Engineering Hydrology

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Module 5
Lecture 61

Derivation of IUH using Nash's Model

Hello all. Welcome back. In the previous lecture, we were discussing about instantaneous unit hydrograph. We have seen how an instantaneous unit hydrograph can be derived by making use of S hydrograph. So, today we will look into another technique, which can be utilized for deriving instantaneous unit hydrograph.

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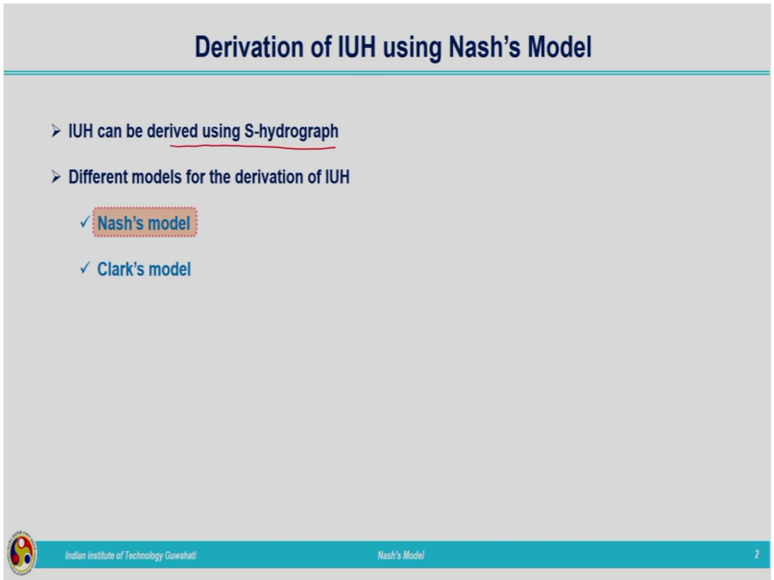

We have already seen IUH can be derived using S hydrograph. IUH is the response of a catchment when it is acted upon by a unit impulse input at the time *t* is equal to 0. So, that response of the catchment is termed this impulse response function and that is termed as the instantaneous unit hydrograph. As we have seen the interrelationships between the different response functions, if unit hydrograph is available to us we can derive the IUH. If S hydrograph is available to us we can derive the IUH. In addition to that few searches in literature you can see there are several methods for the derivation of instantaneous unit hydrographs. So, commonly, two techniques are used first one is Nash model and the second one is Clark model.

Here in this lecture I am going to discuss about Nash model. This is a simple model and this can be utilized for the derivation of instantaneous unit hydrograph. Why do we want to have instantaneous unit hydrograph? In that case, we do not have to bother about the duration. Otherwise, for deriving the unit hydrograph or direct runoff hydrograph we need to have a specific effective rainfall having a duration *D* and corresponding to that we need to have the streamflow records, but always it is not easy to get a uniform rainfall for a duration of *D*. In such cases, if the duration is not having any importance that will be better. So, if we are having the instantaneous unit hydrograph with us we can derive the unit hydrographs and corresponding direct runoff hydrographs from any duration effective rainfall. So, here I am going to discuss about a technique or a model which is known as Nash model which can be utilized for deriving instantaneous unit hydrograph.

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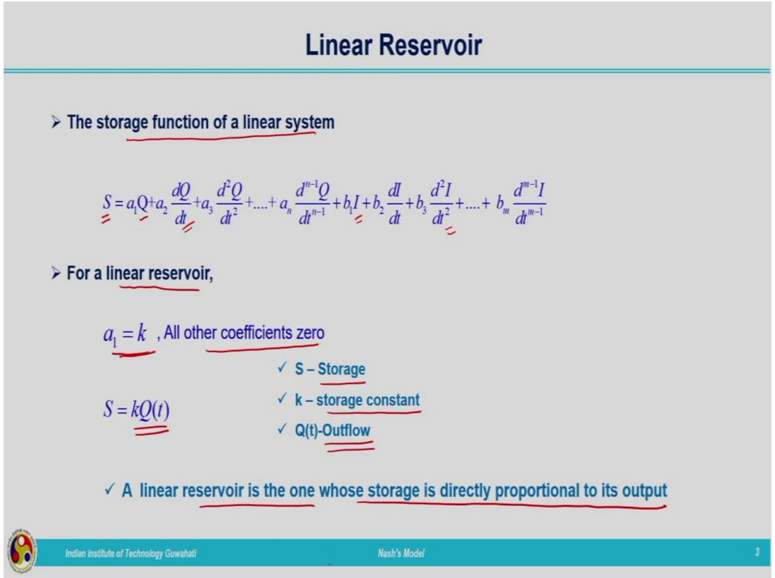

So, for that we need to go back to our basics on linear system. We have discussed in detail about Chow and Kaulandaiswamy model. In that we have seen the storage function of a linear system can be expressed as a function of outflow, inflow, and derivatives of outflows and inflows, that is,

$$S = a_1Q + a_2\frac{dQ}{dt} + a_3\frac{d^2Q}{dt^2} + ... + a_n\frac{d^{n-1}Q}{dt^{n-1}} + b_1I + b_2\frac{dI}{dt} + b_3\frac{d^2I}{dt^2} + ... + b_m\frac{d^{m-1}I}{dt^{m-1}}$$

Depending on the characteristics of the system, this storage function will be having some of the terms presented over here in the storage function. For simplifying the equation, we will be neglecting certain terms based on certain assumptions. In that way we can assume a concept termed as linear reservoir. For a linear reservoir, the coefficient $a_1 = k$ that is here we are having S is equal to $a_1 Q + a_2 \frac{dQ}{dt} + ...$ and other terms. So, in the storage function we are going to assume $a_1 = k$, all other coefficients are assumed to be 0. So, we can write our storage function *S* is equal to

$$S = kQ(t)$$

Where *S* is the storage function, *k* is storage coefficient and *Q*(t) is the outflow, all these parameters that is *I(t), Q(t), S* all are functions of time. And here what we are assuming, we are having a storage function which can be considered as proportional to the outflow or output that is *Q*. So, *S* is directly proportional to *Q* is the assumption which we are making in the case of linear reservoir that is why the proportionality coefficient is replaced by a coefficient $a_1 = k$, we can write $S = kQ(t)$. So, a linear reservoir is the one whose storage is directly proportional to its output, $S = kQ(t)$ that means *k* is a coefficient, so *S* can be written as directly proportional to outflow *Q*. So, in case of any problem related to linear reservoir even if the condition is not given to you, you can assume that the storage is equal to *kQ*, *k* is the storage coefficient and *Q* is the outflow.

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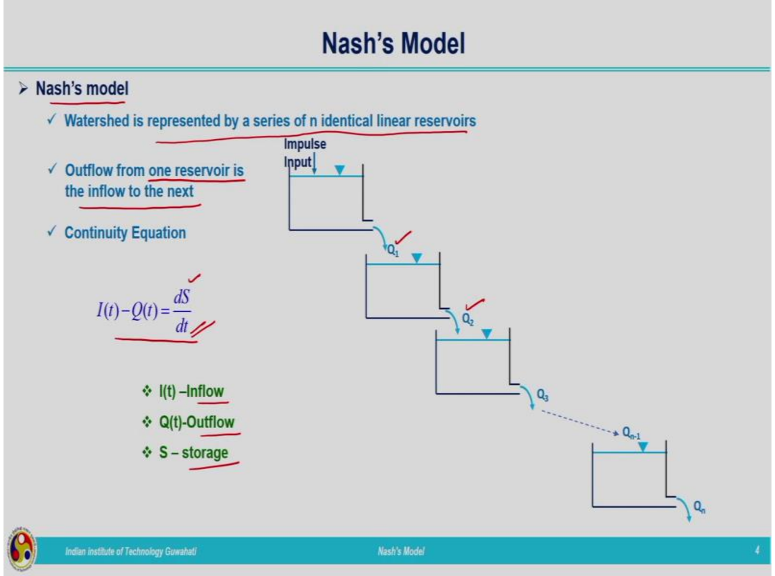

Now, we are moving on to Nash model. According to Nash, the catchment is assumed as consisting of a series of *n* number of linear reservoirs. So, the watershed is represented by a series of *n* identical linear reservoirs and outflow from one reservoir is the inflow into the next one, that is if we are considering catchment as such consisting of *n* number of linear reservoirs. So, the reservoir which is at the upstream end or the upper part of the catchment will be producing an outflow that is the inflow to the next reservoir and that continues up to n^{th} reservoir.

We can schematically represent a catchment as a combination of *n* identical linear reservoir like this. Let this be the first reservoir. The water surface is marked over here and it is acted upon by an impulse input. Here in this case, we are going to derive the instantaneous unit hydrograph. What is instantaneous unit hydrograph? The system is acted upon by an impulse input instantaneously at time t is equal to τ and the response of the catchment is represented by the impulse response function, in hydrology we will be calling that impulse response function as instantaneous unit hydrograph. So, here in this case, we are going to consider the catchment is consisting of *n* number of linear reservoirs which are connected in series and the impulse input is acted upon at the first reservoir. So, the first reservoir is having an impulse input and the outflow from this first reservoir will be the input to this second reservoir. This will be continuing like this, that is Q_1 is the input to the second reservoir, Q_2 is the input to the third reservoir and Q_3 will be the input to the fourth reservoir that way we are having n number of linear reservoirs.

Why I am specifying linear reservoir? If we are making use of the principle of linear reservoir, we can assume $S = kQ(t)$, S is proportional to the outflow from the reservoir. So, this Q_3 goes to Q_4 and finally, we are having the n^{th} reservoir, n^{th} reservoir will be having an input Q_{n-1} which will be producing a response or output as Q_n. So, this is the schematic representation of a catchment for which we are assuming the catchment is consisting of series of n linear reservoirs.

So, this way we can assume that first reservoir is acted upon by the impulse input that is continuing like that, outflow from the first reservoir is the inflow to the second one that continues up to n^{th} reservoir, Q_{n-1} is the input to the n^{th} reservoir and the outflow from the n^{th} reservoir is Q_n. Q_n is nothing but the instantaneous unit hydrograph that is what we are going to derive today. So, we are going to make use of our continuity equation, this is very much familiar to you. Continuity equation is given by

$$I(t) - Q(t) = \frac{dS}{dt}$$

Here in this equation, $I(t)$ is the inflow, $Q(t)$ is outflow and S is the storage function. We know the storage function $S = kQ(t)$. So, here we are having the term corresponding to $\frac{dS}{dt}$. Now, we will go for differentiating the expression for S.

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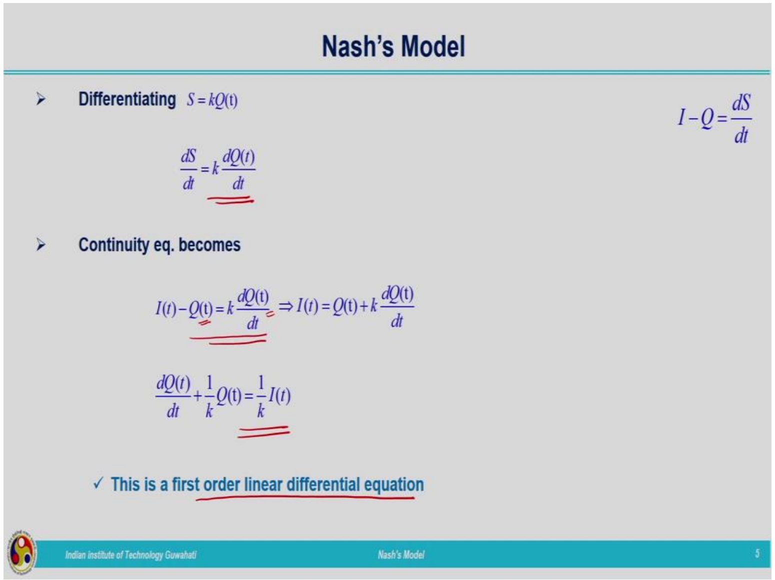

So, when we differentiate $S = kQ(t)$ we will get

$$\frac{dS}{dt} = k\frac{dQ(t)}{dt}$$

After this what we will do we will substitute this $\frac{dS}{dt}$ in our continuity equation, this is the procedure we will be using in any of the rainfall runoff model. Continuity equation is represented by

$$I(t) - Q(t) = \frac{dS}{dt}$$

The complexity is coming in the storage function which is represented by the transfer function. So, this *S* can be considered as linear, nonlinear depending upon the knowledge about the system, we can derive the mathematical expression related to that particular system. Here we are considering the storage function *S* is directly proportional to *Q* considering the system as a linear reservoir. So, the same procedure can be utilized in any of the rainfall runoff models for carrying out the hydraulic analysis.

So, $\frac{dS}{dt}$ we have found the expression, this we will be substituting in the continuity equation. So, our continuity equation becomes

$$I(t) - Q(\mathrm{t}) = k\frac{dQ(\mathrm{t})}{dt}$$

What is our aim? Our aim is to determine *Q* that is certain input is there that is acting on the catchment, some output will be there, for example, some amount of rainfall is occurring on the catchment, after considering all the hydrologic processes or some of the hydrologic processes, which will be taking place within the catchment, finally, we will be getting the runoff at the outlet, that runoff computation which we are intending for which is representing by *Q*. So, we need to compute the value corresponding to *Q*. What we will do? Similar terms that is here we are having *Q(t)* here we are having $\frac{dQ(\mathrm{t})}{dt}$, similar terms will be taken towards one side. So, we can write

$$I(t) = Q(\mathrm{t}) + k\frac{dQ(\mathrm{t})}{dt}$$

We will do some rearrangements with the terms

$$\frac{dQ(t)}{dt} + \frac{1}{k}Q(\mathrm{t}) = \frac{1}{k}I(t)$$

Now, we need to find out this solution of this differential equation, this is a first order linear differential equation. We have considered our continuity equation in that we have substituted the expression for $\frac{dS}{dt}$ time rate of change of storage is written in terms of derivative of the storage function. After that, we have rearranged the continuity equation for finding out the solution that is for finding out the outflow from the linear reservoir represented by *Q(t)*. So, the equation is now in the form of first order linear differential equation. You must be already knowing how to find out the solution for an ordinary linear differential equation. You will be already knowing different techniques for finding out the solution of an ODE.

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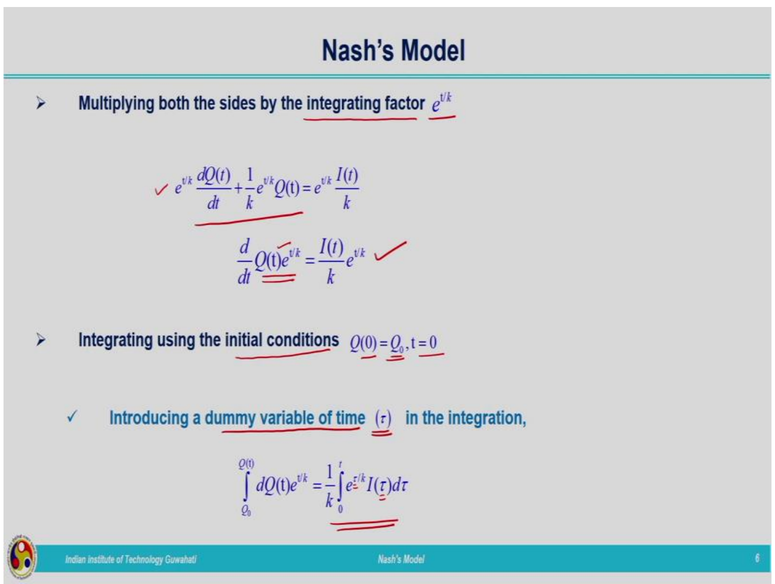

So, we will be multiplying both sides of the equation by integrating factor $e^{t/k}$. Then the equation takes the form like this,

$$e^{t/k}\frac{dQ(t)}{dt}+\frac{1}{k}e^{t/k}Q(\mathrm{t})=e^{t/k}\frac{I(t)}{k}$$

Just we have multiplied both the sides with $e^{t/k}$ for getting the solution easily. You can look at the left-hand side, left hand side is given by $e^{t/k}\frac{dQ(t)}{dt}+\frac{1}{k}e^{t/k}Q(\mathrm{t})$. So, the left-hand side can be written as

$$\frac{d}{dt}Q(\mathrm{t})e^{t/k}=\frac{I(t)}{k}e^{t/k}$$

You apply product rule for this expression and you will get this left-hand side. So,

$$\frac{d}{dt}Q(\mathrm{t})e^{t/k}=\frac{I(t)}{k}e^{t/k}$$

Now, for finding out the solution, we will integrate this equation. When integrating we need to substitute the limits. So, for that we are going to make use of the initial conditions $Q(0)=Q_0$ i.e., at time t is equal to 0 Q is equal to Q_0. So, we can integrate the above equation. Before integrating we are going to introduce a dummy variable of time, only for time variable we are substituting a dummy variable τ in the integration equation. So, the above equation can be modified like this

$$\int_{Q_0}^{Q(\mathrm{t})} dQ(\mathrm{t})e^{t/k}=\frac{1}{k}\int_0^t e^{\tau/k}I(\tau)d\tau$$

We have replaced t by τ. This is just a dummy variable of integration.

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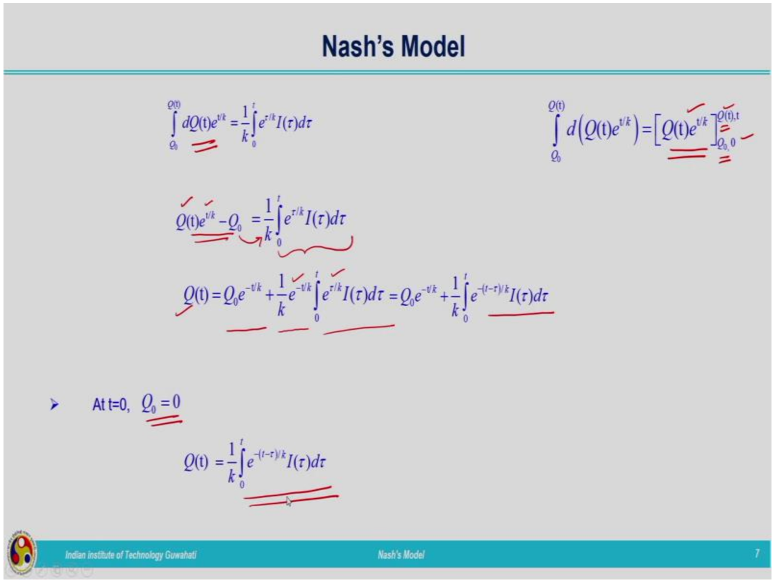

Now, you consider the left-hand side $\int_{Q_0}^{Q(t)} dQ(t)e^{t/k}$. So, when we integrate that it will be taking the form of

$$\int_{Q_0}^{Q(t)} d\left(Q(t)e^{t/k}\right) = \left[Q(t)e^{t/k}\right]_{Q_0,\,0}^{Q(t),t}$$

So, the left-hand side after integration it will be taking the form as shown above.

When we substitute these limits in this $Q(t)e^{t/k}$, upper limit if we substitute it will be $Q(t)e^{t/k}$ and lower limit if we substitute it will become $Q_0e^{\frac{0}{k}}$ i.e., Q_0e^0 which is equal to Q_0. So, the left-hand side will be taking the form $Q(t)e^{t/k} - Q_0$ that is equal to

$$Q(t)e^{t/k} - Q_0 = \frac{1}{k}\int_0^t e^{\tau/k} I(\tau)d\tau$$

Now, what we will do our intention is to find out *Q(t)*, we will take Q_0 to the right-hand side and also $e^{t/k}$ to the right hand side. So, *Q(t)* will be taking the form

$$Q(\mathrm{t}) = Q_0 e^{-\mathrm{t}/k} + \frac{1}{k} e^{-\mathrm{t}/k} \int_0^t e^{\tau/k} I(\tau) d\tau$$

Here you can see we were having $e^{\mathrm{t}/k}$ on the left-hand side. First what we are doing we are taking *Q_0* to the right-hand side, so, *Q_0* plus this term has come and then we are taking $e^{\mathrm{t}/k}$ to the right-hand side then it will be taking the form $e^{-\mathrm{t}/k}$. So, that way we have written the expression for *Q(t)*. So, the above equation can be modified again as

$$Q(t) = Q_0 e^{-\mathrm{t}/k} + \frac{1}{k} \int_0^t e^{-(t-\tau)/k} I(\tau) d\tau$$

These two terms $e^{-\mathrm{t}/k}, e^{-\tau/k}$, so these two combine together to form a single term and we got the expression like this. Now, at time $t = 0; Q_0 = 0$. For a system which is starting from rest initially $Q_0 = 0$, after that some impulses are acted upon that and the response is coming. So, at time $t = 0; Q_0 = 0$, that is the case with a system which is starting from rest. But in some other case, it may not be at rest initially. In that case what is the value corresponding to *Q_0* that has to be substituted. Here, we are considering $Q_0 = 0$ at $t = 0$. So, our *Q(t)* will be taking the form

$$Q(t) = \frac{1}{k} \int_0^t e^{-(t-\tau)/k} I(\tau) d\tau$$

This is the expression for Q(t). Here, we are considering an impulse input.

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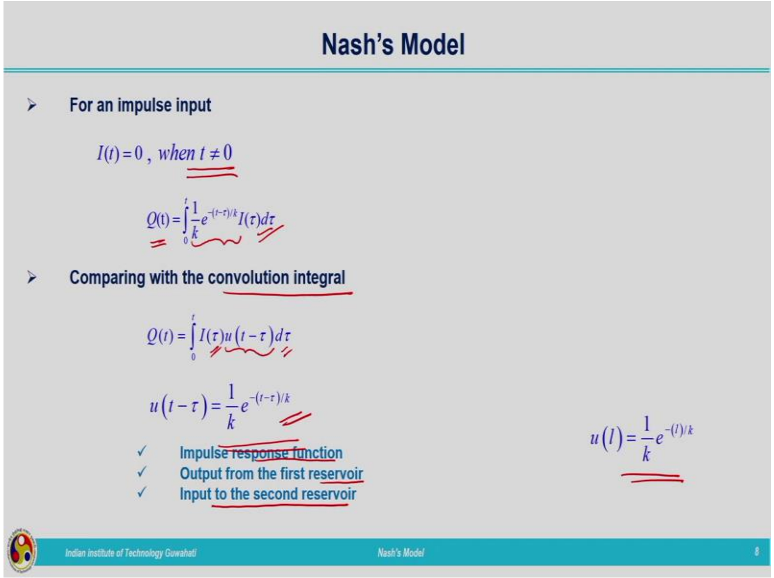

For an impulse input $I(t)=0$ when $t \neq 0$, what is the characteristics of unit impulse input? An impulse input which is having a value of unity is acted on the system at time *t* is equal to 0 instantaneously, at all other times it will be equal to 0, that is what is written over here $I(t)=0$ when $t \neq 0$, $I(t) \neq 0$ when $t=0$.

Now, let us look into our equation for *Q(t)* which we have derived. *Q(t)* is equal to

$$Q(\mathrm{t})=\int_0^t \frac{1}{k}e^{-(t-\tau)/k}I(\tau)d\tau$$

We are having the convolution integral which is the response of a system from a continuous pulses of impulse inputs. So, we will compare the output *Q(t)* of t with the convolution integral, i.e.,

$$Q(t)=\int_0^t I(\tau)u(t-\tau)d\tau$$

So, you can find out that

$$u(t-\tau)=\frac{1}{k}e^{-(t-\tau)/k}$$

Compare these two equation. We are having $I(\tau)d\tau$ and here the extra term coming is $u(t-\tau)$ that is nothing but our impulse response function. Here, instead of $u(t-\tau)$ we are having the value corresponding to $\frac{1}{k}e^{-(t-\tau)/k}$. So, we can write

$$u(t-\tau)=\frac{1}{k}e^{-(t-\tau)/k}$$

Here $u(t-\tau)$ is the impulse response function. A system is acted upon by an impulse input at time tau the response from the system after a time τ that is at t minus tau is given by $u(t-\tau)$, that is nothing but our impulse response function represented by the output from the first reservoir. So, this is the output from the first reservoir. According to Nash model, this output is the input to the second reservoir. So, this expression can be simplified in this form

$$u(l)=\frac{1}{k}e^{-(l)/k}$$

So, this is the outflow from the first reservoir. So, that can be taken as the inflow to the second reservoir.

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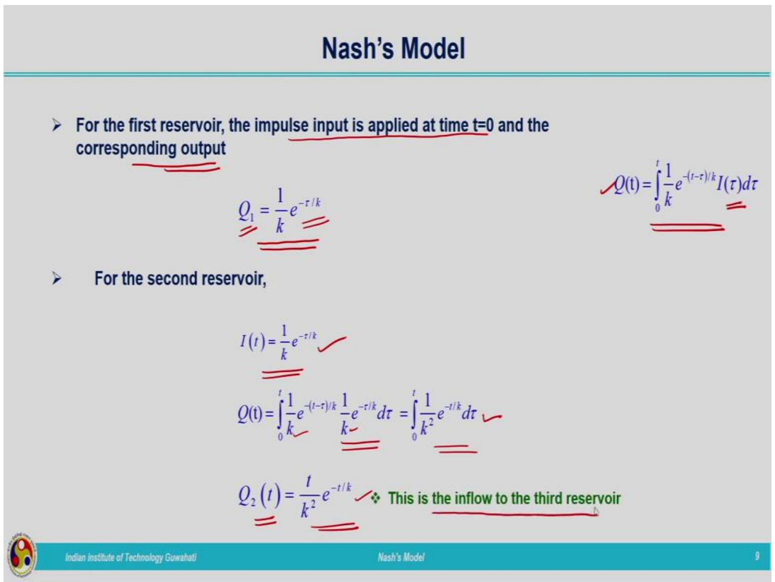

That is what we are going to do. This is our output expression

$$Q(\mathrm{t}) = \int_0^t \frac{1}{k} e^{-(t-\tau)/k} I(\tau) d\tau$$

and we can write for the first reservoir the impulse input is applied at time *t* is equal to 0 and the corresponding output will be the input to the second reservoir. So, in terms of τ itself, we are writing Q_1 is equal to, that is we are not making use of the terminology *u* here. Output from the first reservoir is written as Q_1, Q_1 is equal to

$$Q_1 = \frac{1}{k} e^{-\tau/k}$$

As the variable of time instead of *l* we are substituting τ itself, any variable can be substituted over here, but we were comfortable with the usage of τ that is why I put

$$Q_1 = \frac{1}{k} e^{-\tau/k}$$

This is our input to the second reservoir that is *I(t)* for the second reservoir is

$$I(t) = \frac{1}{k} e^{-\tau/k}$$

Now, what we will do we will substitute for $I(t)$, in the case of second reservoir.

So, *Q(t)* is equal to

$$Q(\mathrm{t}) = \int_0^t \frac{1}{k} e^{-(t-\tau)/k} \frac{1}{k} e^{-\tau/k} d\tau$$

So, this can be simplified as

$$Q(\mathrm{t}) = \int_0^t \frac{1}{k^2} e^{-t/k} d\tau$$

So, we can write with the notation 2 that is the output from the second reservoir $Q_2(t)$ will be

$$Q_2(t) = \frac{t}{k^2} e^{-t/k}$$

So, the expression for Q(t) when you integrate, you will get the expression for $Q_2(t)$. Now, this is the outflow from the second reservoir that is the inflow to the third one. So, this is the inflow to the third reservoir.

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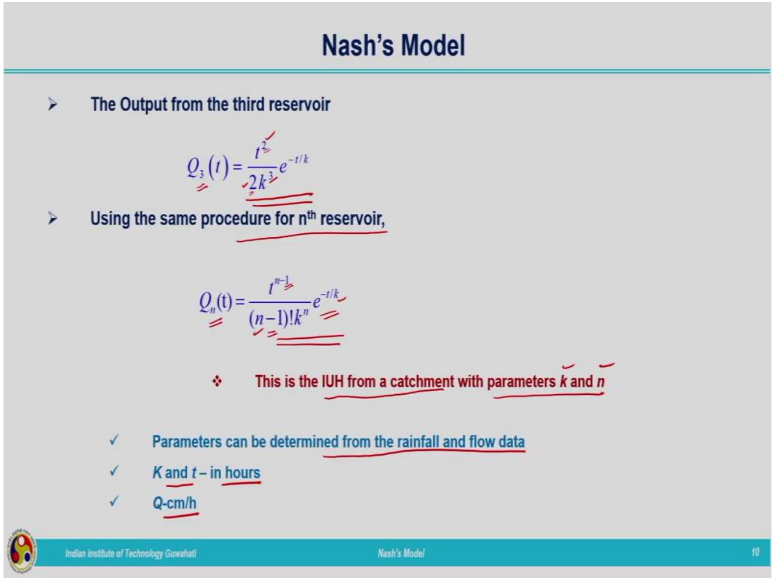

So, the output from the third reservoir can be written as $Q_3(t)$ is equal to

$$Q_3(t) = \frac{t^2}{2k^3} e^{-t/k}$$

Step by step when you do the integration you can get this expression. When it goes to the fourth reservoir this term will be 3 and this will be 3 and k to the power of 4 will be coming. So, using the same procedure, we will go up to n^{th} reservoir, we can write $Q_n(t)$ as

$$Q_n(t) = \frac{t^{n-1}}{(n-1)!k^n} e^{-t/k}$$

So, compare these two equations for understanding more $Q_3(t)$, this is the output from the third reservoir. Here, we are talking about the output from the $^{-th}$ reservoir that is $Q_n(t)$ and t to the power of 2 that is for the third reservoir it is 2. Here, definitely it will be t to the power of n minus 1. And here 2, in the denominator we are having 2, next term it will be 2 into 3, 2 into 3 into 4 that way it goes on up to $n-1$ when we reach the n^{th} reservoir. So, this will be $(n-1)!$, k^n,

k to the power of 3 will be replaced by k^n, $e^{-t/k}$. So, this is the output from the n^{th} reservoir. So, this is nothing by the instantaneous unit hydrograph from a catchment with parameters k and n. Here, k is representing the storage coefficient or reservoir coefficient and n is the number of reservoirs which we are considering. So, these two values will be varying depending on the characteristics of the catchment. The parameters can be determined from the rainfall and flow data. k and t are in hours and Q is in centimeters per hour, because we were discussing about 1-centimeter rainfall instantaneously acted in the case of an impulse input. So, this expression is giving us Q in the unit of centimeters per hour. So, if you want to get the Q in meter cube per second you need to make suitable conversions.

So, the outflow from the n^{th} reservoir we have computed and that is the response of the system. So, initially how we have considered, we have considered the catchment consisting of n number of series of reservoirs, outflow from first one is the inflow to the second one, that way we have proceeded and we have made use of the continuity equation and the concept of linear reservoir $S = kQ(t)$ for deriving this equation.

So, for the derivation of IUH we can make use of this formula Q_n and that IUH can be utilized for deriving the unit hydrograph and the corresponding direct runoff hydrographs from different effective rainfalls. So, here there is two things to be known, that is determination of n and k. So, these parameters n and k can be determined by making use of the effective rainfall hydrograph and the direct runoff hydrograph so that I am not discussing in this lecture, for that we need to make use of some probabilistic concepts. So, as of now, how the IUH can be derived and what is the IUH ordinate that is represented by Q_n that we have found out by making use of this expression given n and k, you can determine the instantaneous unit hydrograph for a particular catchment.

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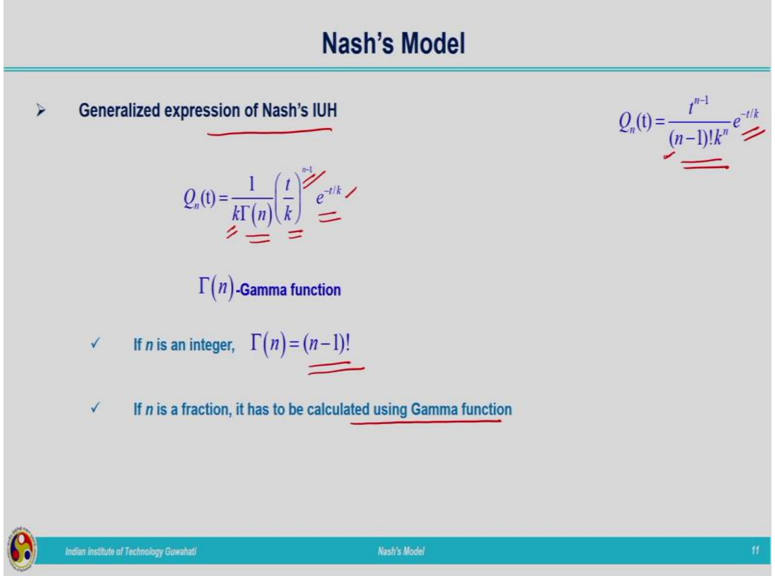

So, now we are going to write the generalized expression for Nash IUH. So, this is the expression which we have derived in the previous slide, i.e.,

$$Q_n(t)=\frac{t^{n-1}}{(n-1)!k^n}e^{-t/k}$$

So, that we are going to make in a generalized form. So, $Q_n(t)$ can be written as

$$Q_n(t)=\frac{1}{k\Gamma(n)}\left(\frac{t}{k}\right)^{n-1}e^{-t/k}$$

So, what we have done *k* is written in terms of *k* multiplied by *k* to the power of *n-1* and *t* and *k* combined together that way we have written $\left(\frac{t}{k}\right)^{n-1}$. Then comes the term that remaining *k* we have separately kept and *(n-1)!* factorial is written as *Γn*. For writing the expression in a generalized form we have made certain modification with the terms and we got the expression like this. Here *Γn* is the gamma function, if *n* is an integer *Γn* can be written as *(n-1)!* that is the one which we have derived and it can happen in such a way that *n* can be a fraction also. So, it has to be calculated using gamma function. Gamma function provides one table corresponding to different values of *n* you can get the value of the gamma function. So, here in the case of gamma *n*, if *n* is an integer, we can calculate it directly without making use of the gamma function table that is by using *(n-1)!* but sometimes it can happen in such a way that *n* can be a fraction. In such cases we have to make use of the gamma function to get the value corresponding to that. Here what we have assumed the catchment is considered as consisting of *n* number of linear reservoirs connected in series. So, this *n* becoming a fraction in the actual condition it is not possible, if we are explaining theoretically we can consider *n* as a fraction. So that way if we are considering *n* as a fraction then we have to go for making use of gamma function and from that we have to find out the value corresponding to Γn. Gamma function in every textbook the table is given, corresponding to that we just have to take the value, it is not a difficult task. If it is not a fraction that is *n* is an integer we can calculate gamma n by making use of the *(n-1)!*. So, that way we can compute the ordinates of the IUH for different t values.

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The references are given. For having more understanding about the topic, please refer these textbooks. Here, I am winding up this lecture. Thank you.