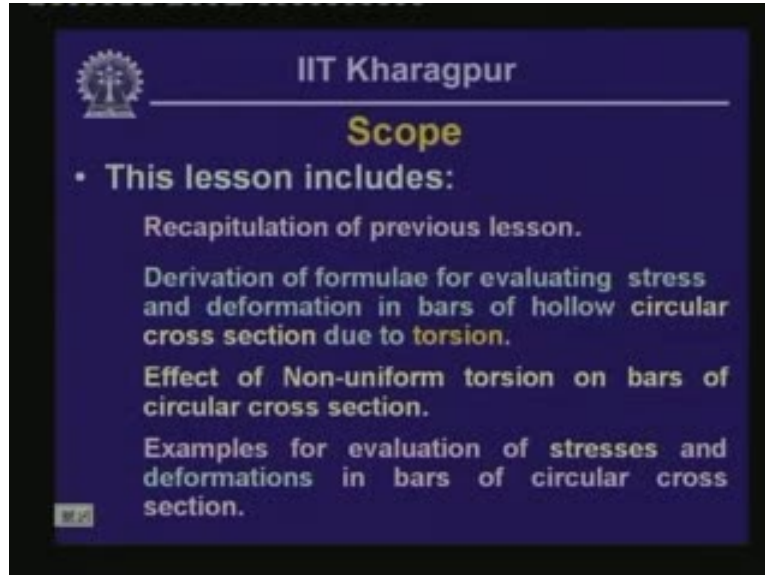
### Specific Instructional Objectives

- After completing this lesson one will be able to:
- Understand the concept of **twisting moment and its effect on bars of hollow circular cross section.**
- Understand the effect of **Non-uniform twisting moment and its effect on bars of circular cross section.**
- Evaluate stresses and deformation in circular bars due to torsion.

After completing this lesson one will be able to understand the concept of twisting moment and its effect on bars of hollow circular cross section, understand the effect of non-uniform twisting moment and its effect on bars of circular cross section. Last time, we applied uniform twisting moment on the bar. The bar was fixed at one end, the twisting moment was applied on the other end and the torsion applied over the bar was uniform in nature.

Now, if there are some bars which may not be uniform there could be variation in the section or there could be uniform section but the torsions applied at different points are of different magnitudes. For that kind of a situation, we will look into the consequence of that kind of loading in the system. This is what the non-uniform twisting moment is and its effect on bars of circular cross section and thereby one should be able to evaluate stresses and deformation in circular bars due to torsion.


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We will be looking into the aspects of the previous lesson. This will be done through answering the questions which I have posed. Also we will be deriving the formulae for evaluating stress and deformation in bars of hollow circular cross section. We will also be looking into the effect of non-uniform torsion on bars of circular cross section and examples for evaluation of stresses and deformation in bars of circular cross sections of varying types.

Now, we will see examples of bars with solid or circular cross sections and the effect of torsion on such bars. Even the cross section could be a hollow like a tubular member. If we have a bar which is consisting of several diameters and if it is subjected to torsion either of same magnitude or different magnitudes then what is the consequence of such systems?

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### Answers to Question Set 4.1


- What is the type of stress a bar encounters due to torsion?
- What is meant by torsional stiffness?
- What will be the stresses and deformation due to torsion if the bar is hollow circular section instead of solid one?

Application

Let us look into the questions. The first question is:

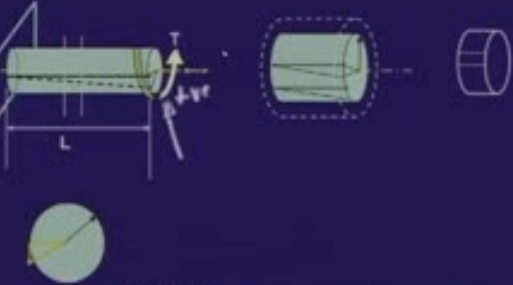
What is the type of stress a bar encounters when it is subjected to a twisting moment or torsion?

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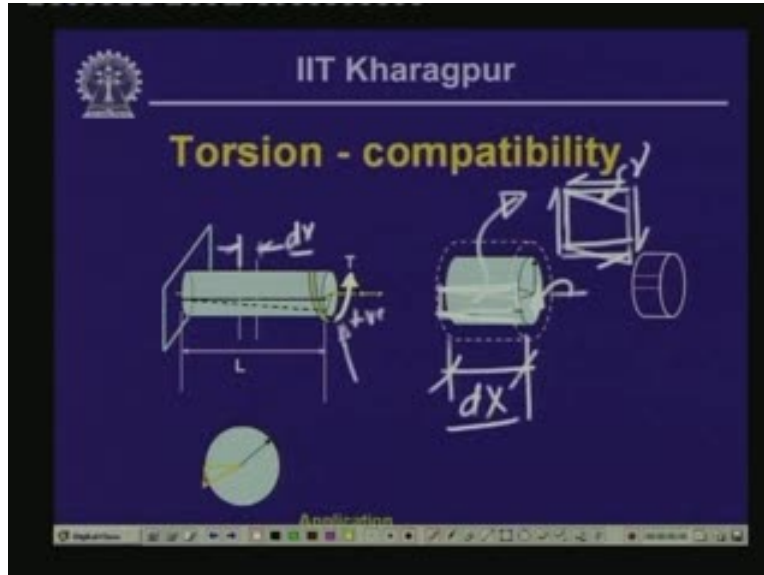
### Torsion - compatibility



Application

Let us look into the aspect which we discussed last time. In this example, the bar has the circular cross section; it is a solid circular bar subjected to a twisting moment  $T$ , and this we had defined as a positive twisting moment because it is acting in an anti-clockwise direction. The thumb is projecting towards the positive  $x$ -axis, which we have denoted as the positive twisting moment.

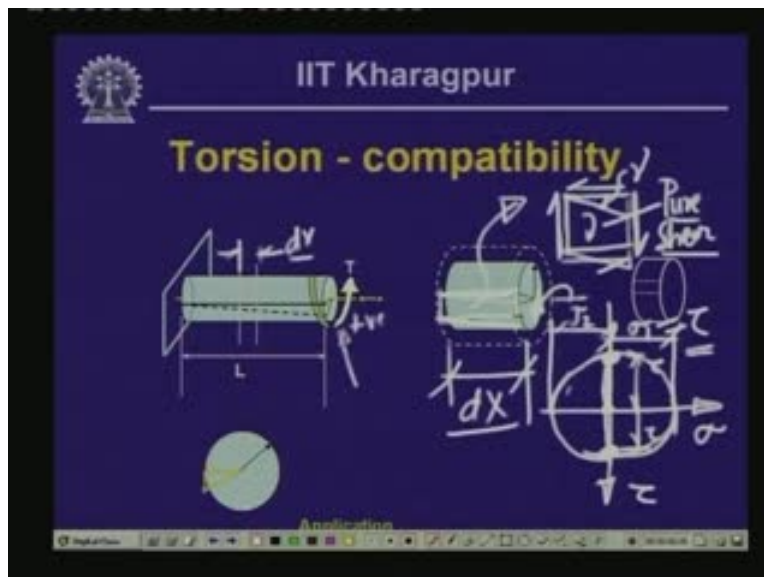
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We already looked into a portion of the bar which is of length  $dx$  and this is the part of that whole bar. Because of the effect of this twisting moment acting on this bar if we take an element on the surface of this particular bar this undergoes moment and thereby an ellipse is formed.

If we look into this particular element, this element undergoes a change in the angle. As we had seen earlier, we call this change in the angle as shearing strain  $\gamma$ , and as per Hooke's law we get the stresses which get induced into the surface called as the shearing stress. Because of the application of the twisting moment in the bar we get shearing stress in the bar and because of this shearing stress is without any application of the normal stress so this is the state of pure shear.

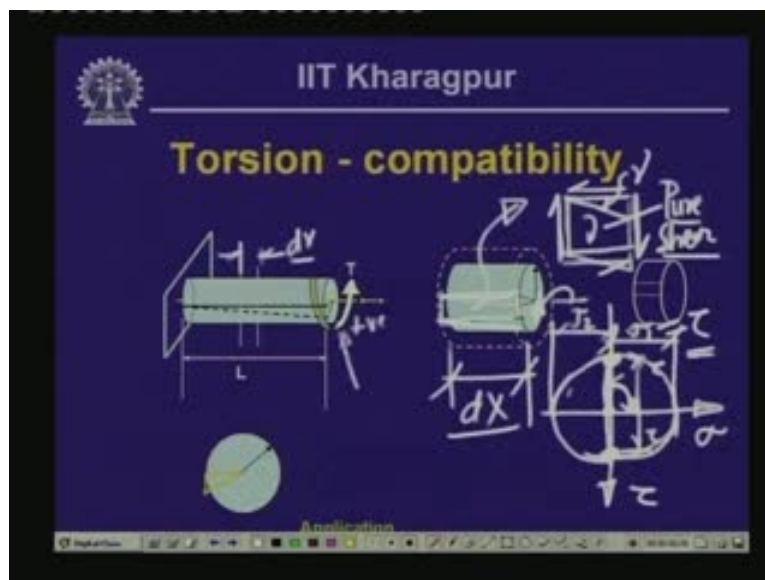
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This state of pure shear can be transformed in terms of Mohr circle to evaluate the maximum normal stress. If we plot this state of stress in the Mohr circle this is the sigma axis and this is the tau axis. And because this is a clockwise moment by this particular shear on this face this is a negative shear as per our notation which is plotted on the top side of the Mohr diagram.

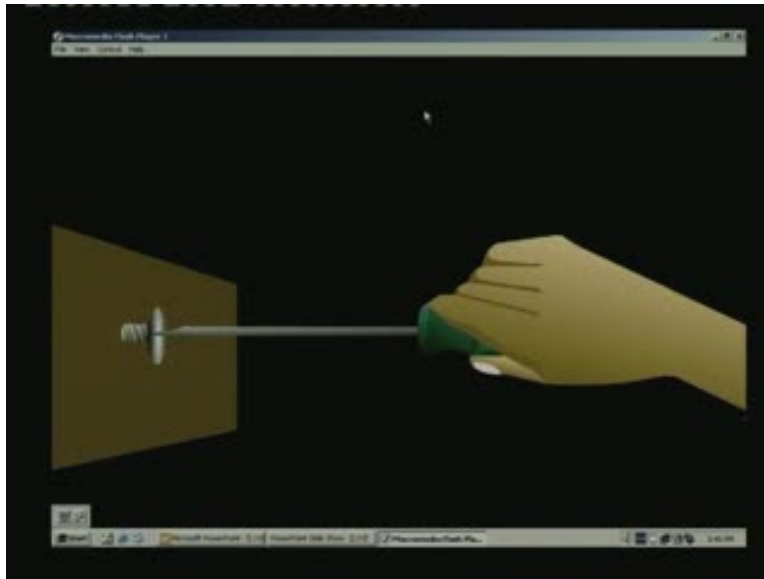
For the other side, we have a positive value which is plotted over here. So this is the value of tau. Now this is the center and if we plot the circle then we get the value of the normal stress that will be acting and this is the value of the maximum normal stress which is  $\sigma_1$  and the minimum normal stress which is  $\sigma_2$ . The values of  $\sigma_1$  and  $\sigma_2$  are equal to tau. Due to the application of the twisting moment, the body is subjected to a shearing stress and thereby we are going to get the normal stress which is also magnitude wise equivalent to the shearing stress.

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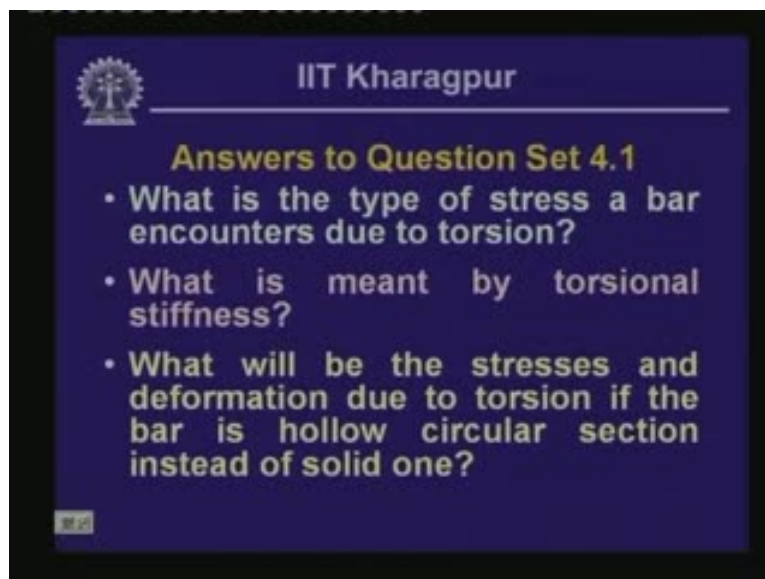
The angle here is 90 degrees so the action of this principle stress will be at an angle of 45 degrees with respect to the shear and that will be the maximum normal stress which is equal to the value of tau.

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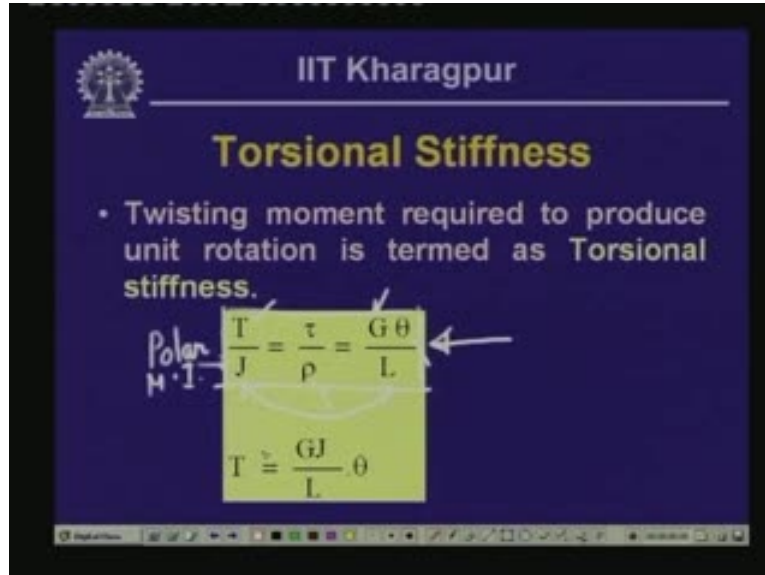
Let us look into the application of such a twisting situation when we move a screw driver. This is the application of a twisting moment. When we are rotating it basically we are applying the twisting moment which is allowing this to rotate and finally moves inside.

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The second question asked was: What is meant by the torsional stiffness?

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The slide features the IIT Kharagpur logo and title at the top. Below the title, a bullet point defines torsional stiffness. A central box contains the torsion formula  $\frac{T}{J} = \frac{\tau}{\rho} = \frac{G\theta}{L}$  and its rearranged form  $T = \frac{GJ}{L} \theta$ . Handwritten annotations include 'Polar M.I.' next to J and arrows pointing to the variables in the formula.

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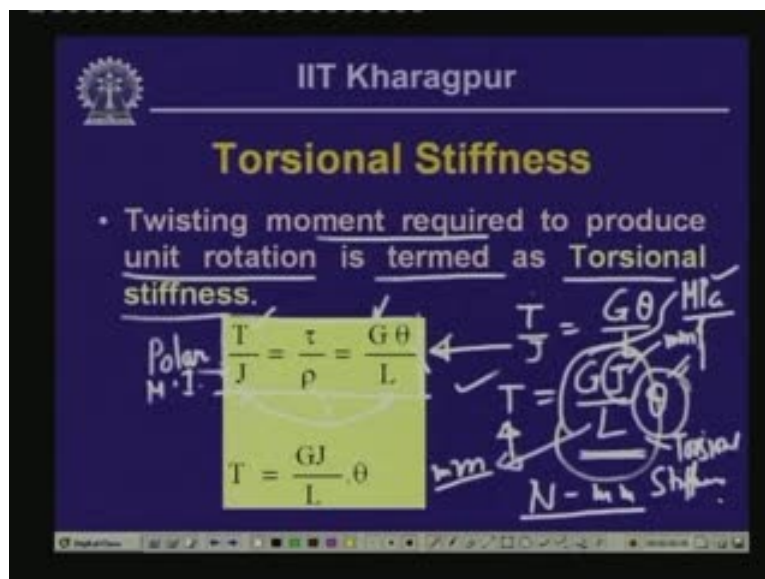
### Torsional Stiffness

- Twisting moment required to produce unit rotation is termed as Torsional stiffness.

$$\frac{T}{J} = \frac{\tau}{\rho} = \frac{G\theta}{L}$$
$$T = \frac{GJ}{L} \theta$$

Let us look into the expression which we had derived last time for the torsional moment. This is the expression which we had derived  $\frac{T}{J} = \frac{\tau}{\rho} = \frac{G\theta}{L}$  where: T is the twisting moment; J is polar moment of inertia; tau is the shearing stress; rho is the radius at any point on the center of the cross section; G is the shear modulus; theta is the rotation and L is the length of the bar. We call this expression a torsion formula. If we equate this to quantity; we are interested in finding out the stiffness because of the torsion. When we apply the twisting moment, the twisting moment required to produce unit rotation is called as the stiffness against this twisting moment.

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This slide is identical to the previous one but includes extensive handwritten annotations. A diagram of a bar under torsion is drawn, showing the relationship between torque T, length L, and angle of twist theta. The torsion formula is repeated, and the stiffness is defined as T/theta = GJ/L. The unit of stiffness is given as N-m/rad.

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### Torsional Stiffness

- Twisting moment required to produce unit rotation is termed as Torsional stiffness.

$$\frac{T}{J} = \frac{\tau}{\rho} = \frac{G\theta}{L}$$
$$T = \frac{GJ}{L} \theta$$

Handwritten notes include:  $\frac{T}{J} = \frac{G\theta}{L} \times \frac{HL^2}{L}$ ,  $T = \frac{GJ}{L} \theta$ , and  $\frac{T}{\theta} = \frac{GJ}{L}$  N-m/rad Stiffness.

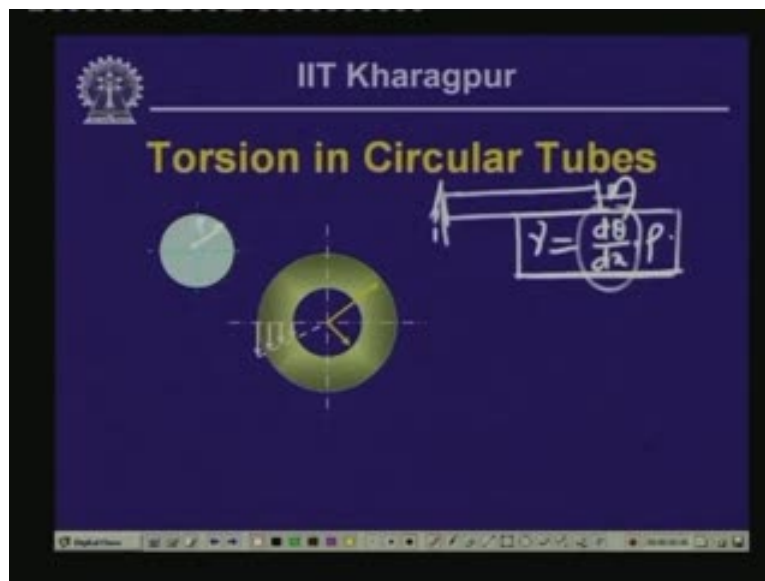


So the twisting moment required to produce unit rotation is termed as the torsional stiffness. If we equate these two quantities  $\frac{T}{J} = \frac{G\theta}{L}$ , we can relate T and theta as  $T = \frac{GJ\theta}{L}$ .

Now, if we make theta as unit, then T is the torsional stiffness. So the torsional stiffness equals  $\frac{GJ}{L}$ . This quantity is designated as the torsional stiffness. Now if we look into the units G is represented in terms of mega Pascal, J is the polar moment of inertia which is mm power 4, and L is the length which is in mm. So we have Newton per mm square and then we have J which is mm power 4 and L is mm. So Newton millimeter is the unit for T which is equivalent to the unit for the twisting moment. This is the stiffness.

Now the third question was: What will be stresses and deformation due to torsion if the bar is a hollow circular section instead of a solid one? This is the aspect we are going to discuss today. So this question will be answered through discussions in this lesson. Let us look into the aspect of how we calculate the effect of the torsion in circular tubes.

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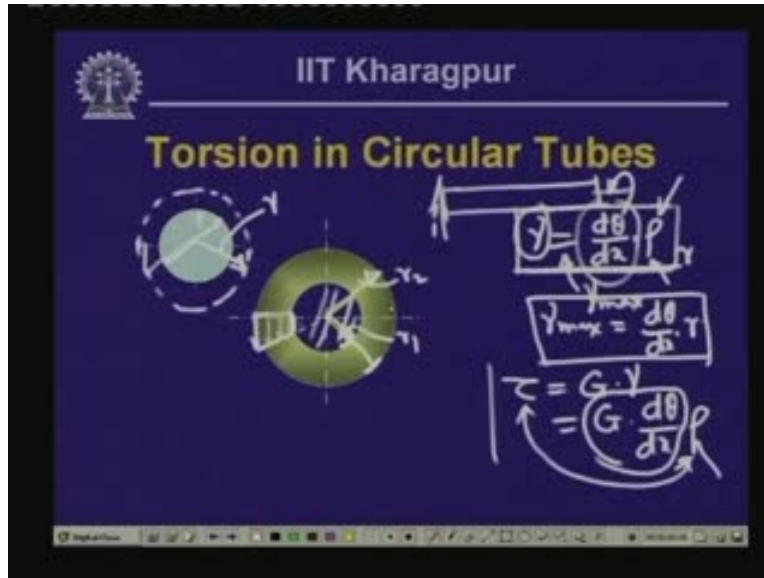


In the previous lesson, we have gone through the fact that if a bar of solid circular section is subjected to a twisting moment, then it is subjected to the shearing strain, which we have seen as  $\gamma = \frac{d\theta}{dx} \rho$ . This is the expression which we had used for the solid shaft and  $\gamma$  is the shearing

strain at any point which is at a radius  $\rho$  and  $\frac{d\theta}{dx}$  is rotation per unit length. This particular expression which we derived for the solid circular section is equally applicable for hollow tubular section as well. This means that as we had seen in case of solid circular shaft, at any point from the center at any radius  $\rho$  we can compute the strain  $\gamma$ . In this case, from the center

till we reach the inner point of the tube there is no material and hence no strain. So the minimum strain that occurs is at the point which is the inside of the tube of this tubular cross section.

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Now this is the center over here. Let us call this internal radius as  $r_1$  and the external radius as  $r_2$ . Coming back to this expression again, the maximum strain that occurs  $\gamma_{\max}$  is at the outer periphery, where if this is the outer periphery of the whole tube, and if we call this radius as  $r$ , then when  $\rho$  becomes  $r$ , that gives us the value of the strain as  $\gamma_{\max} = \frac{d\theta}{dx} r$ . This is what

we get in case of solid circular shaft. Now when you come to this tubular form, the first material it encounters from the center is at the inner radius. So the minimum strain that occurs is at this particular point and the maximum strain occurs at the outer radii. So these are the points and as we have seen from this expression that  $\gamma$  vary linearly with  $\rho$ . So the strain varies linearly with the radius, so as the stress from our Hooks law that  $\tau = G\gamma$ .

Since  $\gamma = \frac{d\theta}{dx} \rho$ ;  $\tau = G \frac{d\theta}{dx} \rho$  again  $G \frac{d\theta}{dx}$  being independent of  $\rho$ ,  $\tau$  varies linearly with  $\rho$ .

So the shearing stress varies linearly with the radius from the center. In case of hollow tube the same theory holds good. Only thing is that since we do not have any material at the center, the first point which encounters stress is at this particular point and then as it goes along the outer periphery, where we have the maximum stress.

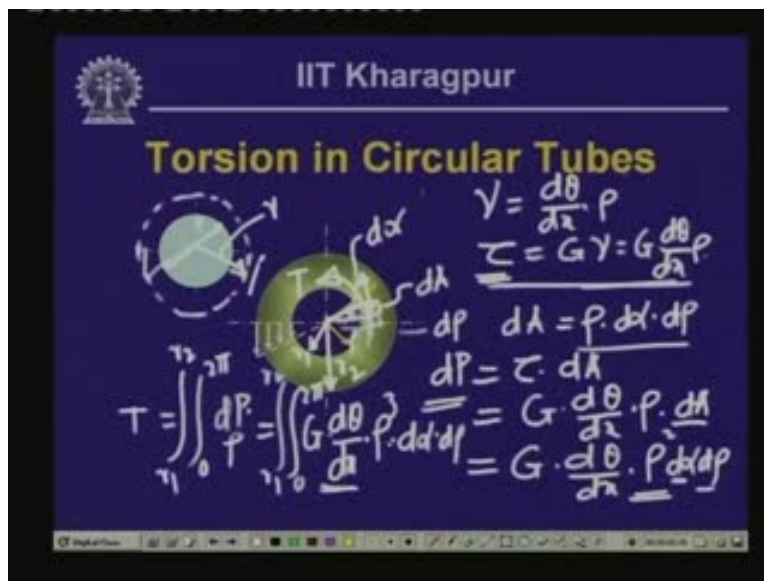
So the variation of the shear stress across thickness is in this form which is proportional to the radius  $\rho$ . This is the distribution of the shear stress in case of tubular section. So the same theory as we had in case of circular shaft holds good even for the tubular section as well. We will relate the different quantities as we had related in the case of solid shaft, like the shear stress to the torsional moment, the radius at which point we like to find out the stress, and then

corresponding quantity of the rotation, the shear modulus and the length of the bar. So let us look into that. How do we compute these relationships in case of tubular sections?

Now, coming back to this expression  $\gamma = \frac{d\theta}{dx} \rho$  we know that the shear stress  $\tau = G\gamma = G \frac{d\theta}{dx} \rho$ .

As we had seen, this is basically from the compatibility criteria. Last time we discussed the derivation of the formulae for the torsion, when the torsion is applied, the corresponding stresses, we had equations of compatibility which is in terms of the rotation and subsequently we had the equations of equilibrium. Equilibrium equations are derived mostly from the resultant stress that is being generated because of the twisting moment. So on any cross section the resultant stress equals the applied twisting moment and we apply the same concept to evaluate what should be the expression or the relationship between the stress and the twisting moment.

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Now, let us consider a small element in this. Let us consider a small element from the center which is at a distance of rho and making a small angle dalpha. So this is the small element having an area dA at a distance of rho and this distance is drho. So let us call this small angle as dalpha and the area dA can be written as  $\rho \times d\alpha \times d\rho$ . Now the force that will be carried by this particular area is equal to the shearing stress times the area.

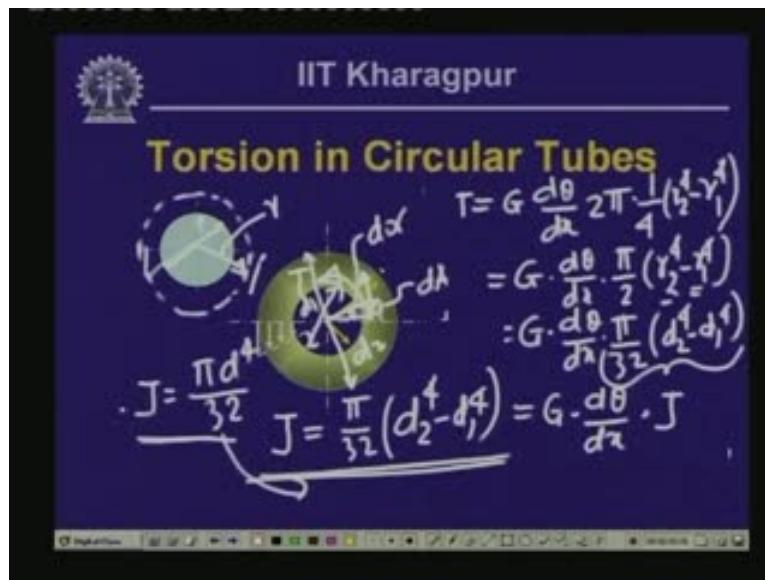
$dP = \tau \times dA$ . If we substitute  $G \frac{d\theta}{dx} \rho$  for tau in the above expression; we get  $dP = G \frac{d\theta}{dx} \rho \times dA$ .

Now if we take the moment and substitute this dA in terms of dalpha and drho, we get  $dP = G \frac{d\theta}{dx} \rho \times \rho \times d\alpha \times d\rho = G \frac{d\theta}{dx} \rho^2 \times d\alpha \times d\rho$ . Now the twisting moment acting in this

section is T and that will be registered by the whole of cross section the  $\int_{r_1}^{r_2} \int_0^{2\pi} dP$  so twisting

moment  $T = \int_{r_1}^{r_2} \int_0^{2\pi} dP$ . Here we have two variables, one is alpha, another is rho. alpha varies from 0 to  $2\pi$  and rho varies from inside  $r_1$  to  $r_2$ . So this is 0 to  $2\pi$  for alpha and from  $r_1$  to  $r_2$  for rho. So this equals  $\int_{r_1}^{r_2} \int_0^{2\pi} G \frac{d\theta}{dx} \rho^2 \times d\alpha \times d\rho$ . Here we have rho and this dP times rho is the twisting moment. Hence T is equal to  $\int_{r_1}^{r_2} \int_0^{2\pi} G \frac{d\theta}{dx} \rho^3 d\alpha \times d\rho$  Now this G, dtheta, dx being constant this is from 0 to  $2\pi$  and from  $r_1$  to  $r_2$ . Now Let us take out G, dtheta, dx and evaluate.

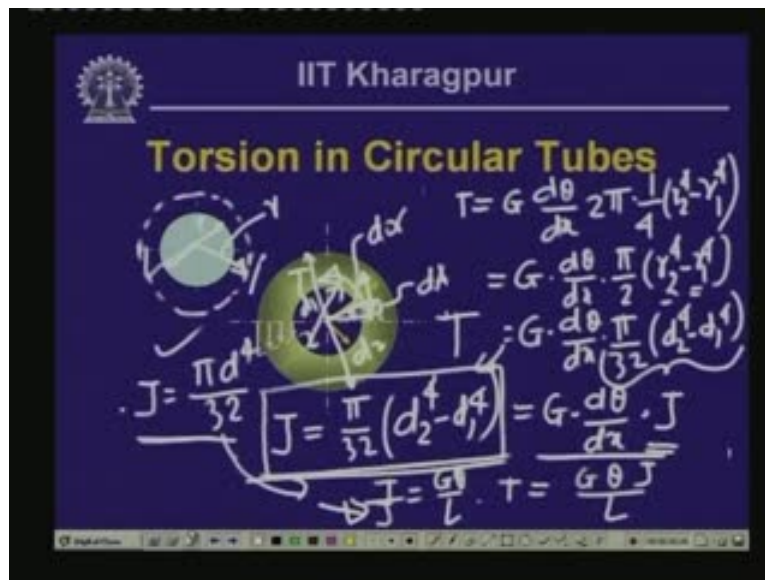
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Now for T, we have  $G \frac{d\theta}{dx} \int_0^{2\pi} d\alpha$  is alpha which is varying from 0 to  $2\pi$  so that is equals to  $2\pi$  and then we have rho cube drho so this is rho power 4 by 4 and if we integrate from  $r_1$  to  $r_2$  that is 1 by 4  $(r_2^4 - r_1^4)$ . So this we can write:  $T = G \frac{d\theta}{dx} 2\pi \times \frac{1}{4} (r_2^4 - r_1^4) = G \frac{d\theta}{dx} \frac{\pi}{2} (r_2^4 - r_1^4)$ . If we write r in terms of diameter then this equals  $G \frac{d\theta}{dx} \frac{\pi}{32} (d_2^4 - d_1^4)$ , since r is d by 2 and  $d_2$  is d outer and  $d_1$  is d inner. Last time, we designated this particular term as J, which is nothing but the polar moment of inertia. So this is J now, if you compare the value of J with respect to the J which we have computed for the solid shaft, J for solid shaft was  $\frac{\pi}{32} d^4$ , here the value of J in case of hollow section where the outer diameter is  $d_2$  and the inner diameter is  $d_1$  then in terms of  $d_2$  and  $d_1$ , J equals  $\frac{\pi}{32} (d_2^4 - d_1^4)$ . So this is similar expression and you can visualize that from the whole one, we are removing the central part. In fact in the case of solid circular shaft when you take out the material because the shearing strain or the stresses are varying from the center to the outer periphery, it is zero at the center and maximum at the outer periphery.

Now if we remove the material from the central portion, the stress is the maximum on the outer periphery and at the same time when we are equating the forces, we are getting the equilibrium equation. If we take the element away from the center we get a larger contribution. The material which is closer to the center has less contribution in comparison to the material which is away from the center. Hence in that sense tubular section should be more effective when resisting the twisting moment and we will look into this subsequently while solving the examples.

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So this is the value of the J and based on this particular expression so this is the twisting moment

T:  $T = G \times \frac{d\theta}{dx} \times J$  and over the length if we integrate we get  $T = \frac{G\theta J}{L}$  and this expression we

had seen last time that  $\frac{T}{J} = \frac{G\theta}{L}$ , only difference is the expression for J is different now for the hollow tube than what we had in case of solid shaft. Now this particular expression for the polar moment of inertia can be written down in a different form as well.

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$$\begin{aligned}
 J &= \frac{\pi}{32} (d_2^4 - d_1^4) \\
 &= \frac{\pi}{32} [(d_2^2)^2 - (d_1^2)^2] = \frac{\pi}{32} (d_2^2 + d_1^2)(d_2^2 - d_1^2) \\
 &= \frac{\pi}{32} (d_2 - d_1)(d_2 + d_1)(d_2^2 + d_1^2) \\
 &= \left( \frac{\pi}{32} \times 2t \cdot 2d \right) [(d_2 + d_1) + (d_2 - d_1)] \\
 &= \frac{\pi}{8} dt \cdot [4d^2 + 4t^2] \\
 &= \frac{\pi dt}{8} \times 4(d^2 + t^2) \\
 &= \frac{\pi dt}{2} (d^2 + t^2) \\
 &= \frac{\pi dt}{4} (d+t)^2
 \end{aligned}$$

Let us look into that J, value of J the polar moment of inertia in case of tubular section is  $\frac{\pi}{32}(d_2^4 - d_1^4)$ . We can simplify a little and many a times we use this expression as  $\frac{\pi}{32}[(d_2^2)^2 - (d_1^2)^2]$ . So this is square of  $d_2^2$  minus square of  $d_1^2$  and this gives us  $\frac{\pi}{32}(d_2^2 + d_1^2)(d_2^2 - d_1^2)$ . Now, further  $(d_2^2 - d_1^2)$  can be split and we can write this as  $\frac{\pi}{32}(d_2 - d_1)(d_2 + d_1)(d_2^2 + d_1^2)$ . Now  $(d_2 - d_1)$  as you can visualize that when we have the tubular form where the outer diameter is  $d_2$  and the inner diameter is  $d_1$ . So from  $d_2$  if we subtract  $d_1$ , we are left with two thicknesses. So this we can write as twice t; if t is the thickness of the tube. Now this term  $(d_1 + d_2)$  we can designate this as average diameter and we can write this as d is equal to  $\frac{(d_1 + d_2)}{2}$ .

Now  $\frac{(d_1 + d_2)}{2}$  is the average diameter d so  $(d_2 + d_1)$  is 2d. Now this particular term  $(d_2^2 + d_1^2)$ , we write in a little different form as  $\frac{(d_2 + d_1)^2 + (d_2 - d_1)^2}{2}$ . Now again as you can see from by writing this  $(d_2^2 + d_1^2)$  term in the form  $(d_2 + d_1)$ , we can write as 2d and  $(d_2 - d_1)$  we can write as 2t. Simplifying this, we get  $\frac{\pi}{8} dt (d_2 + d_1)^2 + (d_2 - d_1)^2$ . Simplifying  $(d_2 + d_1)$  term as 2d and 2d square means  $4d^2$  plus  $(d_2 - d_1)$  is 2t so this is  $4t^2$  by 2.

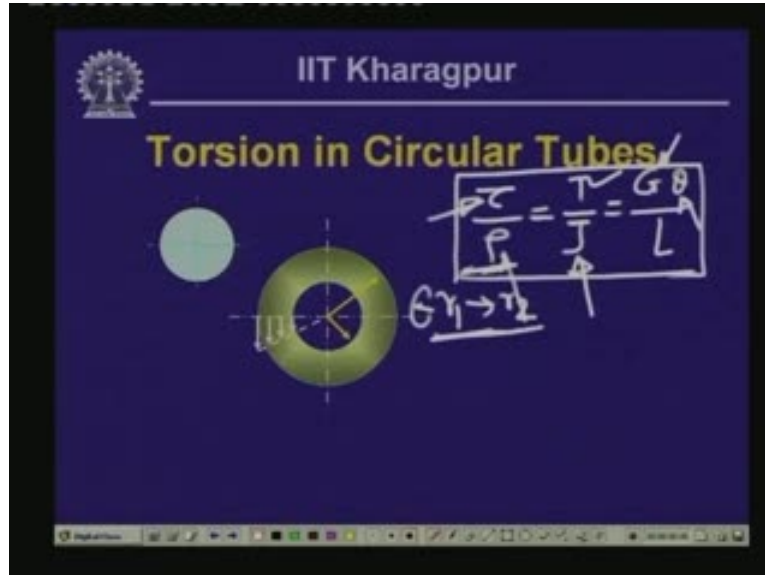
This comes out as  $\frac{\pi}{8} dt \left( \frac{4d^2 + 4t^2}{2} \right)$ . This gives us value of  $\frac{\pi}{4} dt (d^2 + t^2)$ . In fact this is simplified form of this particular expression J, many times, if we can represent the diameter in terms of average diameter we can use this expression, for evaluating the polar moment of inertia instead of  $\frac{\pi}{32} (d_2^4 - d_1^4)$ . Many a times we use shaft or a bar hollow bar the thickness of which could be significantly small in comparison to the radius of that particular tube in that particular case, the term  $t^2$  will be very small in comparison to  $d^2$ . If we neglect that and if we write approximately  $(d^2 + t^2)$  term  $\approx d^2$  then we get this as  $\frac{\pi}{4} d^3 t$ .

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$$\begin{aligned}
 J &= \frac{\pi}{32} (d_2^4 - d_1^4) \\
 &= \frac{\pi}{32} [(d_2^2)^2 - (d_1^2)^2] = \frac{\pi}{32} (d_2^2 + d_1^2)(d_2^2 - d_1^2) \\
 &= \frac{\pi}{32} (d_2 + d_1)(d_2 - d_1)(d_2^2 + d_1^2) \\
 d &= \frac{d_1 + d_2}{2} \\
 t &= \frac{d_2 - d_1}{2} \\
 &= \left( \frac{\pi}{32} \times 2t \cdot 2d \right) \left[ \frac{(d_2 + d_1)(d_2^2 + d_1^2)}{(d_2 + d_1) + (d_2 - d_1)} \right] \\
 &= \frac{\pi}{8} dt \cdot \left[ \frac{4d^2 + 4t^2}{2} \right] \\
 &= \frac{\pi dt}{8} \times 4 (d^2 + t^2) \\
 &= \frac{\pi dt}{2} (d^2 + t^2) \\
 &= \frac{\pi dt}{4} (d^2 + t^2) \quad \text{or} \quad \frac{\pi d^3 t}{4}
 \end{aligned}$$

Now sometimes for a thin tube, we approximate the polar moment of inertia using this expression  $\frac{\pi}{4} d^3 t$ . So this gives us the value of the J, the polar moment of inertia either in the form  $\frac{\pi}{32} (d_2^4 - d_1^4)$  or  $\frac{\pi}{4} dt (d^2 + t^2)$  where d is the average diameter; t is the thickness; or for a thin tube we can even use  $\frac{\pi}{4} d^3 t$  for evaluating the polar moment of inertia.

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Once you compute the polar moment of inertia, then, rest of the expression we have is  $\frac{\tau}{\rho} = \frac{T}{J} = \frac{G\theta}{L}$ . Now we can use this equation for evaluating the stresses. The terms remain as defined earlier, like tau is the shearing stress, rho is the radial distance. Here the rho becomes effective between  $r_1$  and  $r_2$ ; below  $r_1$  there is no material and hence it is not effective. T is the twisting moment, J is the polar moment of inertia, G is the shear modulus, theta is the angle of twist, angle of twist over the whole length of the bar, and L is the length of the bar. So the same expression we used for the solid circular shaft is equally applicable for the tubular section as well, only difference is that the polar moment of inertia is to be computed in a different form than the solid circular shaft.

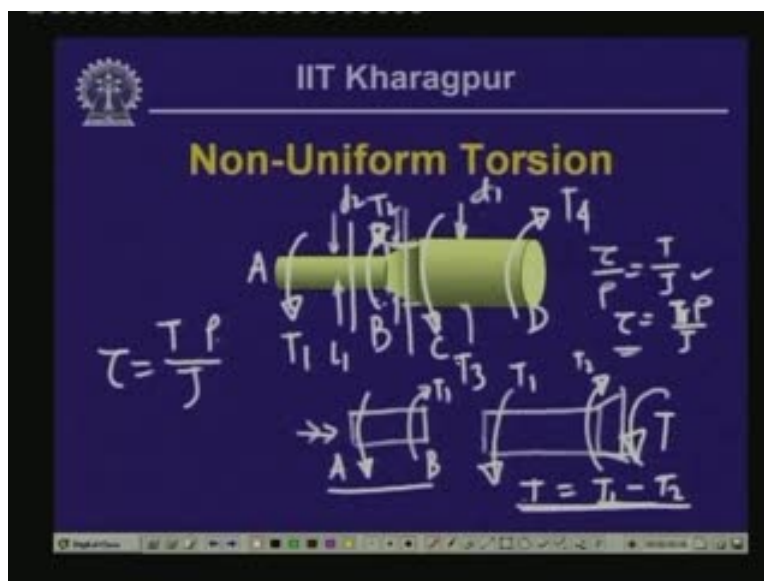


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Having looked into the effect of torsion on a tubular section let us look into that if we have a section which is not uniform. So long we discussed the effect of torsion on a bar which is uniform, whether the shaft is solid or hollow the bar was uniform. Now if we have a non uniform bar, where over the length of the bar the diameter varies and if it is subjected to a twisting moment at one point, or if we have the twisting moment acting at different point, at different magnitudes then what is the consequence of that twisting moment over the bar? Or how to compute the stresses and the deformation in such a bar where the bar is not uniform or where there is variation in the twisting moment?

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Let us take a bar where the length is varying and we have the change in diameter. Let us call this diameter as  $d_1$ , here this is  $d_2$  where this is varying. Let us assume, this is acted on by various twisting moment at different points. Let us call this bar as A, B, C, and D. Now at these points A, B, C, and D different twisting moments  $T_1, T_2, T_3$  and  $T_4$  are acting. When there is twisting moment acting on A, if we draw the free body diagram between A and B, the twisting moment  $T_2$  is acting at the point B. Now between A and B we have only the twisting moment  $T_1$  acting.

If I take a section here and draw the free body diagram between A and B, then we have the part AB, this is just before B, before the application of  $T_2$ . So we have twisting moment acting on this and so far as the application is concerned this is positive twisting moment. As we have defined here, we have the twisting moment  $T_1$  acting into it which will be opposing this. So this is  $T_1$  over the length AB. If we say length AB is  $L_1$ , we can compute the value of the stress from our relationship  $\frac{\tau}{\rho} = \frac{T}{J}$  over the length L. This gives us a stress  $\tau = \frac{T\rho}{J}$  and this T here equals  $T_1$  for the present case. Over the length AB this will be the stress.

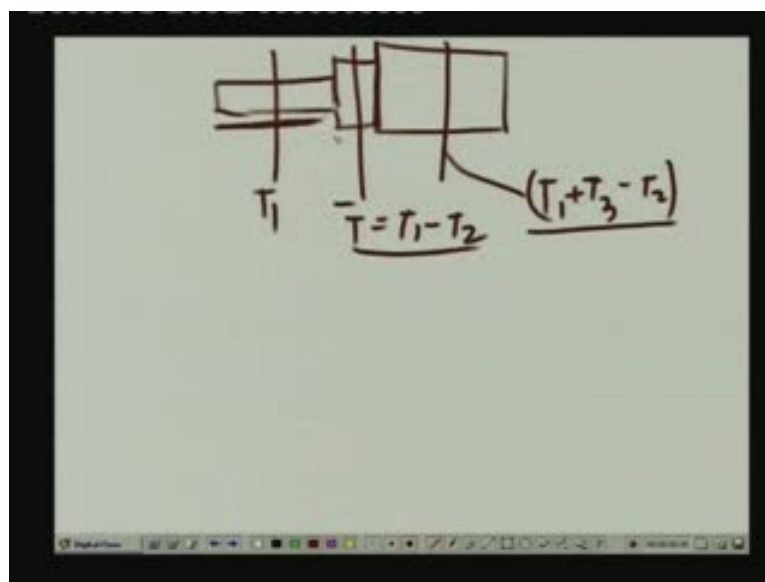
If we consider a segment between B and C and if we cut off then the twisting moment that will be acting here is  $T_1$ , here we have  $T_2$  as per our notation  $T_1$  is positive and  $T_2$  is negative. If we call the resulting twisting moment as T, T is  $T_1 - T_2$ . So for this resulting twisting moment T that will be acting between B and C, we compute the stress again over the length BC. This is the twisting moment T which equals  $T_1 - T_2$  into rho by J, assuming uniform diameter here. We will discuss this aspect at a later time when it is varying. If we have a uniform diameter over that particular stretch between B and C then the stress is the resulting twisting moment which is  $\frac{(T_1 - T_2)\rho}{J}$ .

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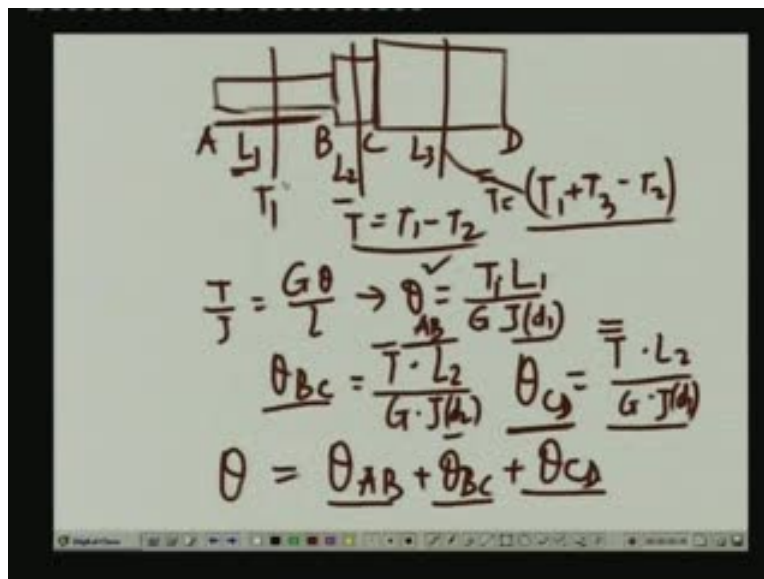
Now, if we take a section between C and D, if we cut off over here then we have  $T_1$ ,  $T_2$  and  $T_3$ .  $T_1$  and  $T_3$  are acting in the same direction and  $T_2$  is in the opposite direction, so the resulting  $T$  equals  $T_1 + T_3 - T_2$ . Since this is in the anticlockwise direction which is positive and we are considering  $T$  also as anticlockwise and positive, so this will be the whole thing and will be negative. So, if this expression becomes negative then it will be in the opposite direction we have assumed and if the whole expression becomes positive then the assumed direction will be same as we have assumed. So this is how we compute the resulting twisting moment and once we have this resulting twisting moment we compute the stress using this expression.

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When we come to the evaluation of the rotation, as you can see we have three different segments over which we got to compute the equivalent of  $T$ , here we have  $T_1$ , here we have  $\bar{T} = T_1 - T_2$  and at this point we have  $T_1 + T_3 - T_2$ . So these are the resulting twisting moments. These twisting moments which are acting over different lengths will have different rotations. Over the first segment where  $T_1$  is acting over length AB, it will undergo a twist. Between B and C there is the resulting effect of the twisting moment  $T_1$  and  $T_2$ . Now  $T_1$  is acting in an anti-clockwise direction and is trying to rotate the shaft in one direction whereas  $T_2$  is acting in the other direction where is trying to nullify this rotation to certain extent so there will be change in the rotation.

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So what we do over the different segments A, B, C, and D. Let us call length AB as  $L_1$ ; length BC as  $L_2$ ; and length CD as  $L_3$ . Now over length  $L_1$  we compute theta, as you know that  $\frac{T}{J} = \frac{G\theta}{L}$  or we can write  $\theta = \frac{TL}{GJ}$ . Now for the first segment AB  $\theta_{AB} = \frac{T_1 L_1}{GJ}$  and  $J$  also will be corresponding to the diameter. Let us call this as  $d_1$  so we will get the value of  $\theta_{AB}$ . Then we compute  $\theta_{BC}$  considering the resulting twisting moment times the length which is  $L_2$ ,  $G$  and  $J$  considering the diameter  $d_2$  over BC. We evaluate  $\theta_{CD}$  from the resulting twisting moment which is  $T = T_1 + T_3 - T_2$  into  $L_3$  and  $GJ$  by  $d_3$ . Now once we compute these three thetas  $\theta_{AB}$ ,  $\theta_{BC}$  and  $\theta_{CD}$ . The final theta will be the sum of these with its appropriate signs that is  $\theta_{AB} + \theta_{BC} + \theta_{CD}$ .

When I am calling this appropriate sign it means that if the twisting moment is in a positive direction it is causing a positive rotation. If the twisting moment is in the negative direction it is causing a negative rotation. So, if at different segments we consider these three different rotations with appropriate signs and sum them together then we will know the net rotation that

the shaft is undergoing which is of varying diameter and varying length between these two points. This is what our contention is to evaluate.

If we look into the evaluation of stress and the deformation because of this non-uniform distribution either in the material diameter wise or change in the twisting values, so far as the stresses are concerned when you compute the stresses for different segments the maximum stress out of all these segments which will give the maximum value will be the governing stress. And rotation-wise between the two ends we will sum them up and the final rotation will be the rotation in the shaft which will be governing. So these are the two aspects which are little different than if the body is uniform and subjected to uniform twisting moment. Here are some examples where we apply some these theories.

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### Example Problem - 1

- A solid aluminum bar of length 1.2m and 25mm diameter is twisted by torques  $T$  acting at the ends. The shear modulus  $G=0.3 \times 10^5$  MPa. (a) Determine the torsional stiffness of the bar; (b) If the angle of twist of the bar is  $5^\circ$ , what is the maximum shear stress?

Diagram: A horizontal bar of length 1.2m with torques  $T$  applied at both ends.

A solid aluminum bar of length 1.2m and 25 mm diameter, is twisted by torques  $T$ , acting at the ends. This means you have a solid aluminum bar which is subjected to a twisting moment and this is our positive twisting moment notation that means we are giving the vectorial notation over here. The shear modulus is given. What you will have to find out is the torsional stiffness of the bar. What will be the torsional stiffness when it is subjected to a twisting moment  $T$ ?

As we have defined the torsional stiffness is the twisting moment required to produce unit rotation and the value is  $\frac{GJ}{L}$ . We have to compute this quantity so that we know the torsional stiffness.

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$$\frac{T}{J} = \frac{G\theta}{L} \rightarrow T = \left(\frac{GJ}{L}\right)\theta$$
$$\frac{GJ}{L} = \frac{0.3 \times 10^5 \times \pi \times 25^4}{32 \times 1200} = \frac{\pi \times 25^4}{32}$$
$$= 958.74 \text{ N-m}$$

Unit analysis:  $\frac{N/mm^2 \times mm^4}{mm} = N-m$


We know that  $\frac{T}{J} = \frac{G\theta}{L}$  and  $T = \frac{GJ}{L}\theta$ . Now as per our definition theta is unit, so the stiffness is  $\frac{GJ}{L}$ . This equals to the value of G given as  $0.3 \times 10^5$ . Now the value of J is the polar moment of

inertia and the diameter is 25 mm. So  $J = \frac{\pi}{32}d^4$  is equal to  $\frac{\pi}{32}25^4 \text{ mm}^4$ . So this is the value of J.

So GJ by L is equal to  $0.3 \times 10^5 \times \frac{\pi}{32}25^4$  and L is equal to 1.2m, so this is 1200 mm and this

gives us a value of 958.74 Newton meter. Now the unit of G is MPa which is Newton per  $mm^2$ , J is  $mm^4$  by L is in mm and the net unit is Newton millimeter. This is what is in Newton meter so this is the torsional stiffness of the bar which is subjected to a twisting moment.

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### Example Problem - 1

- A solid aluminum bar of length 1.2m and 25mm diameter is twisted by torques T acting at the ends. The shear modulus  $G=0.3 \times 10^5$  MPa. (a) Determine the torsional stiffness of the bar; (b) If the angle of twist of the bar is  $5^\circ$ , what is the maximum shear stress?

The next step is to calculate that if the angle of twist of the bar is  $5^\circ$ , if the angle of twist of the bar is limited to  $5^\circ$  then what is the maximum shear stress?

Now when the same bar is rotated and if the angle of twist between the length is restricted to  $5^\circ$  then what will be the maximum shear stress that will be generated on the surface of the bar.

We know that  $\frac{T}{J} = \frac{\tau}{\rho} = \frac{G\theta}{L}$ . So the stress  $\tau = \frac{T\rho}{J}$  and this also equals  $\frac{G\rho\theta}{L}$ . If we equate these

two, we get the shearing stress  $\tau = \frac{G\rho\theta}{L}$  and G is equal to  $0.3 \times 10^5$ , rho is  $\frac{25}{2}$  and theta is  $5^\circ$ . So

in terms of radians we write this as  $\frac{5\pi}{180}$  divided by L which is 1200. So this

equals  $\frac{0.3 \times 10^5 \times 25 \times 5 \times \pi}{2 \times 1200 \times 180}$  MPa or Newton per mm square is the shearing stress tau. By

simplifying this we get tau is equal to 27.27 MPa.

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The image shows handwritten calculations on a whiteboard. At the top, the formula for shear stress is given as  $\tau = \frac{T \cdot r}{J}$ , which is also written as  $\tau = \frac{G \theta}{L}$ . Below this, the shear stress is calculated as  $\tau = \frac{T \cdot r}{J} = \frac{G \theta r}{L}$ . The values are substituted as  $\tau = \frac{0.3 \times 10^5 \times 25 \times 5 \pi / 180}{2 \times 1200 \times 180}$  MPa. The final result is  $\tau = 27.27$  MPa.

So here the value of shear modulus is given, the radius  $\rho$  is given which is  $\frac{d}{2}$ ,  $\theta$  is limited to  $5^\circ$  and length of the member is known. So we have computed the stress in terms of this limiting value and the maximum stress that will be generated on the surface equals to 27.27 MPa.

(Refer Slide Time: 47:09)


The slide is from IIT Kharagpur and is titled "Example Problem - 2". It contains the following text:

• A hollow aluminum tube used in a roof structure has an outside diameter of 100 mm and an inside diameter of 80 mm. The tube is 2.5m long.  $G=28\text{GPa}$ . (a) If the tube is twisted in pure torsion by torques acting at the ends, what is the angle of twist when the maximum shear stress is 50 MPa? (b) What diameter is required for a solid shaft to resist the same torque with the same maximum stress? (c) What is the ratio of the weight of the hollow tube to the weight of the solid tube?

Let us look into another example. In this example, we have a hollow aluminum tube. It is a tubular section, so we can make a comparison between solid section and a tubular section.



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### Example Problem - 2

- A hollow aluminum tube used in a roof structure has an outside diameter of 100 mm and an inside diameter of 80 mm. The tube is 2.5m long.  $G=28\text{GPa}$ . (a) If the tube is twisted in pure torsion by torques acting at the ends, what is the angle of twist when the maximum shear stress is 50 MPa? (b) What diameter is required for a solid shaft to resist the same torque with the same maximum stress? (c) What is the ratio of the weight of the hollow tube to the weight of the solid tube?

A hollow aluminum tube used in a roof structure has an outside diameter of 100 mm, and inside diameter of 80 mm. The tube is 2.5m long and the shear modulus is given. If the tube is twisted in pure torsion by torques acting at the ends, then what is the angle of twist when the maximum shear stress is 50 MPa? We will have to find out the angle of twist when the maximum shear stress is limited to 50 MPa?

Secondly, what diameter is required for a solid shaft to resist the same torque with the same maximum stress so that we can make a comparison between the two? What is the ratio of the weight of the hollow tube to the solid tube?

So let us look into this particular example so that we can make a comparison between the two.

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Handwritten calculations on a whiteboard:

$$\begin{aligned}d_{out} &= 100 \text{ mm} & d_{in} &= 80 \text{ mm} \\L &= 2.5 \text{ m} & G &= 28 \text{ GPa} \\ \checkmark \quad \tau &= 50 \text{ MPa} & \theta &= ? \\ \frac{\tau}{\rho} &= \frac{G\theta}{L} \rightarrow \theta = \frac{\tau \cdot L}{G \cdot \rho} \\ &= \frac{50 \times 2500}{28 \times 10^3 \times 50} = 0.0898 \text{ rad} \\ &= 0.0898 \times \frac{180}{\pi} \\ &= \underline{5.12^\circ}\end{aligned}$$

Now, the outside diameter  $d_{out}$  is equal to 100 mm and  $d_{in}$  is equal to 80 mm;  $L$  the length of the member is 2.5m;  $G$  given as 28 GPa;  $\tau$  is to be limited to 50 MPa and corresponding to that we will have to find out the rotation. So now that we know that  $\frac{\tau}{\rho} = \frac{G\theta}{L}$ . We are taking this relationship because shear stress is limited we will have to find out theta. So this gives us  $\theta = \frac{L\tau}{G\rho}$  and if we substitute the values  $\tau$  is limited to 50 MPa, length of the member is 2500 mm,  $G$  is 28GPa into 10 cube. So  $\rho$  is the maximum shear stress that will occur on the outside phase and the outside diameter is 100 so the  $\rho$  here is 50. This gives us a value of theta is equal to  $0.0898 \text{ radians}$  and when we compute theta in terms of degree this is equal to  $0.0898 \times \frac{180}{\pi}$ . This will give us degree which is  $5.12^\circ$ . This is the maximum rotation that we can expect when we apply the twisting moment and limit our stress to 50 MPa.

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Handwritten derivation on a whiteboard:

$$T = \frac{\tau \cdot J}{\rho}$$

$$J = \frac{\pi}{32} (d_2^4 - d_1^4)$$

$$= 5.8 \times 10^6 \text{ mm}^4 \checkmark$$

$$T = \frac{50 \times 5.8 \times 10^6}{50}$$

$$= 5.8 \times 10^6 \text{ N-mm}$$

$$\frac{T}{J} = \frac{\tau}{\rho} \rightarrow T = \frac{\tau \cdot J}{\rho}$$

$$T = \frac{\tau \cdot \pi d^4}{32 \times d/2} = \frac{\tau \pi d^3}{16}$$

What will be the diameter if we go for a solid section instead of a hollow one?

Now for that part we need to compute the value of twisting moment  $\frac{T}{J} = \frac{\tau}{\rho}$ . We know that  $\frac{\tau J}{\rho}$

will give us the value of the twisting moment and J for this section is  $\frac{\pi}{32} (d_2^4 - d_1^4)$  which is

$5.8 \times 10^6 \text{ mm}^4$ . If you compute the value of T, tau we know which is 50MPa and rho is the maximum outer diameter corresponding to the maximum stress and if we substitute we get:

$T = \frac{50 \times 5.8 \times 10^6}{50}$  is equal to  $5.8 \times 10^6$  N-mm, which is the twisting moment. We have to find

the diameter corresponding to this twisting moment from  $\frac{T}{J} = \frac{\tau}{\rho}$  and from this if you write

that  $T = \frac{\tau J}{\rho}$ . Here the J is  $\frac{\pi}{32} d^4$  and rho is  $\frac{d}{2}$ . So this gives us  $T = \frac{\tau \pi}{16} d^3$ . So this is the value of T

in terms of the stress. We need to compute is the value of diameter d from this expression.

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The image shows handwritten calculations on a whiteboard. At the top, the equation  $d^3 = \frac{16T}{\tau \cdot \pi} = \frac{16 \times 5.8 \times 10^6}{50 \times \pi}$  is written. Below it, the result  $d = 84 \text{ mm.}$  is boxed. To the right, there are two circular diagrams: the top one is a hollow shaft with an outer diameter of 100 mm and an inner diameter of 80 mm; the bottom one is a solid shaft with a diameter of 84 mm. Below the diagrams, the relationship  $\frac{T}{J} = \frac{\tau}{\rho} \rightarrow T = \frac{\tau \cdot J}{\rho}$  is written, followed by  $T = \frac{\tau \cdot \pi d^4}{32 \times d/2} = \frac{\tau d^3}{16}$ .

Now  $d^3 = \frac{16T}{\tau \pi}$  is equal to  $\frac{16 \times 5.8 \times 10^6}{50 \times \pi}$ . From this we compute  $d$  is equal to 84 mm. This is the diameter which we get for a solid shaft or a solid bar. Tubular bar has outer diameter of 100 mm and inner diameter of 80 mm as opposed to a solid section which is 84 mm diameter. Now if we like to compare the weight of these two elements, the ratio between the weight of hollow and the solid will be in the ratio of their radii, because if we take per unit length and if we take the same material then naturally the unit weight is the same so the ratio of the area will give their weight ratio.

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The image shows handwritten calculations on a whiteboard. The text reads: "Ratio of wt. =  $\frac{A_{\text{tube}}}{A_{\text{solid}}}$ ". Below this, the calculation is shown as  $= \frac{\pi/4 (100^2 - 80^2)}{\pi/4 (84)^2}$ , which simplifies to  $= \underline{\underline{0.51}}$ .

So to find the ratio of their weights, this is equal to the area of the tube divided by the area of

solid is equal to  $\frac{\pi}{4}(100^2 - 80^2)$  is equal to 0.51. So you see that for resisting the same amount of stress under the same twisting moment we have the area of the hollow cross section which is 50% of the solid area. So we are making savings in terms of the material and this is more effective when you are resisting the twisting moment or applying the twisting moment.

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### Example Problem - 3

- A stepped shaft ABCD consisting of solid circular segments is subjected to three torques as shown in figure.  $G=80$  GPa. (a) Calculate maximum shear stress in the shaft; (b) Calculate the angle of twist at D.

We have another example. A stepped shaft ABCD consisting of solid circular segments is subjected to three torques as shown and the value of  $G$  is provided. We will have to calculate the maximum shear stress in the shaft and calculate the angle of twist at D.


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### Example Problem - 4

- A solid circular bar **ABC** consists of two segments as shown in figure. What is the allowable torque if the **shear stress is not to exceed 32 MPa** and the **angle of twist between the ends of the bar is not to exceed  $1.0^\circ$** ?  $G=100 \text{ GPa}$ .




55:05

We have another example which is a solid circular bar ABC consisting of two segments as shown. What is the allowable torque in the entire shaft if the shear stress is not to exceed 32 MPa and the angle of twist between the ends of the bar not to exceed  $1^\circ$ ?

Here two criteria are set that the stress cannot exceed 32 MPa and the total angle of twist cannot exceed  $1^\circ$ . Here you have two different diameters of the shaft but it is subjected to one twisting moment. You will have to find out what is the allowable torque that you can apply on this particular shaft if the stress and the rotation are limited?

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### Summary

This lesson included:

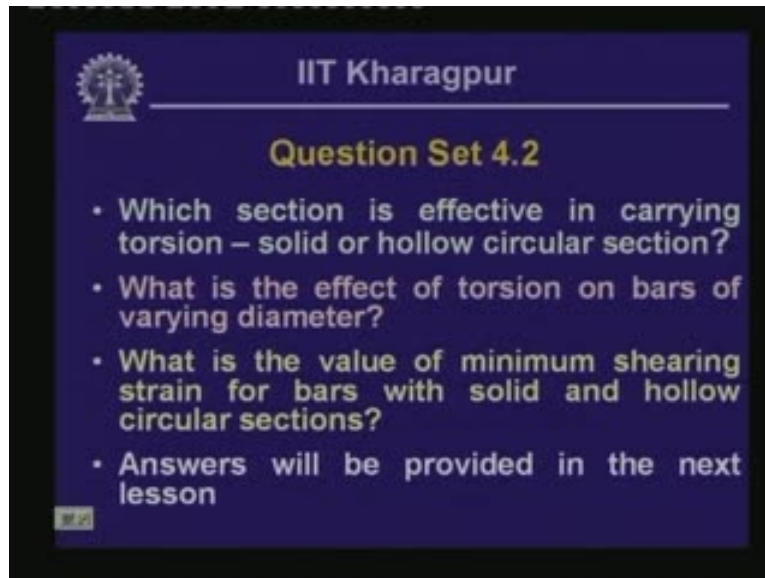
- Concept of torsion in a bar of hollow circular cross section.
- Concept of stresses and deformation in a bar of circular cross section due to non-uniform torsion.
- Examples to evaluate stresses, strains and deformation in bars of circular cross section due to torsion.

55:55

Summary of this lesson:

We discussed the concept of torsion in a bar of hollow circular cross section. In the previous lesson we discussed about a bar having a solid cross section. In this particular lesson we discussed the effect of torsion on a bar where the cross section is a tubular one and not a solid one. We discussed concept of stresses and deformation in a bar of circular cross section due to non-uniform torsion. We also looked at some examples to evaluate stresses strains and deformation in bars of circular cross section due to torsion.

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Questions:

1. Which section is effective in carrying torsion; whether a solid one or hollow circular one?
2. What is the effect of torsion on bars of varying diameter?
3. What is the value of minimum shearing strain for bars with solid and hollow circular sections?